ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 7, pp. 634–646.

Reachable Configurations of a Two-Link Manipulator under Time-Optimal Control with Angular Velocity Constraints

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March 28, 2025
August 26, 2025
November 20, 2025

Abstract—This paper considers the problem of constructing reachable domains for a planar two-link manipulator with a statically balanced second link under time-optimal control of its motion with angular velocity constraints for the links. The optimal controls belong to the class of relay regimes and contain the minimum total number of switchings sufficient to satisfy the boundary conditions: the system is at rest at the initial and final time instants. A method is developed to find reachable domains in the manipulator's configuration plane under the proposed controls with angular velocity constraints.

Keywords: reachable domain, two-link manipulator, optimal control, angular velocity constraints

DOI: 10.7868/S1608303226070067

1. INTRODUCTION

Two-link manipulators are robots widely used in various branches of modern industry, independently and as part of multi-link systems where two links perform the main motions during industrial operations. The development of efficient programmed control regimes for such manipulators remains a topical problem. If reducing the manipulator's cycle time accelerates the industrial process, it is reasonable to design time-optimal control regimes. Much research has been devoted both to mechanical models of two-link manipulators [1–9] and to their dynamic interaction with drives [10–13]. Note that when designing such regimes, one should consider not only the speed of operations but also the constraints imposed on the angular velocities of the links. Exceeding the permissible limits of angular velocities may cause mechanical damage, loss of system stability, and reduced accuracy of task fulfilment. The issues of optimizing time-optimal controls with angular velocity constraints were addressed in [14, 15]. At the same time, accounting for angular velocity constraints possibly increases the number of switchings in the optimal control structure and leads to time intervals during which the manipulator or the motor operates at limiting angular velocities. To implement such optimal controls, it is necessary to calculate the switching instants of the control action precisely. However, due to inevitable computational errors, an increase in the number of switchings can significantly affect control accuracy.

Thus, a key problem is to construct reachable domains on the manipulator's configuration plane that are implemented under optimal controls with a minimum number of switchings. At the same time, limiting angular velocities should be achieved only at the control switching instants, depending on the final configuration. Below, we solve this problem for the two-link manipulator model with a statically balanced second link [16–18]. Note that in the context of time-optimal control, simplified and complete models of manipulators with two degrees of freedom were also considered in [19–25].

2. MANIPULATOR MODEL AND PROBLEM STATEMENT

Consider a mechanical system consisting of two absolutely rigid equal-length links G_1 and G_2 connected by a joint O_2 (a two-link manipulator, see Fig. 1). Link G_1 is connected to a fixed base by joint O_1. The joints are ideal and cylindrical, and their axes are parallel to each other. A gripper is attached to the end of the second link at point O_3. By assumption, the linear dimensions of the gripper are much smaller than the lengths of the links; in the analysis of transport motions, the gripper will be treated as a material point. The manipulator is controlled by two independent drives D_1 and D_2. Drive D_1 carries out the interaction of the first link with the base; drive D_2, the interaction between links G_1 and G_2. The control functions in this manipulator model, denoted by M_1 and M_2, are the torques about axes O_1 and O_2 developed by drives D_1 and D_2, respectively.

The above system performs plane-parallel motion in a horizontal plane perpendicular to the axes of joints O_1 and O_2.

In the case where the center of mass C of the second link G_2 lies on the axis of the second joint (i.e., the second link of the manipulator is statically balanced), the system dynamics are described by the Lagrange equations [1]

$$\left(I_1 + m_2 L^2\right) \ddot{\varphi}_1 = M_1 - M_2, \qquad I_2 \, \ddot{\varphi}_2 = M_2. \tag{2.1}$$

Here, φ_1 and φ_2 stand for the angles between the axis O_1x and the lines O_1O_2 and O_2O_3, respectively; $L = |O_1O_2| = |O_2O_3|$ is the length of the first/second link; I_1 and I_2 are the moments of inertia of links G_1 and G_2 about the axes of joints O_1 and O_2, respectively; finally, m_2 is the mass of link G_2.

By assumption, the manipulator's links are designed to make over one full revolution in both positive and negative directions. Depending on the industrial purpose, this allows, e.g., moving the end O_3 of the manipulator along a spiral trajectory in the workspace.

Note that in time-optimal problems, equations (2.1) are considered not only as a basic model for studying the full equations of a manipulator with a displaced center of mass of the second link [2–9] but also separately [16–18].

Problem. It is required to determine the laws of variation of the control torques $M_1 = M_1(t)$ and $M_2 = M_2(t)$, and construct the reachable domains of the manipulator's final rest states, under

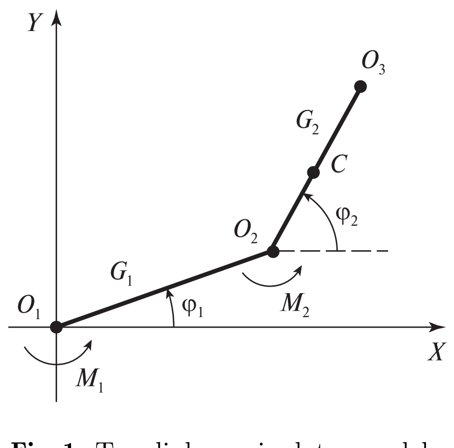

Fig. 1. Two-link manipulator model.

which system (2.1) will move from the initial rest state

$$\varphi_1(0) = \varphi_1^0, \quad \dot{\varphi}_1(0) = 0, \qquad \varphi_2(0) = \varphi_2^0, \quad \dot{\varphi}_2(0) = 0 \tag{2.2}$$

to any final rest state

$$\varphi_1(T) = \varphi_1^T, \quad \dot{\varphi}_1(T) = 0, \qquad \varphi_2(T) = \varphi_2^T, \quad \dot{\varphi}_2(T) = 0 \tag{2.3}$$

within the constructed domain in the minimum time T while satisfying

$$|M_1(t)| \le M_1^0, \quad |M_2(t)| \le M_2^0 \tag{2.4}$$

(control constraints) and

$$|\dot{\varphi}_1(t)| \le a_1, \quad |\dot{\varphi}_2(t)| \le a_2 \tag{2.5}$$

(angular velocity constraints) during the entire motion.

In the dimensionless variables (with the primes subsequently omitted)

$$\begin{aligned} &t' = \left[M_2^0/\left(m_2L^2\right)\right]^{1/2} t, \qquad I_i' = I_i/\left(m_2L^2\right), \quad M_i' = M_i/M_2^0, \\ &a_i' = \left[M_2^0/\left(m_2L^2\right)\right]^{-1/2} a_i, \quad \varphi_i' = \varphi_i - \varphi_i^0, \qquad {\varphi'}_i^{0,T} = \varphi_i^{0,T} - \varphi_i^0, \quad i = 1,2, \end{aligned} \tag{2.6}$$

the relations (2.1)–(2.5) take the simplified form

$$(I_1+1)\,\ddot{\varphi}_1 = M_1 - M_2, \quad I_2\,\ddot{\varphi}_2 = M_2, \tag{2.7}$$

$$\varphi_i(0) = \dot{\varphi}_i(0) = 0, \quad i = 1,2, \tag{2.8}$$

$$\varphi_i(T) = \varphi_i^T, \quad \dot{\varphi}_i(T) = 0, \qquad i = 1,2, \tag{2.9}$$

$$|M_1(t)| \le M_1^0, \quad |M_2(t)| \le 1, \tag{2.10}$$

$$|\dot{\varphi}_1(t)| \le a_1, \quad |\dot{\varphi}_2(t)| \le a_2. \tag{2.11}$$

3. OPTIMAL CONTROLS WITHOUT ANGULAR VELOCITY CONSTRAINTS

Using the results of [1, 9], the solution of the time-optimal problem for system (2.7)–(2.10) without the angular velocity constraints (2.11)) can be represented as follows:

$$M^* = \begin{cases} (M_1^*, M_2) & \text{if } (\varphi_1^T, \varphi_2^T) \in \Phi_1 = \bigcup\limits_{\alpha=0}^{1} \Phi_1^{(\alpha)} \\ (M_1, M_2^*) & \text{if } (\varphi_1^T, \varphi_2^T) \in \Phi_2 = \bigcup\limits_{\alpha=0}^{1} \Phi_2^{(\alpha)}, \end{cases} \tag{3.1}$$

$$M^* = (M_1^*, M_2^*) \quad \text{if } (\varphi_1^T, \varphi_2^T) \in \Phi_0 = \bigcup_{\alpha,k=0}^{1} \Phi_0^{(\alpha,k)}.$$

According to (3.1), we have:

1. If $(\varphi_1^T, \varphi_2^T) \in \Phi_1 = \bigcup\limits_{\alpha=0}^{1} \Phi_1^{(\alpha)}$, where

$$\Phi_1^{(\alpha)} = \left\{ \begin{matrix} (\varphi_1^T, \varphi_2^T)\colon (-1)^\alpha\,(I_1+1)\left[I_2\left(M_1^0 + (-1)^\alpha\right)\right]^{-1}\varphi_1^T \\ < \varphi_2^T < (-1)^\alpha\,(I_1+1)\left[I_2\left(M_1^0 - (-1)^\alpha\right)\right]^{-1}\varphi_1^T, \\ (-1)^\alpha\,\varphi_1^T > 0 \end{matrix} \right\}, \ \alpha = 0,1, \tag{3.2}$$

then the optimal control M_1^* is a relay function with one switching point of the form

$$\begin{gathered} M_1^*(t) = M_1^0 \,\mathrm{sign}\left\{(t_0 - t)\left[(I_1+1)\,\varphi_1^T + I_2\,\varphi_2^T\right]\right\}, \\ t_0 = T/2, \quad T = 2\left(\left|(I_1+1)\,\varphi_1^T + I_2\,\varphi_2^T\right|/M_1^0\right)^{1/2}, \end{gathered} \tag{3.3}$$

and $M_2(t)$ is any admissible control, in particular, a relay function with two switching points of the form

$$M_2(t) = \begin{cases} (-1)^\gamma, & t \in [0, t_1) \cup [t_2, T], \\ (-1)^{\gamma+1}, & t \in [t_1, t_2), \end{cases} \qquad \gamma = 0, 1, \tag{3.4}$$

$$\begin{gathered} t_1 = (-1)^\gamma I_2 \varphi_2^T/T + T/4, \qquad t_2 = (-1)^\gamma I_2 \varphi_2^T/T + 3T/4, \\ T = 2\left(\left|(I_1+1)\varphi_1^T + I_2\varphi_2^T\right|/M_1^0\right)^{1/2}, \quad T = 2(t_2 - t_1), \quad 0 \le t_1 \le t_2 \le T. \end{gathered}$$

2. If $(\varphi_1^T, \varphi_2^T) \in \Phi_2 = \bigcup\limits_{\alpha=0}^{1} \Phi_2^{(\alpha)}$, where

$$\Phi_2^{(\alpha)} = \left\{\begin{aligned} &(\varphi_1^T, \varphi_2^T)\colon -(-1)^\alpha \left[I_2(M_1^0 + (-1)^\alpha\right](I_1+1)^{-1}\varphi_2^T \\ &< \varphi_1^T < (-1)^\alpha (I_1+1)\left[I_2\left(M_1^0 - (-1)^\alpha\right)\right]^{-1}\varphi_2^T, \\ &(-1)^\alpha \varphi_2^T > 0 \end{aligned}\right\}, \ \alpha = 0, 1, \tag{3.5}$$

then, conversely,

$$M_2(t) = M_2^*(t) = \mathrm{sign}\left[(t_0 - t) I_2 \varphi_2^T\right], \quad t_0 = T/2, \quad T = 2\left(\left|I_2\varphi_2^T\right|\right)^{1/2}, \tag{3.6}$$

and $M_1(t)$ is any admissible control, in particular, a relay function with two switching points of the form

$$M_1(t) = \begin{cases} (-1)^\gamma, & t \in [0, t_1) \cup [t_2, T], \\ (-1)^{\gamma+1}, & t \in [t_1, t_2), \end{cases} \qquad \gamma = 0, 1, \tag{3.7}$$

$$\begin{gathered} t_1 = (-1)^\gamma \left[(I_1+1)\varphi_1^T + I_2\varphi_2^T\right]/\left(M_1^0 T\right) + T/4, \\ t_2 = (-1)^\gamma \left[(I_1+1)\varphi_1^T + I_2\varphi_2^T\right]/\left(M_1^0 T\right) + 3T/4, \\ T = 2\left(\left|I_2\varphi_2^T\right|\right)^{1/2}, \ \ T = 2(t_2 - t_1), \ \ 0 \le t_1 \le t_2 \le T. \end{gathered}$$

3. If $(\varphi_1^T, \varphi_2^T) \in \Phi_0 = \bigcup\limits_{\alpha,k=0}^{1} \Phi_0^{(\alpha,k)}$, where

$$\Phi_0^{(\alpha,k)} = \left\{\begin{aligned} &(\varphi_1^T, \varphi_2^T)\colon \varphi_2^T = (-1)^k (I_1+1)\left[I_2\left(M_1^0 - (-1)^k\right)\right]^{-1}\varphi_1^T, \\ &(-1)^\alpha \varphi_1^T > 0 \end{aligned}\right\}, \tag{3.8}$$

$$\alpha, k = 0, 1,$$

then $M_1^*(t)$ and $M_2^*(t)$ are relay functions with one switching point of the form

$$\begin{gathered} M_1^*(t) = (-1)^\alpha M_1^0 \mathrm{sign}\,(t_0 - t), \quad M_2^*(t) = (-1)^k \mathrm{sign}\,(t_0 - t), \\ t_0 = T/2, \ \ \alpha, k = 0, 1. \end{gathered} \tag{3.9}$$

In (3.9), T is determined by the equivalent formulas (3.3) or (3.6).

4. CONSTRUCTION OF REACHABLE DOMAINS OF THE TWO-LINK MANIPULATOR UNDER OPTIMAL CONTROLS WITH ANGULAR VELOCITY CONSTRAINTS

We introduce the following notation:

$$R(M^*) = \bigcup_{i=0}^{2} R_i(M^*), \tag{4.1}$$

$$\begin{gathered} R_0(M^*) = \bigcup_{\alpha,k=0}^{1} R_0^{(\alpha,k)}(M^*), \quad R_i(M^*) = \bigcup_{\alpha=0}^{1} R_i^{(\alpha)}(M^*), \\ R_0^{(\alpha,k)}(M^*) = \left\{(\varphi_1^T,\varphi_2^T) \in \Phi_0^{(\alpha,k)}:\ |\dot\varphi_1| \le a_1,\ |\dot\varphi_2| \le a_2\right\}, \\ R_i^{(\alpha)}(M^*) = \left\{(\varphi_1^T,\varphi_2^T) \in \Phi_i^{(\alpha)}:\ |\dot\varphi_1| \le a_1,\ |\dot\varphi_2| \le a_2\right\}, \end{gathered}$$

where the control M^* is given by (3.1).

According to (4.1), $R(M^*)$ is the domain on the configuration plane $(\varphi_1^T,\varphi_2^T)$ where the optimal regimes (3.1) without violating the angular velocity constraints (2.11).

As shown by the constructed optimal solutions for the piecewise-constant control laws (3.1), for the control torques $M_1(t)$ and $M_2(t)$ with two switching instants, there are two control regimes $\gamma = 0{,}1$ equivalent, in terms of time optimality, together with the one-switching control. Therefore, it suffices to consider only the regimes with $\gamma = 0$.

Due to the invariance of system (2.7)–(2.9) with respect to the change of variables $\varphi_i \to -\varphi_i$, $\varphi_1^T \to -\varphi_1^T$, $\varphi_2^T \to -\varphi_2^T$, and $M^* \to -M^*$, the domains $R_0^{(0,0)}$ and $R_0^{(1,0)}$, $R_0^{(0,1)}$ and $R_0^{(1,1)}$, $R_1^{(0)}$ and $R_1^{(1)}$, as well as $R_2^{(0)}$ and $R_2^{(1)}$, are pairwise symmetric about the origin. Hence, we will focus only on the domains $R_0^{(0,0)}$, $R_0^{(0,1)}$, $R_1^{(0)}$, and $R_2^{(0)}$.

The domains $R_i^{(0)}$, $i = 1,2$, are constructed using the following procedure. Based on the analysis of the functions $\dot\varphi_i(\varphi_i)$, $i = 1,2$, obtained under the optimal control regimes (3.2), (3.4), (3.6), (3.7) transferring system (2.7) from (2.8) to (2.9) for each degree of freedom, the angular velocity of the ith link achieves the maximum absolute value at the time instant t_0 if this link is rotated with the one-switching control, and at the time instants t_1 or t_2 if it is rotated with the two-switching control. Consequently, conditions (2.11) (or (4.1)) are not violated under the following relations:

$$\begin{aligned} &\max_{t\in[0,T]} |\dot\varphi_1(t)| = |\dot\varphi_1(t_j)| \le a_1, \quad j = 0,1,2, \quad 0 < t_1 \le t_0 \le t_2, \\ &\max_{t\in[0,T]} |\dot\varphi_2(t)| = |\dot\varphi_2(t_j)| \le a_2, \quad j = 0,1,2, \quad 0 < t_1 \le t_0 \le t_2. \end{aligned} \tag{4.2}$$

Substituting $t_0 = t_0(\varphi_1^T,\varphi_2^T)$, $t_1 = t_1(\varphi_1^T,\varphi_2^T)$, and $t_2 = t_2(\varphi_1^T,\varphi_2^T)$ from (3.3), (3.4), (3.6), and (3.7) into conditions (4.2) yields two inequalities in φ_1^T and φ_2^T, which define the domains

$$\begin{aligned} D_1^{(0)}(a_1) &= \left\{(\varphi_1^T,\varphi_2^T) \in \Phi_1^{(0)}:\ |\dot\varphi_1| \le a_1\right\}, \\ D_1^{(0)}(a_2) &= \left\{(\varphi_1^T,\varphi_2^T) \in \Phi_1^{(0)}:\ |\dot\varphi_2| \le a_2\right\}, \\ D_2^{(0)}(a_1) &= \left\{(\varphi_1^T,\varphi_2^T) \in \Phi_2^{(0)}:\ |\dot\varphi_1| \le a_1\right\}, \\ D_2^{(0)}(a_2) &= \left\{(\varphi_1^T,\varphi_2^T) \in \Phi_2^{(0)}:\ |\dot\varphi_2| \le a_2\right\}. \end{aligned} \tag{4.3}$$

Then the required domains are given by

$$R_1^{(0)} = D_1^{(0)}(a_1) \cap D_1^{(0)}(a_2), \qquad R_2^{(0)} = D_2^{(0)}(a_1) \cap D_2^{(0)}(a_2). \tag{4.4}$$

Let us proceed to the domains $R_0^{(0,0)}$, $R_0^{(0,1)}$, $R_1^{(0)}$, and $R_2^{(0)}$.

4.1. Construction of $R_0^{(0,0)}$ *and* $R_0^{(0,1)}$

In the domains $\Phi_0^{(0,0)}$ and $\Phi_0^{(0,1)}$, representing a pair of half-lines starting from the origin, both control torques M_1 and M_2 have one switching point (3.9) each. If the problem parameters satisfy the relation

$$a_1 (I_1 + 1) \left(M_1^0 - 1\right)^{-1} < a_2 I_2, \tag{4.5}$$

then the angular velocity $\dot\varphi_1$ achieves its maximum absolute value a_1 earlier than $\dot\varphi_2$ achieves a_2. This conclusion follows from the simple integration of system (2.7), (2.8) up to the time instant $t = t_0$ (when $\dot\varphi_1$ hits the constraint).

In this case, $R_0^{(0,0)}$ and $R_0^{(0,1)}$ consist of two segments: one endpoint coincides with the origin $(0, 0)$, and the other endpoints have the coordinates

$$A = (\varphi_1^A, \varphi_2^A), \qquad B = (\varphi_1^B, \varphi_2^B), \tag{4.6}$$

$$\begin{aligned} \varphi_1^A &= a_1^2 (I_1 + 1) \left(M_1^0 - 1\right)^{-1}, & \varphi_2^A &= a_1^2 (I_1 + 1)^2 I_2^{-1} \left(M_1^0 - 1\right)^{-2}, \\ \varphi_1^B &= a_1^2 (I_1 + 1) \left(M_1^0 + 1\right)^{-1}, & \varphi_2^B &= -a_1^2 (I_1 + 1)^2 I_2^{-1} \left(M_1^0 + 1\right)^{-2}. \end{aligned}$$

When transferring system (2.7), (2.8) to points A and B at the switching instant $t = t_0$, the equality $\dot\varphi_1 = a_1$ and the inequality $\dot\varphi_2 < a_2$ hold simultaneously.

In the case of the reverse inequality (4.5),

$$a_1 (I_1 + 1) \left(M_1^0 - 1\right)^{-1} > a_2 I_2, \tag{4.7}$$

on the contrary, the angular velocity $\dot\varphi_2$ achieves its maximum absolute value a_2 earlier. Then the corresponding domains $R_0^{(0,0)}$ and $R_0^{(0,1)}$ consist of two segments: one endpoint coincides with the origin $(0, 0)$, and the other endpoints have the coordinates

$$C = (\varphi_1^C, \varphi_2^C), \qquad D = (\varphi_1^D, \varphi_2^D), \tag{4.8}$$

$$\begin{aligned} \varphi_1^C &= a_2^2 I_2^2 \left(M_1^0 - 1\right) (I_1 + 1)^{-1}, & \varphi_2^C &= a_2^2 I_2, \\ \varphi_1^D &= a_2^2 I_2^2 \left(M_1^0 + 1\right) (I_1 + 1)^{-1}, & \varphi_2^D &= -a_2^2 I_2^2. \end{aligned}$$

When transferring system (2.7), (2.8) to points C and D at the switching instant $t = t_0$, the equality $\dot\varphi_2 = a_2$ and the inequality $\dot\varphi_1 < a_1$ hold simultaneously.

4.2. Construction of $R_1^{(0)}$

Let $(\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)}$. Then, with single integration of system (2.7), (2.8) on the interval $[0, T]$ under the corresponding controls (3.3), (3.4) ($\gamma = 0$), we obtain the functions $\dot\varphi_i(\varphi_i)$, $i = 1, 2$, shown in Fig. 2.

According to the calculations,

$$|\dot\varphi_1(t)| \le \max_{t \in [0, T_1]} |\dot\varphi_1(t)| = |\dot\varphi_1(t_0)|, \tag{4.9}$$

$$|\dot\varphi_2(t)| \le \max_{t \in [0, T_1]} |\dot\varphi_2(t)| = \begin{cases} |\dot\varphi_2(t_1)|, & \varphi_2^T > 0, \\ |\dot\varphi_2(t_2)|, & \varphi_2^T < 0. \end{cases} \tag{4.10}$$

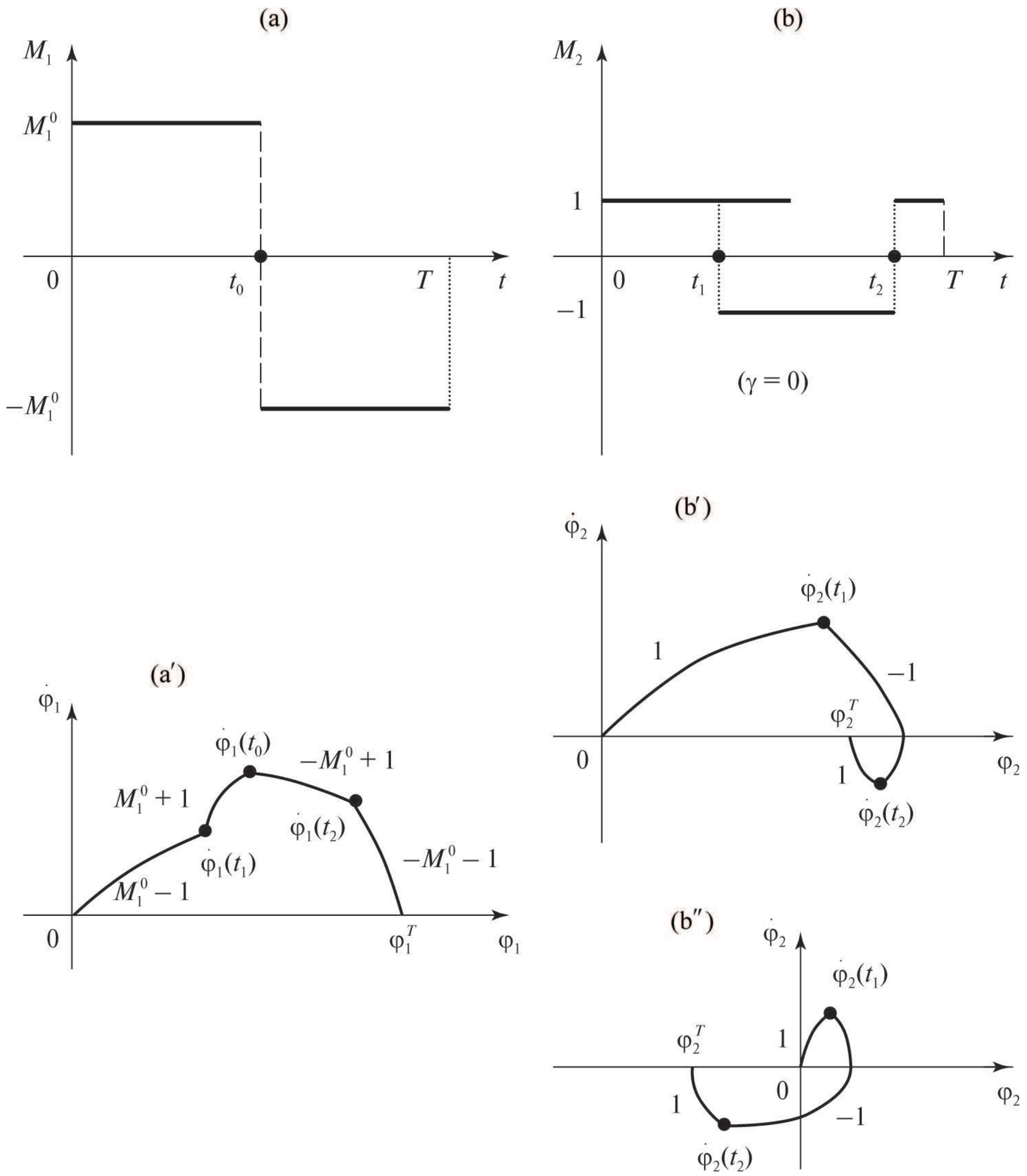

Fig. 2. If $(\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)}$, then (a) M_1 and (b) M_2 are relay functions with one and two switching points, respectively; (a′) $\dot\varphi_1(\varphi_1)$ and (b′), (b″) $\dot\varphi_2(\varphi_2)$ are the corresponding phase plots.

Considering the expressions for T, t_1, and t_2 in terms of t_0 (see (3.3) and (3.4)), $\dot\varphi_1(t_0)$, $\dot\varphi_2(t_1)$, and $\dot\varphi_2(t_2)$ in (4.9) and (4.10) can be represented as

$$\dot\varphi_1(t_0) = -I_2\,(I_1+1)^{-1}\,\varphi_2^T t_0^{-1} + M_1^0 (I_1+1)^{-1} t_0 > 0, \tag{4.11}$$

$$\dot\varphi_2(t_1) = \left(\varphi_2^T t_0^{-1} + I_2^{-1} t_0\right)/2 > 0 \quad \text{for} \quad \varphi_2^T > 0, \tag{4.12}$$

$$\dot\varphi_2(t_2) = \left(\varphi_2^T t_0^{-1} + I_2^{-1} t_0\right)/2 < 0 \quad \text{for} \quad \varphi_2^T < 0. \tag{4.13}$$

After substituting t_0 from (3.3) into (4.1), the inequality $|\dot\varphi_1| \le a_1$ becomes

$$\varphi_2^T \ge M_1^0 I_2^{-1} a_1^{-2} \left(\varphi_1^T\right)^2 - (I_1+1)\, I_2^{-1} \varphi_1^T. \tag{4.14}$$

On the plane $(\varphi_1^T, \varphi_2^T)$, inequality (4.14) defines a domain with the following intersection with the domain $\Phi_1^{(0)}$ (3.2):

$$D_1^{(0)}(a_1) = \left\{ (\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)} \colon \begin{array}{l} -\dfrac{I_1+1}{I_2(M_1^0+1)}\varphi_1^T < \varphi_2^T < \dfrac{I_1+1}{I_2(M_1^0-1)}\varphi_1^T, \quad \varphi_1^T > 0 \\ \varphi_2^T \geq M_1^0 I_2^{-1} a_1^{-2} (\varphi_1^T)^2 - (I_1+1) I_2^{-1} \varphi_1^T, \quad \varphi_1^T > 0 \end{array} \right\}. \tag{4.15}$$

Due to (4.10), (4.12), and (4.13), the second inequality $|\dot{\varphi}_2| \leq a_2$ from (4.1) reduces to $t_0^2 - 2a_2 I_2 t_0 + I_2 \left|\varphi_2^T\right| \leq 0$, a quadratic inequality in t_0. Therefore, we have

$$t_0^- \leq t_0 \leq t_0^+, \qquad t_0^{+,-} = a_2 I_2 \pm \sqrt{a_2^2 I_2^2 - I_2 \left|\varphi_2^T\right|}. \tag{4.16}$$

Using the expression (3.3) for t_0, inequality (4.16) can be transformed to

$$\begin{aligned} &\varphi_1^{T(-)} \leq \varphi_1^T \leq \varphi_1^{T(+)}, \\ &\varphi_1^{T(+)} = M_1^0 (I_1+1)^{-1} \left(a_2 I_2 + \sqrt{a_2^2 I_2^2 - I_2 \left|\varphi_2^T\right|} \right)^2 - I_2 (I_1+1)^{-1} \varphi_2^T, \\ &\varphi_1^{T(-)} = M_1^0 (I_1+1)^{-1} \left(a_2 I_2 - \sqrt{a_2^2 I_2^2 - I_2 \left|\varphi_2^T\right|} \right)^2 - I_2 (I_1+1)^{-1} \varphi_2^T. \end{aligned} \tag{4.17}$$

Within $\Phi_1^{(0)}$, the domain $D_1^{(0)}(a_2)$ is defined as follows:

$$D_1^{(0)}(a_2) = \left\{ (\varphi_1^T, \varphi_2^T) \in \Phi_1^{(0)} \colon \varphi_1^{T(-)} \leq \varphi_1^T \leq \varphi_1^{T(+)} \right\}, \tag{4.18}$$

where $\varphi_1^{T(+)}$ and $\varphi_1^{T(-)}$ are given by (4.17).

According to (4.4), we have

$$R_1^{(0)} = D_1^{(0)}(a_1) \cap D_1^{(0)}(a_2). \tag{4.19}$$

4.3. Construction of $R_2^{(0)}$

Let $(\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}$. Then, with single integration of system (2.7), (2.8) on the interval $[0, T]$ under the corresponding controls (3.6), (3.7) ($\gamma = 0$), we obtain the functions $\dot{\varphi}_i(\varphi_i)$, $i = 1, 2$, shown in Fig. 3.

Similar to the previous case, calculations yield the following results:

$$|\dot{\varphi}_1| \leq \max_{0 \leq t \leq T_2} |\dot{\varphi}_1(t)| = \begin{cases} |\dot{\varphi}_1(t_1)|, & \varphi_1^T > 0, \\ |\dot{\varphi}_1(t_2)|, & \varphi_1^T < 0, \end{cases} \tag{4.20}$$

$$|\dot{\varphi}_2| \leq \max_{0 \leq t \leq T_2} |\dot{\varphi}_2(t)| = |\dot{\varphi}_2(t_0)|, \tag{4.21}$$

where, in view of the expressions (3.6), (3.7) for T, t_1, and t_2 in terms of t_0,

$$\dot{\varphi}_1(t_1) = \left(M_1^0 - 1\right)(I_1+1)^{-1} \left\{ \left[(I_1+1)\varphi_1^T + I_2 \varphi_2^T \right] / \left(M_1^0 t_0 \right) + t_0 \right\} / 2 > 0, \tag{4.22}$$

$$\dot{\varphi}_1(t_2) = \left(M_1^0 + 1\right)(I_1+1)^{-1} \left\{ \left[(I_1+1)\varphi_1^T + I_2 \varphi_2^T \right] / \left(M_1^0 t_0 \right) - t_0 \right\} / 2 < 0, \tag{4.23}$$

$$\dot{\varphi}_2(t_0) = I_2^{-1} t_0 > 0. \tag{4.24}$$

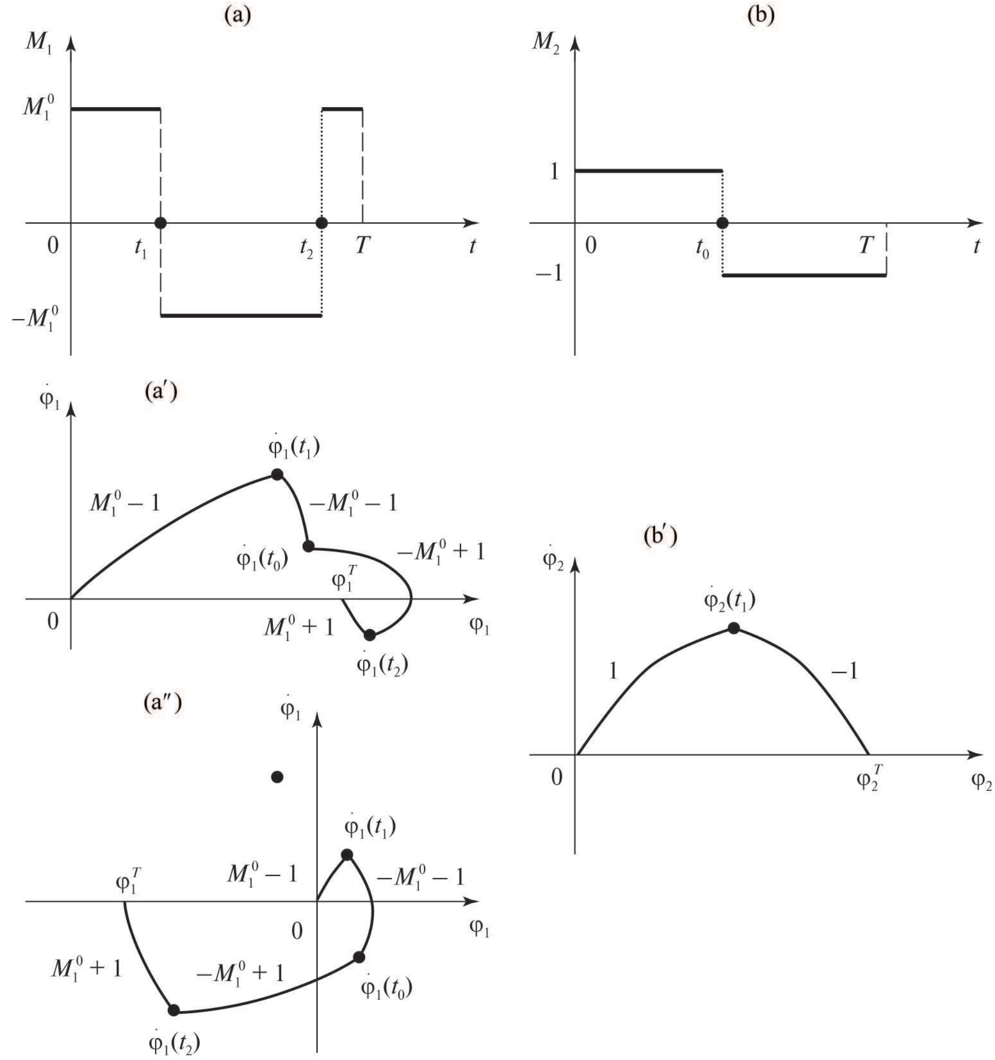

Fig. 3. If $(\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}$, then (a) M_1 and (b) M_2 are relay functions with two and one switching points, respectively; (a′), (a″) $\dot{\varphi}_1(\varphi_1)$ and (b′) $\dot{\varphi}_2(\varphi_2)$ are the corresponding phase plots.

Based on the expression (3.6) for t_0 and (4.20)–(4.24), the inequalities $|\dot{\varphi}_1| \le a_1$ and $|\dot{\varphi}_2| \le a_2$ from (4.1) can be written as

$$\varphi_1^T \le -I_2\,(I_1+1)^{-1}\left(M_1^0+1\right)\varphi_2^T + 2a_1M_1^0\left(M_1^0-1\right)^{-1}\sqrt{I_2\varphi_2^T}, \qquad \varphi_1^T > 0, \tag{4.25}$$

$$\varphi_1^T \ge I_2\,(I_1+1)^{-1}\left(M_1^0-1\right)\varphi_2^T - 2a_1M_1^0\left(M_1^0+1\right)^{-1}\sqrt{I_2\varphi_2^T}, \qquad \varphi_1^T < 0, \tag{4.26}$$

$$\varphi_2^T \le I_2 a_2^2. \tag{4.27}$$

According to (4.13) and (4.25)–(4.27), we have

$$D_2^{(0)}(a_1) = \left\{ (\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}\colon \begin{array}{ll} \varphi_1^T \le -\dfrac{I_2(M_1^0+1)}{I_1+1}\varphi_2^T + \dfrac{2a_1M_1^0}{M_1^0-1}\sqrt{I_2\varphi_2^T}, & \varphi_1^T > 0 \\ \varphi_1^T \ge \dfrac{I_2(M_1^0-1)}{I_1+1}\varphi_2^T - \dfrac{2a_1M_1^0}{M_1^0+1}\sqrt{I_2}\varphi_2^T, & \varphi_1^T < 0 \end{array} \right\}, \tag{4.28}$$

$$D_2^{(0)}(a_2) = \left\{ (\varphi_1^T, \varphi_2^T) \in \Phi_2^{(0)}\colon\ \varphi_2^T \le I_2^{-1}a_2^2 \right\}. \tag{4.29}$$

Hence, from (4.4) it follows that

$$R_2^{(0)} = D_2^{(0)}(a_1) \cap D_2^{(0)}(a_2). \tag{4.30}$$

To construct the domains $R_1^{(0)}$, $R_2^{(0)}$, $R_0^{(0,0)}$, and $R_0^{(0,1)}$, the manipulator was assumed to have the numerical dimensional parameters

$$\begin{gathered} L = 1\,\mathrm{m}, \quad m_2 = 10\,\mathrm{kg}, \quad I_1 = I_2 = (10/3)\,\mathrm{kg}\cdot\mathrm{m}^2, \\ M_1^0 = 2\,\mathrm{N}\cdot\mathrm{m}, \quad M_2^0 = 1\,\mathrm{N}\cdot\mathrm{m}. \end{gathered} \tag{4.31}$$

The links of such a manipulator are identical homogeneous rods. After passing to the dimensionless parameters (2.6), from (4.31) we obtain

$$L = 1, \quad m_2 = 1, \quad I_1 = I_2 = 1/3, \quad M_1^0 = 2, \quad M_2^0 = 1. \tag{4.32}$$

Let the dimensionless values of the angular velocities be

$$\begin{aligned} &(A)\;\; a_1 = 1,\; a_2 = 5.4, \\ &(B)\;\; a_1 = 1,\; a_2 = 0.75. \end{aligned} \tag{4.33}$$

In the case (4.33) (A), condition (4.5) holds; and the case (4.33) (B) corresponds to condition (4.17). According to the graphical plots, $D_{1,2}^{(0)}(a_1) \subset D_{1,2}^{(0)}(a_2)$ for the values (4.33) (A), and $D_{1,2}^{(0)}(a_2) \subset D_{1,2}^{(0)}(a_1)$ for the values (4.33) (B).

Consequently, from (4.19) and (4.30) it follows that

$$\begin{aligned} R_{1,2}^{(0)} &= D_{1,2}^{(0)}(a_1) \quad \text{in the case } (A), \\ R_{1,2}^{(0)} &= D_{1,2}^{(0)}(a_2) \quad \text{in the case } (B). \end{aligned} \tag{4.34}$$

In Fig. 4, the boundary of the domain (4.19) defined by the parabola (4.14) (marked with number (1)) intersects the lines $\Phi_0^{(0,0)}$ and $\Phi_0^{(0,1)}$ at points A and B (4.6), respectively. At each point of the boundary of (4.19), the conditions $\dot\varphi = a_1$ and $|\dot\varphi_2| < a_2$ hold. The curves given by equalities (4.25) and (4.26) (marked with numbers (2) and (3)) join on the axis φ_2^T and also intersect the lines $\Phi_0^{(0,0)}$ and $\Phi_0^{(0,1)}$ at points A (4.6) and B' (the centrally symmetric point to B (4.6)). Due to the aforementioned central symmetry, the boundaries of the domains $R_1^{(1)}$ and $R_2^{(1)}$ are marked by numbers $(1)'$, $(2)'$, and $(3)'$.

Remark. Depending on the value of a_2 in the case (4.33) (A), the junction point of curves (2) and (3) on the axis φ_2^T may lie below (Fig. 4) or above the line $\varphi_2^T = I_2 a_2^2$. In the latter situation, the boundary of the domain $R_2^{(0)}$ will contain a segment of this line.

According to Fig. 4, the domain R (4.1) narrows near the origin $(0,0)$, which can be explained as follows. The first link reaches point B (closer to the origin) with a greater acceleration, since the torque $M_1^0 + 1$ speeds it up faster, whereas at point A (farther from the origin) its acceleration is smaller due to the weaker action of the torque $M_1^0 - 1$. In both cases, $\dot\varphi_1$ achieves its maximum value earlier than $\dot\varphi_2$; moreover, with a greater acceleration this time instant occurs even earlier, further restricting the possible final states by shifting them closer to $(0,0)$. Thus, the greater acceleration the first link has, the closer to the origin the final point will be.

In Fig. 5, corresponding to the case (4.33) (B), numbers (1) and $(1)'$ mark the centrally symmetric boundaries of the domains $R_1^{(0)} = D_1^{(0)}(a_2)$ and $R_1^{(1)} = D_1^{(1)}(a_2)$, while numbers (2) and $(2)'$

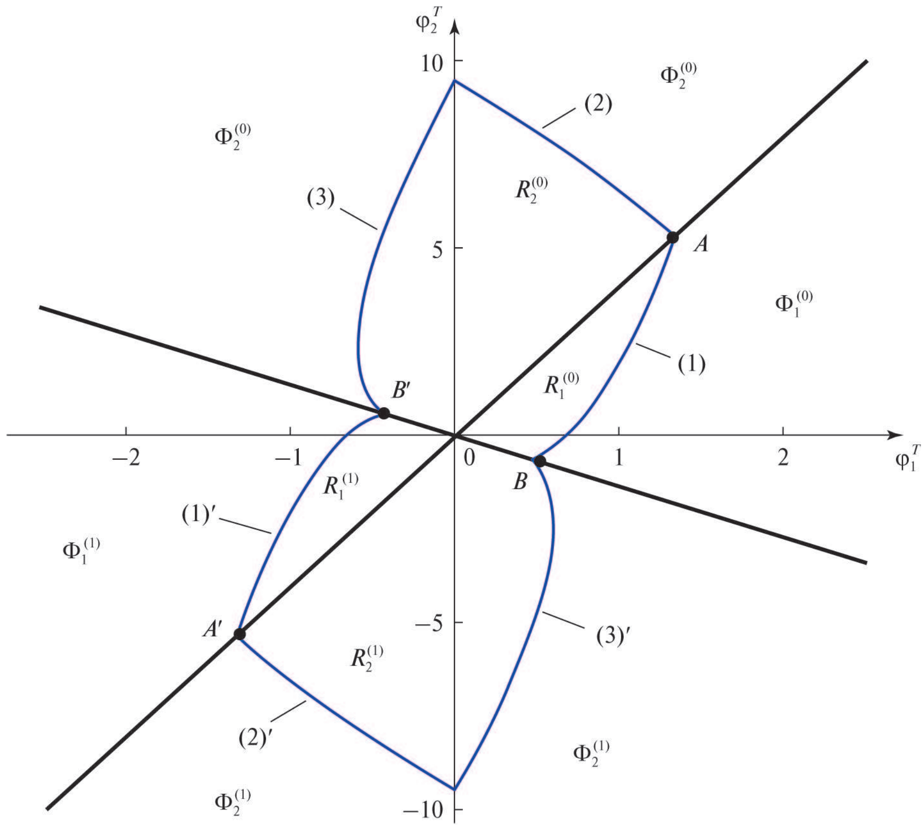

Fig. 4. The domains of final configurations reachable under the optimal regimes (3.1) with condition (4.5).

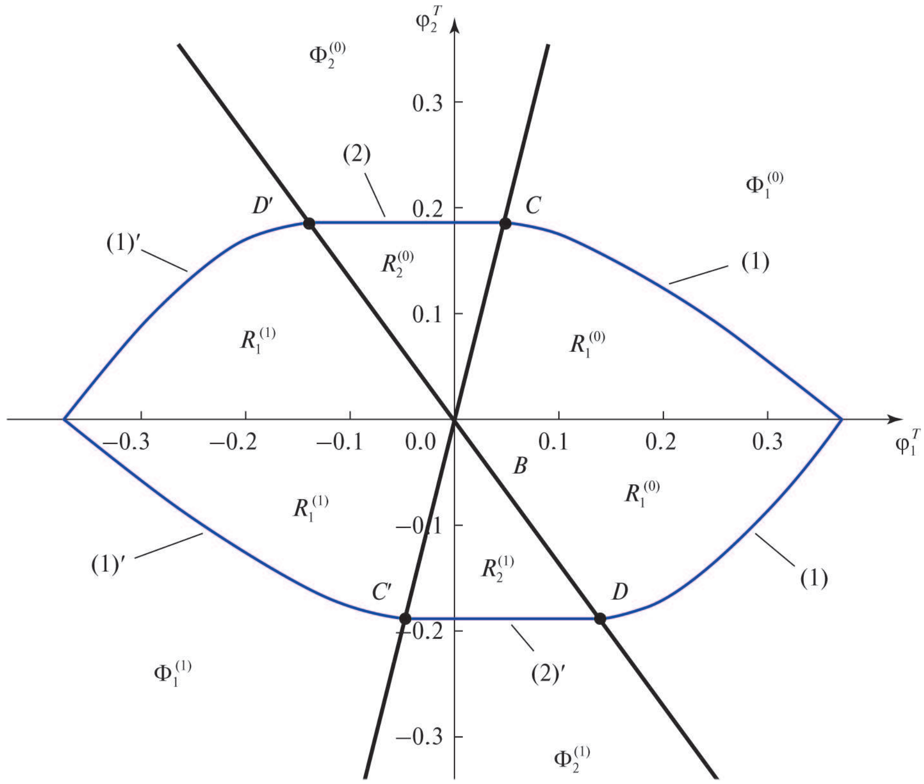

Fig. 5. The domains of final configurations reachable under the optimal regimes (3.1) with condition (4.7).

mark the boundaries of the domains $R_2^{(0)} = D_2^{(0)}(a_2)$ and $R_2^{(1)} = D_2^{(1)}(a_2)$, representing segments of the straight lines $\varphi_2^T = I_2 a_2^2$ and $\varphi_2^T = -I_2 a_2^2$. At each point of the boundary of (4.19), the conditions $\dot{\varphi} = a_1$ and $|\dot{\varphi}_2| < a_2$ hold. The line $\varphi_2^T = I_2 a_2^2$ intersects the lines $\Phi_0^{(0,0)}$ and $\Phi_0^{(1,0)}$ at points C (4.8) and D', the symmetric point to D (4.8), and the line $\varphi_2^T = -I_2 a_2^2$ intersects the lines $\Phi_0^{(0,1)}$ and $\Phi_0^{(1,1)}$ at points D (4.8) and C', the centrally symmetric point to C (4.8).

5. CONCLUSIONS

This paper has considered the model of a two-link manipulator with a statically balanced second link and angular velocity constraints. For this model, an algorithm has been developed to construct reachable domains on the configuration plane under time-optimal controls with a minimum number of switchings. Such controls ensure the manipulator's motion from an initial rest state to a given final rest state without violating angular velocity constraints; moreover, their limiting values can be achieved only at the switching instants, depending on the final configuration of the manipulator.

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This paper was recommended for publication by L.B. Rapoport, a member of the Editorial Board