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Game-Theoretic Centrality of Directed Graph Vertices

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Abstract—The paper considers a game theory approach to calculating the centrality value of the vertices in a directed graph, based on the number of vertex occurrences in fixed length paths. It is proposed to define vertex centrality as a solution of a cooperative game, where the characteristic function is given as the number of simple paths of fixed length in subgraphs corresponding to coalitions. The concept of integral centrality is introduced as the value of a definite integral of the payoff function. It is shown that this centrality measure satisfies the Boldi-Vigna axioms.

Keywords: graph theory, centrality, directed graph, cooperative game

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1. INTRODUCTION

When solving a large number of applied problems, it is necessary to analyse the network describing the system under consideration. In this case, the network is represented as a graph. There is a wide range of methods to study the structural properties of a graph; one of the key methods is to calculate the centrality value of the graph vertices. Centrality allows to identify the most significant vertices, to estimate how well they are located in the graph, how they influence the processes occurring in the network.

There are several approaches to calculating the centrality of vertices in a graph. The simplest in terms of calculation is degree centrality [1], which generally shows how many neighbours a vertex has. When analyzing directed graphs, the numbers of incoming and outgoing connections (in-degree centrality and out-degree centrality) [2] can be taken into account. Historically, one of the first approaches is betweenness centrality [3, 4]. This measure of centrality is based on calculating the number of shortest paths connecting all pairs of vertices in the graph. The centrality of a vertex in this case is determined by the number of paths passing through the considered vertex. Another common method is closeness centrality [5–7], where the most central vertex is closest to other vertices in the network.

In recent years, game-theoretic centrality measures [8–14] that consider groups of vertices as coalitions of players have become increasingly widespread. Such an approach allows us to take into account the group influence of vertices on the system.

This paper presents a game-theoretic approach to computing the centrality value of vertices of a directed graph based on the number of occurrences of a vertex in directed paths of different lengths, including cycles.

2. MYERSON VALUE AS A CENTRALITY MEASURE IN DIRECTED GRAPHS

Let us define a cooperative game $\Gamma = \langle N, v\rangle$ on the directed graph $G = (N, E)$, where N is the set of vertices and E is the set of edges. In this game, N is the set of players on which there is a characteristic function $v(K)$, $K \subset N$, equal to the number of simple paths of fixed length $k = 1, 2, \ldots$ in the subgraph generated by the set of players K. As shown in [11], a cooperative game solution in Shapley–Myerson form [12] can be used to rank vertices in an undirected graph. A similar approach can be applied to directed graphs.

Theorem 1. *The Myerson value for a player $i \in N$ in a cooperative game on a directed graph G, where the characteristic function $v(K)$ is defined as the number of directed simple paths of fixed length k in the subgraph generated by the set $K \subset N$, can be found by the formula*

$$X_i = \frac{n_k(i)}{k+1}, \tag{1}$$

where $n_k(i)$ is the number of simple paths of length k passing through vertex i.

The proof of this theorem for directed graphs is similar to the proof for undirected graphs. We will give it here. To prove the statement it is enough to prove the fulfilment of two axioms [12].

Proof. *Axiom 1.* If S is a connected component of the graph G and $v(S)$ is the payoff of the S coalition, then

$$\sum_{i=1}^{|S|} X_i = v(S).$$

For simplicity, assume that the graph G is connected. In this particular case $v(N)$ is the number of paths (directed) of length k in graph G. Let us renumber all paths $l \in \{1, 2, \ldots, v(N)\}$. We define $\delta_l(i)$ as follows. Let's assume $\delta_l(i) = 1$ if vertex i is in path l and 0, otherwise.

Then

$$\sum_{i=1}^{n} X_i = \frac{1}{k+1}\sum_{i=1}^{n} n_k(i) = \frac{1}{k+1}\sum_{i=1}^{n}\sum_{l=1}^{v(N)} \delta_l(i) = \frac{1}{k+1}\sum_{l=1}^{v(N)}\sum_{i=1}^{n} \delta_l(i).$$

Each path consists of $k+1$ different vertices $(i_1, \ldots, i_{k+1})$. Hence $\sum\limits_{i=1}^{n} \delta_l(i) = k+1$. Therefore,

$$\sum_{i=1}^{n} X_i = \frac{1}{k+1}\sum_{l=1}^{v(N)}\sum_{i=1}^{n} \delta_l(i) = v(N).$$

Thus, Axiom 1 is true. Let us proceed to Axiom 2.

Axiom 2. For any pair of players $i, j \in N$ the payoff changes by the same amount when adding or removing an edge in the graph G.

Let, for example, $ij \in E(G)$. Let us remove this edge (in this case it's a directed edge). Then all paths of length k that previously passed through edge ij will be subtracted when counting paths simultaneously from $n_k(i)$ and $n_k(j)$ in the new graph $G - ij$. Hence,

$$X_i(G) - X_i(G - ij) = X_j(G) - X_j(G - ij).$$

Thus Axiom 2 is also true, which proves the theorem.

3. EXAMPLES OF CALCULATING THE MYERSON VALUE FOR DIRECTED GRAPHS

As can be seen from Theorem 1, the Myerson value is defined in terms of the number of simple paths of a given length. The problem of computing the number of simple paths through a vertex is non-trivial. Here we show how to compute this number for paths of lengths 2 and 3. We restrict ourselves to directed graphs without bidirectional edges. For the calculations we need the adjacency matrix and its degrees 2 and 3.

Statement 1. *Let A be the adjacency matrix of the directed graph G, and A^2 be its square. Then the number of appearances of vertex i in simple paths of length 2 $n_2(i)$ can be calculated by the formula*

$$n_2(i) = \sum_{k=1}^{n} \left(a_{ik}^{(2)} + a_{ki}^{(2)} \right) + \sum_{k=1}^{n} \sum_{j=1}^{n} a_{ki} a_{ij}.$$

The first expression corresponds to the number of all simple paths of length 2 starting or ending at the considered vertex i. The second expression considers paths in which vertex i lies in the middle of the path.

Statement 2. *Let A be the adjacency matrix of the directed graph G, and A^2, A^3 be the square and cube of the matrix A. Then the number of occurrences of vertex i in simple paths of length 3 $n_3(i)$ can be calculated by the formula*

$$n_3(i) = \sum_{\substack{k=1 \\ k \neq i}}^{n} \left(a_{ik}^{(3)} + a_{ki}^{(3)} \right) + \sum_{k=1}^{n} a_{ki} \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{ij}^{(2)} + \sum_{k=1}^{n} a_{ki}^{(2)} \sum_{\substack{j=1 \\ j \neq k}}^{n} a_{ij}.$$

Here the first term corresponds to the number of all simple paths of length 3 starting or ending at vertex i. The second term takes into account paths in which vertex i lies in the second position in the path, and the third takes into account paths in which vertex i lies in the third position in the path.

Example 1. Let us illustrate this formula on the example of a directed graph G_1 of 6 vertices (Fig. 1) with the adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Let us write out, for example, paths of length $d = 3$ passing through vertex 3. There are 8 such paths in total:

3654 3612

1365 5361

6132 6134 6532 6534

The first line lists the paths starting at vertex 3. The second line lists paths with vertex 3 in the second place, and the third line lists vertex 3 in the third place. There are no other paths with vertex 3.

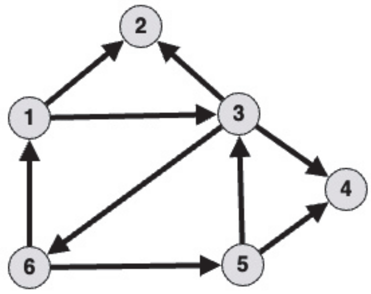

Fig. 1. Graph G_1.

Now let's calculate the square and cube of the adjacency matrix.

$$A^2 = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 2 & 1 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 2 & 0 & 2 & 0 & 2 \end{pmatrix}.$$

Using the formula from statement 2 we find

$$n_3(3) = \sum_{\substack{k=1 \\ k \neq 3}}^{6} \left(a_{3k}^{(3)} + a_{k3}^{(3)}\right) + \sum_{k=1}^{6} a_{k3} \sum_{\substack{j=1 \\ j \neq k}}^{6} a_{3j}^{(2)} + \sum_{k=1}^{6} a_{k3}^{(2)} \sum_{\substack{j=1 \\ j \neq k}}^{6} a_{3j}$$

$$= \left(a_{32}^{(3)} + a_{34}^{(3)}\right) + \left(a_{13}a_{35}^{(2)} + a_{53}a_{31}^{(2)}\right) + \left(a_{63}^{(2)}a_{32} + a_{63}^{(2)}a_{34}\right) = 2 + 2 + 4 = 8.$$

This coincides with the number of simple paths of length 3 passing through vertex 3.

4. INTEGRAL CENTRALITY

Above we considered on a directed graph $G = (N, E)$, $|N| = n$, a cooperative game of n persons $\Gamma = \langle N, v(K)\rangle$, where the characteristic function $v(K)$ was defined as the number of directed paths of fixed length d passing through the considered vertex in the subgraph generated by the coalition K. By varying the path length d, the characteristic function can be defined more generally as a polynomial:

$$v(K) = \sum_{i \in K} \sum_{d=1}^{n-1} n_d(i) r^d, \ \ r \in [0, 1],$$

where $n_d(i)$ is the number of simple paths of length d passing through vertex i.

The value of r can be determined by analogy with Jackson's approach [15], where players receive r for creating a direct link, a coalition receives r^2 for creating a path of length 2, and so on. Here, the players forming the link get r for making a *pair* in paths of length 1, r^2 for making a *triple* in paths of length 2, etc.

Similarly to Section 2, it can be shown (see also [13]) that the distribution of coalition payoffs among players according to the Myerson value is of the form:

$$X_i(r) = \frac{n_1(i)}{2} r + \frac{n_2(i)}{3} r^2 + \cdots + \frac{n_{n-1}(i)}{n} r^{n-1} = \sum_{d=1}^{n-1} \frac{n_d(i)}{d+1} r^d.$$

By choosing a particular value of r, the value of the payoff function $X_i(r)$ for all players can be obtained. To eliminate the step of choosing the value of r, the values of the definite integral of the function defining the payoff function over the variable r on the segment $[0, 1]$ can be used in the ranking. The payoff functions are polynomial functions, which makes it easy to write expressions for determining centrality:

$$I_i = \int_0^1 X_i(r) dr = \int_0^1 \sum_{d=1}^{n-1} \frac{n_d(i)}{d+1} r^d \, dr = \sum_{d=1}^{n-1} \frac{n_d(i)}{(d+1)^2}. \tag{2}$$

Definition 1. The value $I_i = \int_0^1 X_i(r) dr = \sum_{d=1}^{n-1} \frac{n_d(i)}{(d+1)^2}$, where $X_i(r)$ is the function specifying the payoff in the cooperative game Γ on the graph, is called the integral centrality of vertex i.

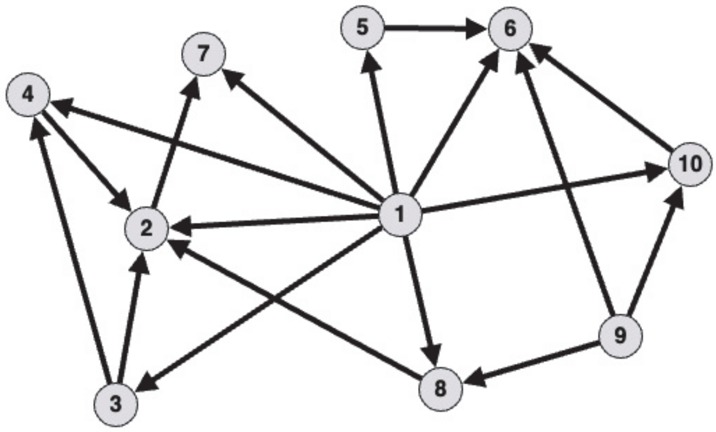

Fig. 2. Math-Net network graph fragment.

Example 2. As an example, consider a fragment of the Math-Net citation graph (Fig. 2). In this case, the presence of an oriented edge ij means that author i refers to the work of author j.

Let us find the integral centrality of the vertices of this graph. We denote by n_k the vector of the occurrences number of the graph vertices in paths of length k.

$$n_1 = (8, 5, 3, 3, 2, 4, 2, 3, 3, 3)\,,$$
$$n_2 = (7, 9, 4, 4, 1, 3, 4, 3, 2, 2)\,,$$
$$n_3 = (4, 6, 3, 3, 0, 0, 5, 2, 1, 0)\,,$$
$$n_4 = (1, 1, 1, 1, 0, 0, 1, 0, 0, 0)\,,$$

all other $n_k, k \geqslant 5$ are zero.

Then the values of integral centrality according to the definition are:

$$I = \frac{n_1}{4} + \frac{n_2}{9} + \frac{n_3}{16} + \frac{n_4}{25} = (3.067, 2.665, 1.421, 1.421, 0.611, 1.333, 1.296, 1.208, 1.034, 0.972).$$

As a result, vertex 1 has the highest centrality, vertex 2 is also important.

Let us calculate the integral centrality of vertices for the cycle and the complete graph. Obviously, due to symmetry, the centrality of all vertices in this case is the same and it is enough to calculate the centrality of one of them.

For all vertices of the $p-$cycle the payoff is defined by the function:

$$X(r) = \sum_{d=1}^{p-1} \frac{n_d}{d+1} r^d = \sum_{d=1}^{p-1} \frac{d+1}{d+1} r^d = \sum_{d=1}^{p-1} r^d.$$

Then the value of integral centrality is written as follows:

$$I_p = \int\limits_0^1 X(r)dr = \int\limits_0^1 \sum_{d=1}^{p-1} r^d dr = \sum_{d=1}^{p-1} \frac{1}{d+1}.$$

For the vertices of $k-$cliques the formula is valid

$$n_d(k) = (d+1)! \binom{k-1}{d}.$$

Indeed, in a path of length d: $l = (i_1, \ldots, i_{d+1})$ a vertex i can be at the first, second,..., $d+1$th place. The remaining d of $k-1$ vertices can be chosen in

$$A_{k-1}^d = \binom{k-1}{d} \times d!$$

ways. Consequently,

$$X(r) = \sum_{d=1}^{k-1} \frac{n_d(k)}{d+1} r^d = \sum_{d=1}^{k-1} d! \binom{k-1}{d} r^d,$$

and then the integral centrality is

$$I_k = \int_0^1 X(r)dr = \int_0^1 \sum_{d=1}^{k-1} d! \binom{k-1}{d} r^d dr = \frac{1}{k} \sum_{d=1}^{k-1} d! \binom{k}{d+1} = \sum_{d=1}^{k-1} \frac{(k-1)!}{(d+1)(k-d-1)!}.$$

5. CENTRALITY AXIOMS

The paper Boldi–Vigna [16] describes a system of centrality axioms based on checking the change of the centrality measure when studying cliques and directed cycles. The following should be checked: whether all vertices of an $k-$clique have the same centrality measure; whether all vertices of a directed $p-$cycle have the same centrality measure; whether vertices of an $k-$clique are more important than vertices of a directed $p-$cycle. Let us give the formulations of the axioms as they are presented in [16, 17].

A1 *(Size Axiom)*. Consider the graph $S_{k,p}$ (Fig. 3) consisting of two components: a $k-$clique and a $p-$cycle. A centrality measure satisfies the size axiom if for each k there exists a number P_k, such that for all $p \geqslant P_k$ in the graph $S_{k,p}$, the centrality of a vertex in a $p-$cycle is strictly greater than the centrality of a vertex in a $k-$clique, and for each p there exists a number K_p, such that for all $k \geqslant K_p$, the centrality of a vertex in a $k-$clique is strictly greater than the centrality of a vertex in a $p-$cycle.

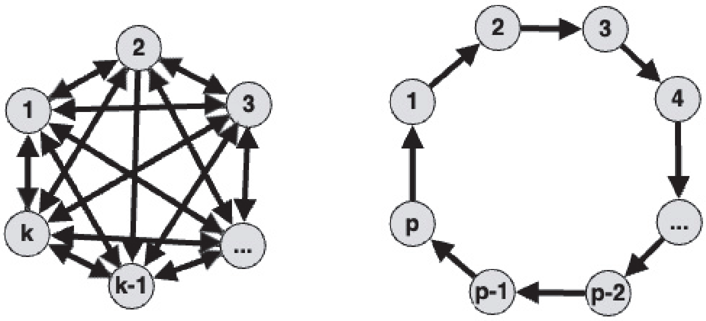

Fig. 3. Graph $S_{k,p}$.

A2 *(Density Axiom)*. Consider the graph $D_{k,p}$ (Fig. 4) consisting of a $k-$clique and a $p-$cycle $(k, p \geqslant 3)$ connected by a bidirectional bridge $x \longleftrightarrow y$, where x is a vertex of the clique and y is a vertex of the cycle. A centrality measure satisfies the density axiom if for $k = p$ the centrality of x is strictly greater than the centrality of y.

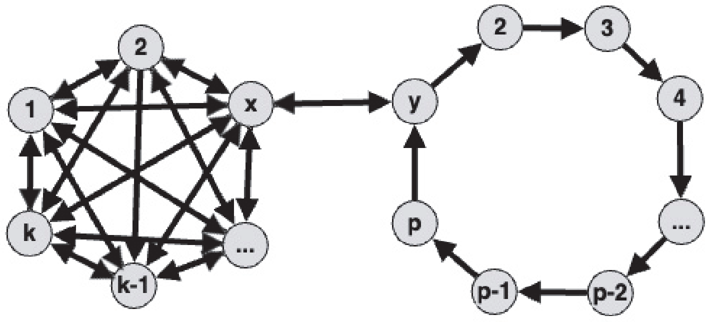

Fig. 4. Graph $D_{k,p}$.

A3 *(Score-Monotonicity Axiom).* A centrality measure satisfies the score-monotonicity axiom if for every graph G and every pair of nodes x and y such that $x \to y$ does not belong to the set of edges E_G of graph G, if we add to G such an edge, then the centrality of y will increase.

Let us add to this system the axiom of connectivity.

A4 *(Connectivity Axiom).* A centrality measure satisfies the connectivity axiom if for any graph G and any two connectivity components G_1 and G_2 of graph G, and for every pair of vertices $x \in G_1$ and $y \in G_2$, the centrality of all vertices in G_2 does not decrease if an edge $x \to y$ is added.

Let us prove that these axioms hold for the integral centrality measure.

A1 To prove this axiom it suffices to show that the centrality of the vertices of both $p-$cycle and $k-$clique increases unboundedly with increasing p and k. In the previous section it was shown that for all vertices of a $p-$cycle the integral centrality is defined by a function:

$$I_p^c = \sum_{d=1}^{p-1} \frac{1}{d+1}.$$

The centrality value of the vertices of the $p-$cycle can be estimated as $I_p = S_p - 1$, where S_p is the sum of the first p terms of the harmonic series. L.Euler obtained an asymptotic expression for the sum of the first n terms of the harmonic series:

$$S_p = \ln p + \gamma + \varepsilon_p,$$

where $\gamma = 0.5772\ldots$ – Euler–Mascheroni constant [18], ε_p – error, $\varepsilon_p \to 0$, $p \to \infty$. Then $I_p^c \to \infty$, $p \to \infty$.

For $k-$clique vertices we have the formula

$$I_k^q = \sum_{d=1}^{k-1} \frac{(k-1)!}{(d+1)(k-d-1)!}.$$

It is easy to see that $I_k^q \geqslant I_k^c$, whence follows the unbounded growth of I_k^q, when $k \to \infty$.

A2. It was noted above that for $k = p$ $I_k^q(x) \geqslant I_k^c(y)$ i.e., centrality of vertex x in the clique is greater than the centrality of vertex y in the cycle. Thus, for any d, the number of directed paths of length d in the clique $n_d^q(x)$ is greater than the number of directed paths of length d in the cycle $n_d^c(y)$. Let us connect vertex x of the clique to vertex y in the cycle by a bidirectional bridge $x \longleftrightarrow y$. Then the number of paths of any length d in both the clique and the cycle increases by the same value. Therefore, the integral centrality of vertex x will still be greater than the centrality of vertex y. The validity of axiom $A2$ is proved.

A3. Axiom $A3$ is obviously true, because after adding the edge $x \to y$, the vertex y will occur in paths ending in it, which will increase the centrality value.

A4. Axiom A4 is true because after adding an edge $x \to y$, the number of directed paths passing through vertices from the graph G_2 can only increase.

Theorem 2. *For integral centrality* (2) *the axioms A1, A2, A3, A4 are fulfilled.*

It is worth noting that according to [16] the axioms A1, A2, A3 are valid without any reservations only for harmonic centrality [19].

6. COMPUTATION OF VERTEX CENTRALITY IN DIRECTED GRAPHS WITH CYCLES

In the proposed approach for determining the centrality of vertices in a graph, the main problem is to compute the number of simple paths (without cycles) of fixed length passing through a given vertex. In [11], a modification of Myerson's value is given for the case when cycles are also considered

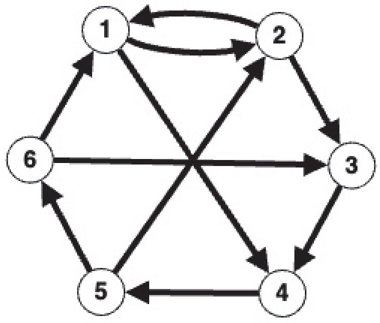

Fig. 5. Graph G_2.

in addition to simple paths. In this case it is possible to obtain quite simple expressions for this characteristic using the elements of the adjacency matrix of the considered graph. For ranking in this case the values $\frac{s_i(k)}{k+1}$ is used, where $s_i(k)$ is the number of appearances of vertex i in paths of length k. Also in [11] proofs of theorems on the number of appearances of vertices in paths of fixed length, including cycles, in undirected graphs are given. Further investigation showed that the similar formula is also valid for the number of occurrences of a vertex in paths in directed graphs.

Theorem 3. *Let A^d be the adjacency matrix of a directed graph G raised to the power d. Then the number of appearances of vertex i in paths of fixed length d (including cycles) $n_d(i)$ can be calculated as*

$$n_d(i) = \sum_{k=1}^{n} \left(a_{ik}^{(d)} + a_{ki}^{(d)} \right) + \sum_{l=1}^{d-1} \left[\sum_{k=1}^{n} a_{ki}^{(l)} \times \sum_{j=1}^{n} a_{ij}^{(d-l)} \right]. \tag{3}$$

Proof. The first sum takes into account the occurrences of vertex i at the beginning and end of paths of length d. The values $a_{ik}^{(d)}$ and $a_{ki}^{(d)}$ – the elements of the A^d matrix – correspond to the number of paths of length d starting in vertex i and ending in it. The second sum allows to take into account the occurrences of the considered vertex in the middle of paths of length d: $a_{ki}^{(l)}$ is an element of the matrix A^l, equal to the number of paths of length l ending in vertex i; $a_{ij}^{(d-l)}$ is an element of the matrix A^{d-l}, describing the number of paths of length $d-l$ starting in the same vertex. By adding their products for all admissible l, we obtain the number of occurrences of vertex i in the middle of paths of fixed length d.

Example 3. Let us illustrate the above formula on the example of a directed graph G_2 of 6 vertices (Fig. 5) with the adjacency matrix A:

$$A = \begin{pmatrix} 0 & 1 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

Let us write out the paths of length $d = 3$. There are 25 such paths in total:

1212	2121	3456	4521	5212	6121
1234	2123	3452	4561	5612	6123
1452	2345		4523	5214	6345
1456	2145		4563	5614	6145
1214				5234	
				5634	

Vertex 2 occurs in paths of length 3 21 times. Let us calculate the number of occurrences of vertex 2 by formula 3:

$$n_3(2) = \sum_{k=1}^{6} \left(a_{2k}^{(3)} + a_{k2}^{(3)}\right) + \sum_{l=1}^{2} \left[\sum_{k=1}^{6} a_{k2}{}^{(l)} \times \sum_{j=1}^{6} a_{2j}^{(3-l)}\right].$$

The computations require the square and cube of the adjacency matrix.

$$A^2 = \begin{pmatrix} 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 1 & 0 & 2 & 0 & 0 \end{pmatrix}, \quad A^3 = \begin{pmatrix} 0 & 2 & 0 & 2 & 0 & 1 \\ 1 & 0 & 1 & 0 & 2 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 \\ 2 & 0 & 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 4 & 0 & 0 \\ 1 & 0 & 1 & 0 & 2 & 0 \end{pmatrix}.$$

$$n_3(2) = (2+2+1+1+2+1) + \left[\sum_{k=1}^{6} a_{k2} \times \sum_{j=1}^{6} a_{2j}^{(2)} + \sum_{k=1}^{6} a_{k2}^{(2)} \times \sum_{j=1}^{6} a_{2j}\right]$$
$$= 9 + [2 \times 3 + 3 \times 2] = 9 + 6 + 6 = 21.$$

For directed acyclic graphs, i.e., directed graphs without directed cycles but allowing parallel paths, the number of occurrences of a vertex in paths of fixed length coincides with the number of simple paths passing through this vertex. Therefore, the value obtained with the help of formula (3) can be used to find the Myerson value in a directed graph by formula (1).

In the case of an arbitrary directed graph, we will define the vertex centrality in the following form

$$X_i(r) = \frac{n_1(i)}{2} r + \frac{n_2(i)}{3} r^2 + \cdots + \frac{n_{n-1}(i)}{n} r^{n-1} = \sum_{d=1}^{n-1} \frac{n_d(i)}{d+1} r^d.$$

The payoff X obviously satisfies the first Myerson axiom [12] (by the way of constructing the payoff and specifying the characteristic function), but it does not satisfy the second axiom (fairness axiom), which states that both players i and j must equally gain or lose benefits when creating or removing the link ij. This condition is not fulfilled due to the inclusion of cycles in the consideration. In general, it is possible that paths of the form $\ldots ijijijiji \ldots$ may appear, which leads to a different number of appearances of vertices i and j depending on the parity of the path length.

Let us illustrate it by means of a counterexample. For this purpose, let us return to the graph G_2 (Example 3). Let's remove the connection 1–4 from this graph (Fig. 6).

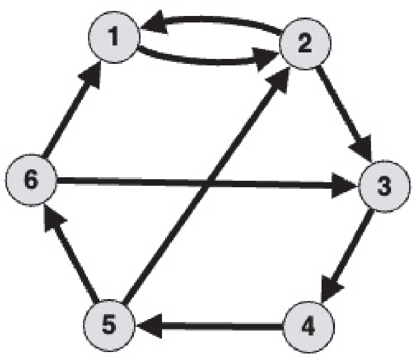

Fig. 6. Graph $G_2 - 14$.

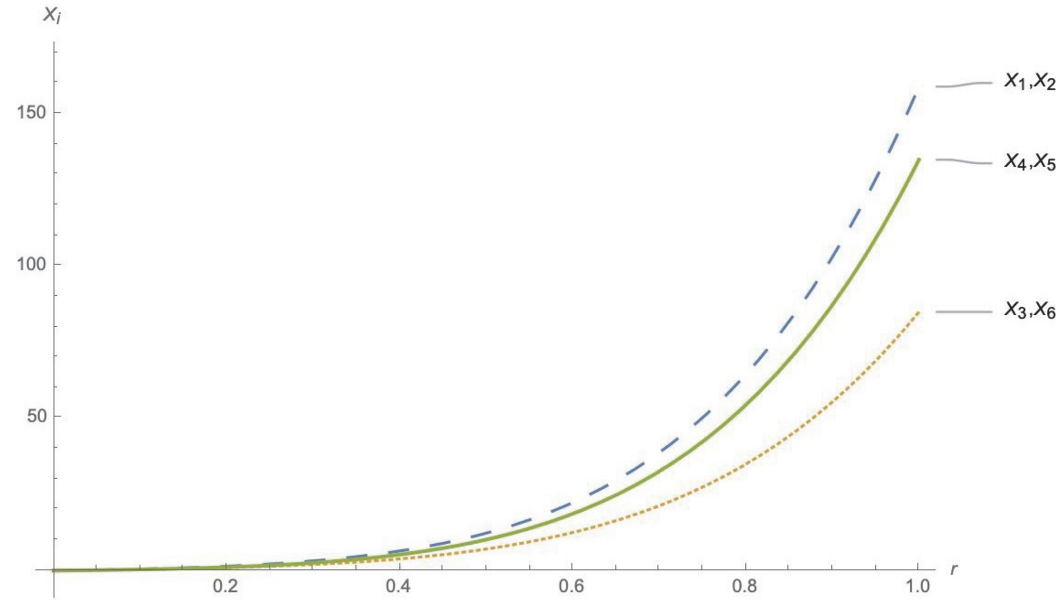

Fig. 7. Plots of the payoff functions.

Let's write down the payoffs of players 1 and 4 before removing the link $(X_1,\ X_4)$ and afterwards $(X_1',\ X_4')$.

$$X_1 = \frac{4}{2}r + \frac{10}{3}r^2 + \frac{21}{4}r^3 + \frac{42}{5}r^4 + \frac{82}{6}r^5$$
$$X_4 = \frac{3}{2}r + \frac{8}{3}r^2 + \frac{18}{4}r^3 + \frac{36}{5}r^4 + \frac{70}{6}r^5$$
$$X_1' = \frac{3}{2}r + \frac{7}{3}r^2 + \frac{13}{4}r^3 + \frac{22}{5}r^4 + \frac{37}{6}r^5$$
$$X_4' = \frac{2}{2}r + \frac{5}{3}r^2 + \frac{11}{4}r^3 + \frac{19}{5}r^4 + \frac{34}{6}r^5$$

Starting from the summand corresponding to the number of occurrences of vertices in paths of length 3, the differences in the substractions $X_1 - X_1'$, $X_4 - X_4'$ appears, which violates the condition of Myerson's axiom.

$$X_1 - X_1' = \frac{1}{2}r + \frac{3}{3}r^2 + \frac{8}{4}r^3 + \frac{20}{5}r^4 + \frac{45}{6}r^5$$
$$X_4 - X_4' = \frac{1}{2}r + \frac{3}{3}r^2 + \frac{7}{4}r^3 + \frac{17}{5}r^4 + \frac{36}{6}r^5$$

By choosing a particular value of r, the value of the payoff function $X_i(r)$ for all players can be obtained. These values can then be used to rank the vertices of the graph, which allows us to introduce another approach to calculating centrality.

Figure 7 shows the plots of the payoff functions for the players in the graph G_2. Let us choose the value $r = \frac{1}{2}$, then the payoff:

$$X = (3.44, 3.44, 2.09, 2.79, 2.79, 2.09)\,.$$

It can be seen that with this approach, vertices 1 and 2 still have the highest centrality values, while vertices 3, 6 have the lowest centrality.

For Example 3, we can write the vector I of integral centrality of the vertices of the graph G_2:

$$I(G_2) = (7.38, 7.38, 4.17, 6.14, 6.14, 4.17)\,.$$

The order of the vertices in the ranking is preserved.

7. CENTRALITY OF GRAPH VERTICES BASED ON TOURNAMENT MATRIX

In [20], a two-stage procedure for ranking the vertices of a graph was proposed, where, at the first stage, the vertices are ranked based on the absolute potentials of the nodes of an electric circuit when current is supplied to all nodes in sequence. At the second stage, a tournament table is constructed and the final ranking is carried out based on the sum of previously found ranks, by analogy with the Borda rule [21].

In this case, the tournament table can be constructed for the values of $n_d(i)$ for different d. Let's compile a tournament table of the vertices of the graph G_2 (Table 1). Assessing centrality based on the total number of occurrences of a vertex in paths of various lengths allows us to conclude that vertices 1 and 2 are the most important. Vertices 3 and 6 have the lowest centrality.

Table 1. Tournament table of graph G_2

Vertex	d					$\sum$
	1	2	3	4	5	
1	4	10	21	42	82	159
2	4	10	21	42	82	159
3	3	6	11	22	43	85
4	3	8	18	36	70	135
5	3	8	18	36	70	135
6	3	6	11	22	43	85

Let us compare these results with the values of degree centrality. Table 2 shows the values of the incoming and outgoing links number in the graph. The highest centrality is also possessed by vertices 1 and 2; when analysing the least centrality vertices, different interpretations are possible depending on the applied problem to be solved. If the directionality of edges is not taken into account, the ranks for degree centrality will coincide with the ranks of vertices of order 2. If the direction is taken into account, vertices 5 and 6 get the same ranks in terms of degree centrality, however, in terms of the involvement of vertices in creating paths in the graph, vertex 5 is considered more important.

Table 2. Estimation of degree centrality for G_2

Vertex	in-degree	out-degree
1	2	2
2	2	2
3	2	1
4	2	1
5	1	2
6	1	2

8. TRANSPORT GRAPH OF PETROZAVODSK CITY

The paper [22] describes the construction of an undirected graph of the transport network of Petrozavodsk. This graph can be considered as directed graph if we take into account the direction of motor transport movement on the road sections corresponding to the edges of the graph. It consists of 1530 vertices and 3781 edges. The values in the graph adjacency matrix are equal to the inverse of the lengths of road segments between the corresponding pairs of vertices.

Let us calculate the number of simple paths of length 3 passing through the vertices of the graph, according to Statement 2. Figure 8 shows a heat map of n_3 values.

Fig. 8. Heat map of n_3 values for vertices in the transport network graph of Petrozavodsk.

Fig. 9. Heat map of integral centrality values of the Petrozavodsk transport network graph vertices.

Next we compute the integral centrality values of the vertices. Since calculating the number of occurrences of a vertex in the paths requires raising the adjacency matrix to power d, to simplify the computation in graphs with a large number of vertices, we can restrict ourselves to considering paths of length less than $n-1$ to estimate centrality. Figure 9 shows a heat map of the vertices of the transport graph for which the integral centrality values have been computed for path lengths up to $d = 100$.

The larger centrality value in the figure corresponds to nodes of larger size and darker colouring. The ranking results correlate with those obtained earlier in [22], where ranking was performed using the PageRank method [23] and the modified Myerson method [11].

9. CONCLUSION

In this paper we propose a number of approaches to computing centrality values of vertices of directed graphs based on the number of occurrences of a vertex in paths of fixed length. It is shown how the Myerson value in a cooperative game, where the characteristic function is defined in terms of the number of simple paths of fixed length in a subgraph generated by a coalition, can be computed for vertices of a directed graph. Formulas are given for finding the number of simple paths of length 2 and 3 through the adjacency matrix. Also, the centrality of a vertex can be found as a solution to a cooperative game where the characteristic function is given in a more general form for different path lengths. We propose to introduce the notion of integral centrality as the value of a certain integral of the payoff function. It is shown that this measure of centrality satisfies the Boldi–Vigna axioms.

In addition, the computation of centrality of vertices of a directed graph with cycles is described.

The proposed approach is tested on the graph of the transport network of Petrozavodsk.

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