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On the Relationship between the Average Characteristics of the Input and Output Fuzzy Signals of a Linear Dynamic System

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Abstract—This paper considers the problem of transforming a fuzzy signal by a linear dynamic system described by a linear differential equation of high order with variable real coefficients, a fuzzy-valued inhomogeneity, and fuzzy initial conditions. By assumption, certain numerical average characteristics of the fuzzy-valued input signal (fuzzy means and fuzzy half-ranges) are known. Based on these characteristics and system parameters, the corresponding numerical average characteristics of the fuzzy signal at the output are established. Methods of fuzzy mathematics and the theory of ordinary differential equations are used. As an application, the model of a series resonant radio circuit with a fuzzy-valued input signal is considered.

Keywords: fuzzy-valued functions, means of fuzzy-valued functions, derivatives of fuzzy-valued functions, fuzzy linear differential equations with variable coefficients

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1. INTRODUCTION

The problem of estimating the output signal of dynamic systems with incomplete or weakly formalized information on the input signal is of significant theoretical and applied importance, particularly for the subsequent mitigation of external disturbances (measurement noise, faults, etc.). The use of fuzzy differential equations is an important modern approach to modeling dynamic systems under possible uncertainties.

This paper addresses the problem of transforming a fuzzy signal by a linear dynamic system. The linear dynamic system is described by a linear fuzzy differential equation of high order with variable real coefficients, a fuzzy-valued inhomogeneity, and fuzzy initial conditions.

The foundations of the theory of fuzzy differential equations were laid in [1–5] and other works. This research area has been actively developing in recent decades. A method for transforming fuzzy linear differential equations to integral fuzzy equations (followed by their solution) was proposed in [6, 7]. Several authors considered an approach involving a reduction of original fuzzy differential equations to differential inclusions [8, 9] and an analysis of the latter.

In recent years, the fuzzy Laplace transform has become a widespread method for solving linear fuzzy differential equations of high order [10, 11]. Many researchers study numerical methods for solving fuzzy differential equations [12–14].

As a rule [15, 16], when dealing with linear fuzzy differential equations, a system of equations for the corresponding α-indices is compiled and solved. Then, it is necessary to verify that the derivatives of the resulting α-indices define the derivative of a fuzzy-valued function. According to demonstrative examples [6, 16], this is not always the case.

This paper utilizes the concept of an ultraweak fuzzy solution [17, 18] as a fuzzy-valued Hausdorff-continuous function whose α-indices satisfy the equations for the α-indices arising from a given fuzzy differential equation.

Various definitions of differentiability of fuzzy-valued functions are considered in the literature; for example, see [2–5]. In Section 3 of this paper, we use the Seikkala derivative [3]; in Section 4, generalized Hukuhara derivatives [5–7] and others.

Note that, in particular, the issue of smoothness of ultraweak Seikkala solutions was discussed in [17, 18].

The standard situation in applications is incomplete information on an input fuzzy signal. More precisely put, the exact membership functions of the corresponding fuzzy numbers are unknown, but some estimates can be provided. Here, we adopt an analogy with the theory of random processes: within a conventional approach, the average characteristics (mean and correlation function) of the output random signal of a linear dynamic system are determined based on those of its input random signal and system parameters [19, Ch. 7]. This approach is motivated by the fact that the random variables corresponding to an input random signal (and even more so, those corresponding to the output one) often have unknown distributions.

In this paper, the numerical average characteristics of an input fuzzy signal, i.e., the means of fuzzy numbers (fuzzy means) and the half-ranges of fuzzy numbers (fuzzy half-ranges), are supposed to be known; in particular, they can be obtained using expert appraisals. Then the following practically important problem naturally arises: taking into account the system parameters, it is required to determine the corresponding numerical average characteristics of a fuzzy signal at the output of a linear dynamic system.

Below, we consider the initial value problem for a fuzzy differential equation and actually determine the above numerical characteristics of its solution from the system parameters, fuzzy initial conditions, and the corresponding characteristics of the input data, without finding the fuzzy-valued solution itself. Note that such an approach for fuzzy differential equations has not been presented before. Fuzzy means commonly serve to defuzzify fuzzy numbers, and fuzzy half-ranges are introduced here and have proved fruitful in both theoretical and applied aspects.

Fuzzy linear differential equations are widely used in various applications. (Among contemporary works, we mention [20–22].) In Section 5, as an application, we describe the model of the response of a radio circuit with constant parameters to a fuzzy input signal.

This paper continues the author's research initiated in [23], where a linear dynamic system with constant real parameters was considered, and the relationship between the means and covariance functions of its input and output bounded fuzzy signals was established.

2. FUZZY NUMBERS AND FUZZY-VALUED FUNCTIONS

A fuzzy number is understood as a fuzzy subset of the universal set of real numbers $\mathbb{R}$ that has a compact support and a normal, convex, and upper semicontinuous membership function; for example, see [24, Ch. 5; 25, Chs. 2, 3].

Here, the support of a fuzzy number $\tilde{z}$ is defined as $Supp\,\tilde{z} = \{x|\mu_{\tilde{z}}(x) > 0\}$. A fuzzy number $\tilde{z}$ is normal if $\max_x \mu_{\tilde{z}}(x) = 1$ and convex if $\mu_{\tilde{z}}(t) \geqslant \min(\mu_{\tilde{z}}(r), \mu_{\tilde{z}}(s))$ for any real numbers $r \leqslant t \leqslant s$.

Let J denote the set of such fuzzy numbers.

Below we will use the interval representation of fuzzy numbers.

As is known, the α-level intervals (α-intervals) of a fuzzy number $\tilde{z} \in J$ with a membership function $\mu_{\tilde{z}}(x)$ are defined as $[\tilde{z}]_\alpha = \{x|\mu_{\tilde{z}}(x) \geqslant \alpha\}$ $(\alpha \in (0,1])$, $[\tilde{z}]_0 = cl\{x|\mu_{\tilde{z}}(x) > 0\}$, where cl indicates the closure of an appropriate set. According to the accepted assumptions, all α-levels of a fuzzy number are closed and bounded intervals of the real axis.

Let z_α^- and z_α^+ denote the left and right bounds of an α-interval: $[\tilde{z}]_\alpha = [z_\alpha^-, z_\alpha^+]$. The values z_α^- and z_α^+ are called the left and right α-indices of a fuzzy number, respectively.

The α-indices of a fuzzy number $\tilde{z} \in J$ has the following properties:

1) $z_\alpha^- \leqslant z_\alpha^+ \ \forall\, \alpha \in [0,1]$.
2) The function z_α^- is bounded, nondecreasing, left-continuous on $(0,1]$ and right-continuous at 0.
3) The function z_α^+ is bounded, nonincreasing, left-continuous on $(0,1]$ and right-continuous at 0.

Conversely, a pair of functions on the interval $[0,1]$ with properties 1–3 defines a fuzzy number whose α-interval has the form $[z_\alpha^-, z_\alpha^+]$.

The sum of fuzzy numbers with indices z_α^-, z_α^+ and u_α^-, u_α^+ is understood as a fuzzy number c with the α-level intervals $[z_\alpha^- + u_\alpha^-, z_\alpha^+ + u_\alpha^+]$. Multiplication by a positive number c is characterized by the α-level intervals $[cz_\alpha^-, cz_\alpha^+]$; multiplication by a negative number c, by the α-level intervals $[cz_\alpha^+, cz_\alpha^-]$. Equality for fuzzy numbers is understood as equality for all the corresponding α-indices $\forall\, \alpha \in [0,1]$. A real number r is treated as a fuzzy number whose α-indices coincide with $r \ \forall\, \alpha \in [0,1]$.

According to [26], the mean value (fuzzy mean) of a fuzzy number $\tilde{z}$ with α-indices $z_\alpha^\pm$ is the real number

$$m_{\tilde{z}} = \frac{1}{2} \int_0^1 (z_\alpha^+ + z_\alpha^-)\, d\alpha.$$

As is known [24], the modal value (mode) of a fuzzy number $\tilde{z}$ is a value z_{Mod} for which $\mu(z_{Mod}) = 1$. For unimodal fuzzy numbers $\tilde{z}$, i.e., those having a unique modal value, the right and left widths are sometimes considered (for example, see [27, p. 11]):

$$(r_{\tilde{z}})_l = \int_0^1 (z_\alpha^+ - z_{Mod})\, d\alpha, \quad (r_{\tilde{z}})_r = \int_0^1 (z_{Mod} - z_\alpha^-)\, d\alpha.$$

We define the half-range of a fuzzy number $\tilde{z}(t)$ (fuzzy half-range) as the real number

$$r_{\tilde{z}} = \frac{1}{2} \int_0^1 (z_\alpha^+ - z_\alpha^-)\, d\alpha.$$

This definition does not require knowledge of the modal value of a fuzzy number and is natural, as a generalization of the concept of radius in interval analysis. Note that $r_{\tilde{z}} = \frac{1}{2}\left((r_{\tilde{z}})_l + (r_{\tilde{z}})_r\right)$.

Example 1. Consider a fuzzy triangular number $\tilde{z}$ characterized by a real-valued triple (a,b,c) with $a < b < c$ defining the membership function

$$\mu_{\tilde{z}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in [a,b] \\[2mm] \dfrac{x-c}{b-c} & \text{if } x \in [b,c] \\[2mm] 0 & \text{otherwise.} \end{cases}$$

In this case, the left and right bounds of the α-interval have the form $z_\alpha^- = (b-a) + a$ and $z_\alpha^+ = -(c-b) + c$, respectively.

As is easily verified, the mean $m_{\tilde{z}}$ of this triangular number is $m_{\tilde{z}} = \frac{1}{4}(2b + a + c)$, and its half-range $r_{\tilde{z}}$ is $r_{\tilde{z}} = \frac{1}{4}(c-a)$.

Below, we will use the Hausdorff distance (metric), defined for fuzzy numbers $\tilde{z}$ and $\tilde{u}$ as [28] $d(\tilde{z},\tilde{u}) = \sup_{0 \leqslant \alpha \leqslant 1} \max\{|z_\alpha^- - u_\alpha^-|,\ |z_\alpha^+ - u_\alpha^+|\}$. Here, $z_\alpha^\pm$ and $u_\alpha^\pm$ are the α-indices of the fuzzy numbers $\tilde{z}$ and $\tilde{u}$, respectively.

Consider a fixed interval $[0, T]$ on the real axis. A mapping $\tilde{z} : [0, T] \to J$ will be called a fuzzy-valued function.

Let a fuzzy-valued function $\tilde{z}(t)$, $t \in [0, T]$, be characterized by a membership function $\mu_{\tilde{z}}(x, t)$. For a fixed number $\alpha \in (0, 1]$, consider the α-interval $[\tilde{z}]_\alpha(t) = \{x \in \mathbb{R} : \mu_{\tilde{z}}(x, t) \geqslant \alpha\}$ and $z_0(\alpha) = cl\{x \in \mathbb{R} : \mu_{\tilde{z}}(x, t) > 0\}$. We denote by $z_\alpha^-(t)$ and $z_\alpha^+(t)$ the left and right bounds of the α-interval, respectively: $[\tilde{z}]_\alpha(t) = [z_\alpha^-(t), z_\alpha^+(t)]$.

The continuity of a function $\tilde{z}(t)$ in t will be understood in the Hausdorff metric, and its boundedness as the existence of a constant $C > 0$ such that $d(\tilde{z}(t), \tilde{\theta}) = \sup_{0 \leqslant \alpha \leqslant 1} \max\{|z_\alpha^-(t)|,\ |z_\alpha^+(t)|\} \leqslant C$ $(\forall\, t \in [0, T])$, where $\tilde{\theta}$ is a fuzzy number with the α-indices $\theta_\alpha^\pm = 0\ \forall\, \alpha \in [0, 1]$, and d means the Hausdorff metric.

Remark 1. The sum of Hausdorff-continuous fuzzy-valued functions and the product of a fuzzy-valued function by a real number are continuous in the Hausdorff metric.

Remark 2. The indices $z_\alpha^-(t)$ and $z_\alpha^+(t)$ of a continuous fuzzy-valued function $\tilde{z}(t)$ are continuous in t for any $\alpha \in [0, 1]$. Moreover, if $\tilde{z}(t)$ is bounded for $t \in [0, T]$, then $z_\alpha^\pm(t)$ is bounded in $t \in [0, T]$ for any $\alpha \in [0, 1]$.

The (Riemann) integral of a continuous fuzzy-valued function $\tilde{z}(t)$ over an interval $[0, T]$ is defined [29] as the fuzzy number $\tilde{g}$ with the α-level intervals $g_\alpha = \int_0^T z_\alpha(t)\, dt$ for any $\alpha \in [0, 1]$. The integral is denoted by $\int_0^T \tilde{z}(t) dt$.

In view of Remark 2, for a continuous function $\tilde{z} : [0, T] \to J$, we have the interval representation $\left[\int_0^T \tilde{z}(t) dt\right]_\alpha = \left[\int_0^T z_\alpha^-(t) dt,\ \int_0^T z_\alpha^+(t) dt\right]$ [29].

Recall the definition of the Seikkala derivative of a fuzzy-valued function.

A fuzzy-valued function $\tilde{z} : [0, T] \to J$ is said to be Seikkala-differentiable (or S-differentiable) at a point $t \in (0, T)$ [3] if its α-indices $z_\alpha^-(t)$ and $z_\alpha^+(t)$ are differentiable and their derivatives $(z_\alpha^-)'(t)$ and $(z_\alpha^+)'(t)$ $\forall\, \alpha \in [0, 1]$ induce a fuzzy number with the α-interval $[\tilde{z}'(t)]_\alpha = [(z_\alpha^-)'(t), (z_\alpha^+)'(t)]$.

For example, a fuzzy-valued function $\tilde{z}(t) = g(t)\tilde{r}$, where $g(t)$ is a scalar differentiable function and $\tilde{r} \in J$ is a given fuzzy number, will be S-differentiable at a point t under the condition $g(t)g'(t) \geqslant 0$ [5].

Remark 3. By definition, the fuzzy S-derivative is additive and positively homogeneous. That is, for fuzzy-valued S-differentiable functions $\tilde{z}(t)$ and $\tilde{w}(t)$, we have $(\tilde{z}(t) + \tilde{w}(t))' = \tilde{z}'(t) + \tilde{w}'(t)$ and $(c\tilde{z}(t))' = c\tilde{z}'(t)$ for any real constant $c \geqslant 0$. Moreover, $[(\tilde{z}(t) + \tilde{w}(t))']_\alpha^\pm = (z_\alpha^\pm)'(t) + (w_\alpha^\pm)'(t)$ and $[(c\tilde{z}(t))']_\alpha^\pm = c(z_\alpha^\pm)'(t)$.

The second S-derivative $\tilde{z}''(t)$ at a point $t \in (0, T)$ is defined as the S-derivative of the first one, i.e., as the fuzzy number $\tilde{z}''(t)$ having the left α-index $(z_\alpha^-)''(t)$ and the right α-index $(z_\alpha^+)''(t)$ $\forall\, \alpha \in [0, 1]$.

The definitions of higher-order S-derivatives are introduced by analogy.

Generalized derivatives of fuzzy-valued functions will be defined and used in Section 4.

3. THE INITIAL VALUE PROBLEM FOR LINEAR DIFFERENTIAL EQUATIONS WITH A FUZZY-VALUED INHOMOGENEITY AND FUZZY INITIAL CONDITIONS

For a linear inhomogeneous fuzzy differential equation on an interval $[0, T]$, consider the initial value problem

$$\tilde{y}^{(n)}(t) + p_1(t)\tilde{y}^{(n-1)}(t) + \cdots + p_n(t)\tilde{y}(t) = \tilde{f}(t) \quad (t \in (0, T]), \tag{1}$$

$$\tilde{y}^{(j)}(0) = \tilde{a}_j \quad (j = 0, \ldots, n-1) \tag{2}$$

with variable real continuous coefficients $p_i(t)$ $(i = 1, \ldots, n)$, a Hausdorff-continuous fuzzy-valued function $\tilde{f} : [0, T] \to J$, and fuzzy initial conditions $\tilde{a}_j$ $(j = 0, \ldots, n-1)$.

We call an n-times continuously S-differentiable fuzzy-valued function $\tilde{y}(t)$ satisfying (1) and (2) a strong solution of problem (1), (2).

Let $y^{\pm}_{\alpha}(t)$, $f^{\pm}_{\alpha}(t)$ $\forall\, t \in [0, T]$ and $(a_j)^{\pm}_{\alpha}$ $(j = 0, \ldots, n-1)$ denote the α-indices of $\tilde{y}(t)$, $\tilde{f}(t)$, and $\tilde{a}_j$, respectively. Then, according to the definition and properties of S-derivatives, problem (1), (2) $\forall\, \alpha \in [0, 1]$ has the interval representation

$$\Big[(y^-_\alpha)^{(n)}, (y^+_\alpha)^{(n)}\Big] + p_1(t)\Big[(y^-_\alpha)^{(n-1)}, (y^+_\alpha)^{(n-1)}\Big] + p_n(t)\,[y^-_\alpha, y^+_\alpha] = [f^-_\alpha(t), f^+_\alpha(t)] \quad (t \in (0, T]), \tag{3}$$

$$\Big[(y^-_\alpha)^{(j)}(0), (y^+_\alpha)^{(j)}(0)\Big] = [(a_j)^-_\alpha, (a_j)^+_\alpha] \quad (j = 0, \ldots, n-1). \tag{4}$$

Note that $f^{\pm}_{\alpha}(t)$ (the α-indices of the continuous fuzzy-valued function $\tilde{f}(t)$) are, continuous real-valued functions of t $\forall\, \alpha \in [0, 1]$ (see Remark 2).

We call a Hausdorff-continuous fuzzy-valued function $\tilde{y} : [0, T] \to J$ with the α-indices satisfying (3), (4) an ultraweak solution of problem (1), (2) [17, 18].

The following result is immediate from the above definitions.

Lemma 1. *Let the coefficients $p_i(t)$ $(t \in [0, T];\ i = 1, \ldots, n)$ of equation* (1) *be continuous real-valued functions, and let the fuzzy-valued function $\tilde{f} : [0, T] \to J$ be continuous in the Hausdorff metric. Then a strong solution of problem* (1), (2) *is an ultraweak solution.*

An ultraweak solution is strong only under the additional assumption of the existence and continuity of all its fuzzy derivatives appearing in equation (1).

According to demonstrative examples [6, 16], the solutions $y^{\pm}_{\alpha}(t)$ of problem (3), (4) may fail to define fuzzy numbers for some (or all) values of t, i.e., at least one of conditions 1–3 of Section 2 is violated for them. On the other hand, even if $y^{\pm}_{\alpha}(t)$ define a fuzzy-valued function, it may be non-differentiable. In this connection, the fuzzy-valued solutions of problem (1), (2) were classified by the author into strong, weak, and ultraweak [17, 18]. Here, a weak solution is understood as an ultraweak one that additionally satisfies the integral equation corresponding to problem (1), (2). Weak solutions are not considered in this paper.

Lemma 2. *Under the hypotheses of Lemma 1, let $\forall\, t \in [0, T]$ $p_i(t) \geqslant 0$ $(i = 1, \ldots, n)$. Then the following problems hold for the α-indices $y^{\pm}_{\alpha}(t)$ of the strong solution $\tilde{y}(t)$ $\forall\, \alpha \in [0, 1]$:*

—for the right indices,

$$(y^+_\alpha)^{(n)}(t) + p_1(t)(y^+_\alpha)^{(n-1)}(t) + \cdots + p_n(t) y^+_\alpha(t) = f^+_\alpha(t) \quad (t \in (0, T]), \tag{5}$$

$$(y^+_\alpha)^{(j)}(0) = (a_j)^+_\alpha, \quad j = 0, \ldots, n-1; \tag{6}$$

—for the left indices,

$$(y^-_\alpha)^{(n)}(t) + p_1(t)(y^-_\alpha)^{(n-1)}(t) + \cdots + p_n(t) y^-_\alpha(t) = f^-_\alpha(t) \quad (t \in (0, T]), \tag{7}$$

$$(y^-_\alpha)^{(j)}(0) = (a_j)^-_\alpha, \quad j = 0, \ldots, n-1. \tag{8}$$

This fact follows from the linearity of equation (1), the properties of arithmetic operations on fuzzy numbers in interval form, and the additivity and positive homogeneity of derivatives (Remark 3).

Thus, in the case of nonnegative coefficients $p_i(t)$ $(i = 1, \ldots, n)$ of equation (1), for all $t \in [0, T]$ the α-indices of an ultraweak solution $\tilde{y} : [0, T] \to J$ of problem (1), (2) satisfy the relations (5), (6) and (7), (8), respectively. That is, the system for the α-indices splits into two independent problems for the right and left indices, respectively.

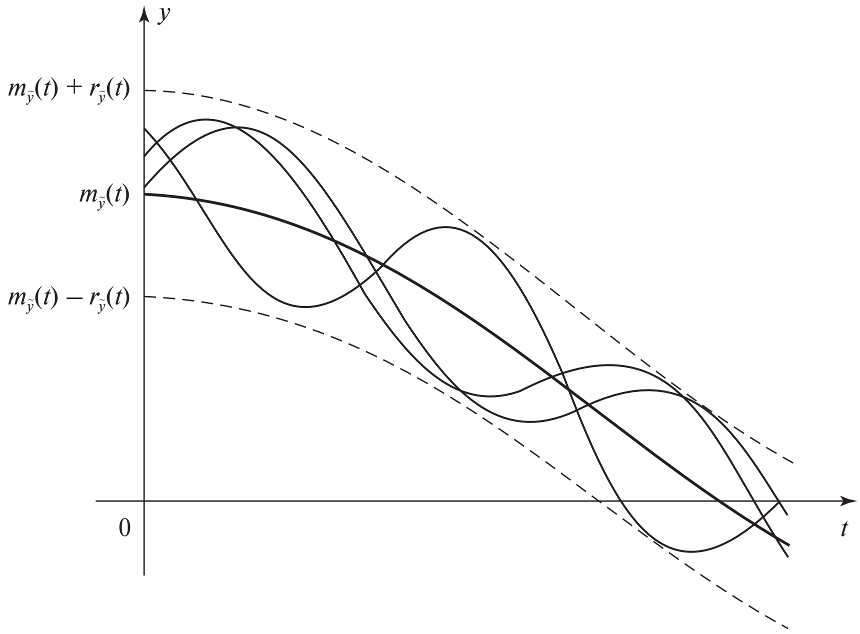

Fig. 1. The geometric interpretation of the mean and half-range of solutions.

Note that according to [30, Ch. 3], under the hypotheses of Lemma 2, for all $\alpha \in [0,1]$ there exist unique solutions of the linear problems (5), (6) and (7), (8).

We denote by

$$m_{\tilde{y}}(t) = \frac{1}{2}\int\limits_0^1 (y_\alpha^+(t) + y_\alpha^-(t))d\alpha, \quad r_{\tilde{y}}(t) = \frac{1}{2}\int\limits_0^1 (y_\alpha^+(t) - y_\alpha^-(t))d\alpha \tag{9}$$

the mean and half-range of an ultraweak solution $\tilde{y}(t)$ of problem (3), (4) for all $t \in (0,T]$. By definition, for any $t \in (0,T]$ and $\alpha \in [0,1]$, an ultraweak solution of problem (1), (2) is associated with the α-interval $[\tilde{y}(t)]_\alpha = [y_\alpha^-(t), y_\alpha^+(t)]$. Consider the interval $\left[\int_0^1 y_\alpha^-(t)d\alpha, \int_0^1 y_\alpha^+(t)d\alpha\right]$. Any element $y(t), t \in (0,T]$, of this interval lies in a band (Fig. 1).

Let us investigate the relationship between the numerical average characteristics of a fuzzy-valued inhomogeneity at the input of a linear fuzzy system and those of the fuzzy-valued function at its output, i.e., the corresponding characteristics of an ultraweak fuzzy-valued solution of problem (1), (2).

Also, we introduce

$$m_{\tilde{f}}(t) = \frac{1}{2}\int\limits_0^1 \left(f_\alpha^+(t) + f_\alpha^-(t)\right) d\alpha, \quad r_{\tilde{f}}(t) = \frac{1}{2}\int\limits_0^1 \left(f_\alpha^+(t) - f_\alpha^-(t)\right) d\alpha, \tag{10}$$

representing the mean and half-range, respectively, of the inhomogeneity $\tilde{f}(t)$ of equation (1) for all $t \in (0,T]$.

Theorem 1. *Under the hypotheses of Lemma 1, the mean* $m_{\tilde{y}}(t)$ (9) *of an ultraweak solution* $\tilde{y}(t)$ *of problem* (1), (2) *satisfies the equation*

$$(m_{\tilde{y}})^{(n)}(t) + p_1(t)(m_{\tilde{y}})^{(n-1)}(t) + \dots + p_n(t)m_{\tilde{y}}(t) = m_{\tilde{f}}(t) \quad (t \in (0,T]), \tag{11}$$

where $m_{\tilde{f}}(t)$ is given by formula (10). *Moreover, the following initial conditions hold:*

$$(m_{\tilde{y}})^{(j)}(0) = m_{\tilde{y}_j} = \frac{1}{2}\left(\int_0^1 ((a_j)^+_\alpha + (a_j)^-_\alpha)\,d\alpha\right) \quad (j = 0, \ldots, n-1). \tag{12}$$

Proof. First, assume that all coefficients of equation (1) are $p_i(t) \geqslant 0$ $(i = 1, \ldots, n)$, $\forall\, t \in [0, T]$. Consider the arithmetic average $A_{\tilde{y}}(\alpha, t) = \frac{1}{2}(y^+_\alpha(t) + y^-_\alpha(t))$. We sum the left and right-hand sides of equalities (5), (7) and divide the resulting relation by 2. Due to the linearity of equations (5), (7), the real-valued function $A_{\tilde{y}}(\alpha, t)$ $\forall\, \alpha \in [0, 1], t \in (0, T]$ satisfies the real differential equation

$$(A_{\tilde{y}})^{(n)}_t(\alpha, t) + p_1(t)(A_{\tilde{y}})^{(n-1)}_t(\alpha, t) + \cdots + p_n(t)A_{\tilde{y}}(\alpha, t) = \frac{1}{2}(f^+_\alpha(t) + f^-_\alpha(t)).$$

In view of the interchangeability of integration with respect to α and differentiation with respect to t for $m_{\tilde{y}}(t)$, we integrate both sides of this equation with respect to α from 0 to 1 to obtain the differential equation (11).

Concerning the initial conditions, note that according to (6), (8),

$$(A_{\tilde{y}}(\alpha, t))^{(j)}\Big|_{t=0} = \frac{1}{2}((a_j)^+_\alpha + (a_j)^-_\alpha) \quad (j = 0, \ldots, n-1);$$

consequently, conditions (12) are valid.

Now we analyze the case where some (or all) coefficients of equation (1) are negative. For example, suppose that $p_i(t) < 0$ for some i at some point $t \in (0, T]$. For the sake of definiteness, let the other coefficients be nonnegative. With $p_i(t) = -q_i(t)$, based on the relation (3), the corresponding expression can be written as

$$p_i(t)\left[(y^-_\alpha)^{(n-i)}, (y^+_\alpha)^{(n-i)}\right] = \left[q_i(t)(-y^+_\alpha)^{(n-i)}, q_i(t)(-y^-_\alpha)^{(n-i)}\right] = q_i(t)\left[(-y^+_\alpha)^{(n-i)}, (-y^-_\alpha)^{(n-i)}\right].$$

Therefore, the equations for the right and left α-indices become

$$(y^+_\alpha)^{(n)} + p_1(t)(y^+_\alpha)^{(n-1)} + \cdots + q_i(-y^-_\alpha)^{(n-i)} + \cdots + p_n(t)y^+_\alpha = f^+_\alpha(t) \tag{13}$$

and

$$(y^-_\alpha)^{(n)} + p_1(t)(y^-_\alpha)^{(n-1)} + \cdots + q_i(-y^+_\alpha)^{(n-i)} + \cdots + p_n(t)y^-_\alpha = f^-_\alpha(t), \tag{14}$$

respectively, where $t \in (0, T]$.

We sum both sides of equalities (13) and (14) and divide by 2. By integrating the resulting relation with respect to α from 0 to 1 and using the designation $p_i(t) = -q_i(t)$, we arrive at equation (11) for the mean $m_{\tilde{y}}(t)$.

Next, it is necessary to study the relationship between the half-range $r_{\tilde{f}}(t)$ (10) of the inhomogeneity and the half-range (9) of an ultraweak solution $\tilde{y}(t)$ of equation (1), (2).

Theorem 2. *Under the hypotheses of Lemma 1, the half-range $r_{\tilde{y}}(t)$* (9) *of an ultraweak solution $\tilde{y}(t)$ of problem* (1), (2) *satisfies the equation*

$$x^{(n)}(t) + |p_1(t)|x^{(n-1)}(t) + \cdots + |p_i(t)|x^{(n-i)}(t) + \cdots + |p_n(t)|x(t) = r_{\tilde{f}}(t), \tag{15}$$

where $t \in (0, T]$, and the right-hand side $r_{\tilde{f}}(t)$ is given by (10). *Moreover, the following initial conditions hold:*

$$(r_{\tilde{y}})^{(j)}(0) = r_{\tilde{a}_j} \equiv \frac{1}{2}\left(\int_0^1 ((a_j)^+_\alpha - (a_j)^-_\alpha)\,d\alpha\right) \quad (j = 0, \ldots, n-1). \tag{16}$$

Proof. First, consider the case of nonnegative coefficients $p_i(t)$ $(i = 1, \ldots, n)$. By subtracting equality (7) from equality (5), due to the linear equations for the half-difference $R_{\tilde{y}}(\alpha, t) = \frac{1}{2}(y_\alpha^+(t) - y_\alpha^-(t))$, we obtain for all $\alpha \in [0, 1]$ and $t \in (0, T]$ the real equation

$$(R_{\tilde{y}})_t^{(n)}(\alpha, t) + p_1(t)(R_{\tilde{y}})_t^{(n-1)}(\alpha, t) + \cdots + p_n(t)R_{\tilde{y}}(\alpha, t) = \frac{1}{2}(f_\alpha^+(t) - f_\alpha^-(t)) \tag{17}$$

with the initial conditions

$$(R_{\tilde{y}}(\alpha, t))^{(j)}\Big|_{t=0} = \frac{1}{2}\left((a_j)_\alpha^+ - (a_j)_\alpha^-\right) \quad (j = 0, \ldots, n-1). \tag{18}$$

Integrating both sides of equation (17) with respect to α from 0 to 1 yields the following real equation for the half-range $r_{\tilde{y}}(t)$:

$$(r_{\tilde{y}})^{(n)}(t) + p_1(t)(r_{\tilde{y}})^{(n-1)}(t) + \cdots + p_n(t)r_{\tilde{y}}(t) = r_{\tilde{f}}(t), t \in (0, T]. \tag{19}$$

In the case of nonnegative coefficients $p_i(t)$, equation (19) coincides with (13).

Moreover, the initial conditions (16) are valid by virtue of (18).

Considering the case of half-ranges, let $p_i(t) < 0$ for some i at some point $t \in (0, T]$, provided that the other coefficients are nonnegative. By subtracting equality (14) from equality (13) and integrating both sides of the resulting relation with respect to α from 0 to 1, we establish the relation

$$x^{(n)}(t) + p_1(t)x^{(n-1)}(t) + \cdots + (-p_i(t))x^{(n-i)}(t) + \cdots + p_n(t)x(t) = r_{\tilde{f}}(t)$$

for the half-ranges (10), where $t \in (0, T]$.

Consequently, in the equation for the half-range, there is a minus sign at the negative coefficient of the original equation. Hence, this coefficient becomes positive. Since $p_i(t)$ can take both positive and negative values at different points $t \in (0, T]$, equation (15) generally holds with the initial conditions (16).

Note that if an ultraweak solution $\tilde{y}(t)$ of problem (1), (2) exists, its α-indices $y_\alpha^\pm(t)$ for any $t \in (0, T]$ will satisfy conditions 1–3 of fuzzy numbers (see Section 2), in particular, $y_\alpha^-(t) \leqslant y_\alpha^+(t)$. Consequently, the half-range (9) of the fuzzy-valued function $\tilde{y}(t)$ is nonnegative. If the solution of problem (15), (16) is a function with negative values at some points, then $\tilde{y}(t)$ is not an ultraweak solution at such points. (In other words, an ultraweak solution does not exist.)

4. THE CASE OF GENERALIZED DERIVATIVES

In recent decades, researchers have actively been using generalized derivatives of fuzzy-valued functions [5–8, 14–16]. Let us formulate the corresponding definition [5].

A set $C = A \ominus B$ is called the Hukuhara difference of sets A and B if $A = B + C$.

Definition [5]. Let $\tilde{f} : (0, T) \to J$ and $t_0 \in (0, T)$. A fuzzy-valued function $\tilde{f}$ is said to be strongly generalized differentiable at a point t_0 if there exists an element $\tilde{f}'(t_0) \in J$ such that either

1) for all sufficiently small $h > 0$, there exist $\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)$, $\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)$, and the limits (in the Hausdorff metric d)

$$\lim_{h \to 0^+} \frac{\tilde{f}(t_0 + h) \ominus \tilde{f}(t_0)}{h} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 - h)}{h} = \tilde{f}'(t_0),$$

or

2) for all sufficiently small $h > 0$, there exist $f(t_0) \ominus f(t_0 + h)$, $f(t_0 - h) \ominus f(t_0)$, and the limits (in the metric d)

$$\lim_{h \to 0^+} \frac{\tilde{f}(t_0) \ominus \tilde{f}(t_0 + h)}{(-h)} = \lim_{h \to 0^+} \frac{\tilde{f}(t_0 - h) \ominus \tilde{f}(t_0)}{(-h)} = \tilde{f}'(t_0).$$

In case (1), one speaks of the generalized (1)-differentiability of $\tilde{f}$ at t_0, denoting the derivative by $D_1^{(1)}\tilde{f}(t_0)$. In case (2), one speaks of the generalized (2)-differentiability of $\tilde{f}$ at t_0, denoting the derivative by $D_2^{(1)}\tilde{f}(t_0)$. As usual, generalized differentiability on an interval is understood as the same property at each point of the interval.

Let a function $\tilde{f}\colon (0,T)\to J\ \forall\,\alpha\in[0,1]$ be associated with the α-intervals $[\tilde{f}]_\alpha(t)=[f_\alpha^-(t), f_\alpha^+(t)]$. Note that by the definition of a Hukuhara difference, the expression $\tilde{f}(t_0+h)\ominus\tilde{f}(t_0)$ is associated with the α-interval $[\tilde{f}(t_0+h)\ominus\tilde{f}(t_0)]_\alpha = [f_\alpha^-(t_0+h)-f_\alpha^-(t_0),\ f_\alpha^+(t_0+h)-f_\alpha^+(t_0)]$. The assumption of the existence of $\tilde{f}(t_0+h)\ominus\tilde{f}(t_0)$ means that these α-intervals induce a fuzzy number, i.e., conditions 1–3 of the α-indices are valid (see Section 2).

Lemma 3 [15]. *Let a fuzzy-valued function $\tilde{f}:(0,T)\to J$ be generalized (1)-differentiable. Then its derivative $D_1^{(1)}\tilde{f}(t)\ \forall\,\alpha\in[0,1]$ is associated with the α-interval $[D_1^{(1)}\tilde{f}(t)]_\alpha=[(f_\alpha^-)'(t),(f_\alpha^+)'(t)]$. Let a fuzzy-valued function $\tilde{f}:(0,T)\to J$ be generalized (2)-differentiable. Then its derivative $D_2^{(1)}\tilde{f}(t)\ \forall\,\alpha\in[0,1]$ is associated with the α-interval $[D_2^{(1)}\tilde{f}(t)]_\alpha = [(f_\alpha^+)'(t),(f_\alpha^-)'(t)]$.*

According to Lemma 3, we arrive at the following.

Corollary 1. *The generalized (1)-differentiability of a fuzzy-valued function $\tilde{f}:(0,T)\to J$ implies its Seikkala differentiability on $(0,T)$.*

A fuzzy-valued function $\tilde{f}:(0,T)\to J$ is said to be twice generalized (n,m)-differentiable (for $n,m=1,2$) at a point t_0 if the first generalized derivative of type n, $D_n^{(1)}\tilde{f}(t)$, exists in a neighborhood of t_0 as a fuzzy-valued function, and it is generalized m-differentiable at t_0. This fact will be denoted by $D_{n,m}^{(2)}\tilde{f}(t_0)$. Higher-order generalized derivatives are defined by analogy.

Due to Lemma 3, we have another result.

Corollary 2. *Let a fuzzy-valued function $\tilde{f}\colon(0,T)\to J$ be characterized by α-intervals $[f_\alpha^-(t), f_\alpha^+(t)]$.*

1. *If $\tilde{f}(t)$ is a twice generalized (1,1)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{1,1}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^-)''(t),(f_\alpha^+)''(t)]$ for its second derivative $D_{1,1}^{(2)}\tilde{f}(t)$.*

2. *If $\tilde{f}(t)$ is a twice generalized (1,2)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{1,2}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^+)''(t),(f_\alpha^-)''(t)]$ for its second derivative $D_{1,2}^{(2)}\tilde{f}(t)$.*

3. *If $\tilde{f}(t)$ is a twice generalized (2,1)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{2,1}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^+)''(t),(f_\alpha^-)''(t)]$ for its second derivative $D_{2,1}^{(2)}\tilde{f}(t)$.*

4. *If $\tilde{f}(t)$ is a twice generalized (2,2)-differentiable fuzzy-valued function on $(0,T)$, then $[D_{2,2}^{(2)}\tilde{f}(t)]_\alpha=[(f_\alpha^-)''(t),(f_\alpha^+)''(t)]$ for its second derivative $D_{2,2}^{(2)}\tilde{f}(t)$.*

Now consider the situation where the derivatives of a fuzzy-valued solution of problem (1), (2) are generalized derivatives of the first or second type. Further analysis deals with a second-order fuzzy differential equation of the form

$$\tilde{y}''(t)+p_1(t)\tilde{y}'(t)+p_2(t)\tilde{y}(t)=\tilde{f}(t)\quad (t\in(0,T]) \tag{20}$$

with continuous real coefficients $p_1,p_2:[0,T]\to\mathbb{R}$, a Hausdorff-continuous fuzzy-valued function $\tilde{f}(t):[0,T]\to J$, and the initial conditions

$$\tilde{y}(0)=\tilde{a},\ \tilde{y}'(t)=\tilde{b}\quad (\tilde{a},\tilde{b}\in J). \tag{21}$$

Note that in this context, four types of equations for the α-indices are possible, corresponding to the types of first and second generalized derivatives.

The following result is immediate from Lemma 3 and Corollary 2.

Lemma 4. *Let the coefficients $p_1, p_2 : [0,T] \to \mathbb{R}$ be continuous real functions nonnegative on $[0,T]$, and let the fuzzy-valued function $\tilde{f}(t) : [0,T] \to J$ be continuous in the Hausdorff metric. Then, depending on the type of the generalized differentiability of a fuzzy-valued solution $\tilde{y}(t)$ of equation* (20)*, the α-indices satisfy the following equations:*

—in the case of generalized (1,1)*-differentiability, the equations*

$$(y_\alpha^+)''(t) + p_1(y_\alpha^+)'(t) + p_2 y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{22}$$

$$(y_\alpha^-)''(t) + p_1(y_\alpha^-)'(t) + p_2 y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]); \tag{23}$$

—in the case of the generalized (1,2)*-differentiability, the equations*

$$(y_\alpha^-)''(t) + p_1(t)(y_\alpha^+)'(t) + p_2(t) y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{24}$$

$$(y_\alpha^+)''(t) + p_1(t)(y_\alpha^-)'(t) + p_2(t) y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]); \tag{25}$$

—in the case of the generalized (2,1)*-differentiability, the equations*

$$(y_\alpha^-)''(t) + p_1(t)(y_\alpha^-)'(t) + p_2(t) y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{26}$$

$$(y_\alpha^+)''(t) + p_1(t)(y_\alpha^+)'(t) + p_2(t) y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]); \tag{27}$$

—finally, in the case of the generalized (2,2)*-differentiability, the equations*

$$(y_\alpha^+)''(t) + p_1(t)(y_\alpha^-)'(t) + p_2(t) y_\alpha^+(t) = f_\alpha^+(t) \quad (t \in (0,T]), \tag{28}$$

$$(y_\alpha^-)''(t) + p_1(t)(y_\alpha^+)'(t) + p_2(t) y_\alpha^-(t) = f_\alpha^-(t) \quad (t \in (0,T]). \tag{29}$$

Moreover, the following initial conditions corresponding to (21) *are valid:*

—in the case of the (1)*-differentiability of $\tilde{y}(t)$,*

$$y_\alpha^\pm(0) = a_\alpha^\pm, (y_\alpha^\pm)'(0) = b_\alpha^\pm; \tag{30}$$

—in the case of the (2)*-differentiability of $\tilde{y}(t)$,*

$$y_\alpha^\pm(0) = a_\alpha^\pm, (y_\alpha^\pm)'(0) = b_\alpha^\mp. \tag{31}$$

A fuzzy-valued function whose α-indices satisfy equations (22), (23), and (30) will be called a (1,1)-ultraweak solution of problem (20), (21); one satisfying equations (24), (25), and (30), a (1,2)-ultraweak solution of problem (20), (21); one satisfying equations (26), (27), and (31), a (2,1)-ultraweak solution of problem (20), (21); finally, one satisfying equations (28), (29), and (31), a (2,2)-ultraweak solution of problem (20), (21).

The following result is true.

Theorem 3. *Under the hypotheses of Lemma 4, regardless of the types of ultraweak solutions of problem* (20), (21)*, the mean* (9) *satisfies the same equation*

$$m_{\tilde{y}}''(t) + p_1(t) m_{\tilde{y}}'(t) + p_2(t) m_{\tilde{y}}(t) = m_{\tilde{f}}(t) \quad (t \in (0,T]) \tag{32}$$

with the initial conditions

$$m_{\tilde{y}}(0) = m_{\tilde{a}} = \frac{1}{2}\int_0^1 (a_\alpha^+ + a_\alpha^-)\,d\alpha, \quad (m_{\tilde{y}})'(0) = m_{\tilde{b}} = \frac{1}{2}\int_0^1 (b_\alpha^+ + b_\alpha^-)\,d\alpha. \tag{33}$$

Indeed, in each case of (n,m)-ultraweak solutions of problem (20), (21) for $n,m=1,2$, we sum the corresponding equalities for the right and left α-indices and divide the resulting relation by 2. By integrating both sides of the latter with respect to α from 0 to 1, we obtain equation (32) for the mean $m_{\tilde{y}}(t)$.

Concerning the initial conditions, note that:

1. In the case of the (1)-differentiability of $\tilde{y}(t)$, formulas (12) and (30) imply (33).
2. In the case of the (2)-differentiability of $\tilde{y}(t)$, formulas (12) and (31) imply (33).

Remark 4. Theorem 3 remains valid if the coefficients $p_i(t)$ of equation (20) have arbitrary signs.

For half-ranges, the following result is true.

Proposition 1. *Under the hypotheses of Lemma 4, depending on the type of a generalized ultraweak solution* $\tilde{y}(t)$ *of equation* (20)*, its half-ranges* (9) *satisfy the following equations:*

—in the case of a generalized $(1,1)$*-ultraweak solution, the equation* $r''_{\tilde{y}}(t)+p_1(t)r'_{\tilde{y}}(t)+p_2(t)r_{\tilde{y}}(t)=r_{\tilde{f}}(t)$ $(t\in(0,T])$;

—in the case of a generalized $(1,2)$*-ultraweak solution, the equation* $-r''_{\tilde{y}}(t)+p_1(t)r'_{\tilde{y}}(t)+p_2(t)r_{\tilde{y}}(t)=r_{\tilde{f}}(t)$ $(t\in(0,T])$;

—in the case of a generalized $(2,1)$*-ultraweak solution, the equation* $-r''_{\tilde{y}}(t)-p_1(t)r'_{\tilde{y}}(t)+p_2(t)r_{\tilde{y}}(t)=r_{\tilde{f}}(t)$ $(t\in(0,T])$;

—finally, in the case of a generalized $(2,2)$*-ultraweak solution, the equation* $r''_{\tilde{y}}(t)-p_1(t)r'_{\tilde{y}}(t)+p_2(t)r_{\tilde{y}}(t)=r_{\tilde{f}}(t)$ $(t\in(0,T])$.

This proposition is obtained by subtracting the corresponding equations for the right and left indices.

In this case, the initial conditions have the form $r_{\tilde{y}}(0)=r_{\tilde{a}}=\frac{1}{2}\int_0^1(a_\alpha^+-a_\alpha^-)d\alpha$ and $r'_{\tilde{y}}(0)=r_{\tilde{b}}=\frac{1}{2}\int_0^1(b_\alpha^+-b_\alpha^-)d\alpha$ in the case of the generalized (1)-differentiability of the solution, or the second condition is replaced by $r'_{\tilde{y}}(0)=-r_{\tilde{b}}$ in the case of the generalized (2)-differentiability of the solution.

5. TRANSFORMATION OF A FUZZY SIGNAL BY A SERIES RESONANT CIRCUIT

Let us pose the following problem: find the response of a radio circuit (Fig. 2) to an input fuzzy signal $\tilde{g}(t)$.

Here, L is the inductance, C is the capacitance, and R is the resistance; $\tilde{g}(t)$ is treated as a voltage source; finally, $\tilde{z}(t)$ is the voltage taken across the resistance. This circuit is called a series resonant circuit. With $\tilde{y}(t)=\frac{1}{R}\tilde{z}(t)$, the circuit is described (e.g.,see [31]) by a second-order

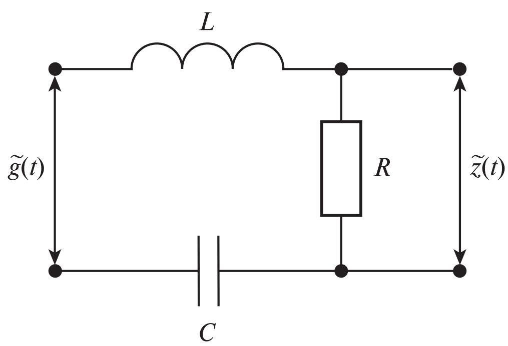

Fig. 2. Series resonant circuit.

differential equation with constant coefficients for the currents $\tilde{y}(t)$ of the form

$$\begin{gathered} a_0\tilde{y}''(t) + a_1\tilde{y}'(t) + a_2\tilde{y}(t) = \tilde{g}'(t), \\ a_0 = L > 0, \;\; a_1 = R > 0, \;\; a_2 = \frac{1}{C} > 0 \quad (t \in (0,T]), \end{gathered} \tag{34}$$

where $\tilde{g} : [0,T] \to J$ is a continuously differentiable (in the Seikkala sense) fuzzy-valued input signal.

Assume that at the initial instant, the system has the state

$$\tilde{y}(0) = \tilde{a}, \quad \tilde{y}'(0) = \tilde{b}, \tag{35}$$

where $a, b \in J$ are fuzzy numbers.

By dividing (34) by $a_0 > 0$, we obtain the equation

$$\begin{gathered} \tilde{y}''(t) + p_1\tilde{y}'(t) + p_2\tilde{y}(t) = \tilde{f}(t) \quad (t \in (0,T]), \\ p_1 := \frac{R}{L}, \quad p_2 := \frac{1}{LC}, \quad \tilde{f}(t) := \frac{1}{L}\tilde{g}'(t). \end{gathered} \tag{36}$$

Proposition 2. *Let the real coefficients p_1 and p_2 be positive, and let the fuzzy-valued function $\tilde{f} : [0,T] \to J$ be continuous in the Hausdorff metric. Let the roots of the characteristic equation $\lambda^2 + p_1\lambda + p_2 = 0$ be real, distinct, and negative, equal to $-\lambda_1$ and $-\lambda_2$, where $\lambda_1 > \lambda_2 > 0$. Then there exists a unique $(1,1)$-ultraweak solution of problem (35), (36) of the form*

$$\tilde{y}(t) = \tilde{a}\gamma_{\tilde{a}}(t) + \tilde{b}\gamma_{\tilde{b}}(t) + \int_0^t K(t-s)\tilde{f}(s)ds, \tag{37}$$

where the functions $\gamma_{\tilde{a}}(t)$ and $\gamma_{\tilde{b}}(t)$ are given by

$$\gamma_{\tilde{a}}(t) = \frac{1}{\lambda_1 - \lambda_2}(\lambda_1 e^{-\lambda_2 t} - \lambda_2 e^{-\lambda_1 t}), \quad \gamma_{\tilde{b}}(t) = \frac{1}{\lambda_1 - \lambda_2}(e^{-\lambda_2 t} - e^{-\lambda_1 t}), \tag{38}$$

and

$$K(t-s) = \frac{1}{\lambda_1 - \lambda_2}(e^{-\lambda_2(t-s)} - e^{-\lambda_1(t-s)}) \tag{39}$$

is the Cauchy function.

Indeed, under the hypotheses of Proposition 2, the α-indices of an $(1,1)$-ultraweak solution of problem (35), (36) satisfy the equations

$$(y_\alpha^\pm)''(t) + p_1(y_\alpha^\pm)'(t) + p_2 y_\alpha^\pm(t) = f_\alpha^\pm(t) \quad (t \in (0,T]) \tag{40}$$

with the initial conditions

$$y_\alpha^\pm(0) = a_\alpha^\pm, \quad (y_\alpha^\pm)'(0) = b_\alpha^\pm. \tag{41}$$

The solution of problem (40), (41) for all $\alpha \in [0,1]$ can be represented as the sum $y_\alpha^\pm(t) = x_\alpha^\pm(t) + z_\alpha^\pm(t)$, where $x_\alpha^\pm(t)$ is the solution of the corresponding homogeneous differential equation $(x_\alpha^\pm)''(t) + p_1(x_\alpha^\pm)'(t) + p_2 x_\alpha^\pm(t) = 0$ with the initial conditions $x_\alpha^\pm(0) = a_\alpha^\pm$ and $(x_\alpha^\pm)'(0) = b_\alpha^\pm$, and $z_\alpha^\pm(t)$ is the solution of the inhomogeneous differential equation $(z_\alpha^\pm)''(t) + p_1(z_\alpha^\pm)'(t) + p_2 z_\alpha^\pm(t) = f_\alpha^\pm(t)$ with the zero initial conditions $z_\alpha^\pm(0) = 0$ and $(z_\alpha^\pm)'(0) = 0$. Obviously, $x_\alpha^\pm(t) = a_\alpha^\pm\gamma_{\tilde{a}}(t) + b_\alpha^\pm\gamma_{\tilde{b}}(t)$, and $z_\alpha^\pm(t)$ can be expressed via the Cauchy function [30, Ch. 3] as

$$z_\alpha^\pm(t) = \int_0^t K(t-s)f_\alpha^\pm(s)ds.$$

By definition, the Cauchy function $K(t-s)$ here is the solution of the real problem $x''(t)+p_1x'(t)+p_2x(t)=0$ with $x(s)=0$ and $x'(s)=1$. This function is given by equality (39). Then we have the formula

$$y_\alpha^{\pm}(t) = a_\alpha^{\pm}\gamma_{\tilde{a}}(t) + b_\alpha^{\pm}\gamma_{\tilde{b}}(t) + \int\limits_0^t K(t-s)f_\alpha^{\pm}(s)ds. \tag{42}$$

Note that the real functions $\gamma_{\tilde{a}}(t)$ and $\gamma_{\tilde{b}}(t)$ (28) in the representation (42) are continuous and positive for $t \geqslant 0$. Furthermore, the Cauchy function (39) is continuous and nonnegative for $t \geqslant s$. Therefore, the functions $y_\alpha^{\pm}(t)$ (42) satisfy conditions 1–3 on the α-indices of fuzzy numbers (Section 2) for all $t \in [0,T]$. Hence, they define the fuzzy-valued function (37). According to the hypotheses of Proposition 2, based on formula (42) and the definition of the Hausdorff metric, the fuzzy-valued function (37) is continuous. And we conclude that (37) is a $(1,1)$-ultraweak solution of problem (35), (36).

The uniqueness of the $(1,1)$-ultraweak solution of problem (35), (36) follows from the uniqueness of the solutions of problems (40), (41) for all $\alpha \in [0,1]$.

Suppose that only the average characteristics are known for the inhomogeneity $\tilde{f}(t)$ of equation (36) and the initial conditions $\tilde{a}$ and $\tilde{b}$ in (35). The following result is true.

Proposition 3. *Under the hypotheses of Proposition 2, the average characteristics* $m_{\tilde{y}}(t)$ *and* $r_{\tilde{y}}(t)$ *of a fuzzy-valued solution of problem* (35), (36) *are expressed through the average characteristics* $m_{\tilde{a}}$, $m_{\tilde{b}}$, $m_{\tilde{f}}$ *and* $r_{\tilde{a}}$, $r_{\tilde{b}}$, $r_{\tilde{f}}$, *respectively, by the formulas*

$$m_{\tilde{y}}(t) = m_{\tilde{a}}\gamma_{\tilde{a}}(t) + m_{\tilde{b}}\gamma_{\tilde{b}}(t) + \int\limits_0^t K(t-s)m_{\tilde{f}}(s)ds, \tag{43}$$

$$r_{\tilde{y}}(t) = r_{\tilde{a}}\gamma_{\tilde{a}}(t) + r_{\tilde{b}}\gamma_{\tilde{b}}(t) + \int\limits_0^t K(t-s)r_{\tilde{f}}(s)ds, \tag{44}$$

where $\gamma_{\tilde{a}}(t)$ *and* $\gamma_{\tilde{b}}(t)$ *are given by* (38), *and* $K(t-s)$ *is given by* (39).

Indeed, let us take the mean $m_{\tilde{y}}(t)$. Due to (40), (41), similar to Theorem 3, the mean $m_{\tilde{y}}(t)$ is the solution of the equation

$$m_{\tilde{y}}''(t) + p_1 m_{\tilde{y}}'(t) + p_2 m_{\tilde{y}}(t) = m_{\tilde{f}}(t) \quad (t \in (0,T]) \tag{45}$$

with the initial conditions

$$m_{\tilde{y}}(0) = m_{\tilde{a}} \equiv \frac{1}{2}\int\limits_0^1 (a_\alpha^+ + a_\alpha^-)d\alpha, \quad m_{\tilde{y}}'(0) = m_{\tilde{b}} \equiv \frac{1}{2}\int\limits_0^1 (b_\alpha^+ + b_\alpha^-)d\alpha. \tag{46}$$

By analogy with the considerations of Proposition 2, we arrive at formula (43).

According to Proposition 1, the half-range $r_{\tilde{y}}(t)$ satisfies the equation

$$r_{\tilde{y}}''(t) + p_1 r_{\tilde{y}}'(t) + p_2 r_{\tilde{y}}(t) = r_{\tilde{f}}(t) \quad (t \in (0,T]) \tag{47}$$

with the initial conditions

$$r_{\tilde{y}}(0) = r_{\tilde{a}} \equiv \frac{1}{2}\int\limits_0^1 (a_\alpha^+ - a_\alpha^-)d\alpha, \quad r_{\tilde{y}}'(0) = r_{\tilde{b}} \equiv \frac{1}{2}\int\limits_0^1 (b_\alpha^+ - b_\alpha^-)d\alpha. \tag{48}$$

And formula (44) is immediate.

Note that by definition, the function $r_{\tilde{y}}(t)$ is nonnegative. In fact, this follows from formula (44) due to the nonnegativity of, first, the numbers $r_{\tilde{a}}$ and $r_{\tilde{b}}$, second, the functions $\gamma_{\tilde{a}}(t)$ and $\gamma_{\tilde{b}}(t) \forall\, t \geqslant 0$, and, third, the functions $r_{\tilde{f}}(s) \forall\, s \geqslant 0$ and $K(t-s) \forall\, t \geqslant s \geqslant 0$.

Consider the case of an infinite interval $[0, \infty)$ of the real axis.

Proposition 4. *Under the hypotheses of Proposition 2, assume also that the right-hand side of equation* (36)*,* $\tilde{f} : [0, \infty) \to J$*, is continuous and bounded in the Hausdorff metric. Then the means* (43) *and* (44) *are bounded for* $t \in [0, \infty)$.

Indeed, let us estimate, e.g., the absolute value $|m_{\tilde{y}}(t)|$ based on formula (33). Taking into account (28), the first term on the right-hand side of (43) can be estimated as $\frac{1}{\lambda_1 - \lambda_2}[|m_{\tilde{a}}|(\lambda_1 - \lambda_2) + 2|m_{\tilde{b}}|]$. To estimate the second term, we make the change of variables $t - s = \sigma$ in the integral; then $\int_0^t K(t-s) m_{\tilde{f}}(s) ds = \int_0^t K(\sigma) m_{\tilde{f}}(t-\sigma) d\sigma$. By assumption and Remark 2, there exists a constant $c_{\tilde{f}}$ such that $|m_{\tilde{f}}(s)| \leqslant c_{\tilde{f}} \forall\, s \geqslant 0$. Then, due to formula (39),

$$\left| \int_0^t K(\sigma) m_{\tilde{f}}(t-\sigma) d\sigma \right| \leqslant \int_0^t |K(\sigma)| |m_{\tilde{f}}(t-\sigma)| d\sigma \leqslant c_{\tilde{f}} \int_0^t |K(\sigma)| d\sigma \leqslant \frac{c_{\tilde{f}}}{\lambda_1 \lambda_2}.$$

And the desired statement for $m_{\tilde{y}}(t)$ follows immediately. The considerations for $r_{\tilde{y}}(t)$ are analogous.

Remark 5. Under the hypotheses of Proposition 4, the solution (43) of equation (45), as well as the solution (44) of equation (47), are Lyapunov stable (with respect to the initial conditions) according to [32, Ch. II]. Moreover, they are asymptotically stable, and stability holds under persistent excitation [32, Ch. II].

Physically, stability is conditioned by the fact that the series resonant circuit with the positive coefficients p_1 and p_2 induces a passive (causal) dynamic system [31].

It should be emphasized that Propositions 2–4 and Remark 5 actually deal with the case of a (1,1)-generalized ultraweak solution of problem (35), (36).

Let us turn to the case of generalized ultraweak solutions of other types in problem (35), (36). For the sake of definiteness, we take a (1,2)-ultraweak twice-generalized solution of problem (35), (36). In this case, the mean $m_{\tilde{y}}(t)$ is a solution of problem (32), (33) (Theorem 3). Consequently, it has the form (43). The other types of ultraweak generalized solutions are considered by analogy.

Concerning the half-range $r_{\tilde{y}}(t)$, note that in the case of a (1,1)-ultraweak solution of problem (47), (48), the function (44) yields a bounded solution as $t \to +\infty$, which is Lyapunov stable.

For example, in the case of a (1,2)-ultraweak solution of problem (35), (36), the equation for $r_{\tilde{y}}(t)$ has the form $r''_{\tilde{y}}(t) - p_1 r'_{\tilde{y}}(t) - p_2 r_{\tilde{y}}(t) = r_{\tilde{f}}(t)$ (Proposition 1). For the corresponding characteristic equation $\lambda^2 - p_1 \lambda - p_2 = 0$, by Vieta's theorem, the roots are real and of opposite signs. This means instability (i.e., physical unrealizability) of the corresponding solution. Other cases are analyzed by analogy. The situation would differ if the original system were replaced, e.g., by a system with feedback or a signal amplification system.

Consider now an important special case of equation (36), i.e., the harmonic oscillator equation (no resistance in the oscillatory circuit, see Fig. 2). We have the following fuzzy differential equation:

$$\tilde{y}''(t) + \omega^2 \tilde{y}(t) = \tilde{f}(t). \tag{49}$$

Here, $\omega > 0$ is a constant determining the oscillation frequency of the solutions of the homogeneous equation

$$y''(t) + \omega^2 y(t) = 0. \tag{50}$$

As is known, the general solution of the homogeneous equation (50) has the form $c_1 \cos \omega t + c_2 \sin \omega t$, where c_1, c_2 are arbitrary constants. In this case, the Cauchy function is given by $K(t-s) = \frac{1}{\omega} \sin \omega(t-s)$. Then, similar to Proposition 3, the average characteristics $m_{\tilde{y}}(t)$ and $r_{\tilde{y}}(t)$ of problem (49), (35) have the form

$$m_{\tilde{y}}(t) = m_{\tilde{a}} \cos \omega t + m_{\tilde{b}} \sin \omega t + \frac{1}{\omega} \int\limits_0^t \sin \omega(t-s) m_{\tilde{f}}(s)\, ds,$$

$$r_{\tilde{y}}(t) = r_{\tilde{a}} \cos \omega t + r_{\tilde{b}} \sin \omega t + \frac{1}{\omega} \int\limits_0^t \sin \omega(t-s) r_{\tilde{f}}(s) ds$$

(with the same designations as in Proposition 3).

6. CONCLUSIONS

In the theory of fuzzy numbers, a widespread method is to estimate fuzzy numbers by real numbers (defuzzification) or by real intervals. In this paper, we have solved the following problem: from an estimate of a fuzzy signal at the input of a linear dynamic system, find the corresponding estimate of the fuzzy signal at its output.

The main estimation parameters have been the fuzzy mean and the fuzzy half-range. We emphasize a fundamental fact established above, i.e., the universal role of the fuzzy mean $m_{\tilde{y}}$ (9) (Theorem 3), which is the same for various types of generalized derivatives of the differential problem describing a given dynamic system. This holds despite the various equations for the α-indices for various differentiability types.

The results of this work can be further developed in the following directions: consideration of the relationship between the modal values of fuzzy signals at the input and output of a linear dynamic system, and elaboration of an interval approach to estimating input and output fuzzy signals.

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