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Theoretical Basis and Methods of Decision Analysis under Uncertainty with Qualitative Estimates of Probabilities and Preferences

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Abstract—The problems of decision-making are considered when preferences are estimated on an ordinal scale, and the possibilities of realizing the values of an uncertain factor are described by a qualitative probability (complete or only partial). Definitions of preference and indifference relations on a set of strategies are introduced. Simple decision rules are proposed that allow comparing strategies by preference. Examples of their application are given.

Keywords: decision-making under uncertainty, attitude to risk, quantitative probability, ordinal utility, preference relations

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1. INTRODUCTION

Practical decision-making problems often require the consideration of uncertain factors whose values are not known in advance. The most common approaches to such problems involve the use of a utility function, i.e., a function whose mathematical expectation models the preferences of the decision maker (DM) [1–3], or risk measures [4–6]. However, constructing a utility function is a non-trivial problem, and even assuming its existence is a strong hypothesis that may lead to paradoxes conclusions inconsistent with common sense or with the DMs expected outcomes [7, 8]. In addition, using the utility function involves using quantitative probability, i.e., the distribution of probabilities over multiple values of an uncertain factor must be set completely [9] or at least partially [10–12]. Risk measures, in turn, face the challenge of selecting an appropriate metric, as no universal measure exists and each has limitations that prevent a comprehensive risk assessment [13]. Furthermore, their calculation also requires knowledge of quantitative probabilities.

However, in complex decision-making problems of a socio-economic, political, or other nature, objective probabilities usually do not exist, necessitating reliance on expert assessments. Psychological research shows that quantitative evaluations are quite challenging for humans, and therefore, they tend to be insufficiently reliable and may contain significant errors [14].

Therefore, the development of methods for analyzing decisions under uncertainty that do not rely on quantitative probabilities is of theoretical and practical interest. On the other hand, in complex problems, the possible consequences of options are also easier and more reliably assessed qualitatively rather than quantitatively, i.e., on an ordinal scale. A method for analyzing such problems was first described in [15]. The purpose of this article is to present such methods in detail, with justification, and to illustrate their “operation”.

2. MATHEMATICAL MODEL OF PROBLEM SITUATION

The following mathematical model of the situation of decision-making under uncertainty is used:

$$\langle S, \Lambda, V, f, \succsim, R\rangle .$$

Here:

S is the set of strategies (plans, alternatives, options, actions, decisions).

$\Lambda = \{\lambda^1, \dots, \lambda^n\}$ is a finite set of values of the *uncertain factor* ($n \geqslant 2$). Its elements are called *states of nature*, or *elementary events*, its subsets are called *events*.

V is the set of *numerical outcome estimates*, defined on an ordinal scale [16]. It is assumed that preferences increase on V: the greater the numerical estimate v, the more preferable it is.

f is the *scoring criterion*—a mapping $S \times \Lambda \to V$, i.e., $v = f(s, \lambda)$ is the numerical estimate of the outcome of using strategy s, if value λ of the uncertain factor is realized. Each strategy s is characterized by a vector of outcomes $x = \left(f\left(s, \lambda^1\right), \dots, f\left(s, \lambda^n\right)\right)$. The set of all vector estimates x (both attainable and hypothetical) is $X = V^n$. Given the assumption about the nature of estimates from V, we consider the scoring criterion to be ordinal.

$\succsim$ is a qualitative probability, complete or only partial (the necessary information on qualitative probability is given in Section 3).

R is a non-strict preference relation of the DM on X: the statement xRy means that vector x is at least as preferable as y. It generates strict preference relation P and indifference relation I as follows: xPy if xRy and not yRx; xIy if both xRy and yRx. The statement xPy means that x is preferred to y, and xIy means that x and y are indifferent (equally preferred). It is assumed that the relation R is a quasi-order (it is reflexive: xRx for any $x \in X$, and transitive: xRy and yRz imply xRx for any $x, y, z \in X$), but only partial, i.e., not every x and y from X are comparable with respect to R: neither xRy nor yRx may hold. In other words, the relations I and P are the symmetric and asymmetric parts of the relation R. For the quasi-order R, the relation P turns out to be a strict partial order (it is irreflexive: yPy is false for any $y \in X$, and is transitive), and the relation I is an equivalence (it is symmetric: xRy implies yRx for any $x, y \in X$, it is reflexive and transitive). The relations R, P, I generate similar relations R_*, P_*, I_* on the set of strategies S as follows:

$$s'R_*s'' \Leftrightarrow x'Rx'', \qquad s'P_*s'' \Leftrightarrow x'Px'', \qquad s'I_*s'' \Leftrightarrow x'Ix'',$$

here $x' = \left(f\left(s', \lambda^1\right), \dots, f\left(s', \lambda^n\right)\right)$ and $x'' = \left(f\left(s'', \lambda^1\right), \dots, f\left(s'', \lambda^n\right)\right)$. The relations R_*, P_*, I_* are directly used to solve the analyzed decision-making problem, taking into account the type of its formulation. For example, if the optimal strategy should be selected, then a set of non-dominated strategies with respect to P_* is constructed, from which the best one must be selected (of course, if it is externally stable).

The relation R is not known in advance: it should be constructed based on given preferences on the set of estimates V and the qualitative probability $\succsim$. Methods for constructing this relation are discussed further in Sections 4 and 5.

3. QUALITATIVE PROBABILITY

Let us present the necessary information about qualitative probability. Let A, B, C denote subsets of the finite set of elementary events, or states of nature $\Lambda = \{\lambda_1, \dots, \lambda_n\}$; these subsets will be called "events". The number of all events in 2^Λ (including Λ itself and the empty set $\varnothing$) is equal to 2^n. Let $N = \{1, \dots, n\}$.

Definition 1 [17]. A binary relation $\succsim$ on a set 2^Λ is called complete qualitative, or complete comparative probability, if it has the following properties, taken as axioms:

C1. Comparability: for any A and B, at least one of the relations $A \succsim B$ or $B \succsim A$ holds.

C2. Transitivity: $A \succsim B$ and $B \succsim C \Rightarrow A \succsim C$.

C3. Nontriviality: $\Lambda \succ \varnothing$ and $A \succsim \varnothing$ for any A.

C4. Additivity: if $A \cap C = B \cap C = \varnothing$, then $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C$.

From properties C1 and C2 it follows that the complete qualitative probability is reflexive and therefore a connected quasi-order. Therefore, its asymmetric part $\succ$ is a (partial) order, and its symmetric part $\sim$ is an equivalence.

Let a quantitative probability Pr be defined on the elementary events as follows: $\Pr(\lambda^1) = p_1, \ldots, \Pr(\lambda^n) = p_n$, so that the probability of event A is equal to $\Pr A = \sum_{j:\lambda^j \in A} p_j$. The quantitative probability *numerically represents* the qualitative probability, or *is consistent* with the qualitative probability, if the following condition is satisfied: for any A, B

$$A \succsim B \Leftrightarrow \Pr A \geqslant \Pr B. \tag{1}$$

According to (1), $A \succ B \Leftrightarrow \Pr A > \Pr B$, $A \sim B \Leftrightarrow \Pr A = \Pr B$ for any A, B.

If a quantitative probability Pr is given, then the relation $\succsim$ generated by it according to (1) satisfies all properties C1–C4, i.e., it is a qualitative probability.

It turned out that any complete qualitative probability can be represented by some quantitative probability only for $n \leqslant 4$: in [18] a counterexample was constructed for $n = 5$.

In the following, to shorten the notation, we will also denote the elementary event λ^i by its number i, and write events A in multiplicative form. For example, the event $A = \{\lambda^1, \lambda^2, \lambda^4\}$ will be written as $\{1, 2, 4\}$ or 124, and instead of $\lambda^2 \in A$ we will write $2 \in A$.

Example 1 [18]. Let $n = 5$. Consider the relation $\succ$ defined by the chain:

$$\begin{aligned} &12\,345 \succ 2345 \succ 1345 \succ 1245 \succ 345 \succ 245 \succ 1235 \succ 235 \succ 145 \\ &\succ 1234 \succ 45 \succ 135 \succ 125 \succ 234 \succ 35 \succ 134 \succ 25 \succ 124 \succ 15 \\ &\succ 34 \succ 24 \succ 123 \succ 5 \succ 23 \succ 14 \succ 4 \succ 13 \succ 12 \succ 3 \succ 2 \succ 1 \succ \varnothing. \end{aligned} \tag{2}$$

It is straightforward to verify that all properties C1–C4 are satisfied, so the relation $\succsim$, defined by (2), is a complete qualitative probability. Suppose there are quantitative probabilities p_1, p_2, p_3, p_4, p_5 that represent this qualitative probability. From the above chain, we distinguish four relations: $4 \succ 13$, $23 \succ 14$, $15 \succ 34$, $134 \succ 25$. They correspond to four strict inequalities:

$$p_4 > p_1 + p_3, \quad p_2 + p_3 > p_1 + p_4, \quad p_1 + p_5 > p_3 + p_4, \quad p_1 + p_3 + p_4 > p_2 + p_5.$$

However, the system of these four inequalities is inconsistent. Indeed, adding the terms on the two sides of these inequalities, we obtain:

$$2p_1 + p_2 + 2p_3 + 2p_4 + p_5 > 2p_1 + p_2 + 2p_3 + 2p_4 + p_5, \text{ i.e., } 0 > 0.$$

In light of the above, it is clear that methods based on the assumption of the existence of quantitative probability (including the methods of utility theory) are, in principle, not applicable to qualitative probability, which cannot be represented quantitatively!

Definition 2 [19]. A binary relation $\succsim$ on a set 2^Λ is called a partial qualitative (or partial comparative) probability if it satisfies axioms C2–C4 and the axiom

C5. Reflexivity: $A \succsim A$ for any A.

A more detailed discussion of the properties of complete and partial qualitative probabilities (these properties are unusual and far from obvious) can be found in [20]. Here we note the following two properties:

C6. If $A_i \succsim B_i, i = 1, \ldots, m, A_i \cap A_j = B_i \cap B_j = \varnothing, i \neq j$, then $A_1 \cup \cdots \cup A_m \succsim B_1 \cup \cdots \cup B_m$.

C7. $A \succsim B \Leftrightarrow \overline{B} \succsim \overline{A}$; therefore $A \succ B \Leftrightarrow \overline{B} \succ \overline{A}$ and $A \sim B \Leftrightarrow \overline{B} \sim \overline{A}$ (the authors have not encountered this property in the literature, and therefore its proof is given in the Appendix).

4. DEFINITION OF PREFERENCE AND INDIFFERENCE RELATIONS

Let us introduce definitions of preference and indifference relations on the set of vector valuations $X = V^n$, taking into account a given qualitative probability $\succsim$ (complete or only partial). We will introduce definitions of the relations $R^{\succsim}$, $P^{\succsim}$, $I^{\succsim}$ on the set of vector estimates $X = V^n$ taking into account a given qualitative probability $\succsim$ axiomatically, i.e., we will specify a system of axioms that have a clear practical meaning (from the point of view of preferences) and therefore these relations must satisfy. Since it makes no sense to take into account states of nature that are clearly not likely to occur when comparing strategies by preference, we will further use the axiom of nontriviality in a stronger formulation for qualitative probability: $A \succ \varnothing$ for any $A \neq \varnothing$.

Axiom 1. The relation $R^{\succsim}$ is reflexive.

Axiom 2. The relation $R^{\succsim}$ is transitive.

If these axioms are satisfied for $R^{\succsim}$, then the relation $P^{\succsim}$ is a strict partial order, and the relation $I^{\succsim}$ is an equivalence.

Axiom 3. If one component of a vector estimate x is greater than the corresponding component of a vector estimate y, and all other corresponding components are equal, then x is preferable to y: if $x_i > y_i$ and $x_j = y_j$, $j \in N = \{1, \ldots, n\}, j \neq i$, then $xP^{\succsim}y$.

The Pareto relation $P^{\varnothing}$ is given by: $xP^{\varnothing}y \Leftrightarrow x_i \geqslant y_i, i \in N, x \neq y$.

According to Axioms 2 and 3, the relation $P^{\succsim}$ satisfies the Pareto axiom: if $x_i \geqslant y_i, i \in N, x \neq y$, then $xP^{\succsim}y$, i.e., $xP^{\varnothing}y \Rightarrow xP^{\succsim}y$.

Let A, B be two events. Let $x, y \in X$ be two vector estimates such that $x_i = y_j, i \in A, j \in B$, i.e., all their components with numbers from A and B are equal to a. Let $b \in V$ such that $b \neq a$, and $x' = (x \parallel x_i = b, i \in A) \in X$ (respectively, $y' = (y \parallel y_j = b, j \in B) \in X$) be the vector estimate obtained from x by replacing all its components with numbers from A with b (respectively, obtained from y by replacing all its components with numbers from B with b). We will say that vector estimates (x and y) are changed in the same way with respect to events A and B if their equal components corresponding to these events are replaced by the same value (i.e., x and y are replaced by x' and y', respectively). Furthermore, if $b > a$, then we will speak of an improvement of x and y (since in this case $x'P^{\varnothing}x$ and $y'P^{\varnothing}y$), and if $b < a$, then, on the contrary, we will speak of their deterioration (since $xP^{\varnothing}x'$ and $yP^{\varnothing}y'$). Note that when $A = \varnothing$ or $B = \varnothing$ the vector estimates x or, respectively, y do not change.

For the non-strict preference relation $R^{\succsim}$ and the relations $P^{\succsim}$ and $I^{\succsim}$ generated by it on the set of vector estimates X according to the meaning of these relations, the following properties are postulated.

Axiom 4. Let two vector estimates be improved in the same way with respect to two equally probable events; if these vector estimates are equal in preference, then they will remain equal in preference; if the first vector estimate is preferable to the second, then it will remain preferable to the second: if $A \sim B \succ \varnothing$ and $a < b$, then $xI^{\succsim}y \Rightarrow x'I^{\succsim}y'$, and $xP^{\succsim}y \Rightarrow x'P^{\succsim}y'$.

Axiom 5. If two vector estimates, the first of which is no less preferable than the second, are improved in the same way with respect to two events, the first of which is more probable than the second, then the first of the resulting vector estimates will be preferable to the second: if $A \succ B$ and $a < b$, then $xR^{\succsim}y \Rightarrow x'P^{\succsim}y'$.

Given that the relation $R^{\succsim}$ is reflexive, transitive, and satisfies Axiom 3, Axioms 4 and 5 define a recursive (inductive) definition of this relation if we start with relations $xI^{\succsim}x$ that are true for any $x \in X$, or with $xP^{\succsim}y$, if $xP^{\varnothing}y$ is true.

Example 2. Let $n = 5$, $V = \{1, 2, \dots, 10\}$ and the complete qualitative probability be given by (2). Let us start with $x = (1, 1, 1, 1, 1)$. Since $2345 \succ 1235$, according to Axiom 5 we have $(1, 2, 2, 2, 2)\, P^{\succsim}\, (2, 2, 2, 1, 2)$. Further, since $245 \succ 123$, we have $(1, 3, 2, 3, 3)\, P^{\succsim}\, (3, 3, 3, 1, 2)$. Taking into account $4 \succ 13$ we obtain $(1, 3, 2, 4, 3)\, P^{\succsim}\, (4, 3, 4, 1, 2)$. Finally, $4 \succ \varnothing$, and therefore $(1, 3, 2, 5, 3)\, P^{\succsim}\, (4, 3, 4, 1, 2)$. It is convenient to represent these calculations as a double chain (arrows are drawn from more preferred vector estimates to less preferred ones, straight lines connect equal or identically preferred vector estimates):

$$\begin{array}{ccccccccc}
(1,1,1,1,1) & \leftarrow & (1,2,2,2,2) & \leftarrow & (1,3,2,3,3) & \leftarrow & (1,3,2,4,3) & \leftarrow & (1,3,2,5,3) \\
 & & \downarrow_{2345\succ 1235} & & \downarrow_{245\succ 123} & & \downarrow_{4\succ 13} & & \downarrow_{4\succ\varnothing} \\
(1,1,1,1,1) & \leftarrow & (2,2,2,1,2) & \leftarrow & (3,3,3,1,2) & \leftarrow & (4,3,4,1,2) & — & (4,3,4,1,2)\,.
\end{array}$$

Such double chains can be called *explanatory*: they clearly show why one vector estimate is preferable to another (or whether they are indifferent). The chain above is *ascending*, since the vector estimates in it improve. From Axioms 1–5, the following statements can be derived, also exploiting the deterioration of vector estimates.

Theorem 1. *Let two vector estimates be changed in the same way with respect to two equally probable events; if these vector estimates are equal in preference, then they will remain equal in preference; if the first vector estimate is preferable to the second, then it will remain preferable to the second: if $A \sim B \succ \varnothing$, then for any a and b, $xI^{\succsim}y \Rightarrow x'I^{\succsim}y'$, and $xP^{\succsim}y \Rightarrow x'P^{\succsim}y'$.*

The proofs of the theorems are given in the Appendix.

Theorem 2. *Let two vector estimates, the first of which is no less preferable than the second, be modified in the same way with respect to two events. If the first vector estimate is improved with respect to the more probable event or deteriorated with respect to the less probable event, then the first of the resulting vector estimates will be preferable to the second: if $(A \succ B$ and $a < b)$ or $(A \prec B$ and $a > b)$, then $xR^{\succsim}y \Rightarrow x'P^{\succsim}y'$.*

Theorems 1 and 2 are generalizations of Axioms 4 and 5 and can also be used to construct double chains. In writing the chains, we specifically use the statement $A \prec B$ instead of $B \succ A$ to further indicate that Theorem 2 is used with deteriorated vector estimates.

Example 3. Let $n = 5$, $V = \{1, 2, \dots, 10\}$ and the complete qualitative probability be given by (2). Let us start with $x = (5, 5, 5, 5, 5)$. Since $1235 \prec 12\,345$, according to Theorem 2 we have $(4, 4, 4, 5, 4)\, P^{\succsim}\, (4, 4, 4, 4, 4)$. Further, since $1235 \prec 245$, we have $(3, 3, 3, 5, 3)\, P^{\succsim}\, (4, 3, 4, 3, 3)$. Taking into account $13 \prec 45$, we obtain $(2, 3, 2, 5, 3)\, P^{\succsim}\, (4, 3, 4, 2, 2)$. Finally, $1 \prec 4$, and therefore $(1, 3, 2, 5, 3)\, P^{\succsim}\, (4, 3, 4, 1, 2)$. We have a chain (this chain is *decreasing*):

$$\begin{array}{ccccccccc}
(5,5,5,5,5) & \rightarrow & (4,4,4,5,4) & \rightarrow & (3,3,3,5,3) & \rightarrow & (2,3,2,5,3) & \rightarrow & (1,3,2,5,3) \\
 & & \downarrow_{1235\prec 12\,345} & & \downarrow_{1235\prec 245} & & \downarrow_{13\prec 45} & & \downarrow_{1\prec 4} \\
(5,5,5,5,5) & \rightarrow & (4,4,4,4,4) & \rightarrow & (4,3,4,3,3) & \rightarrow & (4,3,4,2,2) & \rightarrow & (4,3,4,1,2)\,.
\end{array}$$

5. CONSTRUCTION OF PREFERENCE AND INDIFFERENCE RELATIONS

The decision rules presented below are based on the development of ideas from the theory of qualitative importance of criteria in multicriteria decision-making problems [21]. To formulate them, we assign to each pair of vector estimates x and y a non-decreasingly ordered set consisting of their components:

$$W\,(x,y) = \{x_1\} \cup \cdots \cup \{x_n\} \cup \{y_1\} \cup \cdots \cup \{y_n\} = \{w_1, \ldots, w_q\},$$

where $q \leqslant 2n$ and depend on x and y, $w_1 < \cdots < w_q$. We will assign $2(q-1)$ sets of component numbers to vector estimates:

$$M_k\,(x) = \{i \in N \mid x_i \geqslant w_k\}, \quad M_k\,(y) = \{i \in N \mid y_i \geqslant w_k\}, \quad k = 2, 3, \ldots, q.$$
$$L_k\,(x) = \{i \in N \mid x_i \leqslant w_k\}, \quad L_k\,(y) = \{i \in N \mid y_i \leqslant w_k\}, \quad k = q-1, q-2, \ldots, 1.$$

Theorem 3. *The relation $xR^{\succsim}y$ is true if and only if all of the following relations are true*

$$M_k\,(x) \succsim M_k\,(y)\,, \quad k = 2, 3, \ldots, q. \tag{3}$$

Furthermore, if in (3) *all $\succsim$ are $\sim$, then $xI^{\succsim}y$. Otherwise* (*when* (3) *contains $\succ$*) *$xP^{\succsim}y$ is true.*

Example 4. Under the conditions of Example 2 compare the vector estimates $x = (1, 3, 2, 5, 3)$ and $y = (4, 3, 4, 1, 2)$. Here $W\,(x, y) = \{1, 2, 3, 4, 5\}, q = 5$. According to (3), we have:

$$w_2 = 2, M_2\,(x) = \{2, 3, 4, 5\} \succ \{1, 2, 3, 5\} = M_2\,(y)\,;$$
$$w_3 = 3, M_3\,(x) = \{2, 4, 5\} \succ \{1, 2, 3\} = M_3\,(y)\,;$$
$$w_4 = 4, M_4\,(x) = \{4\} \succ \{1, 3\} = M_4\,(y)\,; \qquad w_5 = 5, M_5\,(x) = \{4\} \succ \varnothing = M_5\,(y)\,,$$

therefore $xP^{\succsim}y$ is true. It is easy to see that in the process of successive use of (3) the double chain from Example 2 is generated.

Due to the reflexivity and transitivity of qualitative probability $\succsim$ the relation $R^{\succsim}$ defined according to (3) is indeed a quasi-order, but in the general case its only partial. However, in the case of $q = 2$ and complete qualitative probability, any two vector estimates, according to (3), are comparable in preference.

Example 5. Let $V = \{1, 2\}$ and complete qualitative probability is given by (2). The complete order of the set X of all thirty-two ($= 2^5$) vector estimates, which are written in multiplicative form for brevity, is as follows:

$$22222\,P\,12222\,P\,21222\,P\,22122\,P\,11222\,P\,12122\,P\,22212\,P\,12212\,P$$
$$P\,21122\,P\,22221\,P\,11122\,P\,21212\,P\,22112\,P\,12221\,P\,11212\,P\,21221\,P$$
$$P\,12112\,P\,22121\,P\,21112\,P\,11221\,P\,12121\,P\,22211\,P\,11112\,P\,12211\,P$$
$$P\,21121\,P\,11121\,P\,21211\,P\,22111\,P\,11211\,P\,12111\,P\,21111\,P\,11111.$$

One can also formulate a decision rule equivalent to (3) and in some sense dual to it.

Theorem 4. *The relation $xR^{\succsim}y$ is true if and only if all of the following relations are true*

$$L_k\,(x) \precsim L_k\,(y)\,, \quad k = q-1, q-2, \ldots, 1. \tag{4}$$

Furthermore, if in (4) *all $\precsim$ are $\sim$, then $xI^{\succsim}y$. Otherwise (when* (4) *contains $\prec$) $xP^{\succsim}y$ is true.*

Considering the sets $M_k\,(x)$ and $L_k\,(x)$ as subsets of the universal set N, we can write: $\overline{M}_{k+1}\,(x) = L_k\,(x)$ and $\overline{M}_{k+1}\,(y) = L_k\,(y)$, $k = 1, \ldots, q-1$, and then in light of property C7 it is immediately clear that conditions (3) and (4) are equivalent.

Example 6. Under the conditions of Example 3 compare the vector estimates $x = (1, 3, 2, 5, 3)$ and $y = (4, 3, 4, 1, 2)$. Here $W(x, y) = \{1, 2, 3, 4, 5\}, q = 5$. According to (4), we have:

$$w_4 = 4, L_4(x) = \{1, 2, 3, 5\} \prec \{1, 2, 3, 4, 5\} = L_4(y);$$
$$w_3 = 3, L_3(x) = \{1, 2, 3, 5\} \prec \{2, 4, 5\} = L_3(y);$$
$$w_2 = 2, L_2(x) = \{1, 3\} \prec \{4, 5\} = L_2(y); \qquad w_1 = 1, L_1(x) = \{1\} \prec \{4\} = L_1(y),$$

so $xP^{\succsim}y$ is true. In the process of successive use of (4) the double chain from Example 3 is generated.

Remark. The relation of non-strict preference on a set of strategies for decision-making problems with qualitative estimates of probabilities and preferences was introduced earlier in [22] similar to the relation of non-strict preference for groups of criteria comparable in importance, but it was weaker than the one considered here, and no analytical decision rule was established for it.

6. CONCLUSION

This article examines a new, previously unexplored formulation of a decision analysis problem, in which preferences are assessed on an ordinal scale, and the probability of realizing the values of an uncertain factor is expressed using qualitative probability. It is not assumed that qualitative probability can be represented quantitatively. An approach to analyzing such problems is developed and effective decision rules are proposed that enable pairwise comparison of strategies by preference and the elimination of unacceptable (dominated) ones.

It is known that the use of qualitative estimates alone typically does not completely resolve the problem of identifying a single optimal strategy, but rather only narrows the set of candidates. This facilitates further analysis (in particular, it requires obtaining fewer quantitative preference estimates, and simpler ones, such as interval estimates). Therefore, obtaining and using only qualitative estimates typically occurs in the early stages of iterative procedures for solving choice problems [20, 23].

The results obtained can be considered a first step toward developing a decision analysis methodology focused on cases in which preferences and event probabilities are assessed qualitatively. Further research could focus on taking into account people's attitudes toward risk, incorporating more general types of preference relations into the results set (e.g., partial relations, which are common to multicriteria problems), and developing more sophisticated scales for representing preference relations (e.g., the first ordered metric scale).

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APPENDIX

A.1. PROOF OF PROPERTY C7

Consider case 1, when $A \cap B = \varnothing$. This implies $A \subseteq \overline{B}$ and $B \subseteq \overline{A}$. Denote $C = \overline{A \cup B} = \overline{A} \cap \overline{B}$. It is clear that $A \cap C = B \cap C = \varnothing$. Therefore, $A \cup C = \left(A \cup \overline{A}\right) \cap \left(A \cup \overline{B}\right) = A \cup \overline{B}$. Then from $A \subseteq \overline{B}$ it follows that $A \cup C = \overline{B}$. Similarly, $B \cup C = \overline{A}$. Since $A \cap C = B \cap C = \varnothing$, according to property C4 we obtain $A \succsim B \Leftrightarrow A \cup C \succsim B \cup C \Leftrightarrow \overline{B} \succsim \overline{A}$.

Now consider the general case 2, when $A \cap B \neq \varnothing$. Denote $D = A \cap B$. According to C4, we have $A \succsim B \Leftrightarrow A \setminus D \succsim B \setminus D$. Because $A \setminus D \cap B \setminus D = \varnothing$, as proved in Case 1, we have $A \setminus D \succsim B \setminus D \Leftrightarrow \overline{B \setminus D} \succsim \overline{A \setminus D}$. Restate $\overline{A \setminus D} = \overline{A} \cup D$ and $\overline{B \setminus D} = \overline{B} \cup D$. Since $\overline{A} \cap D = \overline{B} \cap D = \varnothing$, it follows from property C4 that $\overline{B} \cup D \succsim \overline{A} \cup D \Leftrightarrow \overline{B} \succsim \overline{A}$. As a result, we obtain the required $A \succsim B \Leftrightarrow \overline{B} \succsim \overline{A}$. Property C7 is proved.

The proofs of Theorems 1 and 2 are much simpler if we use Theorem 3. Therefore, we first give the proof of Theorem 3 that does not use Theorems 1 and 2.

A.2. PROOF OF THEOREM 3

Introduce the operator for parallel improvement of vector estimates:

$$\Uparrow [A, B, a, b], \quad \text{where} \quad A, B \in 2^{\Lambda}, \quad a, b \in \mathbb{R}, \quad a < b.$$

The domain of an operator is the set of pairs of vector estimates (x, y) such that $x_i = y_j = a$, $i \in A$, $j \in B$. The result of applying this operator to a pair (x, y) is the pair of vector estimates (x', y') such that $x' = (x \parallel x_i = b, i \in A)$, $y' = (y \parallel y_j = b, j \in B)$. Therefore, this operator preserves the statements of Axioms 4 and 5 of the relation $R^{\succsim}$. For any vector estimates x and y the relation $xR^{\succsim}y$ is true if and only if there exists a vector estimate z and a superposition of improvement operators

$$(x, y) = \Uparrow [A_L, B_L, a_L, b_L] \ldots \Uparrow [A_1, B_1, a_1, b_1] \, (z, z), \tag{A.1}$$

where $A_l \succsim B_l$, $l = 1, \ldots, L$. It is clear that the components of the vector estimate z must satisfy the conditions $z_i \leqslant x_i$ and $z_i \leqslant y_i, i \in N$. For convenience, we will further assume that $z_i \geqslant w_1 = \min_{j \in N}\{x_j, y_j\}$ is satisfied for all $i \in N$. Otherwise, the values of the corresponding components can be increased using the operator $\Uparrow [\{i\}, \{i\}, z_i, w_1]$.

The sequence of application of operators (A.1) corresponds to a double chain, starting with the vector estimate z. Therefore, from now on, an expression of the form (A.1) will also be called a double chain.

Comparison of vector estimates x and y using decision rule (3) can be represented as a superposition of improvement operators of the following form:

$$(x, y) = \Uparrow [M_q(x), M_q(y), w_{q-1}, w_q] \ldots \Uparrow [M_2(x), M_2(y), w_1, w_2] \, (e, e), \tag{A.2}$$

where $e = (w_1, \ldots, w_1)$. The existence of the double chain (A.2) proves the sufficient condition of Theorem 3.

Let us proceed to the proof of necessity. Let a double chain of the form (A.1) be given. Denote $U = W(x, y) \cup \{a_1\} \cup \cdots \cup \{a_L\} \cup \{b_1\} \cup \cdots \cup \{b_L\} = \{u_1, \ldots, u_r\}$, where $r \leqslant 2n + 2L$, $u_1 < \cdots < u_r$. Because, as assumed, $z_i \geqslant w_1, i \in N$, we should have $u_1 = w_1$. Besides, we have $u_r = w_q$, since the quantities $a_l, b_l, l = 1, \ldots, L$, cannot exceed the finite values of the components of the vector estimates x and y. Thus, the boundaries of the non-uniform grids U and $W(x, y)$ coincide, but in the general case U can be finer than $W(x, y)$, i.e., $W(x, y) \subseteq U$.

Construct a double chain of the form (A.2) using (A.1), following the steps:

1) Move from the initial vector estimate z to e.

2) Divide all operators into operators with a step along the adjacent nodes of the non-uniform grid U.

3) Sort the operators in ascending order of the step number k on the grid U.

4) Combine operators with matching k into a common operator.

5) Combine the operators within a step along the grid $W(x, y)$ into a common operator.

Step 1. The pair (z, z) can easily be obtained from (e, e) by successively applying the improvement operators $\Uparrow[\{i\}, \{i\}, w_1, z_i]$ for all $i \in N$ such that $z_i > w_1$. Recall that in (A.1) the assumption was made that $z_i \geqslant w_1, i \in N$. We substitute the resulting expression into (A.1) instead of (z, z).

Step 2. The increment interval in an arbitrary improvement operator can be divided according to the following rule:

$$\Uparrow[A, B, a, b] = \Uparrow[A, B, c, b] \Uparrow[A, B, a, c], \text{where } a < c < b. \tag{A.3}$$

Using property (A.3), each operator $\Uparrow[A, B, u_k, u_t], k < t$, in the existing chain can be expressed through operators with steps along the grid U:

$$\Uparrow[A, B, u_k, u_t] = \Uparrow[A, B, u_{t-1}, u_t] \ldots \Uparrow[A, B, u_{k+1}, u_{k+2}] \Uparrow[A, B, u_k, u_{k+1}].$$

Step 3. Consider any two adjacent operators in the resulting chain:

$$\ldots \Uparrow[C, D, u_{t-1}, u_t] \Uparrow[A, B, u_{k-1}, u_k] \ldots$$

If $k \leqslant t$, we leave the operators in the same sequence. The case $k > t$ is possible only when $(A \cup B) \cap (C \cup D) = \varnothing$, otherwise the result of the operator $\Uparrow[A, B, u_{k-1}, u_k]$ will be outside the domain of the operator $\Uparrow[C, D, u_{t-1}, u_t]$. Because these two operators affect the non-intersecting sets of components of the vector estimates, we can permute these two operators in the overall sequence, without affecting the overall result.

Using the considered permutation of adjacent operators, we sort the entire sequence of operators in ascending order of k.

Step 4. For each $k = 2, \ldots, r$, consider all operators that increase the components of the vector estimates from u_{k-1} to u_k. Due to the sorting performed in the previous step, all these operators form a subsequence:

$$\Uparrow[A_{km}, B_{km}, u_{k-1}, u_k] \ldots \Uparrow[A_{k2}, B_{k2}, u_{k-1}, u_k] \Uparrow[A_{k1}, B_{k1}, u_{k-1}, u_k].$$

Consider the interval $w_{t-1} < u_k \leqslant w_t$ on a coarser grid $W(x, y)$. For each $k = 2, \ldots, r$, it exists, since $w_1 < u_2$ and $u_r = w_q$. It turns out that during the transformation (A.1) from the vector estimate e to the vector estimate x, the components with numbers from the sets $A_{k1}, \ldots, A_{km}$ increased from u_{k-1} to u_k, and we have $x_i \geqslant u_k$. At the same time the sets $A_{k1}, \ldots, A_{km}$ are pairwise disjoint, since the same component of the vector estimate cannot be increased from u_{k-1} to u_k more than once. Since the final values of the components of the vector estimate x belong to $W(x, y)$, we have $x_i \geqslant w_t$. And this by definition means that $N_t(x) = A_{k1} \cup A_{k2} \cup \cdots \cup A_{km}$. Similarly, $N_t(y) = B_{k1} \cup B_{k2} \cup \cdots \cup B_{km}$ and the sets $B_{k1}, \ldots, B_{km}$ are pairwise disjoint.

Then, from $A_{ki} \succsim B_{ki}$, $i = 1, \ldots, m$, according to property C6 of qualitative probability (see Section 3), it follows that $N_t(x) \succsim N_t(y)$. And the considered subsequence of operators can be folded into one: $\Uparrow[N_t(x), N_t(y), u_{k-1}, u_k]$.

Step 5. The previous reasoning is the same for all steps $[u_{k-1}, u_k]$ of the grid U that fall inside the step $[w_{t-1}, w_t]$ of the grid $W(x, y)$, namely, $[u_{k_{t-1}}, u_{k_{t-1}+1}]$, $[u_{k_{t-1}+1}, u_{k_{t-1}+2}], \ldots, [u_{k_t-1}, u_{k_t}]$, where $u_{k_{t-1}} = w_{t-1}$, $u_{k_t} = w_t$. Moreover, the entire interval $[w_{t-1}, w_t]$ must be covered by these steps of the grid U, since in the vector estimates x and y the components with numbers from $N_t(x)$ and $N_t(y)$, respectively, eventually increased from w_{t-1} to w_t. Therefore, the operators $\Uparrow[N_t(x), N_t(y), u_{k-1}, u_k]$ for all $k = k_{t-1} + 1, \ldots, k_t$ can be folded into one operator according to property (A.3):

$$\Uparrow[N_t(x), N_t(y), w_{t-1}, w_t] = \Uparrow[N_t(x), N_t(y), u_{k_t-1}, u_{k_t}] \ldots \Uparrow[N_t(x), N_t(y), u_{k_{t-1}}, u_{k_{t-1}+1}].$$

The required chain (A.2) is obtained. The necessary condition of Theorem 3 is proved.

Lemma 1. *If $A = B \succ \varnothing$, then for any a and b, $xI^{\succsim}y \Rightarrow x'I^{\succsim}y'$, and $xP^{\succsim}y \Rightarrow x'P^{\succsim}y'$.*

A.3. PROOF OF LEMMA 3

The condition of Lemma 1 for $a < b$ follows from Axiom 4. Suppose that $xR^{\succsim}y$ holds for $A = B$ and $a > b$. Consider the union of sets $W = W(x, y) \cup W(x', y')$. It differs from $W(x, y)$ by adding the number b if it was not in $W(x, y)$. And it differs from $W(x', y')$ by adding the number a if it was not in $W(x', y')$. Let $W = \{w_1, \ldots, w_q\}, w_1 < \cdots < w_q$. Adding elements to $W(x, y)$ does not affect the fulfillment of the conditions of the decision rule (3), therefore, according to Theorem 3 from $xR^{\succsim}y$ it follows that

$$M_k(x) \succsim M_k(y), \quad k = 2, 3, \ldots, q. \tag{A.4}$$

In the case of $xI^{\succsim}y$ in (A.4), $M_k(x) \sim M_k(y)$ is true for all k, and in the case of $xP^{\succsim}y$ in (A.4), $M_k(x) \succ M_k(y)$ is true for at least one k.

Denote $w_{k_a} = a$, $w_{k_b} = b$. Note that for $k \leqslant k_b$ and $k > k_a$ the set equalities $M_k(x') = M_k(x)$ and $M_k(y') = M_k(y)$ hold. And for $k_b < k \leqslant k_a$ we have $M_k(x') \cup A = M_k(x)$, $M_k(x') \cap A = \varnothing$ and $M_k(y') \cup A = M_k(y)$, $M_k(y') \cap A = \varnothing$. Then, according to the additivity axiom, from $M_k(x) \sim M_k(y)$ it follows that $M_k(x') \sim M_k(y')$, and from $M_k(x) \succ M_k(y)$ it follows that $M_k(x') \succ M_k(y')$. Therefore, according to Theorem 3 taking into account (A.4) in the case $xI^{\succsim}y$ we obtain $x'I^{\succsim}y'$, and in the case $xP^{\succsim}y$ we obtain $x'P^{\succsim}y'$. Lemma 1 is proved.

A.4. PROOF OF THEOREM 1

For $a < b$ the statement of Theorem 1 follows from Axiom 4. Suppose that $xR^{\succsim}y$ holds for $a > b$. Let us deteriorate the vector estimates x and y with respect to the same event $C = A \cup B$: $x'' = (x \parallel x_i = b, i \in C)$, $y'' = (y \parallel y_i = b, i \in C)$. Then, according to Lemma 1 $x''R^{\succsim}y''$ holds. Note that the vector estimates x' and y' can be obtained by improving in the same way (from b to a) the vector estimates x'' and y'' with respect to the events $B \setminus A$ and $A \setminus B$, respectively. But according to the additivity axiom, $A \sim B$ implies $B \setminus A \sim A \setminus B$. Then according to Axiom 4 we obtain $x'R^{\succsim}y'$. Theorem 1 is proved.

A.5. PROOF OF THEOREM 2

For $A \succ B$ and $a < b$ the statement of Theorem 2 follows from Axiom 5. Suppose that $xR^{\succsim}y$ holds for $A \prec B$ and $a > b$. Let us deteriorate the vector estimates x and y with respect to the same event $C = A \cup B$: $x'' = (x \parallel x_i = b, i \in C)$, $y'' = (y \parallel y_i = b, i \in C)$. Then, according to Lemma 1, $x''R^{\succsim}y''$ holds. Note that the vector estimates x' and y' can be obtained by improving in the same way (from b to a) the vector estimates x'' and y'' with respect to the events $B \setminus A$ and $A \setminus B$, respectively. But according to the additivity axiom, $A \prec B$ implies $B \setminus A \succ A \setminus B$. Then according to Axiom 5 we obtain $x'P^{\succsim}y'$. Theorem 2 is proved.

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