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# Control of the Triaxial Orientation of a Rigid Body under a Delay in the Feedback Law

**A. Yu. Aleksandrov**

*St. Petersburg State University, St. Petersburg, Russia*
*e-mail: a.u.aleksandrov@spbu.ru*

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**Abstract**—A rigid body under the action of a linear dissipative torque and an essentially nonlinear restoring one is considered. It is assumed that there is a constant delay in the restoring torque. Using a special construction of the Lyapunov–Krasovsky functional, it is proved that the given orientation of the body is asymptotically stable for any value of delay. In addition, the case is investigated where the body is subject to PID-like controller with linear differential part and essentially nonlinear integral and proportional ones. The admissible values of the controller parameters are determined, for which it solves the problem of triaxial stabilization. The results of numerical modeling are presented, confirming the established theoretical conclusions.

*Keywords*: rigid body, triaxial orientation, delay, Lyapunov–Krasovskii functional, PID controller

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## 1. INTRODUCTION

Control of the rotational motion of a rigid body is an actual problem of nonlinear mechanics. It has wide applications in the orientation of satellites, aircraft, underwater vehicles, and robotic systems [1–3]. One important special case of this problem is triaxial stabilization of a rigid body [1, 2]. A large number of papers have been devoted to this case (see, e.g., [1–6] and the bibliography cited therein). Various approaches to constructing stabilizing controls have been proposed. It should be noted that rigid body orientation control is a nonlinear problem, since the configuration space for it is not a linear space [7].

When designing controlled systems, it is often necessary to take into account delay that occurs in the feedback loop [8, 9]. Furthermore, delay can be intentionally introduced into the system to ensure stability or sufficient smoothness of transient processes [9–13]. The presence of delay significantly complicates stability analysis. The main approaches to solving this problem are the Razumikhin method and the Lyapunov–Krasovskii functional method [8, 9].

Some approaches to the synthesis of controls that ensure thriaxial stabilization of a rigid body taking into account a constant feedback delay were proposed in [7, 14–16]. Linear control torques were constructed, and the conditions for asymptotic stability of a given body orientation were formulated in terms of the solvability of systems of linear matrix inequalities, which result in rather stringent constraints on the system parameters and the magnitude of the delay. In [17, 18], linear controls of PID-like type were used to solve the thriaxial stabilization problem.

In this paper, we first consider a rigid body subject to a linear dissipative torque and an essentially nonlinear restoring torque. Nonlinear controls are more robust with respect to delays and nonstationary disturbances than linear ones (see [19–21]). In some cases, they allow for faster convergence of the norm of the closed-loop state vector and a reduction in the steady-state error compared to linear algorithms [22].

It is assumed that the restoring torque contains a constant delay. Using the Lyapunov–Krasovskii functional method, conditions are found that guarantee the asymptotic stability of a given body orientation for any delay value. Next, we consider the case where the body is subject to a PID controller with its differential part being linear and its integral and proportional parts being significantly nonlinear. Admissible values of the controller parameters are determined for which it solves the problem of triaxial body stabilization.

## 2. PROBLEM STATEMENT

Consider a rigid body rotating around a fixed point $O$ coinciding with its mass center. Let $\omega(t)$ denote a vector of angular velocity and $Oxyz$ be principal central axes of inertia of the body.

Assume that two right triplets of mutually orthogonal unit vectors $r_1$, $r_2$, $r_3$ and $s_1, s_2, s_3$ are given, where the vectors $r_1$, $r_2$, $r_3$ are fixed in the frame $Oxyz$ and the vectors $s_1, s_2, s_3$ are fixed in the inertial space.

The angular motion of the body under the impact of a control torque $M_u$ is modeled by the Euler equations

$$\Theta\dot{\omega}(t) + \omega(t) \times (\Theta\omega(t)) = M_u, \tag{1}$$

whereas, for the vectors $s_1, s_2, s_3$, we have the Poisson kinematic equations

$$\dot{s}_i(t) = -\omega(t) \times s_i(t), \qquad i = 1, 2, 3. \tag{2}$$

Here $\Theta = \mathrm{diag}\,\{A, B, C\}$ is the inertia tensor of the body in the axes $Oxyz$ (see [1, 2]).

The problem of triaxial stabilization of the body consists in the design of a torque $M_u$ under which the system (1), (2) admits asymptotically stable equilibrium position

$$\omega = 0, \qquad s_i = r_i, \quad i = 1, 2, 3. \tag{3}$$

In [2, 4], it is shown that the required control can be chosen in the form of a sum of linear dissipative and restoring torques: $M_u = M_d + M_r$. Here

$$M_d = -D\omega(t), \tag{4}$$

$M_r = -L(t)$, $D$ is a constant positive definite matrix,

$$L(t) = \alpha_1 s_1(t) \times r_1 + \alpha_2 s_2(t) \times r_2, \tag{5}$$

$\alpha_1, \alpha_2$ are positive coefficients. In [19], it was suggested to use essentially nonlinear restoring torque

$$M_r = -f^{\nu}(t)L(t), \tag{6}$$

instead of linear one, where the vector $L(t)$ is still determined by the formula (5), $\nu = \mathrm{const} > 0$,

$$f(t) = \frac{1}{2}\left(\alpha_1\|s_1(t) - r_1\|^2 + \alpha_2\|s_2(t) - r_2\|^2\right),$$

and $\|\cdot\|$ is the Euclidean norm of a vector. The robustness of such a control with respect to nonstationary perturbations with zero mean values was proved.

In the present work, we will study conditions of the triaxial stabilization in the case where the body is under the influence of a linear torque of dissipative forces (4) and essentially nonlinear restoring torque, and there is a constant delay $\tau > 0$ in the restoring torque. Thus, the dynamical Euler equations can be rewritten in the form

$$\Theta\dot{\omega}(t) + \omega(t) \times (\Theta\omega(t)) = -f^{\nu}(t-\tau)L(t-\tau) - D\omega(t). \tag{7}$$

In addition, consider the case where PID controller of the form

$$M_u = -\beta f^{\nu}(t)L(t) - D\omega(t) - \int\limits_0^t f^{\nu}(\xi)G(t-\xi)L(\xi)d\xi \tag{8}$$

acts on the body, where $\beta$ is a constant coefficient, $G(\zeta)$ is a continuous for $\zeta \in [0, +\infty)$ matrix. We will define admissible values of the controller parameters for which it provides the existence and the asymptotic stability of the equilibrium position (3) for the closed-loop system.

## 3. STABILIZATION WITH THE AID OF A DELAYED FEEDBACK LAW

Let the system (2), (7) be given. We will assume that initial functions $\phi(\xi)$ for it belong to the space $C([-\tau, 0], \mathbb{R}^{12})$ of continuous functions with the uniform norm $\|\phi\|_\tau = \max_{\xi\in[-\tau,0]} \|\phi(\xi)\|$.

**Theorem 1.** *For any constant positive delay $\tau$, the system* (2), (7) *admits asymptotically stable equilibrium position* (3).

**Proof.** It is obvious that (3) is an equilibrium position of the closed-loop system. To prove the asymptotic stability of the equilibrium position, we choose a Lyapunov–Krasovskii functional as follows:

$$\begin{gathered}
V(\omega(t), s_{1t}, s_{2t}) = f(t) + \frac{\gamma}{2}\omega^\top(t)\Theta\omega(t) + L^\top(t)D^{-1}\Theta\omega(t) \\
+ \int\limits_{t-\tau}^{t} (\lambda(\xi+\tau-t)+\mu)(\|s_1(\xi)-r_1\|^{2\nu+2} + \|s_2(\xi)-r_2\|^{2\nu+2})d\xi \\
-L^\top(t)D^{-1}\int\limits_{t-\tau}^{t} f^{\nu}(\xi)L(\xi)d\xi,
\end{gathered} \tag{9}$$

where $\gamma, \lambda, \mu$ are positive parameters, $s_{jt}$ are restrictions of the corresponding components of the solution of the system: $s_{jt} : \xi \mapsto s_j(t+\xi)$, $\xi \in [-\tau, 0]$, $j = 1, 2$.

We obtain

$$\begin{gathered}
f(t) + c_1\gamma\|\omega(t)\|^2 - c_2(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\|\omega(t)\| \\
+\mu\int\limits_{t-\tau}^{t} (\|s_1(\xi)-r_1\|^{2\nu+2} + \|s_2(\xi)-r_2\|^{2\nu+2})d\xi \\
-c_3(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\int\limits_{t-\tau}^{t}\left(\|s_1(\xi)-r_1\|^{2\nu+1} + \|s_2(\xi)-r_2\|^{2\nu+1}\right)d\xi \\
\le V(\omega_t, s_{1t}, s_{2t}) \le f(t) + c_2(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\|\omega(t)\| \\
+ c_4\gamma\|\omega(t)\|^2 + (\lambda\tau+\mu)\int\limits_{t-\tau}^{t} (\|s_1(\xi)-r_1\|^{2\nu+2} + \|s_2(\xi)-r_2\|^{2\nu+2})d\xi \\
+ c_3(\alpha_1\|s_1(t)-r_1\| + \alpha_2\|s_2(t)-r_2\|)\int\limits_{t-\tau}^{t}\left(\|s_1(\xi)-r_1\|^{2\nu+1} + \|s_2(\xi)-r_2\|^{2\nu+1}\right)d\xi,
\end{gathered}$$

where $c_l > 0$, $l = 1, 2, 3, 4$.

We differentiate the functional (9) with respect to the system (2), (7) and obtain

$$\dot V = -\gamma\omega^\top(t)D\omega(t) - f^\nu(t)L^\top(t)D^{-1}L(t) - L^\top(t)D^{-1}\Big(\omega(t)\times(\Theta\omega(t))\Big) + \gamma f^\nu(t-\tau)\omega^\top(t)L^\top(t-\tau)$$

$$+\Big(\alpha_1(\omega(t)\times s_1(t))\times r_1 + \alpha_2(\omega(t)\times s_2(t))\times r_2\Big)^\top D^{-1}\left(\int\limits_{t-\tau}^{t} f^\nu(\xi)L(\xi)d\xi - \Theta\omega(t)\right)$$

$$-\lambda\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi + (\lambda\tau+\mu)(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2})$$

$$-\mu(\|s_1(t-\tau)-r_1\|^{2\nu+2}+\|s_2(t-\tau)-r_2\|^{2\nu+2}) \le (c_5-\gamma c_6)\,\|\omega(t)\|^2 - c_7 f^\nu(t)\|L(t)\|^2$$

$$+\gamma c_8\|\omega(t)\|(\|s_1(t-\tau)-r_1\|^{2\nu+1}+\|s_2(t-\tau)-r_2\|^{2\nu+1}) + c_9\|\omega(t)\|^2(\|s_1(t)-r_1\|+\|s_2(t)-r_2\|)$$

$$+c_{10}\|\omega(t)\|\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+1}+\|s_2(\xi)-r_2\|^{2\nu+1})d\xi - \lambda\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi$$

$$+(\lambda\tau+\mu)(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2}) - \mu(\|s_1(t-\tau)-r_1\|^{2\nu+2}+\|s_2(t-\tau)-r_2\|^{2\nu+2}),$$

where $c_5,\dots,c_{10}$ are positive coefficients.

We choose and fix a number $\epsilon\in(0,1)$. It is known [23], that there exists $\delta>0$ such that

$$\|L(t)\|^2 \ge \epsilon\left(\alpha_1^2\|s_1(t)-r_1\|^2+\alpha_2^2\|s_2(t)-r_2\|^2\right) \tag{10}$$

for $\|s_1(t)-r_1\|+\|s_2(t)-r_2\|<\delta$.

Using the relation (10) and Jensen's inequality (see [8, p. 87]), it is easy to show that if values of the parameters $\lambda,\mu,\delta$ are sufficiently small and $\gamma$ is sufficiently large, then the estimates

$$\frac{1}{2}\left(f(t)+\mu\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi + c_1\gamma\|\omega(t)\|^2\right)$$

$$\le V(\omega_t,s_{1t},s_{2t}) \le 2\left(f(t)+c_4\gamma\|\omega(t)\|^2 + (\lambda\tau+\mu)\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right), \tag{11}$$

$$\dot V \le -\frac{1}{2}\left(\gamma c_6\|\omega(t)\|^2 + \epsilon\, c_7 f^\nu(t)\left(\alpha_1^2\|s_1(t)-r_1\|^2+\alpha_2^2\|s_2(t)-r_2\|^2\right) + \lambda\int\limits_{t-\tau}^{t}(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right) \tag{12}$$

hold for $\|s_{1t}-r_1\|_\tau+\|s_{2t}-r_2\|_\tau<\delta$. Hence (see [8, p. 53, 54]), the equilibrium position (3) is asymptotically stable. The proof is completed.

*Remark 1.* Unlike known conditions of the triaxial stabilization which were obtained with the aid of linear controls with delay [7, 14–16], Theorem 1 guarantees that, in the case of essentially nonlinear restoring torque, the asymptotic stability of the required equilibrium position occurs for any $\tau > 0$. However, it should be noticed that the delay increasing results in the decreasing the value of $\delta$ that determines the region in which estimates (11), (12) are valid. Thus, for a large delay, a control law may not ensure the asymptotic stability if the required position of the axes of the body-fixed coordinate system is sufficiently "far" from the current one. In such situations, to stabilize the program orientation, one should increase values of the parameters $\alpha_1, \alpha_2$ in the control torque.

## 4. STABILIZATION WITH THE AID OF A NONLINEAR PID CONTROLLER

Let the control torque be constructed by the formula (8). Consider the system composed of the kinematic equations (2) and dynamical Euler equations of the following form:

$$\Theta\dot{\omega}(t) + \omega(t) \times (\Theta\omega(t)) = -\beta f^{\nu}(t)L(t) - D\omega(t) - \int_0^t f^{\nu}(\xi)G(t-\xi)L(\xi)d\xi. \tag{13}$$

Every solution of this system is defined by the initial conditions: $\omega(\xi + t_0) = w(\xi)$, $s_i(\xi + t_0) = \psi_i(\xi)$ for $\xi \in [-t_0, 0]$, $i = 1, 2, 3$. Here $t_0 \geq 0$, whereas the functions $w(\xi)$, $\psi_i(\xi)$ belong to the space $C([-t_0, 0], \mathbb{R}^3)$ of continuous functions $\varphi(\xi)$ with the norm $\|\varphi\|_{[-t_0,0]} = \max_{\xi\in[-t_0,0]} \|\varphi(\xi)\|$.

*Assumption 1.* The matrix $G(\zeta)$ possesses the following properties:

1) $\displaystyle\int_0^{+\infty} \|G(\xi)\|d\xi < +\infty,$

2) $\displaystyle\int_\zeta^{+\infty} \|G(\xi)\|d\xi \leq \sigma\|G(\zeta)\|$ for $\zeta \geq 0$, where $\sigma = \text{const} > 0$.

*Remark 2.* Assumption 1 is satisfied in the case when $G(\zeta) = \exp(-a\zeta)\tilde{G}$, where $a$ is a positive constant, $\tilde{G}$ is a constant matrix. It should be noted that PID controllers with such kernels are widely applied in control theory (see, e.g., [8, 24]).

Let $\Xi = \beta I + \int_0^{+\infty} G(\zeta)d\zeta$, where $I$ is the identity matrix.

*Assumption 2.* The matrix $\Xi D + D\Xi^\top$ is positive definite.

**Theorem 2.** *If Assumptions 1 and 2 are satisfied, then the system* (2), (13) *admits asymptotically stable equilibrium position* (3).

**Proof.** It is obvious that (3) is an equilibrium position of the closed-loop system. To prove its asymptotic stability, we construct a Lyapunov–Krasovskii functional in the form

$$\begin{aligned} V(t, \omega(t), s_{1t}, s_{2t}) &= f(t) + \frac{\gamma}{2}\omega^\top(t)\Theta\omega(t) + L^\top(t)D^{-1}\Theta\omega(t) \\ &+ \mu\int_0^t\int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi \\ &- L^\top(t)D^{-1}\int_0^t\int_{t-\xi}^{+\infty} G(\zeta)d\zeta f^{\nu}(\xi)L(\xi)d\xi, \end{aligned} \tag{14}$$

where $\gamma, \mu$ are positive parameters, $s_{jt}$ are restrictions of the corresponding components of a solution of the system, $s_{jt} : \xi \mapsto s_j(t + \xi)$, $\xi \in [-t, 0]$, $j = 1, 2$.

The estimates

$$f(t) + c_1\gamma\|\omega(t)\|^2 - c_2(\alpha_1\|s_1(t) - r_1\| + \alpha_2\|s_2(t) - r_2\|)\|\omega(t)\|$$
$$+ \mu\int_0^t \int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi - c_3(\alpha_1\|s_1(t) - r_1\|$$
$$+ \alpha_2\|s_2(t) - r_2\|)\int_0^t \int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+1} + \|s_2(\xi) - r_2\|^{2\nu+1})d\xi$$
$$\le V(t, \omega_t, s_{1t}, s_{2t}) \le f(t) + c_2(\alpha_1\|s_1(t) - r_1\| + \alpha_2\|s_2(t) - r_2\|)\|\omega(t)\|$$
$$+ c_4\gamma\|\omega(t)\|^2 + \mu\int_0^t \int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi$$
$$+ c_3(\alpha_1\|s_1(t) - r_1\| + \alpha_2\|s_2(t) - r_2\|)\int_0^t \int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+1} + \|s_2(\xi) - r_2\|^{2\nu+1})d\xi,$$

$$\dot V \le (c_5 - \gamma c_6)\|\omega(t)\|^2 - f^\nu(t)L^\top(t)D^{-1}\Xi L(t)$$
$$- \mu\int_0^t \|G(t-\xi)\|(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi +$$
$$+ \mu\int_0^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(t) - r_1\|^{2\nu+2} + \|s_2(t) - r_2\|^{2\nu+2})$$
$$+ \gamma\beta c_7\|\omega(t)\|(\|s_1(t) - r_1\|^{2\nu+1} + \|s_2(t) - r_2\|^{2\nu+1})$$
$$+ c_8\|\omega(t)\|^2(\alpha_1\|s_1(t) - r_1\| + \alpha_2\|s_2(t) - r_2\|)$$
$$+ \gamma c_9\|\omega(t)\|\int_0^t \|G(t-\xi)\|(\|s_1(\xi) - r_1\|^{2\nu+1} + \|s_2(\xi) - r_2\|^{2\nu+1})d\xi$$
$$+ c_{10}\|\omega(t)\|\int_0^t \int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+1} + \|s_2(\xi) - r_2\|^{2\nu+1})d\xi,$$

are valid for the functional (14) and its derivative with respect to the system (2), (13), where $c_1, \ldots, c_{10}$ are positive coefficients.

As well as in the proof of Theorem 1, we choose a number $\epsilon \in (0, 1)$ and find $\delta > 0$ such that (10) holds for $\|s_1(t) - r_1\| + \|s_2(t) - r_2\| < \delta$. Taking into account Assumption 2, we obtain

$$f^\nu(t)L^\top(t)D^{-1}\Xi L(t) \ge \epsilon\, c_{11}(\|s_1(t) - r_1\|^{2\nu+2} + \|s_2(t) - r_2\|^{2\nu+2})$$

for $\|s_1(t) - r_1\| + \|s_2(t) - r_2\| < \delta$, where $c_{11} = \text{const} > 0$.

In addition, using Assumption 1 and the Hölder inequality, we obtain

$$\int_0^t \int_{t-\xi}^{+\infty} \|G(\zeta)\|d\zeta(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi$$
$$\le \sigma\int_0^t \|G(t-\xi)\|(\|s_1(\xi) - r_1\|^{2\nu+2} + \|s_2(\xi) - r_2\|^{2\nu+2})d\xi,$$

$$\int_0^t \|G(t-\xi)\|\|s_j(\xi) - r_j\|^{2\nu+1}d\xi \le c_{12}\left(\int_0^t \|G(t-\xi)\|\|s_j(\xi) - r_j\|^{2\nu+2}d\xi\right)^{\frac{2\nu+1}{2\nu+2}}, \quad j = 1, 2,$$

where $c_{12} = \text{const} > 0$.

With the aid of the derived estimates, it is easy to show that if values of $\delta$ and $\mu$ are sufficiently small and $\gamma$ is sufficiently large, then the inequalities

$$\frac{1}{2}\left(f(t)+\mu\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi+c_1\gamma\|\omega(t)\|^2\right)$$

$$\le V(t,\omega_t,s_{1t},s_{2t})\le 2\left(f(t)+c_4\gamma\|\omega(t)\|^2\right.$$

$$\left.+\mu\int\limits_0^t\int\limits_{t-\xi}^{+\infty}\|G(\zeta)\|d\zeta(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right),$$

$$\dot V\le-\frac{1}{2}\left(\gamma c_6\|\omega(t)\|^2+\epsilon\, c_{11}(\|s_1(t)-r_1\|^{2\nu+2}+\|s_2(t)-r_2\|^{2\nu+2})\right.$$

$$\left.+\mu\int\limits_0^t\|G(t-\xi)\|(\|s_1(\xi)-r_1\|^{2\nu+2}+\|s_2(\xi)-r_2\|^{2\nu+2})d\xi\right)$$

hold for $\|s_{1t}-r_1\|_{[-t,0]}+\|s_{2t}-r_2\|_{[-t,0]}<\delta$. Hence (see [9, p. 211, 212]), the equilibrium position (3) is asymptotically stable. The proof is completed.

*Remark 3.* Theorem 2 states that the considered controller can provide the asymptotic stability of the program orientation even in the case where its proportional part is not restoring, i.e., where $\beta\le 0$. The destabilizing effect of this part is compensated via an appropriate choice of the integral term.

Indeed, if $\beta\le 0$, then one can choose the matrix $G(\zeta)$ in the form $G(\zeta)=ag(\zeta)I$, where $a$ is a positive parameter, $g(\zeta)$ is a scalar continuous for $\zeta\in[0,+\infty)$ function possessing the following properties:

a) $\int\limits_0^{+\infty}|g(\xi)|d\xi<+\infty$,

b) $\int\limits_\zeta^{+\infty}|g(\xi)|d\xi\le\sigma|g(\zeta)|$ for $\zeta\ge 0$, where $\sigma=\mathrm{const}>0$,

c) $\int\limits_0^{+\infty}g(\xi)d\xi>0$.

In this case, Assumption 1 is satisfied, whereas Assumption 2 comes down to the fulfillment of the inequality $\beta+a\int\limits_0^{+\infty}g(\zeta)d\zeta>0$, which is valid for sufficiently large value of the parameter $a$.

## EXAMPLE

Consider a rigid body with principal central moments of inertia $A=20$, $B=25$, $C=35$, and choose the vectors $r_1=(1/\sqrt{2},1/\sqrt{2},0)^\top$, $r_2=(-1/\sqrt{2},1/\sqrt{2},0)^\top$, $r_3=(0,0,1)^\top$ as a program orientation. From here on, all physical quantities have dimensions in the SI system.

First, perform numerical simulation in the case where the angular motion of the body is described by the system (2), (7), and after that compare its results with those obtained for the system (2), (13). Let $\alpha_1=\alpha_2=1$, $\nu=0,3$, $D=2I$, $\tau=1$, $\beta=1$, $G(\zeta)=\exp(-2\zeta)I$.

In both cases, the following initial conditions were chosen for the numerical simulation: $t_0=0$, $\omega(t)=(0.3,-0.4,0.4)^\top$, $s_1(t)=(0,0,1)^\top$, $s_2(t)=(2/\sqrt{5},1/\sqrt{5},0)^\top$, $s_3(t)=(-1/\sqrt{5},2/\sqrt{5},0)^\top$ for

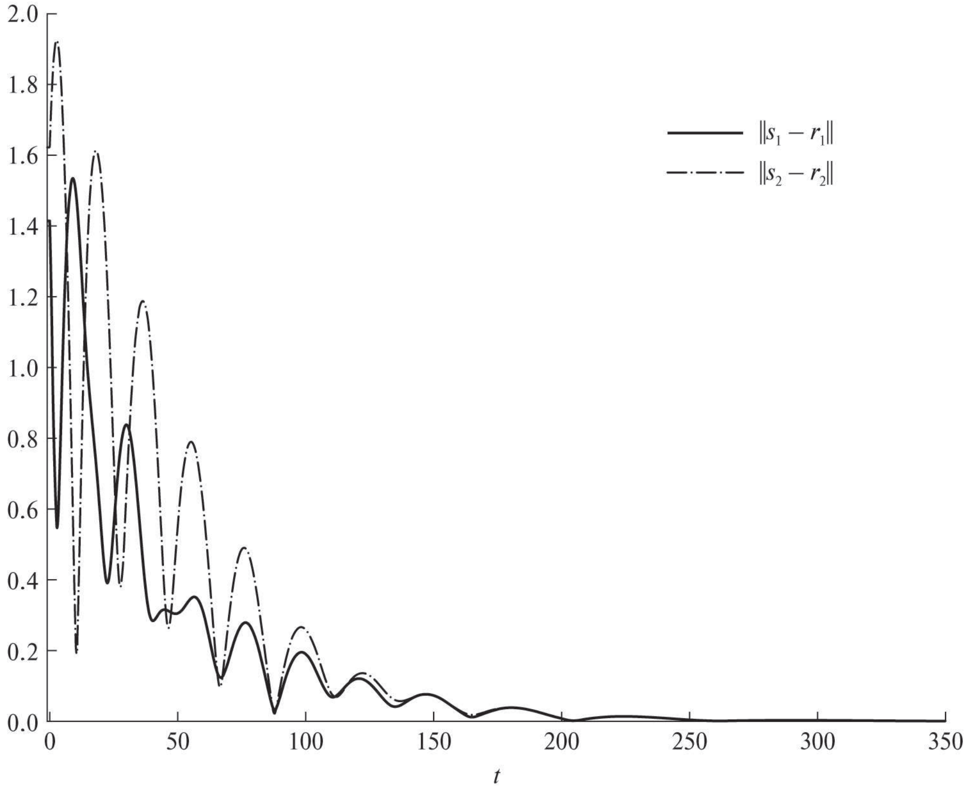


**Fig. 1.** Results of the numerical simulation under using the delay feedback law.

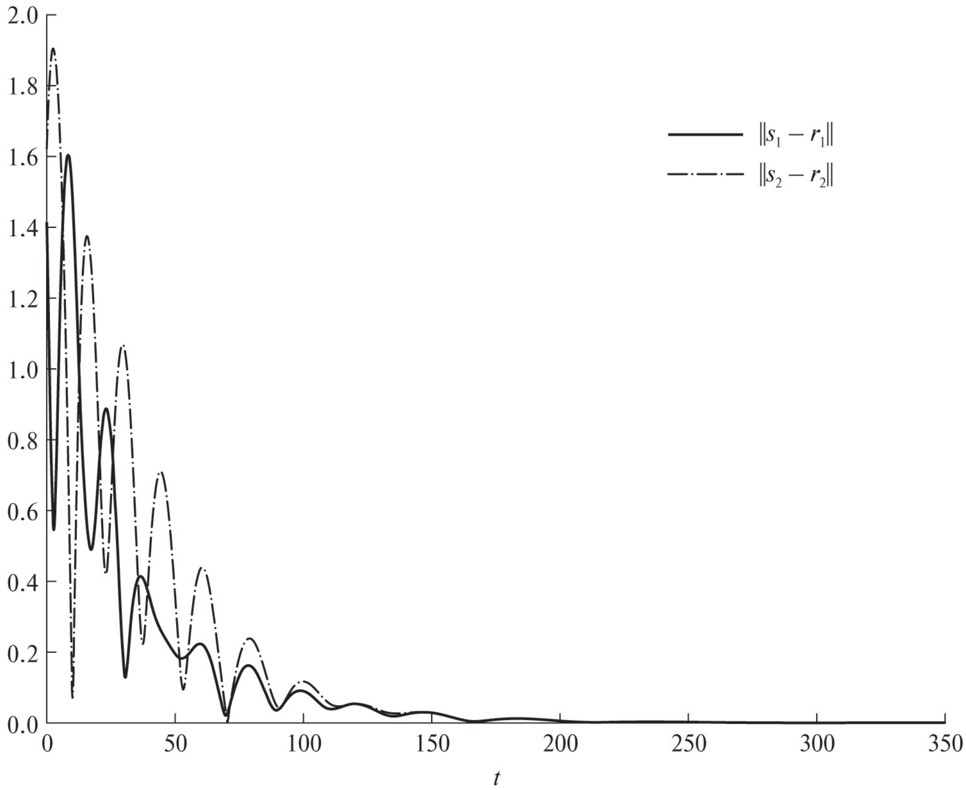


**Fig. 2.** Results of the numerical simulation under using the PID controller.

$t \in [-1, 0]$. In Figs. 1 and 2 the dependence of the values $\|s_1 - r_1\|$, $\|s_2 - r_2\|$ on time is shown. The simulation results demonstrate the asymptotic stability of the program orientation. However, under the using of PID controller, transient processes become smoother and their duration is reduced.

## 5. CONCLUSION

Two approaches to the design of control torques ensuring the triaxial stabilization of a rigid body are considered. The first one is based on the use of linear torque of dissipative forces and essentially nonlinear restoring torque containing a constant delay. It is proved that, unlike previously constructed linear control torques, the control proposed in this paper is robust with respect to delay. The second approach is based on the use of a PID controller with linear differential component and essentially nonlinear integral and proportional ones. Admissible values of the controller parameters for which it solves the triaxial stabilization problem are found. The established results imply that the asymptotic stability of program orientation can be ensured even when the proportional component of the controller is not restorative. The destabilizing effect of this component is compensated for by appropriately choosing the integral term.

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