ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 4, pp. 396–411.
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Russian Text © The Author(s), 2024, published in Avtomatika i Telemekhanika, 2024, No. 4, pp. 61–80.

NONLINEAR SYSTEMS

# An Extension of the Feedback Linearization Method in the Control Problem of an Inverted Pendulum on a Wheel

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Received November 27, 2023
Revised February 11, 2024
Accepted March 4, 2024

**Abstract**—This paper continues previous studies on designing stabilizing control laws for a mechanical system consisting of a wheel and a pendulum suspended on its axis. The control objective is to simultaneously stabilize the vertical position of the pendulum and a given position of the wheel. The difficulty of this problem is that the same control is used to achieve two targets, i.e., stabilize the pendulum angle and the wheel rotation angle. Previously, the output feedback linearization method was applied to this problem. The sum of the pendulum angle and the wheel rotation angle was taken as the output. For the closed loop system to be not only asymptotically stable in the output but also to have asymptotically stable zero dynamics, a dissipative term was added to the output-stabilizing control law. Below, a two-parameter modification of this law is described. Along with the dissipative term, we introduce a positive factor. The more general parameterization allows stabilizing this system in the cases where the control law proposed previously appeared ineffective. The properties of the new control law are investigated, and the attraction domain is estimated. The estimation procedure is reduced to checking the feasibility of linear matrix inequalities.

*Keywords*: asymptotic stabilization, inverted pendulum, estimation of the attraction domain, linear matrix inequalities

**DOI:** 10.31857/S0005117924040043

## 1. INTRODUCTION

The mechanical system considered in this paper consists of a wheel and a pendulum suspended on its axis. The wheel rolls on a flat surface whose intersection with the vertical plane forms the $\xi$-axis (Fig. 1).

This system, as well as its sister system known as cart and pole apparatus (a cart with an inverted pendulum mounted on it), was studied in many publications on control theory; for example, see [1–11]. A list of related research works and an analysis of the state-of-art in this area can be found in [1]. Note that this system attracts interest as a nonlinear, unstable, and nonminimum-phase system when examining different control design methods.

Many researchers investigated control design to stabilize the vertical position of a pendulum in the linearized system; for example, see [2, 3, 8]. A nonlinear controller is easily designed using the output feedback linearization method with the pendulum angle as the output [4]. However, this solution does not settle the complete state stabilization problem since the zero dynamics remain unstable and the position of the wheel center is not stabilized. An approach to constructing the

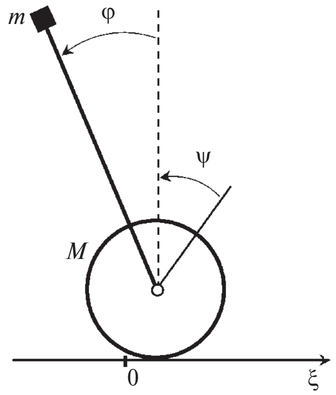


**Fig. 1.** The diagram of a pendulum on a wheel.

so-called virtual outputs was developed in [12]: stabilization of the virtual outputs ensures state stabilization. This approach is difficult to apply in a general form.

Some approaches to solving the problem in a nonlinear statement were briefly reviewed in [1]. Teel used the theory of small gains [9]. A solution based on the time-optimal control design was described in [10]. For the global stabilization problem, Srinivasan et al. proposed a combined control law under which, for large initial deviations, the pendulum control ensures reaching the local stabilizability domain [11]. Estimation of such a domain is of great importance. Obviously, an estimate that depends on the constructed control law and the chosen Lyapunov function can be conservative. It is topical to obtain a maximum-size estimate in the class defined by the Lyapunov function parameters.

This problem is addressed below, in continuation of the previous studies [1]. As was demonstrated therein, with the sum of the pendulum angle and the reduced wheel rotation angle taken as the output, the control stabilizing this output will make the closed loop system stable, albeit not asymptotically. The zero state (the trivial equilibrium) is stabilized by adding to the control law a term proportional to the difference in the angular velocities of the wheel and the pendulum; this term is interpreted as the viscous friction torque in the wheel axle. Also, the attraction domain of the equilibrium state was estimated using a specially constructed Lyapunov function consisting of a quadratic part and a nonlinear term [1]. The parameters of this Lyapunov function were calculated by solving a sequence of linear matrix inequalities (LMIs).

In this paper, we propose a two-parameter modification of the control law obtained by the output feedback linearization method. Along with the dissipative term, interpreted as the viscous friction torque at the suspension point, we introduce a positive factor at the control law. The more general parameterization allows stabilizing the system in the cases where the control law proposed previously appeared ineffective. An example is given to illustrate this fact.

## 2. MODEL OF THE SYSTEM

We use the mathematical model described in [2] with the notations introduced in [1], including the change of the time variable. The positive value of the angles is counted counterclockwise.

The following notations are used:

$\xi$ is the position of the wheel center on the horizontal axis in Fig. 1;

$\varphi$ is the angular deviation of the pendulum from the vertical axis (the pendulum angle);

$l$ is the pendulum length;

$\psi$ is the angle between the vertical and some distinguished wheel radius, with the zero value of $\psi$ corresponding to the zero value of $\xi$; $\psi = -\frac{\xi}{r}$;

$m$ is the mass lumped at the end of the pendulum;

$M$, $J$, and $r$ are the mass, moment of inertia, and radius of the wheel, respectively;

$\theta = \psi \frac{r}{l}$ is the reduced wheel rotation angle;

$U$ is the torque developed by the drive and applied between the pendulum and the wheel;

$u = \frac{U}{mgl}$, where $g$ is the acceleration of gravity;

$t$ is the time variable, and $\tau = t\sqrt{g/l}$ is the new dimensionless independent variable;

$'$ is the derivative with respect to the variable $\tau$.

Also, we denote the angular velocities by $\omega = \varphi'$, $\delta = \theta'$, and $x = (\varphi, \omega, \theta, \delta)^{\mathsf{T}}$. Applying the Lagrangian formalism and the independent variable $\tau$ yields the motion equations

$$\begin{aligned} \varphi' &= \omega, \\ \omega' &= f_1(x) + h_1(x)u, \\ \theta' &= \delta, \\ \delta' &= f_2(x) + h_2(x)u, \end{aligned} \tag{1}$$

where

$$\begin{aligned} f_1(x) &= \frac{\sin\varphi}{d}[-\omega^2\cos\varphi + (1+\beta)], \\ f_2(x) &= \frac{\sin\varphi}{d}(\omega^2 - \cos\varphi), \\ h_1(x) &= \frac{1}{d}(\cos\varphi + 1 + \beta), \\ h_2(x) &= \frac{1}{d}(-\cos\varphi - 1), \\ \beta &= \frac{M + J/r^2}{m}, \\ d &= \beta + \sin^2\varphi. \end{aligned} \tag{2}$$

(For details, see [1].)

Denoting $f = (\omega, f_1, \delta, f_2)^{\mathsf{T}}$ and $h = (0, h_1, 0, h_2)^{\mathsf{T}}$, we write system (1) as

$$x' = f + hu. \tag{3}$$

(For the sake of simplicity, the dependence on $x$ is omitted here.)

## 3. CONTROL DESIGN TO STABILIZE THE TRIVIAL EQUILIBRIUM OF SYSTEM (1)

We choose the system output

$$y = \varphi + \theta \tag{4}$$

and design a control law for system (1) that ensures the asymptotic stability with respect to this output. The corresponding control design method was described, e.g., in [4, Chapter 12]. The control law is given by

$$\begin{aligned} u^*(x) &= -\frac{\lambda^2 y + 2\lambda y' + F}{H} \\ &= -\frac{d\lambda^2(\varphi+\theta) + 2d\lambda(\omega+\delta) + \sin\varphi\,[(1-\cos\varphi)(\omega^2+1) + \beta\,]}{\beta}, \end{aligned} \tag{5}$$

$$F = \frac{\sin\varphi}{d}\left[(1-\cos\varphi)(\omega^2+1) + \beta\,\right], \quad H = \frac{\beta}{d}. \tag{6}$$

In (5), the parameter $\lambda$ is the desired exponential decay rate for the closed-loop system output. This output will satisfy a linear second-order differential equation with the characteristic polynomial root $-\lambda$ of multiplicity 2.

System (1) closed by the control law (5) takes the form

$$x' = f(x) + h(x)u^*(x), \tag{7}$$

where

$$f(0) = 0,\ \ u^*(0) = 0,\ \ \left.\frac{\partial h(x)}{\partial x}\right|_{x=0} = 0.$$

Applying the linearization procedure to system (7) in a neighborhood of zero yields

$$x' = \Phi x,\ \ \Phi = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0} + h(0)\left.\frac{\partial u^*(x)}{\partial x}^{\mathsf{T}}\right|_{x=0}. \tag{8}$$

We have

$$\Phi = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{(\beta+2)\lambda^2+1}{\beta} & -2\lambda\left(\dfrac{\beta+2}{\beta}\right) & -\lambda^2\left(\dfrac{\beta+2}{\beta}\right) & -2\lambda\left(\dfrac{\beta+2}{\beta}\right) \\ 0 & 0 & 0 & 1 \\ \dfrac{2\lambda^2+1}{\beta} & 4\dfrac{\lambda}{\beta} & 2\dfrac{\lambda^2}{\beta} & 4\dfrac{\lambda}{\beta} \end{bmatrix}. \tag{9}$$

Let us make the linear change of variables $\zeta = Sx$, where

$$S = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \end{bmatrix}. \tag{10}$$

In other words,

$$\zeta_1 = \varphi,\ \ \zeta_2 = \omega,\ \ \zeta_3 = y,\ \ \zeta_4 = y'.$$

In the new variables, matrix (9) takes the form

$$\Phi_\zeta = S\Phi S^{-1} = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -\dfrac{1}{\beta} & 0 & -\lambda^2\left(\dfrac{\beta+2}{\beta}\right) & -2\lambda\left(\dfrac{\beta+2}{\beta}\right) \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\lambda^2 & -2\lambda \end{bmatrix}. \tag{11}$$

The characteristic polynomials of the matrices $\Phi$ and $\Phi_\zeta$ coincide. Due to the block-triangular form of matrix (11), its spectrum consists of those of the two diagonal blocks of dimensions $2 \times 2$. Thus, the eigenvalues of matrix (11) are

$$\left\{-\frac{i}{\sqrt{\beta}}, \frac{i}{\sqrt{\beta}}, -\lambda, -\lambda\right\}, \tag{12}$$

where $i$ indicates the imaginary unit. Like $\Phi$, the matrix $\Phi_\zeta$ has a pair of pure imaginary roots and a multiple negative root. Under this distribution of the roots, the closed loop system is asymptotically stable with respect to the output and has zero dynamics that are not asymptotically stable.

Let us design a control law ensuring the asymptotic stability of system (1). In the previous studies, the dissipative term

$$u^{**}(x) = u^{*}(x) - k(\omega - \delta), \;\; k > 0, \tag{13}$$

was added to the control law (5), interpreted as viscous friction at the junction point of the pendulum and the wheel [1].

This paper considers a more general parametric expansion of the control law (5). In addition to the gain $k$, which can be of arbitrary sign, we introduce a factor $s$. The new control law is given by

$$u^{***}(x) = su^{*}(x) - k(\omega - \delta). \tag{14}$$

In the new variables, the matrix of system (1) closed by the control law (14) and linearized in a neighborhood of zero has the form

$$\Phi_s = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \dfrac{(\beta+1)(1-s)}{\beta} - \dfrac{s}{\beta} & -2k\dfrac{\beta+2}{\beta} & -\dfrac{(\beta+2)\lambda^2}{\beta}s & (k-2\lambda s)\dfrac{(\beta+2)}{\beta} \\ 0 & 0 & 0 & 1 \\ 1-s & -2k & -\lambda^2 s & k - 2\lambda s \end{bmatrix}. \tag{15}$$

The characteristic polynomial of this matrix is

$$\begin{gathered} N(\mu, s) = \det(\mu I - \Phi_s) \\ = \mu^4 + \left(2\lambda s + k\frac{4+\beta}{\beta}\right)\mu^3 + \left(\lambda^2 s + \frac{\beta+2}{\beta}(s-1) + \frac{1}{\beta}\right)\mu^2 + \frac{2\lambda s - k}{\beta}\mu + \frac{\lambda^2}{\beta}s. \end{gathered} \tag{16}$$

Under which values of the parameters $s$ and $k$ does the polynomial (16) have roots with negative real parts? To answer this question, we apply the Liénard–Chipart criterion; for example, see [14, Section 3.5]. We compile the Hurwitz matrix

$$\begin{pmatrix} 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} & 0 & 0 \\ 1 & \lambda^2 s + \dfrac{\beta+2}{\beta}(s-1) + \dfrac{1}{\beta} & \dfrac{\lambda^2}{\beta}s & 0 \\ 0 & 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} & 0 \\ 0 & 1 & \lambda^2 s + \dfrac{\beta+2}{\beta}(s-1) + \dfrac{1}{\beta} & \dfrac{\lambda^2}{\beta}s \end{pmatrix}. \tag{17}$$

The polynomial (16) is Hurwitz if and only if its coefficients are positive and the third principal minor of matrix (17) is positive. Due to the positivity of $\lambda$ and $\beta$, the coefficients of this polynomial are positive under the conditions

$$-2\lambda s\frac{\beta}{\beta+4} < k < 2\lambda s \tag{18}$$

and

$$s > \frac{\beta+1}{\lambda^2\beta + \beta + 2} \doteq \bar{s}. \tag{19}$$

For the third principal minor of matrix (17), the positivity condition takes the form

$$\det\begin{pmatrix} 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} & 0 \\ 1 & \lambda^2 s + \dfrac{\beta+2}{\beta}(s-1) + \dfrac{1}{\beta} & \dfrac{\lambda^2}{\beta}s \\ 0 & 2\lambda s + k\dfrac{4+\beta}{\beta} & \dfrac{2\lambda s - k}{\beta} \end{pmatrix} > 0. \tag{20}$$

After necessary transformations, we obtain

$$\frac{\beta+2}{\beta^3}\Big\{k^2\Big[\beta+2-(\beta+4)(1+2\lambda^2)s\Big] - 4\lambda sk + 4\lambda s^2\Big[\lambda\beta(s-1)+k(2-\lambda^2\beta)\Big]\Big\} > 0.$$

(Hereinafter, the computer algebra system Maxima is used [13].) Reducing by the positive factor $(\beta+2)/\beta^3$, we finally write this condition as

$$k^2\Big[\beta+2-(\beta+4)(1+2\lambda^2)s\Big] + 4k\lambda s\Big[s(2-\lambda^2\beta)-1\Big] + 4\lambda^2 s^2\beta(s-1) > 0. \tag{21}$$

The closed loop linearized system can be asymptotically stable even without the dissipative term in the control law (14) (i.e., for $k = 0$.) In particular, the following result is valid.

**Lemma 1.** *For $k = 0$, the polynomial* (16) *is Hurwitz for any*

$$s > 1. \tag{22}$$

The proof of Lemma 1 is provided in the Appendix. Thus, asymptotic stability with respect to the output $y$ turns into asymptotic stability with respect to the state under the control law (5) with a gain exceeding 1.

The terms in (21) are grouped as a quadratic polynomial in $k$. We denote its coefficients by

$$c_0 = \beta+2-(\beta+4)(1+2\lambda^2)s, \quad c_1 = 4\lambda s\Big[s(2-\lambda^2\beta)-1\Big], \\ c_2 = 4\lambda^2 s^2\beta(s-1). \tag{23}$$

For the sake of brevity, the dependence of the coefficients on $s$ is omitted. We denote by $s_0$ the value for which $c_0$ vanishes:

$$s_0 = \frac{\beta+2}{(\beta+4)(1+2\lambda^2)}. \tag{24}$$

Note that

$$c_0 < 0 \quad \text{for} \quad s > s_0. \tag{25}$$

Then we have the following result.

**Lemma 2.** *For any $\lambda$ and $\beta > 0$,*

$$s_0 < \bar{s} < 1. \tag{26}$$

*In other words, condition* (19) *implies the inequality $c_0 < 0$.*

The proof is postponed to the Appendix.

Now let us investigate the asymptotic stability domain of system (15) in the parameter space $(s, k)$. We denote this domain by $\Omega$. Note that the boundary point (1, 0) does not belong to the domain $\Omega$. By Lemma 1, the segment $s > 1$, $k = 0$ belongs to $\Omega$. How far can this domain be extended by choosing $k \neq 0$ under different $s$ satisfying condition (19)? The answer to this question is given below.

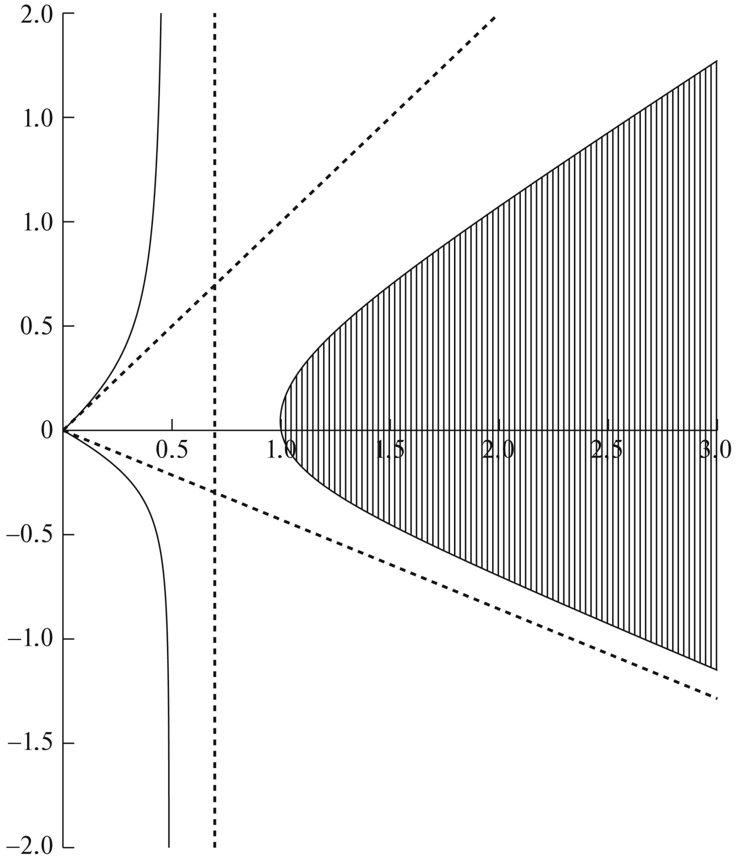


**Fig. 2.** The domain $\Omega$ corresponding to $\beta = 3$ and $\lambda = 0.5$.

**Theorem 1.** *A point* $(s, k)$ *belongs to the domain* $\Omega$ *if and only if* (25) *holds jointly with conditions* (18), (19),

$$c_1^2 - 4c_0c_2 > 0, \tag{27}$$

*and*

$$\frac{-c_1 + \sqrt{c_1^2 - 4c_0c_2}}{2c_0} < k < \frac{-c_1 - \sqrt{c_1^2 - 4c_0c_2}}{2c_0}. \tag{28}$$

*For* $s = 1$, *a point* $(1, k)$ *belongs to the domain* $\Omega$ *if and only if*

$$0 < k < \frac{2\lambda(1 - \lambda^2\beta)}{1 + \lambda^2(\beta + 4)}. \tag{29}$$

This theorem is proved in the Appendix.

Inequality (29) coincides with the previous estimate of the gain $k$ ensuring the asymptotic stability of the linearized system closed by the control law (13); see [1].

Figure 2 shows an example of the domain $\Omega$. Here, the slanted dashed lines represent the boundary of the domain (18) whereas the vertical dashed line the boundary of the domain (19). The solid thin line indicates the boundary of the definitional domain of inequality (21). According to the figure, this domain has three parts with curvilinear boundaries. Only one of them satisfies the conditions of Theorem 1, thus being the domain $\Omega$ (see shading with vertical lines).

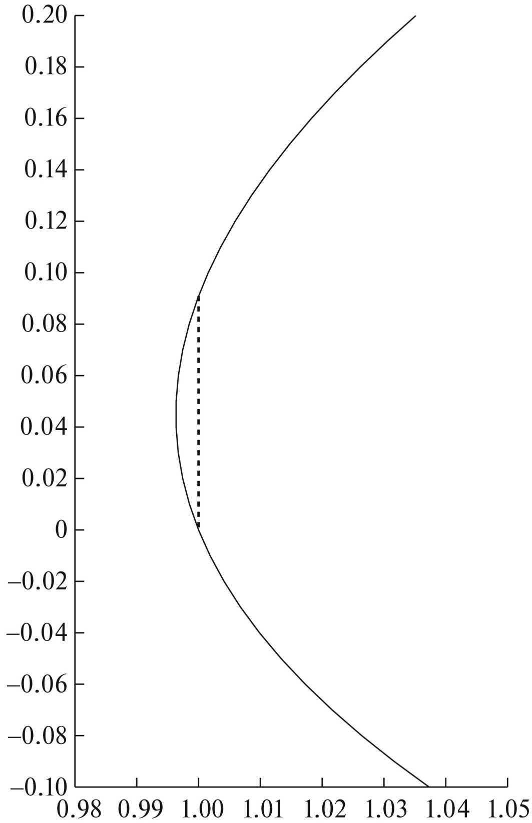


**Fig. 3.** Comparing the domain $\Omega$ and its part shown by dashed line. This part corresponds to $s = 1$ and is defined by inequality (29).

For comparison, Fig. 3 presents an enlarged fragment of the domain $\Omega$ and the stability domain defined by inequality (29) for $s = 1$ and the same values of the parameters $\beta$ and $\lambda$. According to the figure, introducing the parameter $s$ into the control law (14) allows extending the stability domain significantly. Moreover, this control law can ensure asymptotic stability for those values of the parameters $\beta$ and $\lambda$ under which stabilization by the control law (13) is impossible.

Now we consider the behavior of system (1) closed by the control law (14) in the entire space instead of a neighborhood of zero. After passing to the variables $\varphi$, $\omega$, $y$, and $y'$, the closed loop system takes the form

$$\begin{aligned}
\varphi' &= \omega, \\
\omega' &= -s\frac{\beta+\cos\varphi+1}{\beta}\left(\lambda^2 y + 2\lambda z\right) - s\frac{1+\omega^2}{\beta}\sin\varphi \\
&\quad - k\frac{\beta+\cos\varphi+1}{\sin^2\varphi+\beta}(2\omega - z) - (s-1)\frac{(\beta+1-\omega^2\cos\varphi)}{\sin^2\varphi+\beta}\sin\varphi, \\
y' &= z, \\
z' &= -s\left(\lambda^2 y + 2\lambda z\right) - k\frac{\beta}{\sin^2\varphi+\beta}(2\omega - z) \\
&\quad - (s-1)\frac{(\omega^2+1)(1-\cos\varphi)+\beta}{\sin^2\varphi+\beta}\sin\varphi.
\end{aligned} \tag{30}$$

System (30) can be written as

$$\zeta' = \Psi(\gamma)\zeta,$$
$$\Psi(\gamma) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ \frac{(\gamma_4-1)}{\beta}s - \gamma_5(s-1) & -2\gamma_3 k & -\gamma_2\lambda^2 s & \gamma_3 k - 2\gamma_2\lambda s \\ 0 & 0 & 0 & 1 \\ -\gamma_6(s-1) & -2\gamma_1 k & -\lambda^2 s & \gamma_1 k - 2\lambda s \end{bmatrix}, \tag{31}$$

where, due to $d = \beta + \sin^2\varphi$,

$$\begin{aligned} \gamma_1 &= \frac{\beta}{d}, \\ \gamma_2 &= \frac{1+\beta+\cos\varphi}{\beta}, \\ \gamma_3 &= \frac{1+\beta+\cos\varphi}{d}, \\ \gamma_4 &= 1 - \frac{\sin\varphi}{\varphi}(1+\omega^2), \\ \gamma_5 &= \frac{1+\beta-\omega^2\cos\varphi}{d}\left(\frac{\sin\varphi}{\varphi}\right), \\ \gamma_6 &= \frac{(1+\omega^2)(1-\cos\varphi)+\beta}{d}\left(\frac{\sin\varphi}{\varphi}\right). \end{aligned} \tag{32}$$

Regarding the variation of the angle $\varphi$, we suppose the following.

*Assumption 1.* On the trajectories of the controlled system (1),

$$|\varphi| \leqslant \varphi_0 < \frac{\pi}{2}, \quad |\omega| \leqslant \omega_0, \tag{33}$$

where $\varphi_0$ and $\omega_0$ are some positive constants.

System (31), equivalent to (30), is nonlinear. Along with this system, we consider the linear time-varying system

$$\zeta' = \Psi(\gamma(\tau))\zeta, \tag{34}$$

where $\gamma_l(\tau)$, $l = 1, \cdots, 6$, represent arbitrary measurable functions of the time variable $\tau$ that are subjected only to the two-sided constraints following from the expressions (32) and Assumption 1 :

$$\begin{aligned} \gamma_1(\tau) &\in \left[\frac{\beta}{d_0}, 1\right], \\ \gamma_2(\tau) &\in \left[\frac{1+\beta+\cos\varphi_0}{\beta}, \frac{2+\beta}{\beta}\right], \\ \gamma_3(\tau) &\in \left[\frac{1+\beta+\cos\varphi_0}{d_0}, \frac{2+\beta}{\beta}\right], \\ \gamma_4(\tau) &\in \left[-\omega_0^2, 1-r_0\right], \\ \gamma_5(\tau) &\in \left[\min\left\{\frac{1+\beta-\omega_0^2}{\beta}, \frac{1+\beta-\omega_0^2\cos\varphi_0}{d_0}r_0\right\}, \frac{\beta+1}{\beta}\right], \\ \gamma_6(\tau) &\in \left[\frac{1-\cos\varphi_0+\beta}{d_0}r_0, \max\left\{1, \frac{(1-\cos\varphi_0)(1+\omega_0^2)+\beta}{d_0}r_0\right\}\right], \end{aligned} \tag{35}$$

with $d_0 = \beta + \sin^2\varphi_0$ and $r_0 = \frac{\sin\varphi_0}{\varphi_0}$. For all possible values of the functions $\gamma_l(\tau)$, the solution set of system (34) is wider than that of the nonlinear system (30). Therefore, the absolute stability requirement for the zero (trivial) solution of system (34) in the class of functions $\gamma_l(\tau)$ subjected to the constraints (35) will also ensure the stability of the trivial solution of system (30). Immersion in a wider class of systems (in the sense of the solution set) yields sufficient conditions for the stability of the trivial equilibrium of system (30). To derive such conditions, we choose a Lyapunov function having a negative derivative for all systems (34), (35) simultaneously.

As a candidate Lyapunov function we choose

$$V(\zeta) = \frac{1}{2}\zeta^{\mathsf{T}}P\zeta + \alpha\left[1 - \cos\varphi + \frac{\beta}{2}\ln\left(1+\omega^2\right) - \frac{\varphi^2}{2} - \frac{\omega^2\beta}{2}\right], \tag{36}$$

which is parameterized by a positive definite matrix $P \succ 0$ and a nonnegative number $\alpha \geqslant 0$. (The signs $\succ$, $\prec$, $\succeq$, and $\preceq$ mean positive and negative definiteness and positive and negative semidefiniteness, respectively.) In the expression (36), the Taylor expansion of the nonlinear term with the factor $\alpha$ starts from the third-order terms and corrects the quadratic form for better consideration of the nonlinear properties of system (34). The expression $1 - \cos\varphi + \frac{\beta}{2}\ln\left(1+\omega^2\right)$ is the first integral of the limit system introduced in [1]. The requirement that the derivative of the function (36) along the trajectories of system (34) be negative definite for all possible values of the functions $\gamma_l(\tau)$, $l = 1, \cdots, 6$, from the intervals (35) will be written as a system of LMIs.

The derivative of the function (36) along the trajectories of system (34) has the form

$$\begin{aligned}
V' &= \zeta^{\mathsf{T}}P\Psi(\gamma)\zeta + \alpha\Big[\sin\zeta_1\zeta_2 - \zeta_1\zeta_2 \\
&\quad + \frac{\beta\zeta_2}{1+\zeta_2^2}\left(\gamma_5(1-s)\zeta_1 - (1+\zeta_2^2)\frac{s}{\beta}\sin\zeta_1 - 2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right) \\
&\quad - \beta\zeta_2\left(\gamma_5(1-s)\zeta_1 - (1+\zeta_2^2)\frac{s}{\beta}\sin\zeta_1 - 2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right)\Big] \\
&= \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta \\
&\quad + \alpha\left[\zeta_2\left(-2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right)\left(\frac{1}{1+\zeta_2^2} - 1\right)\beta - \zeta_1\zeta_2\gamma_4\right. \\
&\qquad\qquad \left. + (s-1)\zeta_1\zeta_2\left(1 - \beta\gamma_5\left(\frac{1}{1+\zeta_2^2} - 1\right) - \gamma_4 - \frac{\sin\zeta_1}{\zeta_1}\right)\right] \\
&= \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta \\
&\quad + \alpha\left[\zeta_2\left(-2k\gamma_3\zeta_2 - \lambda^2\gamma_2 s\zeta_3 + (k\gamma_3 - 2\lambda\gamma_2 s)\zeta_4\right)\left(\frac{1}{1+\zeta_2^2} - 1\right)\beta - \zeta_1\zeta_2\gamma_4\right. \\
&\qquad\qquad \left. + (s-1)\zeta_1\zeta_2\left(\zeta_2^2\frac{\sin\zeta_1}{\zeta_1} - \beta\gamma_5\left(\frac{1}{1+\zeta_2^2} - 1\right)\right)\right] \\
&= \zeta^{\mathsf{T}}\Psi(\gamma)P\zeta \\
&\quad + \alpha\left[\zeta_2\left(-2k\gamma_9\zeta_2 - \lambda^2\gamma_8 s\zeta_3 + (k\gamma_9 - 2\lambda\gamma_8 s)\zeta_4\right)\beta - \gamma_4\zeta_1\zeta_2 + \gamma_7(s-1)\zeta_1\zeta_2\right],
\end{aligned} \tag{37}$$

where

$$\gamma_0 = -\frac{\zeta_2^2}{1+\zeta_2^2},$$
$$\gamma_7 = \frac{\omega^2 \sin\varphi}{\varphi}\left(1 + \frac{\beta}{(1+\omega^2)}\frac{(\beta + 1 - \omega^2\cos\varphi)}{d}\right),$$
$$\gamma_8 = \gamma_0\gamma_2,$$
$$\gamma_9 = \gamma_0\gamma_3.$$

For the values defined above, we have the two-sided estimates

$$\begin{aligned}
\gamma_0 &\in \left[-\frac{\omega_0^2}{1+\omega_0^2}, 0\right], \\
\gamma_7 &\in \left[0, \max\left\{\omega_0^2 r_0\left(1 + \frac{\beta\left(\beta+1-\omega_0^2\cos\varphi_0\right)}{(1+\omega_0^2)(\beta+\sin^2\varphi_0)}\right), \frac{\omega_0^2(2+\beta)}{(1+\omega_0^2)}\right\}\right], \\
\gamma_8 &\in \left[-\frac{\omega_0^2(2+\beta)}{\beta(1+\omega_0^2)}, 0\right], \\
\gamma_9 &\in \left[-\frac{\omega_0^2(2+\beta)}{\beta(1+\omega_0^2)}, 0\right].
\end{aligned} \tag{38}$$

Due to the expressions (35), the parameters $\gamma_2$ and $\gamma_3$ achieve their maximum values simultaneously. Therefore, the parameters $\gamma_8$ and $\gamma_9$ achieve the minimum value $-\frac{\omega_0^2(2+\beta)}{\beta(1+\omega_0^2)}$ and the maximum value 0 simultaneously. In this case, it suffices to keep the single parameter $\gamma_8$ in the expression (37) and write

$$V' = \zeta^\mathsf{T}\Psi(\gamma)P\zeta + \alpha\left[\beta\gamma_8\zeta_2\left(-2k\zeta_2 - \lambda^2 s\zeta_3 + (k - 2\lambda s)\zeta_4\right) + \left(\gamma_7(s-1) - \gamma_4\right)\zeta_1\zeta_2\right]. \tag{39}$$

In the next section, we represent the condition $V' < 0$ in terms of LMIs.

## 4. ESTIMATING THE ATTRACTION DOMAIN OF THE TRIVIAL EQUILIBRIUM

We write the Lyapunov function (36) as

$$V(\zeta) = \frac{1}{2}\zeta^\mathsf{T} Q(\alpha)\zeta + \alpha\left[1 - \cos\varphi + \frac{\beta}{2}\ln\left(1+\omega^2\right)\right] \geqslant \frac{1}{2}\zeta^\mathsf{T} Q(\alpha)\zeta, \tag{40}$$

where $Q(\alpha)$ denotes the matrix

$$Q(\alpha) = P - \alpha\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \beta & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix}. \tag{41}$$

By imposing the inequality

$$Q(\alpha) \succeq \varepsilon I,$$

where $\varepsilon > 0$ is sufficiently small, from (40) we conclude that the function $V(\zeta)$ is positive definite. Further, the expression (39) for $V'$ affinely depends on the arbitrarily varying parameters $\gamma_l$,

$l = 1, \cdots, 8$, each ranging in a closed interval. Hence, the vector $\gamma$ ranges in the Cartesian product of eight closed intervals. This set, denoted by $\Gamma \subset R^8$, is convex and has 256 extreme points obtained by equating the arbitrarily varying parameters $\gamma_l$, $l = 1, \cdots, 8$, to their minimum and maximum values on the closed intervals (35) and (38). The quadratic form (39), whose matrix affinely depends on the parameters $\gamma$, is negative definite for all $\gamma \in \Gamma$ if and only if it is negative definite at the extreme points of this set, i.e., at the vectors $\gamma_i$. Therefore, the condition $V' < 0$ is equivalent to a system of 256 LMIs, each corresponding to one of the vectors $\gamma^i$, $i = 1, \ldots, 256$. For all possible values $\gamma^i$, we obtain the system of LMIs (one LMI of high dimensions)

$$P\Psi(\gamma^i) + \Psi^T(\gamma^i)P - \alpha Y(\gamma^i) \preceq 0, \tag{42}$$

where

$$Y(\gamma^i) = \begin{bmatrix} 0 & \gamma_4^i - \gamma_7^i(s-1) & 0 & 0 \\ \gamma_4^i - \gamma_7^i(s-1) & 4\beta\gamma_8^i k & \beta\gamma_8^i\lambda^2 s & \beta\gamma_8^i(2\lambda s - k) \\ 0 & \beta\gamma_8^i\lambda^2 s & 0 & 0 \\ 0 & \beta\gamma_8^i(2\lambda s - k) & 0 & 0 \end{bmatrix}.$$

The resulting system of LMIs may be infeasible for given values $\varphi_0$ and $\omega_0$. We introduce a parameter $a \in [0, 1]$ and choose

$$\varphi_0(a) = a\frac{\pi}{2}, \;\; \omega_0(a) = a\bar{\omega}$$

for $\varphi_0$ and $\omega_0$, where $\bar{\omega} = \sqrt{e^{\frac{4}{\beta}} - 1}$. Each value of $\varphi_0(a)$ and $\omega_0(a)$ is associated with the corresponding limits of the intervals (35) and (38) and, consequently, with the 256 vectors $\gamma^i(a)$. By the expressions for these limits, the vectors $\gamma^i(a)$ continuously depend on $a$. For $a = 0$, the lower and upper limits of the intervals coincide. Therefore, for $a = 0$ we have

$$\gamma^i(0) \doteq \gamma^0 = \left[1, \frac{2+\beta}{\beta}, \frac{2+\beta}{\beta}, 0, \frac{1+\beta}{\beta}, 1, 0, 0\right]^T.$$

Further, $\Psi(\gamma^0) = \Phi_s$ due to the matrix formulas (15) and (31). According to Theorem 1, the system of LMIs (42) is feasible for sufficiently small $a > 0$.

Let $a^*$ be the supremum of those $a$ for which the system of LMIs

$$\begin{aligned} P\Psi(\gamma^i(a)) + \Psi^T(\gamma^i(a))P - \alpha Y(\gamma^i(a)) &\preceq 0, \\ Q(\alpha) &\succeq \varepsilon I, \\ \mathrm{tr(Q(\alpha))} &= 1 \end{aligned} \tag{43}$$

is feasible with respect to the variables $P$ and $\alpha$. The linear equation $\mathrm{tr(Q(\alpha))} = 1$ (the unit trace of the matrix $Q(\alpha)$) has been added to normalize the solution: otherwise, the feasibility set of the LMIs would be a cone and, together with any solution $P$ and $\alpha$, $\sigma P$ and $\sigma\alpha$ would be another solution for any $\sigma > 0$, including both arbitrarily large and arbitrarily small values.

The value $a^*$ is obtained by successively checking the feasibility of (43) for an increasing numerical sequence $\{a\}$.

Thus, under Assumption 1, where $\varphi_0 = a^*\frac{\pi}{2}$ and $\omega_0 = a^*\bar{\omega}$, the Lyapunov function (36) has a negative definite derivative along the trajectories of system (31). If there exists a constant $c > 0$ such that the set

$$\Omega_c = \{\zeta : V(\zeta) \leqslant c\} \tag{44}$$

is inscribed in the set

$$\Pi_0 = \{\zeta : |\varphi| \leqslant \varphi_0,\ |\omega| \leqslant \omega_0\}, \tag{45}$$

then any trajectory of the closed loop system (31) evolving from the interior of the set $\Omega_c$ remains inside it at any time due to the negativity of the derivative $V'$. As a result, if

$$\Omega_c \subset \Pi_0, \tag{46}$$

Assumption 1 will hold along the entire trajectory of the closed loop system (31) evolving from the interior of the set $\Omega_c$. In other words, we have established the following result.

**Theorem 2.** *Let the conditions of Theorem 1 be satisfied and the value $a^*$ be chosen as the supremum of those $a$ for which the LMI* (43) *is feasible. If the constant $c$ is such that condition* (46) *holds, then the set $\Omega_c$ is the asymptotic stability domain of system* (1) *closed by the control law* (14).

A method for finding a constant $c$ that ensures condition (46) was described in [1].

## 5. AN EXAMPLE OF CONSTRUCTING THE ATTRACTION DOMAIN

Consider an example corresponding to the value $\beta = 3$. Let the parameters of the control law (14) be $\lambda = 0.578$, $k = 0$, and $s = 1.5$ to ensure the conditions of Theorem 1. This example is interesting because for the chosen values of the parameters $\beta$ and $\lambda$, condition (29) fails, i.e., the mechanical system under consideration cannot be stabilized by the control law (13) under any value of the parameter $k$. However, this system can be stabilized by the control law (14) with $s > 1$ and $k = 0$. Checking the feasibility of the LMI (43) for the maximum possible value $a^*$ yields the following parameters of the Lyapunov function:

$$P = \begin{bmatrix} 0.068 & -0.040 & 0.037 & 0.167 \\ -0.040 & 0.128 & -0.070 & -0.247 \\ 0.037 & -0.070 & 0.070 & 0.175 \\ 0.167 & -0.247 & 0.175 & 0.734 \end{bmatrix}, \quad \alpha = 0.000017,$$

achieved at $a^* = 0.349$.

Using Theorem 2, the invariant attraction domain of the closed loop system is constructed in the same way as described in [1]. Omitting the details, we present the final value: $c = 0.0036466$. The feasibility of LMIs was checked in Scilab [15].

Figure 4 shows the trajectories of the closed loop system in the coordinates $\theta$ (the abscissa axis) and $\delta$ (the ordinate axis). The combined control law was applied. If the state of the system does not fall into the attraction domain $\Omega_c$, then the control law described in [1, Section 5] is applied. The condition $V(\zeta^*) = c$ is the criterion for crossing the boundary of the domain $\Omega_c$. Switching to the control law (14) occurs upon reaching the domain $\Omega_c$. In the figures, the angular variables are measured in degrees whereas the angular velocity in degrees per unit of the dimensionless time $\tau$.

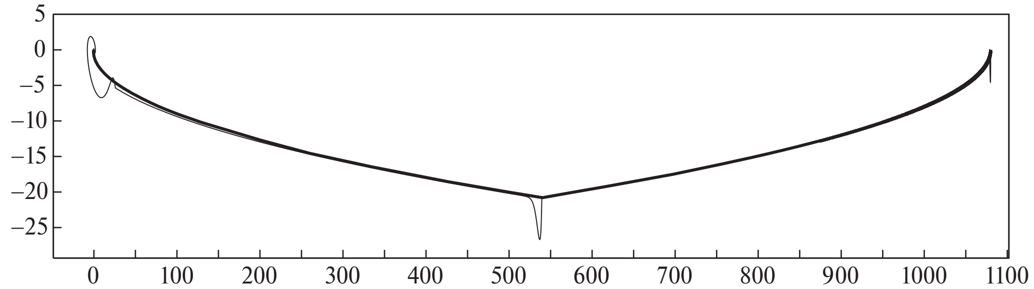


**Fig. 4.** Trajectories of the closed loop system in the coordinates $\theta$ (the abscissa axis) and $\delta$ (the ordinate axis).

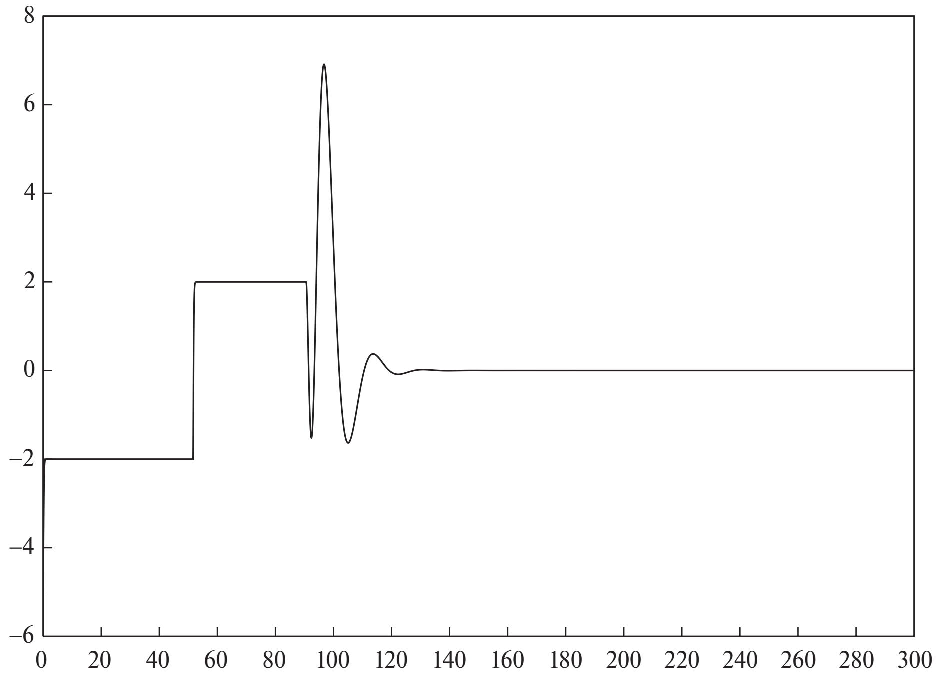


**Fig. 5.** Graph of the variable $\varphi$.

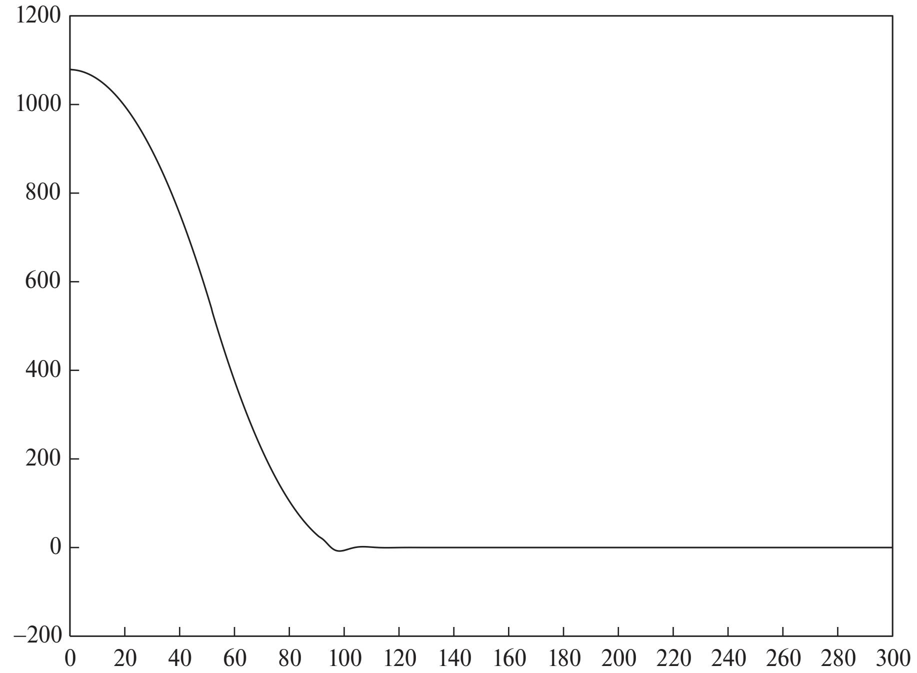


**Fig. 6.** Graph of the variable $\theta$.

The thin line shows the trajectory of system (1) closed by the combined control law for $\bar{\varphi} = 2°$ (see formula (5.2) in [1]) and the initial conditions $\varphi(0) = -5°$, $\omega(0) = 0$, $\theta(0) = 1080°$, and $\delta(0) = 0$. The thick line shows the optimal trajectory of system (5.5) under the control law (5.6) of [1].

The graphs of the angles $\varphi$ and $\theta$ are demonstrated in Figs. 5 and 6, respectively, where the abscissa axis is the dimensionless time $\tau$. According to Fig. 5, at the initial stage with the control law (5.2) of [1], the variable $\varphi$ stabilizes first at $-2°$ and then at $2°$; subsequently, at an approximate time of $\tau \sim 90$, switching to the control law (14) occurs, and the trivial equilibrium is asymptotically stabilized.

## 6. CONCLUSIONS

This paper has considered the problem of stabilizing the vertical position of an inverted pendulum on a wheel. A two-parameter extension has been proposed for the control law simultaneously stabilizing the pendulum angle (its deviation from the vertical line) and the wheel rotation angle. The stabilization problem has been solved by the output feedback linearization method, with the sum of the pendulum angle and the wheel rotation angle taken as the output. In contrast to the previous studies [1], an additional factor has been introduced along with a dissipative term. A numerical example has been provided to show that the new control law allows stabilizing the system in the case where it cannot be stabilized by the previous control law. The attraction domain of the trivial equilibrium has been estimated. This estimate has been constructed by solving a system of linear matrix inequalities.

## *APPENDIX*

**Proof of Lemma 1.** For $k = 0$, condition (18) takes the form $s > 0$ whereas condition (21) the form

$$4\lambda^2 s^2 \beta(s-1) > 0.$$

Due to the first inequality and the positivity of $\beta$, it follows that $s > 1$. The proof of Lemma 1 is complete.

**Proof of Lemma 2.** By definition (19) of the value $\bar{s}$ and $\beta > 0$, we directly establish that $\bar{s} < 1$. Straightforward algebraic transformations in Maxima [13] yield

$$\bar{s} - s_0 = \frac{\lambda^2\beta^2 + 8\lambda^2\beta + 8\lambda^2 + \beta}{(\beta+4)(1+2\lambda^2)(\lambda^2\beta+\beta+2)},$$

and the desired conclusion is obvious.

**Proof of Theorem 1.** Note that the lines $k = 2\lambda s$ and $k = -2\lambda s\frac{\beta}{\beta+4}$, which determine the boundary of the definitional domain of inequality (18), have no intersection with the boundary of the (possibly non-simply-connected) domain given by (20). Indeed, substituting $k = 2\lambda s$ into the left-hand side of inequality (20) leads to the contradiction

$$-8\lambda^4 s^3(\beta+2) > 0.$$

By analogy, we arrive at a contradiction when substituting the equality $k = -2\lambda s\frac{\beta}{\beta+4}$ into (20):

$$-\frac{8\beta\lambda^2 s^2(\beta+2)}{(\beta+4)^2} > 0.$$

Thus, the boundary of the domain $\Omega$ is formed by the points $(s, k)$ satisfying condition (19) and turning the left-hand side of inequality (20) to zero. According to Lemma 2, for these values of $s$ the coefficient $c_0$ in (23) takes negative values, i.e., the dependence of the left-hand side of (21) on the variable $k$ is an inverted parabola. Then the closed interval of values $k$ for which inequality (21) holds is of the form (28) provided the positive determinant of the quadratic inequality (21).

For $s = 1$, we have $c_2 = 0$, and inequality (21) takes the form

$$-k^2[2 + 2\lambda^2(\beta+4)] + k(4\lambda - 4\lambda^3\beta) > 0,$$

which implies inequality (29). The proof of Theorem 1 is complete.

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*This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board*