ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 7, pp. 613–633.

The Detectability of Time-Varying Differential-Algebraic Equations with Scalar Output

A. A. Shcheglova

Matrosov Institute for System Dynamics and Control Theory, Siberian Branch, Russian Academy of Sciences, Irkutsk, Russia
e-mail: shchegl@icc.ru

October 10, 2025

December 11, 2025

December 24, 2025

Abstract—This paper considers time-varying linear and nonlinear systems of differential-algebraic equations with a scalar observable output. In the linear case, sufficient detectability conditions are established, and a state observer is designed. For systems with a controlled input of a special form, a stabilization procedure using dynamic output feedback is proposed. Based on a linear approximation, detectability conditions are derived for a nonlinear system.

Keywords: differential-algebraic equations, detectability, stabilizability

DOI: 10.7868/S1608303226070057

1. INTRODUCTION

Consider a system of ordinary differential equations of the form

$$A(t)\frac{d}{dt}x(t) + B(t)x(t) + U(t)u(t) = 0, \tag{1.1}$$

$$y(t) = c(t)^{\top}x(t), \quad t \in T = [0, +\infty), \tag{1.2}$$

where $A(t)$ and $B(t)$ are known matrices of dimensions $(n \times n)$; $U(t)$ is a given matrix of dimensions $(n \times l)$; $x(t)$ and $c(t)$ are the desired and given n-dimensional vector functions, respectively; $u(t)$ denotes the l-dimensional controlled input, and $y(t)$ is the scalar observable output; finally, $^{\top}$ indicates transpose. By assumption, the elements of the matrices $A(t)$, $B(t)$, and $U(t)$, as well as the components of the vectors $u(t)$ and $c(t)$, are continuously differentiable on T sufficiently many times. The case

$$\det A(t) \equiv 0, \quad t \in T,$$

is allowed.

In particular, such systems are called differential-algebraic equations (DAEs) or descriptor systems.

The classes of DAEs studied in this paper include applied problems from various fields of science and technology: the theory of electrical circuits [1], biology [2], mechanics [3], chemistry [4], economics [5], information security [6], etc.

Since the 1990s, there has been a significantly growing interest of researchers in the design of observers for differential-algebraic equations, in both linear [1, 3, 4, 7, 8] and nonlinear [9–20] settings.

Most works on linear systems analyze the time-invariant case. There exist relatively few results on nonlinear systems. Moreover, as a rule, a quasilinear formulation of the problem is considered under assumptions ensuring a special structure of the system or other rather rigid constraints. Here are some examples of such requirements: the linear part does not depend on time [9–11]; the system has a linear dominant [12, 13] or a triangular form [14]; the nonlinearity of the system satisfies a strict global Lipschitz condition [10, 12, 13].

A literature review on the design of state observers for nonlinear descriptor systems was provided in [11], including a discussion of the types of nonlinearities. The local stability of an observer for a semi-explicit system with a simple structure was considered in [9]. DAEs with rectangular descriptors were studied in [6, 15]. Observer design using linearization and methods for simplifying the internal structure of the system was described in [16]. The most common approach is to establish the existence of observers via the solvability of certain linear matrix inequalities (LMIs) [17–20].

Currently, the issue of obtaining detectability conditions and constructing state observers for time-varying DAEs in the general case remains open. This paper is devoted to linear and nonlinear systems satisfying fairly general requirements on the internal structure. In Sections 2–4, we deal with linear systems. The structural form from Section 2 is employed to derive detectability conditions and design a state observer in Section 3. This observer is used in Section 4 to propose a stabilization procedure for systems with a special controlled input. Section 5 considers the internal structure of nonlinear DAEs of the form

$$f\left(t, x(t), \frac{d}{dt}x(t)\right) = 0,$$

where $\det \frac{\partial f(t,\alpha,\beta)}{\partial \beta} = 0$ in the domain of definition. Based on these results, in Section 6, we establish sufficient detectability conditions under the linear approximation and also the form of the state observer in the nonlinear case.

2. AUXILIARY RESULTS ON THE STRUCTURE OF LINEAR DAES

For some $r : 0 \le r \le n$, let us associate with the matrix coefficients of system (1.1) the following objects: the matrices

$$\mathcal{B}_r(t) = \operatorname{col}\left(B(t),\ B'(t),\ \dots,\ B^{(r)}(t)\right), \tag{2.1}$$

$$\mathcal{A}_r(t) = \operatorname{col}\left(C_0^0 A(t),\ C_1^0 A'(t) + C_1^1 B(t),\ \dots,\ C_r^0 A^{(r)}(t) + C_r^1 B^{(r-1)}(t)\right) \tag{2.2}$$

of dimensions $(n(r+1) \times n)$, the matrix

$$\Lambda_r(t) = \begin{pmatrix} O & O & \dots & O \\ C_1^1 A(t) & O & \dots & O \\ C_2^1 A'(t) + C_2^2 B(t) & C_2^2 A(t) & \dots & O \\ \vdots & \vdots & \dots & \vdots \\ C_r^1 A^{(r-1)}(t) + C_r^2 B^{(r-2)}(t) & C_r^2 A^{(r-2)}(t) + C_r^3 B^{(r-3)}(t) & \dots & C_r^r A(t) \end{pmatrix} \tag{2.3}$$

of dimensions $n(r+1) \times nr$, and the matrix

$$\mathcal{D}_r(t) = \begin{pmatrix} \mathcal{B}_r(t) & \mathcal{A}_r(t) & \Lambda_r(t) \end{pmatrix} \tag{2.4}$$

of dimensions $n(r+1) \times n(r+2)$. Throughout this paper, $C_i^j = \frac{i!}{j!(i-j)!}$ are the binomial coefficients, and

$$\phi'(t) = \frac{d}{dt}\phi(t), \quad \phi^{(j)}(t) = \left(\frac{d}{dt}\right)^j \phi(t).$$

We make the following assumptions:

A1) $\operatorname{rank}\Lambda_r(t) = \lambda = \text{const} \quad \forall t \in T$.

A2) For all $t \in T$, the matrix $\mathcal{D}_r(t)$ has an invertible submatrix $\mathcal{M}_r(t)$ of order $n(r+1)$ containing λ columns of the matrix $\Lambda_r(t)$ and all n columns of the matrix $\mathcal{A}_r(t)$.

Let Q denote a column permutation matrix such that

$$\mathcal{B}_r(t)Q = \begin{pmatrix} \bar{\mathcal{B}}_1(t) & \bar{\mathcal{B}}_2(t) \end{pmatrix}, \tag{2.5}$$

where the block $\bar{\mathcal{B}}_2(t)$ ($\bar{\mathcal{B}}_1(t)$) is included (not included, respectively) in the matrix $\mathcal{M}_r(t)$. Let d denote the number of columns in $\bar{\mathcal{B}}_2(t)$. Accordingly,

$$B(t)Q = \begin{pmatrix} B_1(t) & B_2(t) \end{pmatrix}, \quad A(t)Q = \begin{pmatrix} A_1(t) & A_2(t) \end{pmatrix}, \tag{2.6}$$

where the blocks $B_1(t)$, $A_1(t)$ and $B_2(t)$, $A_2(t)$ have pairwise dimensions $n \times (n-d)$ and $n \times d$, respectively.

Under Assumptions A1 and A2, there exists an operator of the form

$$\mathcal{R} = \sum_{j=0}^{r} R_j(t) \left(\frac{d}{dt}\right)^r, \tag{2.7}$$

with the coefficients given by

$$\begin{pmatrix} R_0(t) & R_1(t) & \dots & R_r(t) \end{pmatrix} = \begin{pmatrix} E_n & O & \dots & O \end{pmatrix} \mathcal{M}_r^{-1}(t), \tag{2.8}$$

where E_n stands for an identity matrix of order n; for details, see [21].

Operator (2.7) transforms system (1.1) to

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1(t) & E_d \\ J_2(t) & O \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \mathcal{R}[U(t)u(t)] = 0, \quad t \in T, \tag{2.9}$$

where $x_1(t) \in \mathbf{R}^{n-d}$, $x_2(t) \in \mathbf{R}^d$,

$$\operatorname{col}(x_1(t), x_2(t)) = Q^{-1}x(t), \tag{2.10}$$

$$\operatorname{col}(J_1(t),\ J_2(t)) = \sum_{j=0}^{r} R_j(t) B_1^{(j)}(t), \tag{2.11}$$

$$\mathcal{R}[U(t)u(t)] = \sum_{j=0}^{r} R_j(t) \left(U(t)u(t)\right)^{(j)} = \begin{pmatrix} R_0(t) & R_1(t) & \dots & R_r(t) \end{pmatrix} \mathcal{U}_r(t)\bar{u}_r(t), \tag{2.12}$$

$$\bar{u}_r(t) = \operatorname{col}\left(u(t), u'(t), \dots, u^{(r)}(t)\right), \tag{2.13}$$

$$\mathcal{U}_r(t) = \begin{pmatrix} C_0^0 U(t) & O & O & \dots & O \\ C_1^0 U'(t) & C_1^1 U(t) & O & \dots & O \\ C_2^0 U^{(2)}(t) & C_2^1 U'(t) & C_2^2 U(t) \dots & O & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ C_r^0 U^{(r)}(t) & C_r^1 U^{(r-1)}(t) & C_r^2 U^{(r-2)}(t) & \dots & C_r^r U(t) \end{pmatrix}. \tag{2.14}$$

Definition 1. By a solution of system (1.1) we mean an n-dimensional vector function $x(t) \in \mathbf{C}^1(T)$ turning (1.1) into an identity upon substitution.

Obviously, for sufficiently smooth input data, any solution of system (1.1) is a solution of system (2.9). To prove the converse, additional requirements should be imposed.

Let us supplement A1 and A2 with the following assumptions:

A3) $\operatorname{rank}\Lambda_{r+1}(t) = \operatorname{rank}\Lambda_r(t) + n \ \ \forall t \in T$.

A4) For all $t \in T$, the matrix $\mathcal{D}_{r+1}(t)$ has an invertible submatrix $\mathcal{M}_{r+1}(t)$ of order $n(r+2)$ containing all columns of the matrix $\mathcal{M}_r(t)$ and also n columns of the matrix $\Lambda_{r+1}(t)$.

Then the operator $\mathcal{R}$ has the left inverse [21]

$$\mathcal{L} = L_0(t) + L_1(t)\frac{d}{dt},$$

where $L_0(t)$ and $L_1(t)$ are matrices of dimensions $(n \times n)$. This means that the operator $\mathcal{L}$, together with a permutation of rows of the desired vector (2.10), transforms system (2.9) to (1.1). In this case, any solution of system (2.9) is a solution of system (1.1).

Thus, Assumptions A1–A4 ensure the equivalence of systems (1.1) and (2.9) in the sense of solutions (up to permutations of components). Obviously, to extract a unique solution of system (2.9) or (1.1), it is necessary to specify initial data in the form

$$x_1(0) = x_0, \tag{2.15}$$

where $x_0 \in \mathbf{R}^{n-d}$.

Under certain conditions, the operator $\mathcal{R}$ (2.7) may have a right inverse.

Lemma 1. *Let:*

1) $A(t)$, $B(t) \in \mathbf{C}^{r+1}(T)$.
2) *Assumptions A1 and A2 be valid.*
3) *For all* $t \in T$,

$$\begin{pmatrix} R_1(t) & R_2(t)\dots R_r(t)\end{pmatrix}\begin{pmatrix} C_1^1 B_1(t) & O & \dots & O \\ C_2^1 B_1'(t) & C_2^2 B_1(t) & \dots & O \\ \vdots & \vdots & \ddots & \vdots \\ C_r^1 B_1^{(r-1)}(t) & C_r^2 B_1^{(r-2)}(t) & \dots & C_r^r B_1(t)\end{pmatrix} = O. \tag{2.16}$$

Then the operator (2.7) *has the right inverse*

$$\mathcal{P} = P_0(t) + P_1(t)\frac{d}{dt}, \tag{2.17}$$

where

$$P_0(t) = \begin{pmatrix} B_2(t) & A_1(t)\end{pmatrix}, \quad P_1(t) = \begin{pmatrix} A_2(t) & O\end{pmatrix},$$

and the matrices $B_1(t)$, $B_2(t)$, $A_1(t)$, *and* $A_2(t)$ *are given by* (2.6).

The proof is postponed to the Appendix.

3. STATE OBSERVER DESIGN FOR THE LINEAR SYSTEM

Let Assumptions A1–A4 be valid. In the output representation (1.2), we make the change of variables

$$x(t) = Q\operatorname{col}(x_1(t), x_2(t)),$$

where Q is the permutation matrix from (2.5). Then

$$y(t) = c_1^\top(t)x_1(t) + c_2^\top x_2(t), \tag{3.1}$$

$c_1(t) \in \mathbf{R}^{n-d}$, and $c_2(t) \in \mathbf{R}^d \ \forall t \in T$.

Definition 2. A system of the form

$$\begin{pmatrix} O & O \\ E_{n-d} & O \end{pmatrix} \begin{pmatrix} \hat{x}_1'(t) \\ \hat{x}_2'(t) \end{pmatrix} + \begin{pmatrix} \hat{J}_1(t) & E_d \\ \hat{J}_2(t) & O \end{pmatrix} \begin{pmatrix} \hat{x}_1(t) \\ \hat{x}_2(t) \end{pmatrix} + \hat{U}(t)\bar{u}_r(t)$$
$$+\, g(t)\left(y(t) - c_1^\top(t)\hat{x}_1(t) - c_2^\top \hat{x}_2(t)\right) = 0,$$
$$\hat{y}(t) = H(t)\mathrm{col}\,(\hat{x}_1(t),\, \hat{x}_2(t)) + \nu(t)y(t) + F(t)\bar{u}_r(t), \quad t \in T,$$

where the matrices $\hat{J}_1(t)$, $\hat{J}_2(t)$, $\hat{U}(t)$, $H(t)$, and $F(t)$ and the vectors $g(t)$ and $\nu(t)$ have the corresponding dimensions, is called a *state observer* for (2.9), (3.1), if

$$\lim_{t\to\infty}\left(\hat{y}(t) - \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix}\right) = 0.$$

We will seek a state observer for system (2.9), (3.1) in the form

$$\hat{x}_2(t) + J_1(t)\hat{x}_1(t) + \mathcal{V}_{r,1}(t)\bar{u}_r(t) = 0, \tag{3.2}$$
$$\hat{x}_1'(t) + J_2(t)\hat{x}_1(t) + \mathcal{V}_{r,2}(t)\bar{u}_r(t) + g(t)\left(y(t) - c_1^\top \hat{x}_1(t) - c_2^\top \hat{x}_2(t)\right) = 0, \tag{3.3}$$
$$\hat{y}_1(t) = \hat{x}_1(t), \quad \hat{y}_2(t) = \hat{x}_2(t), \tag{3.4}$$

where $g(t) : T \to \mathbf{R}^{n-d}$;

$$\mathrm{col}\,(\mathcal{V}_{r,1}(t),\, \mathcal{V}_{r,2}(t)) = \begin{pmatrix} R_0(t) & R_1(t) & \ldots & R_r(t) \end{pmatrix} \mathcal{U}_r(t),$$

with $\mathcal{U}_r(t)$ defined by (2.14) and the matrices $\mathcal{V}_{r,1}(t)$ and $\mathcal{V}_{r,2}(t)$ having dimensions $d \times l(r+1)$ and $(n-d) \times l(r+1)$, respectively; finally, $J_1(t)$ and $J_2(t)$ are the matrix coefficients from (2.9) that are calculated using formulas (2.6), (2.8), and (2.11).

Then the residuals

$$e_1(t) = \hat{x}_1(t) - x_1(t), \quad e_2(t) = \hat{x}_2(t) - x_2(t) \tag{3.5}$$

satisfy the system

$$e_2(t) + J_1(t)e_1(t) = 0, \tag{3.6}$$
$$e_1'(t) + J_2(t)e_1(t) - g(t)\left(c_1^\top(t)e_1(t) + c_2^\top(t)e_2(t)\right) = 0. \tag{3.7}$$

From (3.6) it follows that

$$e_2(t) = -J_1(t)e_1(t). \tag{3.8}$$

Substituting (3.8) into (3.7) yields the equation for $e_1(t)$:

$$e_1'(t) + \left(J_2(t) - g(t)\left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)\right)e_1(t) = 0. \tag{3.9}$$

Suppose the existence of a vector function $g(t)$ such that $\lim\limits_{t\to\infty} e_1(t) = 0$ for any solution of system (3.9). In this case, the nonsingular system

$$x_1'(t) + J_2(t)x_1(t) = 0, \tag{3.10}$$
$$y(t) = \left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)x_1(t) \tag{3.11}$$

is said to be *detectable* [22, p. 313], and there exists its observer of the form

$$\hat{x}_1'(t) + J_2(t)\hat{x}_1(t) + g(t)\left(y(t) - \left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)\hat{x}_1(t)\right) = 0, \tag{3.12}$$

$$\hat{y}(t) = \hat{x}_1(t). \tag{3.13}$$

Under an additional assumption that the matrix $J_1(t)$ has a nonpositive characteristic (Lyapunov) exponent, i.e.,

$$\chi[J_1(t)] = \overline{\lim_{t\to\infty}} \frac{1}{t} \ln \|J_1(t)\| \le 0, \tag{3.14}$$

equality (3.8) implies the limit relation $\lim\limits_{t\to\infty} e_2(t) = 0$. Thus, system (3.2)–(3.4) is a state observer for system (2.9), (3.1) as well.

Under Assumptions A1–A4, systems (1.1) and (2.9) have the same solutions. This means that system (3.2)–(3.4) is a state observer also for system (1.1), (1.2). Thus, the following result is true.

Proposition 1. *Under Assumptions A1–A4 and* (3.14), *there exists a state observer of the form* (3.2)–(3.4) *for the DAE* (1.1), (1.2) *iff system* (3.10), (3.11) *has an observer of the form* (3.12), (3.13).

Definition 3. The observation system (1.1), (1.2) is said to be detectable if there exists a vector function $g(t)$ such that system (3.9) is asymptotically stable.

Proposition 2. *Under Assumptions A1–A4 and* (3.14), *system* (1.1), (1.2) *is detectable iff system* (3.10), (3.11) *is detectable.*

To proceed, we formulate a detectability condition for system (3.10), (3.11) [22, p. 314]. To this end, it is necessary to define the observability matrix

$$\mathcal{S}(t) = \mathrm{col}\left(s_0(t),\, s_1(t),\, \ldots, s_{n-d-1}(t)\right), \tag{3.15}$$

where $s_0(t) = c_1^\top(t) - c_2^\top(t)J_1(t)$ and $s_i(t) = s_{i-1}'(t) - s_{i-1}(t)J_2(t), \ \ i = \overline{1, n-d}$.

For an invertible matrix $\mathcal{S}(t)$, we introduce the functions

$$(\omega_1(t),\, \omega_2(t),\, \ldots,\, \omega_{n-d}(t)) = s_{n-d}(t)\mathcal{S}^{-1}(t). \tag{3.16}$$

Lemma 2. *Let:*

1) $J_2(t) \in \mathbf{C}^{n-d-1}(T)$ *and* $c_1^\top(t) - c_2^\top(t)J_1(t) \in \mathbf{C}^{n-d}(T)$.
2) $\mathcal{S}(t)$ *be a Lyapunov matrix.*
3) *Each function* $\omega_i(t) \in \mathbf{C}^{i-1}(T)$ $(i = \overline{1, n-d})$ *be bounded on* T *together with its derivatives up to order* $i-1$ *inclusive.*

Then system (3.10), (3.11) *is detectable.*

Remark 1. According to the proof of the detectability theorem for a linear system [22, p. 314], the vector function $g(t)$ in (3.12) can be chosen so that it is bounded on T, and system (3.12) is reducible in the Lyapunov sense.

Based on Propositions 1 and 2 and Lemma 2, we obtain sufficient conditions for the detectability of the DAE (1.1), (1.2).

Theorem 1. *Let:*

1) $A(t), B(t) \in \mathbf{C}^{n+r-d}(T)$ $(n-d \ge 1)$, $c(t) \in \mathbf{C}^{n-d}(T)$, *and* $U(t), u(t) \in \mathbf{C}^r(T)$.
2) *Assumptions A1–A4 and* (3.14) *be valid.*
3) *Conditions 2 and 3 of Lemma 2 hold.*

Then system (1.1), (1.2) *is detectable and has an observer of the form* (3.2)–(3.4).

4. STABILIZABILITY OF THE LINEAR SYSTEM USING A STATE OBSERVER

Within the hypotheses of Lemma 1, consider the DAE

$$A(t)x'(t) + B(t)x(t) + P_0(t)b(t)u(t) + P_1(t)\left(b(t)u(t)\right)' = 0, \tag{4.1}$$

where $P_0(t)$ and $P_1(t)$ are the coefficients of the operator (2.17), which is right inverse to the operator (2.7); $b(t)$ is a given n-dimensional vector function; finally, $u(t)$ is a scalar controlled input.

Under definite conditions, the system

$$A(t)x'(t) + B(t)x(t) + \beta_1(t)u(t) + \beta_2(t)u(t)' = 0 \tag{4.2}$$

with sufficiently smooth n-dimensional vector functions $\beta_1(t)$ and $\beta_2(t)$ can be represented in the form (4.1).

Lemma 3. *Let:*

1) $A(t), B(t) \in \mathbf{C}^{r+1}(T)$.
2) *Assumptions 2 and 3 of Lemma 1 be valid.*
3) $\beta_1(t) \in \mathbf{C}^{r}(T)$.
4) *For all* $t \in T$,

$$\beta_2(t) = P_1(t)\mathcal{R}\left[\beta_1(t)\right]. \tag{4.3}$$

Then system (4.2) *can be written in the form* (4.1), *where* $b(t) \in \mathbf{C}^1(T)$ *is given by*

$$b(t) = \mathcal{R}\left[\beta_1(t)\right], \tag{4.4}$$

and the operator $\mathcal{R}$ *is defined in* (2.7) *and* (2.8).

The proof is provided in the Appendix.

Let us introduce the observable output (1.2) for system (4.1) and design a stabilizing control for (4.1).

By applying the operator (2.7) to both sides of equation (4.1), we obtain the system

$$x_2(t) + J_1(t)x_1(t) + b_1(t)u(t) = 0, \tag{4.5}$$

$$x_1'(t) + J_2(t)x_1(t) + b_2(t)u(t) = 0 \tag{4.6}$$

with the output (3.1), where $\operatorname{col}\left(b_1(t), b_2(t)\right) = b(t)$, $b_1(t) \in \mathbf{R}^d$, and $b_2(t) \in \mathbf{R}^{n-d}$.

A state observer for (4.5), (4.6) will be constructed in the form (3.2)–(3.4), where

$$\mathcal{V}_{r,1}(t)\bar{u}_r(t) = b_1(t)u(t), \quad \mathcal{V}_{r,2}(t)\bar{u}_r(t) = b_2(t)u(t).$$

A stabilizing control for the DAE (4.5), (4.6) will be found in the class of dynamic output feedback

$$u(t) = k^{\top}(t)\hat{x}_1(t); \tag{4.7}$$

consequently,

$$x_2(t) + J_1(t)x_1(t) + b_1(t)k^{\top}(t)\hat{x}_1(t) = 0,$$
$$x_1'(t) + J_2(t)x_1(t) + b_2(t)k^{\top}(t)\hat{x}_1(t) = 0.$$

By introducing the residuals (3.5), we arrive at the system

$$x_2(t) = -\left(J_1(t) + b_1(t)k^\top(t)\right) x_1(t) - b_1(t)k^\top(t)e_1(t), \tag{4.8}$$

$$x_1'(t) + \left(J_2(t) + b_2(t)k^\top(t)\right) x_1(t) + b_2(t)k^\top(t)e_1(t) = 0. \tag{4.9}$$

$$e_2(t) = -J_1(t)e_1(t), \tag{4.10}$$

$$e_1'(t) + \left(J_2(t) - g(t)\left(c_1^\top(t) - c_2^\top(t)J_1(t)\right)\right) e_1(t) = 0. \tag{4.11}$$

The control (4.7) will be stabilizing if system (4.8)–(4.11) is asymptotically stable. The latter condition particularly implies the stabilizability of system (4.6) and the detectability of system (3.10), (3.11).

Definition 4. System (4.6) is said to be *stabilizable* if there exists a feedback control

$$u(t) = k^\top(t)x_1(t) \tag{4.12}$$

such that the closed-loop system

$$x_1'(t) + \left(J_2(t) + b_2(t)k^\top(t)\right) x_1(t) = 0 \tag{4.13}$$

is asymptotically stable.

To formulate stabilizability conditions for system (4.6), we introduce the vectors

$$\varphi_0(t) = b_2(t), \quad \varphi_i = J_2(t)\varphi_{i-1}(t) - \varphi_{i-1}'(t), \quad i = \overline{1, n-d},$$

and the controllability matrix

$$\Phi(t) = \begin{pmatrix} \varphi_0(t) & \varphi_1(t) & \ldots & \varphi_{n-d-1}(t) \end{pmatrix}.$$

If $\Phi(t)$ is invertible, it is possible to define the function

$$\psi(t) = \Phi^{-1}(t)\varphi_{n-d}(t) = \operatorname{col}\left(\psi_1(t),\, \psi_2(t), \ldots,\, \psi_{n-d}(t)\right).$$

Lemma 4 [22, p. 302]. *Let:*
1) $J_2(t) \in \mathbf{C}^{n-d-1}(T)$ *and* $b_2(t) \in \mathbf{C}^{n-d}(T)$.
2) $\Phi(t)$ *be a Lyapunov matrix.*
3) *Each function* $\psi_i(t) \in \mathbf{C}^i(T)$ $(i = \overline{1, n-d})$ *be bounded on* T *together with its derivatives up to order* $i-1$ *inclusive.*

Then there exists a row

$$k^\top(t) = \left(k_1(t),\, \ldots,\, k_{n-d}(t)\right) K(t) \tag{4.14}$$

such that the closed-loop system (4.13) *is asymptotically stable. In* (4.14), $K(t)$ *is a Lyapunov matrix, and each function* $k_i(t) \in \mathbf{C}^{i-1}(T)$ *is bounded on* T *together with its derivatives up to order* $i-1$ *inclusive.*

Remark 2. According to Lemma 4, the vector function $k(t)$ in the feedback control (4.12) is bounded, hence

$$\chi[k(t)] = 0. \tag{4.15}$$

Remark 3. If $J_2(t) \in \mathbf{C}^{n-d}(T)$ and $b_2(t) \in \mathbf{C}^{n-d+1}(T)$ in system (4.6), then the stabilizing control (4.12) can be designed so that $k(t) \in \mathbf{C}^1(T)$ [22, pp. 166, 248, 299].

Theorem 2. *Let:*

1) $A(t), B(t) \in \mathbf{C}^{n+r-d}(T)$ $(n-d \geq 1)$, $c(t) \in \mathbf{C}^{n-d}(T)$, $b_2(t) \in \mathbf{C}^{n-d+1}(T)$, *and* $b_1(t) \in \mathbf{C}^1(T)$.
2) *Assumptions A1–A4 and* (3.14) *be valid.*
3) *Conditions 1 and 3 of Lemma 1, as well as conditions 2 and 3 of Lemma 2, hold.*
4) *Conditions 2 and 3 of Lemma 4 hold.*
5) $\chi\left[b_1(t)\right] \leq 0$.

Then system (4.1) *is stabilizable by the feedback control* (4.7).

The proof can be found in the Appendix.

5. NECESSARY INFORMATION ON THE STRUCTURE OF NONLINEAR DAES

In this section, we consider the nonlinear system with a scalar observable output of the form

$$f(t, x(t), x'(t)) = 0, \tag{5.1}$$

$$y(t) = c^\top(t)x(t) + \rho(t, x(t)) \;\; t \in T, \tag{5.2}$$

where the components of the function $f(t,\alpha,\beta): \mathcal{W} \to \mathbf{R}^n$ are continuously differentiable with respect to each argument sufficiently many times in the domain $\mathcal{W} = \{(t,\alpha,\beta) \in \mathbf{R}^{2n+1} : t \in T, \|\alpha\| < K_0, \|\beta\| < K_1\}$; $\rho(t,\alpha): \hat{\mathcal{W}} \to \mathbf{R}$, $\hat{\mathcal{W}} = \{(t,\alpha) \in \mathbf{R}^{n+1} : t \in T, \|\alpha\| < K_0\}$, and $c(t): T \to \mathbf{R}^n$. By assumption,

$$\begin{gathered} f(t,0,0) = 0, \\ \rho(t,0) = 0 \;\; \forall t \in T, \\ \det \frac{\partial f(t,\alpha,\beta)}{\partial \beta} = 0 \;\; \forall (t,\alpha,\beta) \in \mathcal{W}. \end{gathered} \tag{5.3}$$

In the nonlinear case, a nonsingular system equivalent to (5.1) in the sense of solutions has to be sought as part of the components of an implicit function satisfying the so-called r-derivative array equations. They consist of system (5.1) and its r total derivatives with respect to $t:$

$$\mathcal{F}_r\left(t, \bar{x}_{r+1}\right) = \begin{pmatrix} f(t, x(t), x'(t)) \\ \dfrac{d}{dt} f(t, x(t), x'(t)) \\ \cdots \\ \left(\dfrac{d}{dt}\right)^r f(t, x(t), x'(t)) \end{pmatrix} = 0, \tag{5.4}$$

where

$$\bar{x}_{r+1} = \left(x,\, x',\, \ldots,\, x^{(r+1)}\right). \tag{5.5}$$

In (5.4), t, x, x', $\ldots$, $x^{(r+1)}$ should be treated as independent variables.

The situation is complicated by the fact that the classical theorem [23, p. 68] ensures the existence of an implicit function only in some neighborhood of the initial point, whereas detectability analysis requires the definition of this function at least for all $t \in T$. To satisfy this requirement, we employ a global version of the implicit function theorem [24].

Let us introduce the matrices

$$\bar{\mathcal{B}}_r\left(t, \bar{x}_{r+1}\right) = \left(\frac{\partial \mathcal{F}_r\left(t, \bar{x}_{r+1}\right)}{\partial x}\right), \quad \bar{\mathcal{A}}_r\left(t, \bar{x}_{r+1}\right) = \left(\frac{\partial \mathcal{F}_r\left(t, \bar{x}_{r+1}\right)}{\partial x'}\right)$$

of dimensions $(n(r+1)\times n)$, the matrix

$$\bar{\Lambda}_r\left(t,\bar{x}_{r+1}\right)=\begin{pmatrix}\dfrac{\partial \mathcal{F}_r\left(t,\bar{x}_{r+1}\right)}{\partial x''} & \dots & \dfrac{\partial \mathcal{F}_r\left(t,\bar{x}_{r+1}\right)}{\partial x^{(r+1)}}\end{pmatrix}$$

of dimensions $(n(r+1)\times nr)$-matrix, and the matrix

$$\bar{\mathcal{D}}_r\left(t,\bar{x}_{r+1}\right)=\left(\bar{\mathcal{B}}_r\left(t,\bar{x}_{r+1}\right)\quad \bar{\mathcal{A}}_r\left(t,\bar{x}_{r+1}\right)\quad \bar{\Lambda}_r\left(t,\bar{x}_{r+1}\right)\right)$$

of dimensions $(n(r+1)\times n(r+2))$.

Condition (5.3) implies

$$\mathcal{F}_r(0,0)=0.$$

We assume the following conditions:

B1) $\operatorname{rank}\bar{\Lambda}_r\left(t,\bar{x}_{r+1}\right)=\lambda=\mathrm{const}$ everywhere in the domain of definition.

B2) At the point $(t,\bar{x}_{r+1})=0$, the matrix $\bar{\mathcal{D}}_r\left(t,\bar{x}_{r+1}\right)$ has an invertible submatrix $\bar{\mathcal{M}}_r\left(t,\bar{x}_{r+1}\right)$ of order $n(r+1)$ containing λ columns of the matrix $\bar{\Lambda}_r\left(t,\bar{x}_{r+1}\right)$ and all n columns of the matrix $\bar{\mathcal{A}}_r\left(t,\bar{x}_{r+1}\right)$. Moreover,

$$\bar{\mathcal{M}}_r\left(t,\bar{x}_{r+1}\right)=\begin{pmatrix}\dfrac{\partial \mathcal{F}_r}{\partial x_2} & \dfrac{\partial \mathcal{F}_r}{\partial x'} & \dfrac{\partial \mathcal{F}_r}{\partial X_1}\end{pmatrix},$$

where

$$x=Q\operatorname{col}\left(x_1,x_2\right), \tag{5.6}$$

$$\operatorname{col}\left(x'',\dots,x^{(r+1)}\right)=Q_r\operatorname{col}\left(X_1,X_2\right), \tag{5.7}$$

Q and Q_r are the corresponding row permutation matrices, $x_1\in\mathcal{W}_1\subseteq\mathbf{R}^{n-d}$, $x_2\in\mathcal{W}_2\subseteq\mathbf{R}^{d}$, $X_1\in\mathcal{X}_1\subseteq\mathbf{R}^{\lambda}$, $X_2\in\mathcal{X}_2\subseteq\mathbf{R}^{d}$, $d=nr-\lambda$, Q is constructed by analogy with the linear case (see (2.5)), and $\mathcal{W}_1$, $\mathcal{W}_2$, $\mathcal{X}_1$, and $\mathcal{X}_2$ denote neighborhoods of zero in the corresponding spaces.

Under conditions B1 and B2 with some additional assumptions [25], an implicit function of the form

$$x_2=f_1(t,x_1), \tag{5.8}$$

$$x_1'=f_2(t,x_1), \tag{5.9}$$

$$x_2'=f_3(t,x_1), \tag{5.10}$$

$$X_1=f_4(t,x_1,X_2) \tag{5.11}$$

defined in the domain $T\times\mathcal{W}_1\times\mathcal{X}_2$ turns system (5.4) into an identity upon substitution.

The $(r+1)$-derivative array equations for (5.1) have the form

$$\mathcal{F}_{r+1}(t,\bar{x}_{r+2})=0.$$

Let us supplement B1 and B2 with the following conditions:

B3) $\operatorname{rank}\bar{\Lambda}_{r+1}\left(t,\bar{x}_{r+2}(t)\right)=\lambda+n=\mathrm{const}$ everywhere on the domain of definition of this matrix.

B4) At the point $(t,\bar{x}_{r+2})=0$, the matrix $\bar{\mathcal{D}}_{r+1}\left(t,\bar{x}_{r+2}(t)\right)$ has an invertible submatrix $\bar{\mathcal{M}}_{r+1}\left(t,\bar{x}_{r+2}\right)$ of order $n(r+2)$ containing all columns with $\bar{\mathcal{M}}_r\left(t,\bar{x}_{r+1}\right)$ and n columns of the matrix $\bar{\Lambda}_{r+1}\left(t,\bar{x}_{r+2}\right)$.

Then

$$\bar{\mathcal{M}}_{r+1}\left(t, \bar{x}_{r+2}\right) = \begin{pmatrix} \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_2} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x'} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_1} \end{pmatrix},$$

$$\bar{\mathcal{D}}_{r+1}\left(t, \bar{x}_{r+2}\right) \begin{pmatrix} Q & O & O \\ O & Q & O \\ O & O & \hat{Q}_r \end{pmatrix} = \begin{pmatrix} \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_1} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_2} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_1'} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_2'} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_1} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_2} \end{pmatrix},$$

$$\operatorname{col}\left(x'', \ldots, x^{(r+2)}\right) = \hat{Q}_r \operatorname{col}\left(\hat{X}_1,\, \hat{X}_2\right),$$

$\hat{Q}_r$ is the corresponding row permutation matrix, $\hat{X}_1(t) \in \hat{\mathcal{X}}_1 \subseteq \mathbf{R}^{\lambda+n}$, and $\hat{X}_2(t) \in \hat{\mathcal{X}}_2 \subseteq \mathbf{R}^{d}$.

For a matrix H of dimensions $(p \times q)$ and a q-dimensional vector h, we introduce

$$\Omega(H) = \max_{\|h\|_{\mathbf{R}^q}=1} \|Hh\|_{\mathbf{R}^p}, \quad \omega(H) = \min_{\|h\|_{\mathbf{R}^q}=1} \|Hh\|_{\mathbf{R}^p}.$$

Hereinafter, the norm of a vector is understood as its Euclidean norm, and the norm of a matrix as its spectral norm.

Theorem 3 [25]. *Let:*
1) *The function $f(t, \alpha, \beta)$ have continuous partial derivatives with respect to its arguments up to order $r + 2$ inclusive in the domain $\mathcal{W}$: $f(t, \alpha, \beta) \in \mathbf{C}^{r+2}(\mathcal{W})$.*
2) *Conditions B1–B4 hold.*
3) *There exist a continuous function $v(s) : T \to T$ such that*

$$v(s) > 0, \quad s \in (0, \infty), \quad \int\limits_1^\infty \frac{ds}{v(s)} = \infty,$$

and for all $(t, x, x', \hat{X}_1, \hat{X}_2) \in \mathcal{W} \times \hat{\mathcal{X}}_1 \times \hat{\mathcal{X}}_2$,

$$\omega\left(\bar{\mathcal{M}}_{r+1}\right) > 0,$$

$$\Omega\begin{pmatrix} \dfrac{\partial \mathcal{F}_{r+1}}{\partial t} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial x_1} & \dfrac{\partial \mathcal{F}_{r+1}}{\partial \hat{X}_2} \end{pmatrix} / \omega\left(\bar{\mathcal{M}}_{r+1}\right) \leq v\left(\left\|\left(x_2, x', \hat{X}_1\right)\right\|\right).$$

Then any solution of system (5.1) *defined in a sufficiently small neighborhood of zero is a solution of the system*

$$x_2(t) = f_1\left(t, x_1(t)\right), \tag{5.12}$$

$$x_1'(t) = f_2\left(t, x_1(t)\right), \quad t \in T, \tag{5.13}$$

and vice versa.

The functions $f_1\left(t, x_1(t)\right)$ and $f_2\left(t, x_1(t)\right)$ are found as some components of the implicit function (5.8)–(5.11) satisfying system (5.4). Moreover,

$$f_i(t, x_1) \in \mathbf{C}^1(T \times \mathcal{W}_1), \tag{5.14}$$

$$f_i(t, 0) = 0, \quad t \in T, \quad i = 1, 2. \tag{5.15}$$

Obviously, to extract a unique solution of system (5.12), (5.13) or (5.1), one should specify the initial data (2.15).

6. THE DETECTABILITY OF NONLINEAR DAES

First, consider a nonlinear system of the form (5.12), (5.13) with the properties (5.14), (5.15) and the observable output

$$y(t) = c_1^\top(t)x_1(t) + c_2^\top(t)x_2(t) + \tilde{\rho}(t, x_1(t), x_2(t)), \tag{6.1}$$

where $c_1(t) : T \to \mathbf{R}^{n-d}$, $c_2(t) : T \to \mathbf{R}^d$, and

$$\tilde{\rho}(t, 0, 0) \equiv 0 \ \ t \in T.$$

We linearize system (5.12), (5.13) to

$$x_2(t) = G_1(t)x_1(t) + r_1(t, x_1(t)), \tag{6.2}$$

$$x_1'(t) = G_2(t)x_1(t) + r_2(t, x_1(t)), \tag{6.3}$$

where $G_1(t) = \frac{\partial f_1}{\partial x_1}(t, 0)$ and $G_2(t) = \frac{\partial f_2}{\partial x_1}(t, 0)$.

Substituting (6.2) into (6.1) yields the output representation

$$y(t) = \Big(c_1^\top(t) + c_2^\top(t)G_1(t)\Big)\, x_1(t) + c_2^\top(t)r_1(t, x_1(t)) + \tilde{\rho}\Big(t, x_1(t), G_1(t) + r_1(t, x_1(t))\Big). \tag{6.4}$$

For system (6.2)–(6.4), we construct a state observer of the form

$$\hat{x}_2(t) = G_1(t)\hat{x}_1(t) + r_1(t, \hat{x}_1(t)), \tag{6.5}$$

$$\begin{aligned}\hat{x}_1'(t) = G_2(t)\hat{x}_1(t) + r_2(t, \hat{x}_1(t)) + g(t)\Big(y(t) - (c_1^\top(t) + c_2^\top G_1(t))\hat{x}_1(t) \\ - c_2^\top(t)r_1(t, \hat{x}_1(t)) - \tilde{\rho}\Big(t, \hat{x}_1(t), G_1(t)\hat{x}_1(t) + r_1(t, \hat{x}_1(t))\Big)\Big),\end{aligned} \tag{6.6}$$

$$\hat{y}_1(t) = \hat{x}_1(t), \ \ \hat{y}_2(t) = \hat{x}_2(t). \tag{6.7}$$

Then the residuals $e_1(t) = \hat{x}_1(t) - x_1(t)$ and $e_2(t) = \hat{x}_2(t) - x_2(t)$ satisfy the equations

$$e_2(t) = G_1(t)e_1(t) + r_1(t, \hat{x}_1(t)) - r_1(t, x_1(t)), \tag{6.8}$$

$$\begin{aligned}e_1'(t) = \Big(G_2(t) - g(t)(c_1^\top(t) + c_2^\top G_1(t))\Big)\, e_1(t) + r_2(t, \hat{x}_1(t)) - r_2(t, x_1(t)) \\ - c_2^\top(t)\left(r_1(t, \hat{x}_1(t)) - r_1(t, x_1(t))\right) + \tilde{\rho}\Big(t, x_1(t), G_1(t)x_1(t) + r_1(t, x_1(t))\Big) \\ - \tilde{\rho}\Big(t, \hat{x}_1(t), G_1(t)\hat{x}_1(t) + r_1(t, \hat{x}_1(t))\Big).\end{aligned} \tag{6.9}$$

Definition 5. System (6.2)–(6.4) is said to have a *state observer* of the form (6.5)–(6.7) if

$$\lim_{t\to\infty} (\hat{y}_i(t) - x_i(t)) = 0, \ \ i = 1, 2.$$

Definition 6. System (6.2)–(6.4) is said to be *detectable* if there exists a vector function $g(t) : T \to \mathbf{R}^{n-d}$ such that system (6.8)–(6.9) is asymptotically stable.

Obviously, the system

$$e_2(t) = G_1(t)e_1(t), \tag{6.10}$$

$$e_1'(t) = \Big(G_2(t) - g(t)(c_1^\top(t) + c_2^\top G_1(t))\Big)\, e_1(t), \ \ t \in T, \tag{6.11}$$

is the first approximation system for (6.8), (6.9). The following stability conditions for (6.10), (6.11) are straightforward from Lemma 2.

Proposition 3. *Let:*
1) $G_1(t) \in \mathbf{C}(T)$, $G_2(t) \in \mathbf{C}^{n-d-1}(t)$, *and* $c_1^\top(t) + c_2^\top(t)G_1(t) \in \mathbf{C}^{n-d}(T)$ $(n-d \geq 1)$.
2) $\|G_1(t)\| \leq \kappa$ $\forall t \in T$, $\kappa = \text{const} > 0$.
3) *Assumptions 2 and 3 of Lemma 2 hold, with*

$$s_0(t) = c_1^\top + c_2^\top(t)G_1(t), \quad s_i = s_{i-1}'(t) + s_{i-1}(t)G_2(t), \quad i = \overline{1, n-d}, \tag{6.12}$$

in (3.15) *and* (3.16).

Then there exists an $(n-d)$*-dimensional vector function* $g(t)$ *such that system* (6.10), (6.11) *is asymptotically stable.*

Using Proposition 3, we derive sufficient conditions for the detectability of system (5.12), (5.13), (6.1).

Theorem 4. *Let:*
1) *The properties* (5.14), (5.15), *and* $\tilde{\rho}(t, x_1, x_2) \in \mathbf{C}(\hat{\mathcal{W}})$ *hold.*
2) *The assumptions of Proposition 3 be valid.*
3) $|\tilde{\rho}(t, \hat{x}_1, \hat{x}_2) - \tilde{\rho}(t, x_1, x_2)| \leq \eta_\rho\,(\|\hat{x}_1 - x_1\| + \|\hat{x}_2 - x_2\|)$, *where* $\eta_\rho > 0$ *is a sufficiently small constant for* x_i *and* $\hat{x}_i - x_i$ *sufficiently close to zero* $(i = 1, 2)$.
4) $\|c_2(t)\| \leq c$, $t \in T$, $c = \text{const} > 0$.
5) $\|r_i(t, \hat{x}_1) - r_i(t, x_1)\| \leq \eta_i \|\hat{x}_1 - x_1\|$ $(i = 1, 2)$, *where* $\eta_i > 0$ *are sufficiently small constants for* $\|x_1\|$ *and* $\|\hat{x}_1 - x_1\|$ *sufficiently close to zero.*

Then system (5.12), (5.13), (6.1) *is detectable and has a state observer of the form* (6.5)–(6.7).

The proof is postponed to the Appendix.

Addressing system (5.1), (5.2), we linearize (5.1) in a neighborhood of the point $(x, x') = (0, 0)$:

$$A(t)x'(t) + B(t)x(t) + h(t, x(t), x'(t)) = 0, \tag{6.13}$$

where

$$A(t) = \frac{\partial f(t, x, x')}{x'}(t, 0, 0), \quad B(t) = \frac{\partial f(t, x, x')}{x}(t, 0, 0).$$

Under Assumptions A1 and A2, there exists the operator (2.7) transforming system (1.1) to (2.9)–(2.14). Applying this operator to equation (6.13) gives

$$x_2(t) + J_1(t)x_1(t) + h_1(t, \bar{x}_{r+1}(t)) = 0, \tag{6.14}$$

$$x_1'(t) + J_2(t)x_1(t) + h_2(t, \bar{x}_{r+1}(t)) = 0, \tag{6.15}$$

where the designations (2.10), (5.5) are used and

$$\begin{pmatrix} h_1(t, \bar{x}_{r+1}(t)) \\ h_2(t, \bar{x}_{r+1}(t)) \end{pmatrix} = \mathcal{R}\left[h(t, x(t), x'(t))\right].$$

Under the conditions of Theorem 3, any solution of system (5.1) defined in a sufficiently small neighborhood of zero is a solution of system (5.12), (5.13), and vice versa. Equations (6.2), (6.3) linearize system (5.12), (5.13) in a neighborhood of the point $x_1 = 0$.

Lemma 5. *Let all conditions of Theorem 3 hold. Then in systems* (6.14), (6.15) *and* (6.2), (6.3),

$$J_1(t) = -G_1(t), \quad J_2(t) = -G_2(t) \;\; \forall t \in T,$$

and the functions $-r_1(t, x_1(t))$ *and* $-r_2(t, x_1(t))$ *are obtained from* $h_1(t, \bar{x}_{r+1}(t))$ *and* $h_2(t, \bar{x}_{r+1}(t))$, *respectively, by substituting the implicit function* (5.8)–(5.11) *that satisfies the* r*-derivative array equations* (5.4).

A similar result was proved in [26].

Theorem 5. *Let all conditions of Theorem 3, as well as conditions 2–4 of Theorem 4, hold. Moreover, let*

$$\lim_{\hat{x}_2 \to x_2,\, \hat{x}' \to x',\, \hat{X}_1 \to X_1,\, \hat{X}_2 \to X_2} \|h_i(t, \hat{\bar{x}}_{r+1}) - h_i(t, \bar{x}_{r+1})\| \le \mu_i \|\hat{x}_1 - x_1\|, \quad i = 1, 2, \tag{6.16}$$

where $\mu_i = \mathrm{const} > 0$ *are sufficiently small for* $\|\hat{x}_1 - x_1\|$ *and* $\|\bar{x}_{r+1}\|$ *from the corresponding small neighborhoods of zero. Let the variables* $\bar{x}_{r+1}$, x_1, x_2, X_1, *and* X_2 *be given by* (5.5)–(5.7).

Then system (5.1), (5.2) *is detectable and has an observer of the form* (6.5)–(6.7).

The proof is provided in the Appendix.

7. AN ILLUSTRATIVE EXAMPLE

Based on Theorem 5, we analyze the detectability of the nonlinear system

$$2x_{(1)}(t) + x_{(3)}(t) + e^{-t} x_{(1)}(t) x_{(2)}(t) = 0, \tag{7.1}$$

$$\Big(1 + x_{(1)}(t)\Big) x'_{(1)}(t) = 0, \tag{7.2}$$

$$x'_{(2)}(t) + x'_{(3)}(t) + 4x_{(2)}(t) + 2x_{(2)}(t) x_{(3)}(t) = 0, \tag{7.3}$$

$$y(t) = \left(1 + \frac{1}{4}\cos t\right) x_{(2)}(t) + 2x_{(3)}(t) + e^{-2t} x_{(1)}(t) x_{(3)}(t), \quad t \in T, \tag{7.4}$$

where

$$x(t) = \mathrm{col}\Big(x_{(1)}(t),\, x_{(2)}(t),\, x_{(3)}(t)\Big).$$

To verify Assumptions B1–B4, we compile the matrix

$$\mathcal{D}_2(t, x, x', x'', x''') = \left(\begin{array}{ccc|ccc|ccc||ccc}
v_1 & v_2 & \boxed{1} & 0 & 0 & 0 & & & & & & \\
x'_{(1)} & 0 & 0 & \boxed{1 + x_{(1)}} & 0 & 0 & & & & & & \\
0 & v_3 & v_4 & 0 & \boxed{1} & 1 & & & & & & \\
\hline
v'_1 & v'_2 & 0 & v_1 & v_2 & \boxed{1} & 0 & 0 & 0 & & & \\
x''_{(1)} & 0 & 0 & 2x'_{(1)} & 0 & 0 & \boxed{1 + x_{(1)}} & 0 & 0 & & & \\
0 & v'_3 & v'_4 & 0 & v_3 & v_4 & 0 & \boxed{1} & 1 & & & \\
\hline\hline
v''_1 & v''_2 & 0 & 2v'_1 & 2v'_2 & 0 & v_1 & v_2 & \boxed{1} & 0 & 0 & 0 \\
x'''_{(1)} & 0 & 0 & 3x''_{(1)} & 0 & 0 & 3x'_{(1)} & 0 & 0 & \boxed{1 + x_{(1)}} & 0 & 0 \\
0 & v''_3 & v''_4 & 0 & 2v'_3 & 2v'_4 & 0 & v_3 & v_4 & 0 & \boxed{1} & 1
\end{array}\right),$$

where $v_1(t) = 2 + e^{-t} x_{(2)}(t)$, $v_2(t) = e^{-t} x_{(1)}(t)$, $v_3(t) = 4 + 2x_{(3)}(t)$, and $v_4(t) = 2x_{(2)}(t)$.

Clearly, Assumption B1 is valid for $r = 1$, i.e.,

$$\mathrm{rank}\,\Lambda_1(t, x, x') = \mathrm{rank} \begin{pmatrix} 1 + x_{(1)} & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} = \lambda = 2,$$

when $x_{(1)}$ satisfies the inequality $|x_{(1)}| < 1$.

The matrix $\mathcal{D}_1(t, x, x', x'')$ of dimensions (6×9), located above the horizontal and to the left of the vertical double lines, has an invertible submatrix $\mathcal{M}_1(t, x, x', x'')$ of order 6, whose columns are marked by boxed elements. Therefore, Assumption B2 is valid as well.

The matrix $\Lambda_2(t, x, x', x'')$ consists of the last six columns of the matrix $\mathcal{D}_2$. Obviously, $\mathrm{rank}\,\Lambda_2(t, x, x', x'') = \lambda + n = 5$ for $|x_{(1)}| < 1$, so Assumption B3 is valid.

The matrix $\mathcal{D}_2(t,x,x',x'',x''')$ also has a submatrix $\mathcal{M}_2(t,x,x',x'',x''')$ of order 9 with the properties formulated in Assumption B4; this submatrix consists of the columns containing the boxed elements. Thus, for $r=1$, Assumptions B1–B4 are valid.

For the first approximation system

$$\begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 1 \end{pmatrix} x'(t) + \begin{pmatrix} 2 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 4 & 0 \end{pmatrix} x(t) = 0,$$

the operator

$$\mathcal{R} = R_0(t) + R_1(t)\frac{d}{dt} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}\frac{d}{dt}$$

is defined; it has the left inverse operator $\mathcal{L} = L_0(t) + L_1(t)$, where $L_0(t) = R_0^{-1}(t)$ and $L_1(t) = -R_1(t)$.

The operator $\mathcal{R}$ transforms system (7.1)–(7.3) to

$$\begin{pmatrix} O & O \\ E_2 & O \end{pmatrix}\begin{pmatrix} x_1'(t) \\ x_2'(t) \end{pmatrix} + \begin{pmatrix} J_1(t) & 1 \\ J_2(t) & O \end{pmatrix}\begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} h_1(t,x,x',x'') \\ h_2(t,x,x',x'') \end{pmatrix} = 0,$$

where $d=1$, $Q=E_3$,

$$x_2(t) = x_{(3)}(t), \quad x_1(t) = \begin{pmatrix} x_{(1)}(t) \\ x_{(2)}(t) \end{pmatrix}, \tag{7.5}$$

$$J_1(t) = \begin{pmatrix} 2 & 0 \end{pmatrix}, \quad J_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & 4 \end{pmatrix},$$

$$h_1(t,x,x',x'') = e^{-t}x_{(1)}(t)x_{(2)}(t), \tag{7.6}$$

$$h_2(t,x,x',x'') = \begin{pmatrix} x_{(1)}(t)x_{(1)}'(t) \\ e^{-t}\left(x_{(1)}(t)x_{(2)}(t) - x_{(1)}'(t)x_{(2)}(t) - x_{(1)}(t)x_{(2)}'(t)\right) \\ + 2x_{(1)}(t)x_{(1)}'(t) + 2x_{(2)}(t)x_{(3)}(t) \end{pmatrix}. \tag{7.7}$$

Next, we verify the conditions of Proposition 3. According to Lemma 5, $G_1(t) = \begin{pmatrix} -2 & 0 \end{pmatrix}$ and $G_2(t) = \begin{pmatrix} 0 & 0 \\ 0 & -4 \end{pmatrix}$. Obviously, Condition 2 holds.

In view of (7.5), from (7.4) we obtain $c_1^\top(t) = \begin{pmatrix} 0 & 1+\frac{1}{4}\cos t \end{pmatrix}$, $c_2^\top(t) = 2$. According to formulas (6.12) and (3.15),

$$S(t) = \begin{pmatrix} -4 & 1+\dfrac{1}{4}\cos t \\ 0 & -\left(4+\dfrac{1}{4}\sin t + \cos t\right) \end{pmatrix}.$$

Since $\det S(t) = 16 + \sin t + 4\cos t \neq 0$ $\forall t \in T$, it is possible to find the inverse matrix

$$S^{-1}(t) = -\frac{1}{16+\sin t + 4\cos t}\begin{pmatrix} 4+\dfrac{1}{4}\sin t + \cos t & 1+\dfrac{1}{4}\cos t \\ 0 & 4 \end{pmatrix}$$

and the functions (3.16)

$$\omega_1(t) = 0, \quad \omega_2(t) = -\frac{64 + 8\sin t + 15\cos t}{16 + \sin t + 4\cos t}.$$

Clearly, conditions 2 and 3 of Lemma 2, as well as condition 3 of Proposition 3, hold.

Let us verify conditions 3 and 4 of Theorem 4, taking the relations (7.5) into account. Obviously, $\|c_2(t)\| = 2$. Since $\tilde{\rho}(t, x_1, x_2) = e^{-2t}x_{(1)}x_{(3)}$,

$$|\tilde{\rho}(t, \hat{x}_1, \hat{x}_2) - \tilde{\rho}(t, x_1, x_2)| \le |x_{(3)}|\,\|\hat{x}_1 - x_1\| + \Big(|\hat{x}_{(1)} - x_{(1)}| + |x_{(1)}|\Big)\,\|\hat{x}_2 - x_2\|,$$

with yields the estimate from condition 3 of Theorem 4 with $\eta_\rho = \max\{\delta_2, \delta_1 + \delta_3\}$, where $|x_{(1)}| \le \delta_1 < 1$, $|x_{(3)}| \le \delta_2$, and $|\hat{x}_{(1)} - x_{(1)}| \le \delta_3$, $\delta_i = \text{const} > 0$, $i = 1, 2, 3$.

Now we address Theorem 5, in particular, condition (6.16). Due to (7.6) and (7.7),

$$\lim_{\hat{x}_2\to x_2,\, \hat{x}'\to x',\, \hat{X}_1\to X_1,\, \hat{X}_2\to X_2} \|h_1(t, \hat{\bar{x}}_{r+1}) - h_1(t, \bar{x}_{r+1})\| = e^{-t}\,|\hat{x}_{(1)}\hat{x}_{(2)} - x_{(1)}x_{(2)}|$$
$$\le \Big(|\hat{x}_{(1)} - x_{(1)}| + |x_{(1)}| + |x_{(2)}|\Big)\,\|\hat{x}_1 - x_1\| \le \eta_1\|\hat{x}_1 - x_1\|,$$

where $\eta_1 = \delta_1 + \delta_3 + \delta_4$, $|x_{(2)}| \le \delta_4$, and $\delta_4 = \text{const} > 0$.

In turn,

$$\lim_{\hat{x}_2\to x_2,\, \hat{x}'\to x',\, \hat{X}_1\to X_1,\, \hat{X}_2\to X_2} \|h_2(t, \hat{\bar{x}}_{r+1}) - h_2(t, \bar{x}_{r+1})\|$$
$$\le \Big(|x_{(1)}| + |x_{(2)}| + 2|x_{(3)}| + |\hat{x}_{(1)} - x_{(1)}| + 4|x'_{(1)}| + |x'_{(2)}|\Big)\,\|\hat{x}_1 - x_1\| \le \eta_2\|\hat{x}_1 - x_1\|,$$

where $\eta_2 = \delta_1 + 2\delta_2 + \delta_3 + \delta_4 + 4\delta_5 + \delta_6$, $|x'_{(1)}| \le \delta_5$, and $|x'_{(2)}| \le \delta_6$ with positive constants and δ_5 and δ_6.

Thus, all conditions of Theorem 5 are satisfied, so system (7.1)–(7.4) is detectable.

For the example under consideration, the first four equations of the 2-derivative array ones can be used to find the functions (5.12), (5.13):

$$x_{(3)}(t) = -x_{(1)}(t)\left(2 + e^{-t}x_{(2)}(t)\right),$$

$$x'_{(1)}(t) = 0, \quad x'_{(2)}(t) = \frac{x_{(2)}(t)\left(e^{-t}x_{(1)}(t)(2x_{(2)}(t) - 1) + 4(x_{(1)}(t) - 1)\right)}{1 - e^{-t}x_{(1)}(t)}.$$

Therefore, it is possible to design the following state observer with the output (6.7) for system (7.1)–(7.4):

$$\hat{x}_{(3)}(t) = -\hat{x}_{(1)}(t)\left(2 + e^{-t}\hat{x}_{(2)}(t)\right),$$

$$\begin{pmatrix} \hat{x}'_{(1)}(t) \\ \hat{x}'_{(2)}(t) \end{pmatrix} = \begin{pmatrix} 0 \\ -4\hat{x}_{(2)}(t) + \dfrac{\hat{x}_{(1)}(t)\hat{x}_{(2)}(t)\left(2e^{-t}\hat{x}_{(2)}(t) - 5e^{-t} - 4\right)}{1 - e^{-t}\hat{x}_{(1)}(t)} \end{pmatrix}$$
$$+\, g(t)\left(y(t) + 4\hat{x}_{(1)}(t) - \left(1 + \frac{1}{4}\right)\hat{x}_{(2)}(t) + 2e^{-t}\hat{x}_{(1)}(t)\hat{x}_{(2)}(t) + e^{-2t}\hat{x}^2_{(1)}(t)\left(2 + e^{-t}\hat{x}_{(2)}(t)\right)\right).$$

The vector function $g(t)$ is intended to ensure the detectability of system (3.10), (3.11) or, equivalently, the asymptotic stability of system (3.9). According to the book [22, pp. 255–257, 283–284, 312–314], one such function is given by

$$g(t) = \Gamma(t)\begin{pmatrix} g_1(t) - \gamma_1 \\ g_2(t) - \gamma_2 \end{pmatrix},$$

where $\gamma_1 = -\xi_1\xi_2$, $\gamma_2 = -(\xi_1 + \xi_2)$, ξ_1 and ξ_2 are specified negative numbers,

$$g_1(t) = \frac{\sin t + 4\cos t}{q(t)} + \left(\frac{\cos t - 4\sin t}{q(t)}\right)^2, \quad g_2(t) = 4 + \frac{4\sin t - \cos t}{q(t)},$$

$$\Gamma(t) = \frac{1}{q(t)} \begin{pmatrix} 1 + \frac{1}{4}\cos t & g_2(t)\left(1 + \frac{1}{4}\cos t\right) - 4 - \cos t - \frac{1}{4}\sin t \\ 4 & 4g_2(t) \end{pmatrix},$$

$$q(t) = 16 + 4\cos t + \sin t.$$

To verify the asymptotic stability of system (3.9), we make the change of variables $e_1(t) = \Gamma(t)\tilde{e}_1(t)$ in (3.9). Then direct substitution yields the system $\tilde{e}_1'(t) = \begin{pmatrix} 0 & -\gamma_1 \\ -1 & -\gamma_2 \end{pmatrix} \tilde{e}_1(t)$, whose matrix has negative eigenvalues ξ_1 and ξ_2. Since $\Gamma(t)$ is a Lyapunov matrix, system (3.9) will also be asymptotically stable.

8. CONCLUSIONS

This paper has been devoted to linear and nonlinear time-varying DAEs of observation under general assumptions imposed on their internal structure.

For the linear system (1.1), we have established existence conditions for the structural form (2.9) obtained by applying the linear differential operator (2.7) to (1.1). This operator has a left inverse, so systems (1.1) and (2.9) share the same set of solutions. Moreover, in (2.9), the differential and algebraic components are already separated, and each component mentioned is nonsingular. Conditions ensuring the existence of the right inverse operator (2.17) for (2.7) have been derived (Lemma 1).

For the structural form (2.9) with a scalar output, a state observer of the form (3.2)–(3.4) has been designed. It can be used to estimate the state of system (1.1), (1.2) as well (Theorem 1). In fact, it suffices to construct an observer for the differential component of the DAE (1.1).

In addition, we have considered system (4.1) with scalar control, which also includes the derivative of the control. What is essential, the right inverse operator (2.17) is employed to construct such a system. Conditions have been justified under which system (4.2) can be represented in the form (4.1) (Lemma 3). For the DAE (4.1), stabilizability conditions based on the state observer (3.2)–(3.4) have been established (Theorem 2). Moreover, the stabilizing feedback control (4.7) includes only the solution of the differential subsystem of the observer.

The nonlinear system (5.1) with the scalar output (5.2) has been analyzed using the structural form (5.12), (5.13): any of its solutions starting in a sufficiently small neighborhood of zero is also a solution of system (5.1), and vice versa. Being already a nonsingular system, this structural form consists of some components of an implicit function satisfying the r-derivative array equations (5.4) (Theorem 3). For the DAE (5.1), (5.2), we have utilized the linear approximation system to obtain sufficient detectability conditions and design a state observer of the form (6.5)–(6.7) (Theorem 5). Note that to construct the observer, it is necessary to find the above implicit function. At the same time, verifying the detectability conditions does not require this procedure: only the properties of the linear approximation are used here. Important auxiliary results are the detectability conditions and the method for designing a state observer for the quasilinear system of the form (5.12), (5.13), (6.1) (Theorem 4).

The above approach can be applied to analyze detectability and construct observers for linear and nonlinear discrete descriptor systems, as well as singular systems with continuous and discrete time.

FUNDING

This work was carried out within the state assignment on the topic "Evolutionary and Dynamic Controlled Systems: Theory, Numerical Methods, and Applications" (scientific topic code FWEW-2026-0011, state registration no. 126021217177-7).

APPENDIX

Proof of Lemma 1. We rearrange the columns of the matrix $\mathcal{D}_r(t)$ (2.1)–(2.4) by multiplying it on the right by the permutation matrix $\operatorname{diag}\{Q, \ldots, Q\}$, where Q is given by (2.5). In doing so, it is necessary to consider the partition structure (2.6). By deleting the columns included in the block $B_1(t)$ from $\mathcal{D}_r(t)$, we obtain the matrix $\bar{\mathcal{D}}_r(t)$ (omitted here due to cumbersomeness).

Note that condition 1 of the lemma ensures $P_0(t), P_1(t) \in \mathbf{C}^r(T)$.

By the construction procedure, the coefficients of the operator $\mathcal{R}$ satisfy the equation

$$\begin{pmatrix} R_0(t) & R_1(t) & \ldots & R_r(t) \end{pmatrix} \bar{\mathcal{D}}_r(t) = \begin{pmatrix} E_n & O & \ldots & O \end{pmatrix} \tag{A.1}$$

for all $t \in T$. On the other hand, by definition, the coefficients of the operators (2.7) and (2.17) are related through the identity

$$\begin{pmatrix} R_0 & R_1 & \ldots & R_r \end{pmatrix} \begin{pmatrix} P_0 & C_0^0 P_1 & O & \ldots & O \\ P_0' & C_1^0 P_1' + C_1^1 P_0 & C_1^1 P_1 & \ldots & O \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ P_0^{(r)} & C_r^0 P_1^{(r)} + C_r^1 P_0^{(r-1)} & C_r^1 P_1^{(r-1)} + C_r^2 P_0^{(r-2)} & \ldots & C_r^r P_1 \end{pmatrix}$$
$$= \begin{pmatrix} E_n & O & \ldots & O \end{pmatrix}. \tag{A.2}$$

Letting $P_0(t) = \begin{pmatrix} B_2(t) & A_1(t) \end{pmatrix}$, $P_1(t) = \begin{pmatrix} A_2(t) & O \end{pmatrix}$ in (A.2) makes equalities (A.1) and (A.2) coincident under identity (2.16).

Proof of Lemma 3. Obviously, for the desired result, the vectors $b(t)$, $\beta_1(t)$, and $\beta_2(t)$ shall satisfy the relations

$$\beta_1(t) = P_0(t)b(t) + P_1(t)b'(t), \tag{A.3}$$

$$\beta_2(t) = P_1(t)b(t), \quad t \in T. \tag{A.4}$$

According to Lemma 1, the operator $P_0(t) + P_1(t)\frac{d}{dt}$ has the left inverse $\mathcal{R}$. By applying this operator to both sides of identity (A.3), we arrive at the representation (4.4). Condition (4.3) is finally obtained by substituting (4.4) into equality (A.4).

Proof of Theorem 2. First, we show that under the conditions of this theorem, there exists a feedback control (4.7) stabilizing system (4.5), (4.6).

Consider equations (4.10), (4.11). Due to assumptions 1 and 3, all conditions of Lemma 2 are satisfied, and there exists a function $g(t) \in \mathbf{C}(T)$ such that

$$\lim_{t\to\infty} e_1(t) = 0. \tag{A.5}$$

In this case, in view of the estimate (3.14), equation (4.10) implies the limit relation $\lim\limits_{t\to\infty} e_2(t) = 0$. Thus, for an appropriate choice of $g(t)$, subsystem (4.10), (4.11) is asymptotically stable.

Now we turn to equations (4.8), (4.9). Under assumptions 1 and 4, Lemma 4 is valid: there exists a vector function $k(t) \in \mathbf{C}(T)$ such that system (4.9) is asymptotically stable. This means that for any continuous function $b_2(t)k^\top(t)e_1(t)$, each solution $x_1(t)$ of system (4.9) has the property

$$\lim_{t\to\infty} x_1(t) = 0. \tag{A.6}$$

In turn, assumption 5 together with conditions (3.14), (4.15) lead to the estimates

$$\chi\left[J_1(t) + b_1(t)k^\top(t)\right] \le 0,\ \ \chi\left[b_1(t)k^\top(t)\right] \le 0;$$

in view of (A.5) and (A.6), it follows that in (4.8), $x_2(t) \to 0$ as $t \to +\infty$.

Based on the above considerations, we conclude that the control (4.7) is stabilizing for system (4.5), (4.6).

The supposed smoothness of the input data of system (4.1) ensures the inclusions $J_2(t) \in \mathbf{C}^{n-d}(T)$ and $b_2(t) \in \mathbf{C}^{n-d+1}(T)$. According to Remark 3, for the control (4.7), the function $k(t)$ can be constructed in the class $\mathbf{C}^1(T)$. Under Assumptions A1–A4, the DAEs (4.1) and (A.5), (4.5) are equivalent in the sense of their solutions; therefore, the feedback control (4.7) will also stabilize system (4.1).

Proof of Theorem 4. According to Proposition 3, there exists a function $g(t) : T \to \mathbf{R}^{n-d}$ such that system (6.11) is asymptotically stable. Moreover, by Remark 1, $g(t)$ can be assumed bounded and system (6.11) reducible in the Lyapunov sense.

Under the above assumptions, in equation (6.9),

$$\begin{aligned} &\Big\| r_2(t,\hat{x}_1(t)) - r_2(t,x_1(t)) - c_2^\top(t)\Big(r_1(t,\hat{x}_1(t)) - r_1(t,x_1(t))\Big) \\ &+ \tilde{\rho}\Big(t, x_1(t), G_1(t)x_1(t) + r_1(t,x_1(t))\Big) - \tilde{\rho}\Big(t,\hat{x}_1(t), G_1(t)\hat{x}_1(t) + r_1(t,\hat{x}_1(t))\Big)\Big\| \\ &\le (\eta_2 + \eta_1(1+c) + \eta_\rho(1+\kappa))\,\|e_1\|. \end{aligned}$$

According to the stability theorem by the first approximation [22, p. 340] (also known as Lyapunov's indirect method in the literature), the trivial solution of system (6.9) is asymptotically stable for sufficiently small constants η_1, η_2, and η_ρ under which $e_1(t)$ varies in a small neighborhood of zero.

In turn, due to the bounded matrix $G_1(t)$ and the estimate $\|r_1(t,\hat{x}_1) - r_1(t,x_1)\| \le \eta_1\|e_1(t)\|$, the vector function $e_2(t)$, as a solution of equation (6.8), vanishes as $e_1(t) \to 0$.

Thus, the trivial solution of system (6.8), (6.9) is asymptotically stable for an appropriate choice of $g(t)$. This means that system (6.2)–(6.4), and hence system (5.12), (5.13), (6.1), is detectable and has an observer of the form (6.5)–(6.7).

Proof of Theorem 5. According to Theorem 3, for sufficiently small initial data (2.15), system (5.1) has a nontrivial solution satisfying equations (5.12) and (5.13). In turn, any solution of system (5.12), (5.13) lying in a sufficiently small neighborhood of zero is a solution of system (5.1). Consequently, systems (5.1), (5.2) and (5.12), (5.13), (6.1) are detectable simultaneously. Recall that (6.1) is another representation for the output (5.2).

Under the conditions of Theorem 3, there exists an implicit function (5.8)–(5.11) satisfying the r-derivative array equations (5.4). Due to assumption 1 of Theorem 3 and condition (5.3), this implicit function has continuous partial derivatives with respect to each argument on its domain of definition and $f_j(t,0) \equiv 0$, $f_4(t,0,0) \equiv 0$, $t \in T$, $j = 1,2,3$.

We substitute the functions (5.8)–(5.11) into condition (6.16). Unlike the constants η_i from assumption 5 of Theorem 4, the constants μ_i may depend on the sizes of the corresponding neighborhoods of zero covering the variables x_2, x_1', x_2', and X_1. As a result of the substitution, these variables are replaced by the functions on the right-hand side of equalities (5.8)–(5.11). According to the aforementioned properties of these functions, one can choose $\|x_1\|$ and $\|X_2\|$ sufficiently close to zero so that the values $|f_j(t, x_1)|$ and $|f_4(t, x_1, X_2)|$ will belong to given small neighborhoods of zero for all $t \in T$. In view of Lemma 5, the estimates of assumption 5 of Theorem 4 hold.

Thus, all assumptions of Theorem 4 are satisfied. According to this theorem, system (5.12), (5.13), (5.2) is detectable and has a state observer of the form (6.5)–(6.7). The same property holds for system (5.1), (5.2).

REFERENCES

1. Dai, L., *Singular Control Systems*, Berlin: Springer, 1989.
2. Assoudi, A. and Yaagoubi, H., Observer Design for a Class of Differential-Algebraic Model of an Aerobic Culture of a Recombinant Yeast, *Bioprocess Biosyst. Eng.*, 2003, vol. 26, pp. 27–35.
3. Hou, M. and Muller, P., Observer Design for Descriptor Systems, *IEEE Trans. Autom. Control*, 1999, vol. 44, no. 1, pp. 164–169.
4. Duan, G.-R., *Analysis and Design of Descriptor Linear Systems*, Advances in Mechanics and Mathematics, vol. 23, New York: Springer, 2010.
5. Silva, M.S. and Lima, T.P., Looking for Nonnegative Solutions of a Leontief Dynamic Model, *Linear Algebra and Its Applications*, 2003, vol. 364, pp. 281–316.
6. Moysis, L., Giakoumis, A., Gupta, M.K., et al., Observers for Rectangular Descriptor Systems with Output Nonlinearities: Application to Secure Communications and Microcontroller Implementation, *Int. J. Dyn. Control*, 2021, vol. 9, no. 2, pp. 530–540.
7. Shields, D.N., Observers for Descriptor Systems, *Int. J. of Control*, 1991, vol. 55, no. 1, pp. 249–256.
8. Nikoukhah, R., Willsky, A.S., and Levy, B.C., Kalman Filtering and Riccati Equations for Descriptor Systems, *IEEE Transactions on Automatic Control*, 1992, vol. 37, no. 9, pp. 1325–1342.
9. Aslund, J. and Frisk, E., An Observer for Non-Linear Differential-Algebraic Systems, *Automatica*, 2006, vol. 42, no. 6, pp. 959–965.
10. Shields, D.N., Observer Design and Detection for Nonlinear Descriptor Systems, *Int. J. of Control*, 2010, vol. 67, no. 2, pp. 153–168.
11. Tripathi, M., Moysis, L., Gupta, M.K., Fragulis, G.F., and Volos, Ch., Observer Design for Nonlinear Descriptor Systems: A Survey on System Nonlinearities, *Curcuits, Systems and Signal Processing*, 2024, vol. 43, pp. 2853–2872.
12. Abbaszadeh, M. and Marquez, H.J., Nonlinear Observer Design for One-Sided Lipschitz Systems, *Proceedings of the 2010 American Control Conference*, IEEE, 2010, pp. 5284–5289.
13. Berger, T., On Observers for Nonlinear Differential-Algebraic Systems, *IEEE Trans. Autom. Control*, 2018, vol. 64, no. 5, pp. 2150–2157.
14. Cong, C., Observer-Based Robust Control of Uncertain Systems via an Integral Quadratic Constraint Approach, *Int. J. Dyn. Control*, 2019, vol. 7, no. 3, pp. 926–939.
15. Darouach, M., Boutat-Baddas, L., and Zerrougui, M., H^∞-Observers Design for a Class of Nonlinear Singular Systems, *Automatica*, 2011, vol. 47, no. 11, pp. 2517–2525.
16. Kidane, N., Yamashita, Y., and Nishitani, H., Observer Based I/O-Linearizing Control of High Index DAE Systems, *Proceedings of the IEEE Conference on Decision and Control*, Denver, USA, 2003, pp. 3537–3542.
17. Gao, Z. and Ho, D.W., State/Noise Estimator for Descriptor Systems with Application to Sensor Fault Diagnosis, *IEEE Trans. Signal Proc.*, 2006, vol. 54, no. 4, pp. 1316–1326.
18. Lu, G. and Ho, D.W., Full-Order and Reduced-Order Observers for Lipschitz Descriptor Systems: the Unified LMI Approach, *IEEE Trans. Circuits Syst., II: Express Briefs*, 2006, vol. 53, no. 7, pp. 563–567.

19. Zhang, J., Swain, A.K., and Nguang, S.K., Simultaneous Estimation of Actuator and Sensor Faults for Descriptor Systems, in *Robust Observer Based Fault Diagnosis for Nonlinear Systems Using MATLAB*, Advances in Industrial Control, Cham: Springer, 2016, pp. 165–197.

20. Zulfiqar, A., Rehan, M., and Abid, M., Observer Design for One-Sided Lipschitz Descriptor Systems, *Appl. Math. Model.*, 2016, vol. 40, no. 3, pp. 2301–2311.

21. Shcheglova, A.A., The Solvability of the Initial Problem for a Degenerate Linear Hybrid System with Variable Coefficients, *Russ. Math.*, 2010, vol. 54, pp. 49–61.

22. Gaishun, I.V., *Vvedenie v teoriyu lineinykh nestatsionarnykh sistem* (Introduction to the Theory of Linear Time-Varying Systems), Minsk: Institute of Mathematics, the National Academy of Sciences of Belarus, 1999.

23. Shilov, G.E., *Matematicheskiy analiz. Funktsii neskol'kikh veshchestvennykh peremennykh* (Mathematical Analysis. Functions of Several Real Variables), Moscow: Nauka, 1972.

24. Cristea, M., A Note on Global Implicit Function Theorem, *Journal of Inequalities in Pure and Applied Mathematics*, 2007, vol. 8, no. 4, art. no. 100.

25. Shcheglova, A.A., On the Unbounded Extendability of Solutions to Nonlinear Differential Algebraic Equations, *Sib. Math. J.*, 2014, vol. 55, no. 4, pp. 768–782.

26. Shcheglova, A.A., The Stability by Linear Approximation of Discrete-Time Nonlinear Singular Systems, *Sib. Math. J.*, 2024, vol. 65, no. 2, pp. 392–410.

This paper was recommended for publication by P.S. Shcherbakov, a member of the Editorial Board