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The Interaction of Economic Agents in Cournot Duopoly Models under Ecological Conditions: A Comparison of Organizational Modes

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Abstract—This paper presents a comparative analysis of the efficiency of organizational modes (information structures) for the interaction of economic agents in static and dynamic Cournot duopoly models. We compare the independent behavior of equal players, their cooperation, and the hierarchy formalized as Germeier games. The efficiency of individual players and the entire society is quantitatively assessed using the private and social relative efficiency indices. The ecological safety conditions of the system are investigated. An organizational and economic interpretation of the results is proposed.

Keywords: Cournot duopoly, Germeier games, relative efficiency indices, cooperative solution, ecological safety

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1. INTRODUCTION

The main organizational modes of economic interaction are the independent behavior, cooperation, and hierarchy of economic agents. The collective outcome of the rational behavior of independent players can be worse than the one obtained by centralized or voluntary cooperation. How much is it worse? To answer this question, researchers introduce a special payoff function measuring quantitatively the (in)efficiency of equilibria. Usually, such measures are defined as the ratio of the payoff function value in some equilibrium to the collectively optimal value. The (in)efficiency of equilibria was extensively in network games, scheduling games, resource allocation games, and other areas [1–5].

Note that these indices reflect the interests of society (economy). From this standpoint, cooperation is always beneficial, and the indices assess only losses from selfish behavior (although hierarchical control can be no less beneficial than cooperation). However, the payoff of an individual player (Leader) or an independent player may exceed his share in the total payoff distribution under cooperation. Hence, it is crucial to study the beneficialness conditions of cooperation from the perspective of social welfare and the interests of individual economic agents.

The viability theory was proposed by J.P. Aubin [6] and was developed in [7, 8]. The idea is that the state vector of a controlled dynamic system must belong to a given domain of the state space (reflecting, e.g., the requirements of ecological equilibrium). For static models, viability conditions are treated as additional constraints.

The Cournot duopoly model is convenient to illustrate conceptually the (in)efficiency of equilibria and analyze viability conditions. In this case, viability is understood as ecological safety. In the static model, the players are supposed to have complete information and, therefore, reach a Nash equilibrium in one step [9]. A dynamic Cournot oligopoly was examined in detail in [10]. The authors [11] considered the bounded rationality of players and derived local stability conditions for Nash equilibria by a Cournot groping procedure in the discrete-time dynamic model.

Cournot oligopoly models were studied in several papers [12–19]. In the first series, M.I. Geras'kin adopted the apparatus of conjectural variations and reflexive games. Nash and Stackelberg equilibria were analyzed, and some applications to the Russian telecommunication market were presented. In the second series, G.I. Algazin and his coauthors also synthesized the approaches of the classical game theory and collective behavior and the concept of reflexive games to Cournot oligopoly models. In the publications mentioned, relative efficiency indices were used.

This paper considers Cournot duopoly models in continuous time, directly generalizing the basic static model. The players interact through their state variables (outputs); the control variables (fixed costs) are defined as piecewise continuous open-loop strategies. These models are studied using standard methods [20, 21] and original numerical algorithms [22–24].

Of course, Cournot duopoly is only a particular example. Generally, the matter concerns interaction systems of active (economic and other) agents described by a normal-form game model [9]. In the static statement, this model has the form

$$g_i(u_1, \ldots, u_n) \to \max, \quad u_i \in U_i, \quad i \in N,$$

with the following notations: $N = \{1, \ldots, n\}$ is a finite set of active agents (players); U_i is the set of admissible actions of player $i \in N$; u_i is a particular action of player $i \in N$; $u = (u_1, \ldots, u_n) \in U = U_1 \times \ldots \times U_n$ is the game outcome (the action profile of all players); $u_{-i} = (u_1, \ldots, u_{i-1}, u_{i+1}, \ldots, u_n)$ is the action profile of all players except player i; $g_i : U \to R$ is the payoff function of player $i \in N$. For the independent selfish behavior of equal players, the game solution is the set of Nash equilibria

$$NE = \Big\{u^{NE} \in U : \ \forall i \in N \ \forall u_i \in U_i \ g_i(u^{NE}) \geq g_i(u_i, u_{-i}^{NE})\Big\}.$$

Cooperation in this model means the joint maximization of the total payoff (the utilitarian social welfare function) $g(u) = \sum_{i \in N} g_i(u)$ by all players. Another interpretation of this behavior mode is the introduction of a centralized manager (social planner) maximizing $g(u)$ over all u_i. Thus, the transition to cooperation makes the game-theoretic model becomes an optimization problem.

In the case of a hierarchy, the Leader with the first-move advantage can inform the Follower (or several Followers) of his action, constant strategy (the Germeier game Γ_1) or feedback strategy by the Follower's control (the Germeier game Γ_2) [25]. In the English-language literature, Cournot oligopolies are studied using the Stackelberg game and the inverse Stackelberg game, respectively, and the solution of the game Γ_1 is called the Stackelberg equilibrium. In the dynamic statement, nothing changes fundamentally from the comparative efficiency analysis standpoint.

The approach proposed below has the following restrictions.

1. Only deterministic models are considered so far. The situation becomes complicated under exogenous uncertainty. It will be the subject of future research.

2. The optimal proportion of centralization to decentralization (F.I. Ereshko) within the uncertainty problem is also not analyzed so far.

3. Of course, the above-mentioned game-theoretic models and information structures do not exhaust the rich variety of organizational modes and control methods for active agents. For example, hierarchical control can be described not only by the games Γ_1 and Γ_2 but also by the game Γ_3. It is possible to consider games with aggregated information (V.S. Aliev and A.F. Kononenko), an intermediate statement between the games Γ_1 and Γ_2. Hierarchical decision-making can be described by extensive-form games, etc. However, the fundamentally different interaction modes are equality, cooperation, and hierarchy; the rest are details.

The contributions of this paper are as follows:

—Original dynamic Cournot duopoly models are constructed and investigated for different information structures.

—An integrated system of collective and individual indices is proposed for the relative efficiency of organizational modes of economic agents.

—This system is used to compare the efficiency of organizational modes of economic agents in Cournot duopoly models with ecological safety conditions.

2. STATIC COURNOT DUOPOLY MODEL

Consider the Cournot duopoly model

$$g_1(u_1, u_2) = (1/2 - u_1 - u_2)u_1 \to \max, \quad 0 \le u_1 \le 1/2, \tag{1}$$

$$g_2(u_1, u_2) = (1/2 - u_1 - u_2)u_2 \to \max, \quad 0 \le u_2 \le 1/2, \tag{2}$$

where u_i denotes the production output of firm i and g_i is its profit. For simplicity, the difference between the closing price and variable costs is assumed to be 1/2, fixed costs are zero, and the slope of the demand line equals 1. The outputs of both firms belong to the interval $[0, 1/2]$.

Table 1 is organized as follows. The columns correspond to various optimality principles in model (1)–(2): the Nash equilibrium in the normal-form game of independent equal agents (NE), the equal imputation under the cooperative behavior of players (C), and the Germeier games Γ_1 (ST) and Γ_2 (IST) with the first player as the Leader. The first row shows the optimal game outcomes; the second row, the corresponding payoffs of players; the third row, the total payoff of both players. All proofs are postponed to the Appendix.

Table 1. Players' payoffs in Cournot duopoly

	NE	C	ST	IST
(u_1, u_2)	$(1/6, 1/6)$	$(1/8, 1/8)$	$(1/4, 1/8)$	$(1/4, 0)$
(g_1, g_2)	$(1/36, 1/36)$	$(1/32, 1/32)$	$(1/32, 1/64)$	$(1/16, 0)$
$g = g_1 + g_2$	1/18	1/16	3/64	1/16

For the comparative analysis of the efficiency of these organizational modes, we introduce a system of private and social relative efficiency indices in an n-player game. Social relative efficiency indices correlate the social welfare values in different organizational modes with its maximum value under the cooperation of players:

$$SCI^{NE} = \frac{g_{\min}^{NE}}{g_{\max}}; \quad SCI^{ST} = \frac{g^{ST}}{g_{\max}};$$
$$g_{\max} = g^C = \max_{x \in X} \sum_{i \in N} g_i(x); \quad g_{\min}^{NE} = \min_{x \in NE} \sum_{i \in N} g_i(x). \tag{3}$$

For determining g^{ST} and g^{IST}, we employ the following considerations. Let ST_i be the solution set of the game Γ_1 (Stackelberg equilibria) for player i as the Leader [9]. Note that any of the players can be the Leader. Therefore, the social payoff under hierarchical control without feedback can be calculated as

$$g^{ST} = \frac{1}{n} \sum_{i \in N} g^{ST_i} = \frac{1}{n} \sum_{i \in N} \sum_{j \in N} g_j^{ST_i}.$$

By analogy, let IST_i be the solution set of the game Γ_2 for Leader i [25]. Then $g^{IST} = \frac{1}{n} \sum_{i \in N} \sum_{j \in N} g_j^{IST_i}$ is the social payoff under hierarchical control with feedback.

Private relative efficiency indices correlate the player's payoffs in different organizational modes with their symmetric payoff under cooperation:

$$K_i^{NE} = \frac{g_{i,\min}^{NE}}{\bar{g}_i^C}; \quad K_i^{ST} = \frac{\gamma_i}{\bar{g}_i^C}; \quad K_i^{IST} = \frac{\tilde{\gamma}_i}{\bar{g}_i^C};$$
$$g_{i,\min}^{NE} = \min_{x \in NE} g_i(x); \quad \bar{g}_i^C = \frac{1}{n} g^C, \quad i \in N. \tag{4}$$

Here, γ_i and $\tilde{\gamma}_i$ denote the payoffs of player i as the Leader in the games Γ_1 and Γ_2, respectively. The payoffs are assumed nonnegative in all cases. Table 2 presents the values of the relative efficiency indices in the Cournot duopoly (1)–(2). The last two cells contain the player's payoffs as the Leader (top) and the Follower (bottom). Under cooperation, the values of the indices are 1.

Table 2. The values of social and private efficiency indices in Cournot duopoly

	NE	ST	IST
SCI	8/9	3/4	1
K_i^L / K_i^F	8/9	1 / 1/2	2 / 0

The analysis of Table 2 leads to two preference systems:

society $\quad C \sim IST \succ NE \succ ST;$

individual $\quad IST^L \succ ST^L \sim C \succ NE \succ ST^F \succ IST^F.$

In any normal-form game, we have $g^C = g_{\max} = \sum_{i \in N} g_{i,\min}^{NE} + \Delta$, where $\Delta \geq 0$ is the emergent (cooperative, synergic) effect. This effect shows the beneficialness of cooperation for society. Thus, in the preference system of society, cooperation always leads to the best outcome. (In the Cournot duopoly, the same maximum payoff is achieved in the game Γ_2.) Therefore, social relative efficiency indices can be called system compatibility indices. The closer the index value is to 1, the higher the system compatibility degree will be.

For private preferences, possible situations are $K_i \geq 1$ and $K_i \leq 1$. If Δ is great, then usually $\bar{g}_i^C > \tilde{\gamma}_i$; hence, cooperation is more beneficial than hierarchy for all players. However, in most applications (including the Cournot duopoly under consideration), Δ takes moderate values and $\bar{g}_i^C < \tilde{\gamma}_i$. As a result, players struggle for leadership. Note that in the hierarchy, the Follower occupies a much less advantageous position.

Now we analyze ecological safety conditions. Let the production pollution be proportional to the total output: $P = \alpha(u_1 + u_2)$. Then the ecological condition can be defined as the constraint $u_1 + u_2 \leq \kappa$, where $\kappa = P^*/\alpha$ and P^* is the maximum admissible pollution level. In the hierarchical game, the leading player is responsible for fulfilling this requirement. In other cases, it is an exogenous constraint verified additionally. Table 3 combines the analysis results.

Table 3. Analysis of ecological safety conditions

Parameter κ	NE	C	ST	IST
$[0, 1/4)$	–	–	–	–
$[1/4, 1/3)$	–	$\{(1/8, 1/8)\}$	–	$\{(1/4, 0)\}$
$[1/3, 3/8)$	$\{(1/6, 1/6)\}$	$\{(1/8, 1/8)\}$	–	$\{(1/4, 0)\}$
$[3/8, \infty)$	$\{(1/6, 1/6)\}$	$\{(1/8, 1/8)\}$	$\{(1/4, 1/8)\}$	$\{(1/4, 0)\}$

The en dash indicates that the corresponding game solution does not exist within the given range of the parameter κ. (In other words, it turns out incompatible with ecological conditions.) In particular, for $\kappa \geq 3/8$, all optimality principles under consideration yield ecologically safe solutions; for $\kappa < 1/4$, the situation changes upside down. The Stackelberg equilibrium is most sensitive to the ecological constraint, whereas the cooperative solution has the highest stability.

3. DIFFERENTIAL COURNOT DUOPOLY MODEL WITH LINEAR DYNAMICS

Consider a dynamic generalization of model (1)–(2) with linear dynamics:

$$J_i = \int\limits_0^T e^{-\rho t}\Big\{\beta\,[D - x_1(t) - x_2(t)]\,x_i(t) - v_i(t)\Big\}dt + e^{-\rho T}x_i(T) \to \max; \qquad (5)$$

$$0 \leq v_i(t) \leq v_{\max};$$

$$\dot{x}_i = a_i v_i(t) - m_i x_i(t), \quad x_i(0) = x_{i0}, \quad i = 1,2. \qquad (6)$$

This model has the following notations: J_i is the profit of player (firm) i over a time T; $v_i(t)$ is the control variable of player i (variable costs) in an admissible range; $v_{\max}$ is the maximum admissible variable costs; $x_i(t)$ is the state variable (output) of player i; the expression in square brackets determines the product price depending on the demand, which is inversely proportional to the total output; a_i is the productivity coefficient of player i; m_i is the output decrease coefficient of player i; β is a scaling factor to ensure equal dimensions; ρ is the discounting factor; T is the game length (planning horizon); finally, D is the demand parameter. The natural assumption is $x_i(t) = 0$ if $v_i(t) = 0$. Thus, the interaction of players (competing firms) is described by their state variables.

We study model (5)–(6) using Pontryagin's maximum principle [20, 21]. Let the players use open-loop strategies with piecewise constant controls. Note that according to [28], a Nash equilibrium exists.

The Hamiltonian function has the form

$$H_i(x_i, v_i, \lambda_i) = (D - x_1 - x_2)x_i - v_i + \lambda_i(a_i v_i - m_i x_i), \quad i = 1, \ldots, n,$$

where $\lambda_i(t)$ is the conjugate variable. Then

$$\frac{\partial H_i}{\partial u_i} = -1 + a_i\lambda_i \begin{cases} \geq 0, & \lambda_i(t) \geq \dfrac{1}{a_i} \\ < 0, & \lambda_i(t) < \dfrac{1}{a_i}, \end{cases}$$

$$\frac{\partial \lambda_i}{\partial t} = -D + 2x_i + x_j + (\rho + m_i)\lambda_i; \quad \lambda_i(T) = 1; \quad i = 1,2, \quad j \neq i.$$

Considering the model structure, the Nash equilibrium strategies are given by

$$v_i^{NE}(t) = \begin{cases} v_{\max}, & \lambda_i(t) \geq \dfrac{1}{a_i} \\ 0, & \lambda_i(t) < \dfrac{1}{a_i}, \quad i = 1,2. \end{cases}$$

Here, the conjugate variables have the form

$$\lambda_i(t) = \left[1 - \frac{D}{\rho + m_i}\left(e^{-T(\rho+m_i)} - e^{-t(\rho+m_i)}\right) - 2\int_t^T A_i(\tau)e^{-2\tau(\rho+m_i)}d\tau\right.$$
$$\left. - \int_t^T A_j(\tau)e^{-\tau(2\rho+m_i+m_j)}d\tau\right] e^{t(\rho+m_i)}, \quad j \neq i, \quad i,j = 1,2;$$
$$A_i(t) = x_{0i} + a_i\int_0^t e^{m_i\tau}v_i^{NE}(\tau)d\tau;$$
$$x_i^{NE}(t) = A_i(t)e^{-m_it} = x_{0i}e^{-m_it} + a_i\int_0^t e^{m_i(\tau-t)}v_i^{NE}(\tau)d\tau.$$

The players' payoffs in this equilibrium are

$$J_i = \int_0^T e^{-\rho t}\Big\{[D - x_1(t) - x_2(t)]x_i(t) - v_i(t)\Big\}dt + e^{-\rho T}x_i(T).$$

Well, there exists a unique Nash equilibrium. Note that the functions $\lambda_i(t)$ and the controls $v_i^{NE}(t)$ are interconnected. Therefore, we studied model (5)–(6) numerically to determine the number of control switching points between values and calculate the agents' payoffs. The calculation was performed by the shooting method. In total, 150 numerical experiments were carried out for two agents. The parameters were varied as follows: D, from 0.5 to 40; m_1 and m_2, from 0.1 to 40; a_1 and a_2, from 1 to 100; x_{10} and x_{20}, from 1 to 50; $v_{\max}$, from 50 to 1000. The results of the experiments are demonstrated below for the case $T = 365$ days and $\rho = 0.001$. The input data table is given in the Appendix. Table 4 presents the calculation results. The values t_1 and t_2 correspond to the control switching instants of the agents.

According to the numerical experiments with a wide range of input functions, the controls on the planning horizon switch at most once; for about half of the input data, they even remained unchanged. At the same time, under small values of the demand parameter D (below 13), there is one control switch on the planning horizon. In different examples, the controls switch from minimum to maximum or vice versa. Moreover, the control switching instant changes depending on the input parameters of the model.

Note that for some input data class, the agents' controls in the Nash equilibrium remain equal to the maximum value over time. In this case, if $x_{i0} = a_iv_{\max}/m_i$, the differential equation has a singularity (attractor).

To confirm the numerical calculation-based conclusions, we performed an analytical study of the model with the controls switching at most once, either from zero to the maximum value or vice versa. We considered the case of two agents with the controls

$$v_i^{NE}(t) = \begin{cases} v_{\max} & \text{if } t \leq t_i \\ 0 & \text{if } t_i \leq t, \ i = 1,2 \end{cases} \quad \text{or} \quad v_i^{NE}(t) = \begin{cases} 0 & \text{if } t \leq t_i \\ v_{\max} & \text{if } t_i \leq t, \ i = 1,2. \end{cases} \tag{7}$$

Four possible combinations of the agents' controls were examined. Below we present the calculations when both agents switch their controls once at different instants from the maximum value to zero, i.e.,

$$v_i^{NE}(t) = \begin{cases} v_{\max} & \text{if } t \leq t_i \\ 0 & \text{if } t_i \leq t, \ i = 1,2. \end{cases} \tag{8}$$

In this case,

$$x_i^{NE}(t) = \begin{cases} (x_{i0} - a_i v_{\max}/m_i)e^{-m_i t} + a_i v_{\max}/m_i & \text{if } 0 \le t \le t_i \\ \left(a_i v_{\max}/m_i + (x_{i0} - a_i v^{\max}/m_i)\, e^{-m_i t_i}\right) e^{-m_i(t-t_i)} & \text{if } t_i \le t \le T,\ i = 1,2. \end{cases}$$

Let us denote

$$\begin{aligned} A_i &= x_{i0} - a_i v_{\max}/m_i, \\ B_i &= a_i v_{\max}/m_i, \\ C_i &= a_i v_{\max}/m_i + (x_{i0} - a_i v_{\max}/m_i)\, e^{-m_i t_i}. \end{aligned}$$

The functions $\lambda_i^{NE}(t)$ are found analytically; assuming $t_1 \le t_2$, they have the form

$$\lambda_i(t) = E_i(t) e^{-(\rho+m_i)(T-t)},$$

where

$$E_i(t) = \begin{cases} E_{i0} & \text{if } \ 0 \le t < t_1 \\ E_{i1} & \text{if } \ t_1 \le t < t_2 \\ E_{i2} & \text{if } \ t_2 \le t \le T; \end{cases}$$

in what follows, $i = 1,2$, $j = 1$ if $i = 2$, and $j = 2$ if $i = 1$:

$$\begin{aligned} E_{i0} = 1 &+ \frac{-D + 2B_i + B_j}{\rho + m_i}\left(1 - e^{(\rho+m_i)(T-t)}\right) \\ &+ \frac{2A_i}{\rho + 2m_i} e^{(\rho+m_i)T}\left(e^{-(\rho+2m_i)T} - e^{-(\rho+2m_i)t}\right) \\ &+ \frac{A_j}{\rho + m_1 + m_2} e^{(\rho+m_i)T}\left(e^{-(\rho+m_1+m_2)T} - e^{-(\rho+m_1+m_2)t}\right); \end{aligned}$$

$$\begin{aligned} E_{11} = 1 &+ \frac{-D + B_2}{\rho + m_1}\left(1 - e^{(\rho+m_1)(T-t)}\right) \\ &+ \frac{2C_1}{\rho + 2m_1} e^{(\rho+m_1)T + m_1 t_1}\left(e^{-(\rho+2m_1)T} - e^{-(\rho+2m_1)t}\right) \\ &+ \frac{A_2}{\rho + m_1 + m_2} e^{(\rho+m_1)T}\left(e^{-(\rho+m_1+m_2)T} - e^{-(\rho+m_1+m_2)t}\right); \end{aligned}$$

$$\begin{aligned} E_{21} = 1 &+ \frac{-D + B_2}{\rho + m_2}\left(1 - e^{(\rho+m_2)(T-t)}\right) \\ &+ \frac{2C_1}{\rho + m_1 + m_2} e^{(\rho+m_2)T + m_1 t_1}\left(e^{-(\rho+m_1+m_2)T} - e^{-(\rho+m_1+m_2)t}\right) \\ &+ \frac{A_2}{\rho + 2m_2} e^{(\rho+m_2)T}\left(e^{-(\rho+2m_2)T} - e^{-(\rho+2m_2)t}\right); \end{aligned}$$

$$\begin{aligned} E_{i2} = 1 &+ \frac{-D}{\rho + m_i}\left(1 - e^{(\rho+m_i)(T-t)}\right) \\ &+ \frac{2C_i}{\rho + 2m_i} e^{(\rho+m_i)T + m_i t_i}\left(e^{-(\rho+2m_i)T} - e^{-(\rho+2m_i)t}\right) \\ &+ \frac{C_j}{\rho + m_1 + m_2} e^{(\rho+m_i)T + m_j t_j}\left(e^{-(\rho+m_1+m_2)T} - e^{-(\rho+m_1+m_2)t}\right). \end{aligned}$$

Then the conjecture (8) is checked for $i = 1,2$:

$$\lambda_i \ge 1/a_i \ \text{ if } \ 0 \le t < t_i; \quad \lambda_i < 1/a_i \ \text{ if } \ t_i \le t \le T. \tag{9}$$

If inequalities (9) hold, the agents' controls will have the form (8).

By analogy, it is checked whether the agents' controls have the form corresponding to any other combination of the controls (7). Such calculations confirmed the results in Table 4.

Table 4. The results of numerical study

Example no.	$v_1(0)$	t_1 (days)	$v_1(\mathrm{T})$	$v_2(0)$	t_2 (days)	$v_2(\mathrm{T})$	J_1	J_2	$x_1(\mathrm{T})$	$x_2(\mathrm{T})$
1	0	–	0	100	75	0	13.4	16.1	2.5	3.2
2	100	30	0	100	30	0	4.7	4.5	1.1	0.1
3	100	30	0	100	30	0	21.3	7.9	1.1	0.1
4	0	–	0	0	–	0	24.2	7.5	1.1	0.1
5	0	–	0	0	–	0	30.2	6.8	3.7	0.5
6	0	–	0	0	–	0	30.7	6.8	3.7	0.5
7	0	–	0	0	–	0	30.7	6.8	3.7	0.5
8	0	–	0	0	–	0	6.7	1.7	0.001	2E-8
9	0	–	0	0	–	0	188.6	85.9	3.7	0.5
10	0	–	0	100	10	0	4.3	5.3	0.02	0.1
11	0	–	0	0	–	0	125	54	3.7	0.5
12	0	–	0	0	–	0	125	54	3.7	0.5
13	0	–	0	0	–	0	17	71	0.0004	0.5
14	0	–	0	0	–	0	146	5	3.7	0
15	0	–	0	0	–	0	125	54	3.7	0.5
16	0	–	0	0	–	0	125	54	3.7	0.5
17	0	–	0	0	–	0	117	5.3	11	0.5
18	0	–	0	0	–	0	76	63	3.7	1.5
19	100	–	100	0	5	100	5.3	76	0	0.5
20	0	10	100	0	–	0	78	48	10	0.5
21	0	10	100	0	30	100	146	5	3.7	0
22	100	10	0	100	30	0	33.8	104.4	0.0006	0.5
23	100	20	0	100	50	0	237	8	4.3	9E-11
24	0	–	0	0	–	0	291	61	7.4	0.5
25	0	–	0	0	–	0	62	23	3.7	0.5
26	0	20	100	0	100	100	7.3	39.5	0.0004	0.5
27	0	10	100	0	40	100	83.6	1.8	3.7	0
28	0	40	300	0	100	300	62	23	3.7	0.5
29	0	30	100	0	70	100	12.5	2.5	0.0004	0
30	0	10	100	0	40	100	83.6	1.8	3.7	0
31	0	20	100	0	70	100	12.5	2.5	0.0004	0
32	0	30	300	0	60	300	7.3	39.5	0.0004	0.5
33	0	50	100	100	–	100	2.4	0.5	0.0001	0
34	100	–	100	0	25	100	14.7	4.3	1.1	0.1
35	0	–	0	0	–	0	71	17	1.1	1E-12
36	0	–	0	0	–	0	70	20	0.001	0.1
37	0	10	100	100	–	100	2.3	0.5	0.0001	0
38	0	–	0	0	–	0	73	1	0.001	1E-12
39	0	15	100	100	–	100	16	0.4	1.1	0
40	0	–	0	0	–	0	28.8	141.7	0.001	1

Under cooperation, the game becomes an optimal control problem, and its solution is similar to that of model (5)–(6) with trivial modifications.

The game-theoretic statements Γ_{1t} and Γ_{2t} were investigated numerically. The solution algorithms were described in [22, 23]. Equilibria for the problems Γ_{1t} and Γ_{2t} were found using the algorithms [22, 23] by the method of qualitatively representative scenarios [24]. The initial sets of qualitatively representative scenarios for the players consist of three elements: the minimum and maximum controls (5) and their arithmetic mean. All elements of the initial set of qualitatively representative scenarios are checked for completeness and redundancy; if necessary, it is narrowed or supplemented with new elements. The calculation results for cooperation and hierarchical control are presented in Table 5.

Table 5. The payoffs of players under different information rules

Example no.	NE		C	ST		IST	
	J_1	J_2	J	J_1	J_2	J_1	J_2
1	13.4	16.1	30.8	16.7	13.3	18.2	11.6
2	4.7	4.5	9.4	5.1	4.2	5.1	4.2
3	21.3	7.9	30.4	22.5	7.5	25	4.6
4	24.2	7.5	32.4	26	6	26	6
5	30.2	6.8	38	33.2	4.3	36.3	1
6	30.7	6.8	38.4	35.3	2.7	37.2	0.7
7	30.7	6.8	38.4	35.3	2.7	37.2	0.7
8	6.7	1.7	9.3	7.2	1.5	8.1	1
9	188.6	85.9	286.6	199.3	76.8	199.3	76.8
10	4.3	5.3	10.7	5.7	4	5.7	4
11	125	54	197.5	139	48	145	39
12	125	54	197.5	139	48	145	39
13	17	71	90.8	46	43	49	40
14	146	5	157.8	152	3.4	152	3.4
15	125	54	187.5	137	46	142	38
16	125	54	187.5	137	46	142	38
17	117	5.3	125.3	120.2	4.7	124.3	0.5
18	76	63	145.3	88	53	97	43
19	5.3	76	86	45	38	46	37
20	78	48	138	92	40	95	34
21	146	5	152.8	148	3.3	149.6	1.5
22	33.8	104.4	143.8	44	97	48.3	92.4
23	237	8	256.7	245	5	247.3	2.2
24	291	61	364.6	302.1	57.3	311.2	47.5
25	62	23	88.5	71	16	74.5	11.6
26	7.3	39.5	51.4	26	23	27.4	20
27	83.6	1.8	91.2	88	2.7	89.1	1.4
28	62	23	88.5	69	17	71.2	14.2
29	12.5	2.5	17	14	2	14	2
30	83.6	1.8	89.4	88	0.2	88	0.2
31	12.5	2.5	17	15	1.3	15.4	0.7
32	7.3	39.5	48.4	29	18.6	30.5	16.9
33	2.4	0.5	3.9	2.9	0.2	2.9	0.2
34	14.7	4.3	19.2	17.5	1.6	18.3	0.8
35	71	17	92.6	78	12.7	81.8	9.4
36	70	20	94.6	74.5	18.4	76.2	17.7
37	2.3	0.5	3.9	3.0	0.2	3.1	0.2
38	73	1	76.7	75.1	0.6	75.1	0.6
39	16	0.4	21.2	19.4	0.3	20.3	0.2
40	28.8	141.7	172.3	33.5	136.5	37.3	134.5

In the dynamic version of the game, the private and social relative efficiency indices are determined by expressions similar to (3)–(4). Their values for model (5)–(6) under different organizational modes are combined in Table 6. The last row of this table shows the average values of the indices.

Table 6. The efficiency indices of players under different information rules

Example no.	NE		ST		IST	
	SCI	K_1/K_2	SCI	K_1^L/K_2^F	SCI	K_1^L/K_2^F
1	0.96	0.87 /1.05	0.97	1.08/0.86	0.96	1.18/0.75
2	0.98	1 /0.95	0.99	1.09/0.89	0.98	1.09/0.89
3	0.96	1.4 /0.51	0.99	1.48/0.33	0.97	1.64/0.3
4	0.98	1.48 /0.46	0.99	1.6/0.37	0.98	1.6/0.37
5	0.97	1.58/0.36	0.98	1.75/0.23	0.98	1.91/0.05
6	0.98	1.59/0.35	0.99	1.83/0.14	0.98	1.94/0.04
7	0.98	1.59/0.35	0.99	1.83/0.14	0.98	1.94/0.04
8	0.9	1.46/0.37	0.94	1.54/0.32	0.93	1.74/0.22
9	0.96	1.32/0.6	0.96	1.39/0.54	0.96	1.39/0.54
10	0.9	0.8/1	0.91	1.08/0.75	0.91	1.08/0.75
11	0.91	1.26/0.55	0.95	1.41/0.49	0.93	1.46/0.33
12	0.91	1.26/0.55	0.95	1.41/0.49	0.94	1.46/0.33
13	0.97	0.38/1.58	0.98	1.01/0.95	0.97	1.08/0.88
14	0.96	1.87/0.06	0.98	1.93/0.04	0.97	1.93/0.04
15	0.95	1.33/0.57	0.97	1.46/0.49	0.96	1.51/0.4
16	0.95	1.33/0.57	0.97	1.46/0.49	0.96	1.51/0.4
17	0.98	1.86/0.08	0.99	1.92/0.08	0.98	1.99/0.01
18	0.96	1.04/0.86	0.97	1.21/0.73	0.96	1.34/0.59
19	0.95	0.12/1.77	0.98	1.05/0.88	0.97	1.07/0.86
20	0.91	1.13/0.7	0.95	1.33/0.58	0.94	1.38/0.49
21	0.99	1.92/0.07	0.99	1.95/0.04	0.99	1.97/0.02
22	0.96	0.47/1.45	0.98	0.61/1.35	0.98	0.67/1.28
23	0.95	1.85/0.06	0.97	1.91/0.04	0.97	1.93/0.02
24	0.96	1.6/0.33	0.99	1.65/0.31	0.98	1.7/0.26
25	0.96	1.40/0.52	0.98	1.61/0.36	0.98	1.69/0.26
26	0.91	0.28/1.54	0.93	1.01/0.89	0.92	1.07/0.78
27	0.94	1.83/0.04	0.97	1.93/0.06	0.96	1.95/0.03
28	0.96	1.4/0.52	0.98	1.56/0.38	0.97	1.61/0.32
29	0.88	1.47/0.29	0.94	1.65/0.24	0.94	1.65/0.24
30	0.96	1.86/0.04	0.98	1.97/0.01	0.98	1.97/0.01
31	0.88	1.47/0.29	0.92	1.76/0.15	0.9	1.81/0.08
32	0.97	0.3/1.63	0.99	1.2/0.77	0.98	1.26/0.7
33	0.74	1.23/0.26	0.79	1.49/0.1	0.77	1.49/0.1
34	0.99	1.53/0.45	0.99	1.82/0.17	0.99	1.91/0.08
35	0.95	1.53/0.37	0.98	1.68/0.27	0.98	1.77/0.2
36	0.95	1.48/0.42	0.98	1.58/0.39	0.99	1.61/0.37
37	0.72	1.18/0.26	0.82	1.54/0.1	0.85	1.59/0.1
38	0.96	1.9/0.03	0.99	1.96/0.02	0.99	1.96/0.02
39	0.77	1.51/0.04	0.93	1.83/0.03	0.97	1.92/0.02
40	0.99	0.33/1.65	0.99	0.39/1.59	0.99	0.43/1.77
Average value	**0.935**	**1.28/0.59**	**0.962**	**1.5/0.42**	**0.957**	**1.56/0.37**

As a result, we obtain the following preference systems:

society $C \succ ST \succ IST \succ NE$;

individual $IST^L \succ ST^L \succ NE^1 \succ C \succ NE^2 \succ ST^F \succ IST^F$.

Thus, the cooperative organizational mode is preferable for society and the Follower. For the Leader, the hierarchical organization of the control system and the information rules of the game Γ_{2t} are preferable [26].

Now, we analyze the impact of ecological requirements on the solutions. Let the ecological safety condition be $x_1(T) + x_2(T) \leq \kappa_T$. Under hierarchical control, it becomes the Leader's responsibility; in other cases, it forms an exogenous constraint to be analyzed additionally. We analyzed the sensitivity of the solutions to this condition; the corresponding results are described in Table 7.

Table 7. Analysis of ecological safety conditions

κ_T	NE (%)	C (%)	ST (%)	IST (%)
12	100	100	100	100
11	97.5	100	97.5	97.5
10	95	100	97.5	95
9	95	100	95	95
8	95	100	95	92.5
7	92.5	100	92.5	92.5
6	92.5	100	92.5	90
5	87.5	100	90	87.5
4	60	100	60	60
3	50	100	60	57.5
2	50	97.5	50	50
1	30	95	30	27.5
0.5	30	95	25	22.5
0.1	15	62.5	15	12.5

Columns 2–5 of this table indicate in percentage the number of simulation experiments where the ecological safety condition was satisfied (under different information rules). The first column of Table 7 presents the values of the parameter κ_T. For great values of κ_T, all optimality principles under consideration yield ecologically safe solutions. Decreasing the value κ_T reduces the number of equilibria satisfying the ecological condition under all optimality principles. From the ecological safety standpoint, the optimality principles are ordered as follows: $C \succ NE \sim T \sim IST$.

4. CONCLUSIONS

The (in)efficiency of equilibria is a widely recognized problem studied in numerous research works. (In)efficiency was quantitatively assessed using several indices reflecting the pessimistic approach (the price of anarchy), the optimistic one (the price of stability), dynamic aspects (the price of information), and the altruistic behavior of an individual (the price of cooperation).

However, these indices analyze the efficiency of equilibria from a society standpoint. In this case, cooperation is the obvious best outcome, and the indices assess only the degree of the system's deviation from the global optimum. Meanwhile, the capability of cooperation depends not only on the interests of society but also on the interests of private economic agents (entrepreneurs, firms, etc.). For example, the Leader's payoff in a hierarchical organization may exceed his share in the cooperative distribution; then cooperation gives way to a struggle for leadership. Therefore, a systematic analysis of the (in)efficiency of equilibria and cooperation beneficialness conditions requires private and social relative efficiency indices.

In addition, it is necessary to consider viability conditions determining requirements for the state of the controlled dynamic system. In particular, these conditions can set ecological constraints on the economic activities necessary for the sustainable development of ecological and economic systems.

In this paper, we have applied a system of private and social relative efficiency indices to studying static and dynamic Cournot duopoly models. In dynamics, the indices have been determined by averaging over the set of computational experiments. As expected, the preference systems for individuals (firms) and society are generally contradictory. The cooperation of players is beneficial to society, a subordinate player (Follower), and ecological safety conditions. For the Leader, it is preferable to choose a hierarchy with the information rules of the Germeier game Γ_2. Moreover, two asymmetrical players have different attitudes to cooperation: for one player, it is more beneficial than independent behavior; for the other, vice versa.

Future research will focus on Cournot duopoly and oligopoly models with the ecological safety condition for other classes of functions (particularly power functions) and the game-theoretic models of Cournot oligopolies in the characteristic function form. Also, other static and dynamic game-theoretic models in the normal and characteristic function form will be considered to compare the efficiency of different interaction modes for active agents [27].

APPENDIX

Let us elucidate the data from Table 1. To find a Nash equilibrium in model (1)–(2), we solve the system $\frac{\partial g_i}{\partial u_i} = 0$, $i = 1, 2$. As a result, $\begin{cases} 1/2 - 2u_1 - u_2 = 0 \\ 1/2 - u_1 - 2u_2 = 0, \end{cases}$ $u_1 = u_2 = 1/6$.

The Hessian matrix for this system, $\begin{Vmatrix} -2 - 1 \\ -1 - 2 \end{Vmatrix}$, is negative definite. Therefore, $u_1^{NE} = u_2^{NE} = 1/6$, $g_1^{NE} = g_2^{NE} = 1/36$. Under cooperation, the players jointly maximize the function $g(\bar{u}) = (1/2 - \bar{u})\bar{u}$, where $\bar{u} = u_1 + u_2$. We have $\frac{\partial g}{\partial \bar{u}} = 1/2 - 2\bar{u} = 0$, $\bar{u} = 1/4$, and $\frac{\partial^2 g}{\partial \bar{u}^2} = -2 < 0$. Therefore, the set of Pareto-optimal cooperative solutions is the singleton $\bar{u}^C = 1/4$; the corresponding equal imputation is $u_1^C = u_2^C = 1/8$ and the payoffs are $g_1^C = g_2^C = 1/32$. Assume that player 1 is a Leader in the Stackelberg sense. From the condition $\frac{\partial g_2}{\partial u_2} = 0$ the optimal response of player 2 has the form $u_2(u_1) = 1/4 - u_1/2$. Substituting it into g_1 gives $g_1(u_1, u_2(u_1)) = (1/4 - u_1/2)u_1$. The condition $\frac{\partial g_1}{\partial u_1} = 0$ yields $u_1 = 1/4$. Since $\frac{\partial^2 g_1}{\partial u_1^2} = -1 < 0$, we obtain $u_1^{ST_1} = 1/4$, $u_2^{ST_1} = u_2(u_1^{ST_1}) = 1/8$, $g_1^{ST_1} = 1/32$, and $g_2^{ST_1} = 1/64$.

Finally, let us solve the game (1)–(2) as the Germeier game Γ_2 [25]. We have

$$u_1^D(u_2) = Arg \max_{0 \le u_1 \le 1/2} g_1(u_1, u_2) = 1/4 - u_2/2,$$

$$u_1^P(u_2) = Arg \min_{0 \le u_1 \le 1/2} g_2(u_1, u_2) \equiv 1/2,$$

$$L_2 = \max_{0 \le u_2 \le 1/2} \left(u_1^P(u_2), u_2 \right) = \max_{0 \le u_2 \le 1/2} (-u_2^2) = 0,$$

$$E_2 = \left\{ u_2 \in U_2 : g_2 \left(u_1^P(u_2), u_2 \right) = L_2 \right\} = \{0\},$$

$$D_2 = \{(u_1, u_2) : g_2(u_1, u_2) > 0\},$$

$$K_2 = \min_{u_2 \in E_2} \max_{0 \le u_1 \le 1/2} g_1(u_1, u_2) = \max_{0 \le u_1 \le 1/2} (1 - u_1)u_1 = 1/16.$$

To find the values $K_1 = \sup_{D_2} g_1(u_1, u_2)$, it is necessary to solve the optimization problem $(1/2 - u_1 - u_2)u_1 \to \max$ subject to the constraints $(1/2 - u_1 - u_2)u_2 > 0$ and $0 \le u_i \le 1/2$. Obviously, $u_2^\varepsilon = \varepsilon$ and $u_1^\varepsilon = 1/4$. Then $K_1 = 1/16 - \varepsilon/4 < K_2$ and, therefore, the ε-optimal strategy of the Leader is $\tilde{u}_1^\varepsilon(u_2) = \begin{cases} 1/4 \text{ if } u_2 = 0 \\ 1/2 \text{ otherwise.} \end{cases}$ In this case, $g_1^{IST_1} = 1/16$ and $g_2^{IST_1} = 0$.

Note that $\bar{u}^{NE} = 1/3$, $\bar{u}^C = 1/4$, $\bar{u}^{ST} = 3/8$, and $\bar{u}^{IST} = 1/4$. Thus, we have arrived at the data from Table 3.

Table. Input data for the numerical solution of the dynamic Cournot duopoly

Example no.	D	m_1	m_2	a_1	a_2	x_{10}	x_{20}	$v_{\max}$
1	10	0.2	0.001	1	5	3	2	100
2	10	1	3	1	5	3	2	100
3	15	1	3	1	5	3	2	100
4	15	1	3	3	7	3	2	100
5	15	1	3	3	7	10	10	100
6	15	1	3	3	7	10	10	500
7	15	1	3	10	15	10	10	100
8	15	10	20	3	7	10	10	100
9	40	1	3	3	7	10	10	100
10	10	5	3	3	5	3	2	100
11	30	1	3	3	7	10	10	100
12	30	1	3	3	7	10	10	300
13	30	10	3	3	7	10	10	100
14	30	1	30	3	7	10	10	100
15	30	1	3	30	7	10	10	100
16	30	1	3	3	70	10	10	100
17	30	1	3	3	7	30	10	100
18	30	1	3	3	7	10	30	100
19	30	30	3	30	7	10	10	100
20	30	30	3	3	70	10	10	100
21	30	1	30	30	7	10	10	100
22	40	10	3	3	7	10	10	100
23	40	1	30	3	7	10	10	100
24	40	1	3	30	7	20	10	100
25	20	1	3	3	7	10	10	100
26	20	10	3	3	7	10	10	100
27	20	1	30	3	7	10	10	100
28	20	1	3	3	7	10	10	300
29	20	10	30	3	7	10	10	100
30	20	1	30	3	70	10	10	100
31	20	10	30	30	7	10	10	100
32	20	10	3	30	7	10	10	300
33	10	10	30	1	5	3	2	100
34	10	1	3	10	50	3	2	100
35	40	1	30	1	5	3	20	100
36	40	10	3	1	5	30	2	100
37	10	10	30	10	5	3	2	100
38	40	10	30	1	50	30	2	100
39	10	1	30	10	50	3	2	100
40	40	10	3	10	5	30	20	100

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