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LINEAR SYSTEMS

An Approach to Parametrization of Suboptimal Anisotropic Controllers

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Abstract—The problem of parametrization of suboptimal and γ-optimal anisotropic controllers in the form of static state feedback for linear discrete-time invariant systems driven by stochastic disturbances with bounded mean anisotropy is considered. In the controller design problem, the solution is based on applying Finsler lemma to the system of inequalities characterizing the boundedness of the anisotropic norm by some number. The relation of the solution of this problem to the solution of the controller design problem for suboptimal and γ-optimal anisotropic controllers via convex optimization without parametrization is pointed out. An example demonstrating the efficiency of the proposed approach to parametrization of anisotropic controllers is provided.

Keywords: anisotropy-based theory, linear systems, discrete-time systems, suboptimal controllers, parametrization

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1. INTRODUCTION

The anisotropy-based control and filtering theory, the foundations of which were laid in the mid-90s by I.G. Vladimirov [1–3], provides a mathematical framework for the analysis of linear discrete-time control systems with random external disturbances and for the synthesis of controllers which are optimal with respect to the mean-square gain criterion under the worst-case disturbance from a certain class. The main difference of this theory from the well-known $\mathcal{H}_2$- and $\mathcal{H}_\infty$-control theories is the manner of describing external disturbances perturbing on the system – the associated stochastic uncertainty is described in information-theoretic terms such as relative entropy and mutual information. The approach proposed by I.G. Vladimirov allows the $\mathcal{H}_2$- and $\mathcal{H}_\infty$-norms of systems to be considered as special cases of the so-called anisotropic norm, and $\mathcal{H}_2$- and $\mathcal{H}_\infty$-controllers as corresponding versions of anisotropic controllers.

Within the framework of anisotropy-based theory, many problems have been solved in various formulations: control and filtering, for stationary (time-invariant) and nonstationary (time-varying) systems, in optimal and suboptimal cases, with multiple criteria, for systems with multiplicative noise and systems with network structure, and many others [4–9]. Along with this, methods for solving these problems have also been developed: if in the first works the results were presented in the form of coupled Riccati and Lyapunov equations general primarily for optimal formulations, then later almost all solutions began to be presented in the form of a convex optimization problem with constraints in the form of linear matrix inequalities and one inequality of a special type. Nevertheless, each of the mentioned cases, when implemented on a computing device, gave only one solution found by a numerical algorithm. In fact, such a controller (even an optimal one) is not always unique, and in the suboptimal formulation the solution is a continuum set. Parametrization of controllers, if possible, is one of the ways to describe the set of solutions.

In this paper, we consider the problem of parametric description of the set of anisotropic controllers in the form of state feedback in suboptimal and γ-optimal formulations. The paper is organized as follows. Section 2 describes the basic concepts of anisotropy-based theory necessary for understanding what follows. Section 3 presents the mathematical statement of the problem to be solved. The main result and accompanying statements are given in Section 4. The proof of the main result is placed in the Appendix. Section 5 gives an example, and Section 6 concludes the paper.

2. PRELIMINARIES

This section provides the basic information necessary for the reader to understand the further material.

2.1. Notations

Let $L \in \mathbb{R}^{m\times n}$ be an arbitrary matrix over the field of real numbers. The transpose of L is denoted by L^{T}. The spectral norm (or the Schatten ∞-norm) of L is defined as $\|L\| = \sigma_{\max}(L)$, where $\sigma_{\max}(L) = \sqrt{\lambda_{\max}(L^{\mathrm{T}}L)}$ is the maximum singular value of L, and $\lambda_{\max}(L^{\mathrm{T}}L)$ is the maximum eigenvalue of $L^{\mathrm{T}}L$. For a symmetric matrix $L = L^{\mathrm{T}}$, the notation $L > 0$ is used to denote its positive definiteness. The notation $L^{\perp}$ is used for the left null-space (or cokernel, or orthogonal complement) of L, and is defined as any matrix satisfying the relations $L^{\perp}L = 0$ and $L^{\perp}L^{\perp\mathrm{T}} \succ 0$, if such exists. The matrix L^{+} denotes the Moore–Penrose pseudoinverse of L.

2.2. Basic Concepts of Anisotropy-Based Theory

Anisotropy-based theory belongs to the branches of automatic control theory dealing with a wide range of control and filtering problems for linear discrete-time stochastic systems, the presence of stochasticity in which is determined mainly by the fact that they operate in the presence of additive random noises and/or external disturbances. Assuming that the statistics of these disturbances are not precisely known, but it is possible to describe in some way the general pattern of information transfer in the process under consideration, a special functional is introduced on the set of spectral densities of the disturbance. Setting the condition that this functional is bounded by some number allows us to isolate a certain class from the set of all external disturbances and further investigate the influence of disturbances from this class on the system, as well as the ability of the control system to attenuate disturbances from such a class.

Let $\{w_k\}_{k\in\mathbb{Z}}$ be a stationary ergodic sequence of n_w-dimensional random vectors. The mean anisotropy of the sequence W is defined as [10]

$$\overline{\mathbf{A}}(W) = \lim_{N\to+\infty} \frac{\mathbf{A}(W_{0:N-1})}{N} = -\frac{1}{2}\ln\det\left(\frac{n_w\mathbf{E}[\widetilde{w}_0\widetilde{w}_0^{\mathrm{T}}]}{\mathbf{E}[|w_0|^2]}\right),$$

where $W_{s:t} = (w_s^{\mathrm{T}}, \ldots, w_t^{\mathrm{T}})^{\mathrm{T}}$ is a vector fragment of the sequence, $\widetilde{w}_0 = w_0 - \mathbf{E}[w_0|\{w_k\}_{k<0}]$ is the steady-state prediction error vector, and $\mathbf{A}(w)$ is the anisotropy of a random vector w, defined as

$$\mathbf{A}(w) = \frac{n_w}{2}\ln\left(\frac{2\pi\mathrm{e}}{n_w}\mathbf{E}[|w|^2]\right) - \mathbf{h}(w),$$

where the differential entropy $\mathbf{h}(w)$ of a random vector w with probability density function $f : \mathbb{R}^{n_w} \to \mathbb{R}_+$ is defined as

$$\mathbf{h}(w) = -\mathbf{E}[\ln(f(w))].$$

Now let $\{z_k\}_{k\in\mathbb{Z}}$ be a sequence linearly dependent on $\{w_j\}_{j\leqslant k}$ vectors generated by a system F, i.e. $Z = F * W$. Assume that the transfer function of F is analytic inside the unit disk on the complex plane and has a finite $\mathcal{H}_\infty$-norm

$$\|F\|_\infty = \sup_{z\in\mathbb{C}\colon |z|<1} \sigma_{\max}(F(z)).$$

Then for the system F the anisotropic norm can be introduced as [2]

$$|\!|\!| F |\!|\!|_a = \sup_{G\colon V\to W,\, \overline{\mathbf{A}}(W)\leqslant a} \frac{\|FG\|_2}{\|G\|_2},$$

where W is a stationary Gaussian sequence formed by passing a standard Gaussian white noise V through a stationary linear system G with transfer function $G(z)$ analytic inside the unit disk and having a finite $\mathcal{H}_2$-norm

$$\|G\|_2 = \sqrt{\frac{1}{2\pi}\int_{-\pi}^{\pi} \operatorname{tr}(\widehat{G}(\omega)(\widehat{G}(\omega))^*)d\omega}.$$

3. PROBLEM STATEMENT

Consider a linear discrete-time stationary system F given by the equations

$$\begin{cases} x_{k+1} = Ax_k + B_u u_k + B_w w_k, \\ z_k \quad\;\, = Cx_k, \end{cases} \tag{1}$$

where $k = 0, 1, \ldots$ is the time index, x_k is an n_x-dimensional state vector ($x_0 = 0$), u_k is an n_u-dimensional control vector, w_k is an n_w-dimensional external disturbance vector, and z_k is an n_z-dimensional regulated output vector. All matrices A, B_u, B_w, C of system (1) have appropriate dimensions and are known, the pair (A, B_w) is controllable, and the pair (A, B_u) is stabilizable. Without loss of generality, we assume that $n_u \leqslant n_x$, $\operatorname{rank}(C) = n_z \leqslant n_x$. It is assumed that the system (1) is subject to a random external disturbance represented by the sequence $W = \{w_k\}_{k\geqslant 0}$ with uncertainty related to unknown stochastic characteristics of the vectors w_k, the quantitative description of which is expressed in terms of an upper bound on the mean anisotropy of the sequence W by some number $a > 0$: $\overline{\mathbf{A}}(W) \leqslant a$. Let the control law have the form of static state feedback

$$u_k = Kx_k, \tag{2}$$

where the matrix $K \in \mathbb{R}^{n_u\times n_x}$ of the controller is to be determined. The closed-loop system F_{cl} obtained by closing system (1) with controller (2) has the form

$$\begin{cases} x_{k+1} = A_{cl}x_k + B_{cl}w_k, \\ z_k \quad\;\, = C_{cl}x_k, \end{cases} \tag{3}$$

where $A_{cl} = A + B_uK$, $B_{cl} = B_w$, $C_{cl} = C$.

The standard problem of γ-optimal anisotropic controller design in the form of static state feedback for system (1) consists in finding a matrix K of the control law (2) such that the corresponding controller is stabilizing and the anisotropic norm of the closed-loop system (3) is bounded from above by the minimal possible number $\gamma_{\min}$, i.e. the conditions

$$\rho(A + B_uK) < 1, \quad |\!|\!| F_{cl} |\!|\!|_a \leqslant \gamma, \quad \gamma \to \min \tag{4}$$

are met. This problem, including more general formulations, has been solved in [4, 11, 12]. The difference between the above formulation and the suboptimal one is the minimization of the upper bound γ: if γ is a fixed number ($\gamma > \gamma_{\min}$), then we have a suboptimal formulation.

Problem 1. Parametrize the set of solutions of the standard problems of synthesis of suboptimal and γ-optimal anisotropic controllers in the form of static state feedback.

The case of state feedback (4) is chosen for simplicity. For the cases of static and dynamic output feedback, under a number of additional requirements on the system, parametrization is also possible.

The practical significance of the problem posed consists of several aspects. First, solving the problem of controller parametrization allows one to form an idea of their set, which may be useful in studying the possibility of introducing additional criteria into the problem that do not lead to an empty set of solutions. Second, when the parameters of the problem (for example, the representation of the mean anisotropy of the external disturbance) vary in real time, the standard way of solving is to recompute the controller, and there is no guarantee that as a result of this operation, relatively small changes in the parameters will lead to relatively small changes in the controller coefficients. In other words, since the problem of finding controller coefficients in the classical approach is solved numerically and the found solution may differ significantly from the previous one, when switching from one controller to another, adverse transients may occur. Among other aspects one can mention, for example, the importance of taking into account rounding-off errors, which cannot be taken into account in the classical way, but for which corresponding results can be obtained within the framework of solving the parametrization problem.

4. PARAMETRIZATION OF SUBOPTIMAL ANISOTROPIC STATE-FEEDBACK CONTROLLERS

As noted, the problem of synthesizing a γ-optimal anisotropic controller has already been solved. Its solution (up to some renotation) is described by the following theorem [12, Theorem 1].

Theorem 1. *For given $a \geqslant 0$ and $\gamma > 0$, for the existence of a stabilizing static state-feedback controller* (2) *for system* (1) *guaranteeing that the anisotropic norm of the closed-loop system* (3) *is bounded by γ, it is sufficient that the system of inequalities*

$$P \succ 0, \quad \Psi \succ 0, \quad \eta^2 \geqslant \gamma^2, \tag{5}$$

$$\begin{bmatrix} P & \star & \star & \star \\ 0_{n_w \times n_x} & \eta^2 I_{n_w} & \star & \star \\ AP + B_u\widetilde{K} & B_w & P & \star \\ CP & 0_{n_z \times n_w} & 0_{n_z \times n_x} & I_{n_z} \end{bmatrix} \succ 0, \tag{6}$$

$$\begin{bmatrix} \eta^2 I_{n_w} - \Psi & \star \\ B_w & P \end{bmatrix} \succ 0, \tag{7}$$

$$(\det(\Psi))^{1/n_w} - e^{2a/n_w}(\eta^2 - \gamma^2) \geqslant 0 \tag{8}$$

is feasible with respect to the scalar variable η^2 and real matrices $P \succ 0_{n_x \times n_x}$, $\Psi \succ 0_{n_w \times n_w}$, and $\widetilde{K} \in \mathbb{R}^{n_u \times n_x}$. In this case, the static controller matrix is given by $K = \widetilde{K}P^{-1}$.

Remark 1. If in Theorem 1 γ is considered as a minimized variable, then we obtain formulation (4) for finding a γ-optimal anisotropic state-feedback controller.

Remark 2. The application of a numerical algorithm based on the conditions of Theorem 1 allows one to find suboptimal anisotropic controllers, but leaves no freedom of choice—the result of the algorithm is a fixed point K in the space of matrices $\mathbb{R}^{n_u \times n_x}$.

Remark 2 represents one of the motivation sides for this research. From the set of controllers that are a solution to some problem, the choice of a particular one may be determined by a number of auxiliary conditions not included in the original problem statement, and neglecting this freedom of choice (transferring the "right to choose" to a computing device) may lead to the fact that a number of secondary characteristics of the controller or closed-loop system will be chosen not in the best way. In addition, when an additional condition is introduced into the problem, it will have to be solved anew instead of choosing a suitable one from the already found set of controllers. Finally, describing the boundaries of the entire set of controllers is a more complete mathematical problem than finding some single solution.

To formulate the main result, we need the following auxiliary statement— a version of Finsler lemma [13, Theorem 2.3.12] (given with renotation consistent with that used in this paper).

Theorem 2. *Let matrices $\Theta \in \mathbb{R}^{n\times n}$, $\Lambda \in \mathbb{R}^{n\times m}$, $\Pi \in \mathbb{R}^{k\times n}$ be given. The following statements are equivalent:*

(1) *there exists a matrix $\widetilde{K}$ such that*

$$\Theta + \Lambda\widetilde{K}\Pi + (\Lambda\widetilde{K}\Pi)^{\mathrm{T}} \succ 0; \tag{9}$$

(2) *the following two conditions hold:*

$$\Lambda^{\perp}\Theta\Lambda^{\perp\mathrm{T}} \succ 0 \quad or \quad \Lambda\Lambda^{\mathrm{T}} \succ 0, \tag{10}$$

$$\Pi^{\mathrm{T}\perp}\Theta\Pi^{\mathrm{T}\perp\mathrm{T}} \succ 0 \quad or \quad \Pi^{\mathrm{T}}\Pi \succ 0. \tag{11}$$

Assume that the above statements hold. Let the matrices Λ and Π are factorized as $\Lambda = \Lambda_L\Lambda_R$ and $\Pi = \Pi_L\Pi_R$, where Λ_L, Λ_R, Π_L, Π_R are arbitrary full-rank matrices. Then all solution matrices $\widetilde{K}$ of inequality (9) have the form

$$\widetilde{K} = \Lambda_R^{+}X\Pi_L^{+} + Z - \Lambda_R^{+}\Lambda_R Z\Pi_L\Pi_L^{+}, \tag{12}$$

where Z is an arbitrary matrix, and

$$X = T\Lambda_L^{\mathrm{T}}\Phi\Pi_R^{\mathrm{T}}(\Pi_R\Phi\Pi_R^{\mathrm{T}})^{-1} + S^{1/2}Y(\Pi_R\Phi\Pi_R^{\mathrm{T}})^{-1/2}, \tag{13}$$

$$S = T - T\Lambda_L^{\mathrm{T}}(\Phi - \Phi\Pi_R^{\mathrm{T}}(\Pi_R\Phi\Pi_R^{\mathrm{T}})^{-1}\Pi_R\Phi)\Lambda_L T, \tag{14}$$

where $Y : \|Y\| < 1$ is an arbitrary matrix with bounded spectral norm, and T is an arbitrary positive definite matrix such that

$$\Phi = (\Theta + \Lambda_L T\Lambda_L^{\mathrm{T}})^{-1} \succ 0. \tag{15}$$

The content of this theorem is omitted. We only note that when it is applied in control problems, in addition to the degrees of freedom associated with the arbitrary choice of matrices Z, Y, and T, there will be one more degree of freedom caused by the value taken by the matrix Θ, which depends, generally speaking, on other variables. More on this will be presented after the main result given below.

4.1. Main Result

Theorem 3. *Let system (1) with matrix B_u be given, and let B_u admit a factorization $B_u = B_{uL}B_{uR}$ with full-rank matrices $B_{uL} \in \mathbb{R}^{n_x\times r}$ and $B_{uR} \in \mathbb{R}^{r\times n_u}$, where $r = \operatorname{rank}(B_u)$, and the pairs (A, B_w) and (A, B_u) are controllable and stabilizable, respectively. Let also two numbers $a \geqslant 0$*

and $\gamma > 0$ be chosen. Then for the existence of a stabilizing static state-feedback controller (2) guarantying $\|F_{cl}\|_a \leqslant \gamma$, it is sufficient that the system of inequalities

$$\begin{bmatrix} P & \star & \star & \star \\ 0_{n_w \times n_x} & \eta^2 I_{n_w} & \star & \star \\ B_u^{\perp} AP & B_u^{\perp} B_w & B_u^{\perp} P B_u^{\perp\mathrm{T}} & \star \\ CP & 0_{n_z \times n_w} & 0_{n_z \times (n_u - r)} & I_{n_z} \end{bmatrix} \succ 0, \tag{16}$$

$$\begin{bmatrix} \eta^2 I_{n_w} - \Psi & \star \\ B_w & P \end{bmatrix} \succ 0, \tag{17}$$

$$(\det(\Psi))^{1/n_w} - e^{2a/n_w}(\eta^2 - \gamma^2) \geqslant 0, \tag{18}$$

$$P \succ 0, \quad \Psi \succ 0, \quad \eta^2 \geqslant \gamma^2 \tag{19}$$

be feasible with respect to the variables P, Ψ, η^2.

If the statements of the theorem hold, then all matrices K of stabilizing controllers (2) guaranteeing $\|F_{cl}\|_a \leqslant \gamma$ are of the form $K = \widetilde{K} P^{-1}$, where

$$\widetilde{K} = B_{uR}^{+} X + (I_{n_u} - B_{uR}^{+} B_{uR}) Z, \tag{20}$$

$$X = T B_{uL}^{\mathrm{T}} \Phi_{31} \Phi_{11}^{-1} + S^{1/2} Y \Phi_{11}^{-1/2}, \tag{21}$$

$$S = T - T B_{uL}^{\mathrm{T}} (\Phi_{33} - \Phi_{31} \Phi_{11}^{-1} \Phi_{31}^{\mathrm{T}}) B_{uL} T, \tag{22}$$

$$\Phi = \underset{i,j=\overline{1,4}}{\mathrm{block}}(\Phi_{ij}) = \begin{bmatrix} P & \star & \star & \star \\ 0_{n_w \times n_x} & \eta^2 I_{n_w} & \star & \star \\ AP & B_w & P + B_{uL} T B_{uL}^{\mathrm{T}} & \star \\ CP & 0_{n_z \times n_w} & 0_{n_z \times n_x} & I_{n_z} \end{bmatrix}^{-1}, \tag{23}$$

$Z \in \mathbb{R}^{n_u \times n_x}$ is an arbitrary matrix, $Y \in \mathbb{R}^{r \times n_x}$ is an arbitrary matrix satisfying $\|Y\| < 1$, and $T \in \mathbb{R}^{r \times r}$ is any positive definite matrix such that $\Phi \succ 0$.

The proof of the theorem is given in the Appendix.

Note that expressions (16)–(19) of Theorem 3 do not depend on the matrix K of the desired controller, i.e., the sufficient conditions for the existence of a suitable anisotropic controller have a closed form. The second part of the theorem, namely formulas (20)–(23), describe in explicit parametric form all anisotropic controllers that are solutions to the problem.

It can be shown that all matrices T from Theorem 3 associated with $\Phi \succ 0$ given by (23) belong to the cone

$$T \succ B_{uL}^{+} \left(A(P^{-1} - C^{\mathrm{T}} C)^{-1} A^{\mathrm{T}} + \eta^{-2} B_w B_w^{\mathrm{T}} - P \right) B_{uL}^{+\mathrm{T}}, \tag{24}$$

i.e. the boundary (in the sense of the order relation $\succ$ on the set of symmetric matrices) value of T can be found explicitly. We also draw attention to the fact that in addition to the degrees of freedom due to the choice of matrices Z, Y, and T, the conditions of Theorem 3 implicitly contain another degree of freedom related to the values that the variables P and η^2 will take during the solution of inequalities (16)–(19), on which the matrices in (20)–(23) depend. If in the framework of Theorem 3 we consider the case when B_u is a full-rank matrix, i.e. $r = n_u$, then condition (16) takes the simplified form

$$\begin{bmatrix} P & \star \\ CP & I_{n_z} \end{bmatrix} \succ 0.$$

Remark 3. If in Theorem 3 the number γ is not given but is one of the variables, and we solve the problem of minimizing the upper bound of the anisotropic norm

$$\gamma^2 \to \min_{P\succ 0, \Psi\succ 0, \eta^2\geqslant\gamma^2>0} \quad \text{subject to (16)–(19)},$$

then we obtain a formulation for the parametrization of a γ-optimal state-feedback anisotropic controller.

We note once again that the obtained parametrization depends on the variables P, Ψ, η^2 (and in the framework of Remark 3 also on γ^2) found at the stage of solving the system of inequalities (16)–(19). Obtaining a parametrization of this system of inequalities is indeed a relevant problem, but it is difficult due to the presence of condition (18), which describes the relation between the variables η^2 and γ^2 and the spectrum of Ψ.

5. EXAMPLE

For academic purposes, we demonstrate the obtained results on a simple example. To this end, consider a system of the form

$$\begin{cases} x_{k+1} = Ax_k + B_u u_k + w_k, \\ z_k \quad = \quad x_k \end{cases}$$

with matrices

$$A = \begin{bmatrix} 0.98 & 0.2 & 0.02 & 0 \\ -0.2 & 0.98 & 0.2 & 0.02 \\ 0.02 & 0 & 0.98 & 0.2 \\ 0.2 & 0.02 & -0.2 & 0.98 \end{bmatrix}, \quad B_u = \begin{bmatrix} 0.02 \\ 0.2 \\ 0 \\ 0 \end{bmatrix}.$$

Assume that the external disturbance $\{w_k\}_{k\geqslant 0}$ acting on the system belongs to the class of sequences of random vectors with mean anisotropy bounded by $a = 0.1$: $\overline{\mathbf{A}}(W) \leqslant 0.1$. It is required to parametrically describe the set of solutions to the problem of constructing a stabilizing control law that ensures $\|F\|_a \leqslant \gamma$. We consider this problem in two cases: in the first case we assume that γ is a given number equal to $\gamma = 6$; in the second, the problem is to synthesize a γ-optimal anisotropic controller, i.e., $\gamma \to \min$. For comparative analysis, we also present the solution of the problems in accordance with Theorem 1 without parametrization. The calculations were performed in Matlab using the Yalmip toolbox for solving convex programming problems. Alternatively, the CVX system for solving convex programming problems can be used.

During the solution in accordance with the formulas of Theorem 1, one of the suboptimal anisotropic controller matrices found numerically at $\gamma = 6$ has the form (here and below, to distinguish controller matrices and other variables, they are assigned to a superscript index in parentheses)

$$K^{(1)} \approx [-7.14 \quad -5.72 \quad 1.95 \quad -5.01],$$

and the γ-optimal anisotropic controller matrix obtained by the same method but in the optimal formulation (see Remark 1) is

$$K^{(2)} \approx [-12.04 \quad -6.23 \quad 3.75 \quad -9.73]. \tag{25}$$

For comparison, the values of the anisotropic norms of the systems closed by the corresponding controllers turned out to be

$$\|F_{cl}(K^{(1)})\|_{0.1} \approx 5.83, \quad \|F_{cl}(K^{(2)})\|_{0.1} \approx 5.39.$$

Next, we successively present the solution of the parametrization problems for suboptimal and γ-optimal anisotropic state-feedback controllers, carried out in accordance with the content of Theorem 3 and Remark 3. The first of them will be presented in sufficient detail, and for the second we will only give the final result.

The found solution of the system of inequalities (16)–(19) with respect to the variables $P \succ 0$, $\Psi \succ 0$, $\eta^2 \geqslant \gamma^2$ are the matrices

$$P^{(3)} \approx \begin{bmatrix} 0.13 & \star & \star & \star \\ -0.09 & 0.93 & \star & \star \\ 0.05 & 0.01 & 0.07 & \star \\ -0.07 & -0.01 & -0.03 & 0.07 \end{bmatrix}, \quad \Psi^{(3)} \approx \begin{bmatrix} 213.52 & \star & \star & \star \\ -2.17 & 233.12 & \star & \star \\ 4.69 & 0.57 & 215.96 & \star \\ -19.68 & -2.1 & -3.7 & 197.32 \end{bmatrix}$$

and the number $\eta^{(3)} \approx 15.37$. Since some of the inverted matrices on the right-hand side of inequality (24) are ill-conditioned, determining the threshold value of T with high accuracy may turn out to be quite laborious. To avoid problems with computing the threshold value of T, in this example we set an overestimated value for it: $T^{(3)} \geq 410$. The required blocks Φ_{ij} of the matrix Φ will be obtained by substituting into (23) the chosen matrix T and the found P and η. Note that due to $B_{uR} = 1$, the variable Z will not participate in the parametrization, i.e., $\widetilde{K} = X$, and the controller matrix itself, given by expression (20) and the substitution $K = \widetilde{K}P^{-1}$, becomes $K = XP^{-1}$. Since it is difficult to visualize the parametrization proposed in Theorem 3 even in the (very simple) example under consideration, we restrict ourselves to the case of variation of the matrix T. To this end, we fix the matrix $\overline{Y} = [0.999 \quad 0 \quad 0 \quad 0]$ and consider the case $Y^{(3)} = \overline{Y}$, the case of the opposite sign matrix (i.e., $Y^{(3)} = -\overline{Y}$), and the case of the zero matrix $Y^{(3)} = 0$ in order to form an idea of the range of admissible values of the controller coefficients. The obtained dependences of the controller coefficients on the parameter T are shown in the figure. For the points "a" ($T^{(3)} = 410$, $Y^{(3)} = \overline{Y}$), "b" ($T^{(3)} = 820$, $Y^{(3)} = -\overline{Y}$), and "c" ($T^{(3)} = 39\,000$, $Y^{(3)} = 0$), we give the coefficients of the corresponding controllers:

$$\begin{aligned} K^{(3,\mathrm{a})} &\approx [-11.79 \quad -6.14 \quad 3.79 \quad -10.19], \\ K^{(3,\mathrm{b})} &\approx [-15.23 \quad -6.47 \quad 4.23 \quad -13.17], \\ K^{(3,\mathrm{c})} &\approx [-13.6 \quad -6.39 \quad 4.11 \quad -11.81]. \end{aligned}$$

The values of the anisotropic norms of the systems closed by them are respectively

$$|\!|\!| F_{cl}(K^{(3,\mathrm{a})}) |\!|\!|_{0.1} \approx 5.4, \quad |\!|\!| F_{cl}(K^{(3,\mathrm{b})}) |\!|\!|_{0.1} \approx 5.45, \quad |\!|\!| F_{cl}(K^{(3,\mathrm{c})}) |\!|\!|_{0.1} \approx 5.41,$$

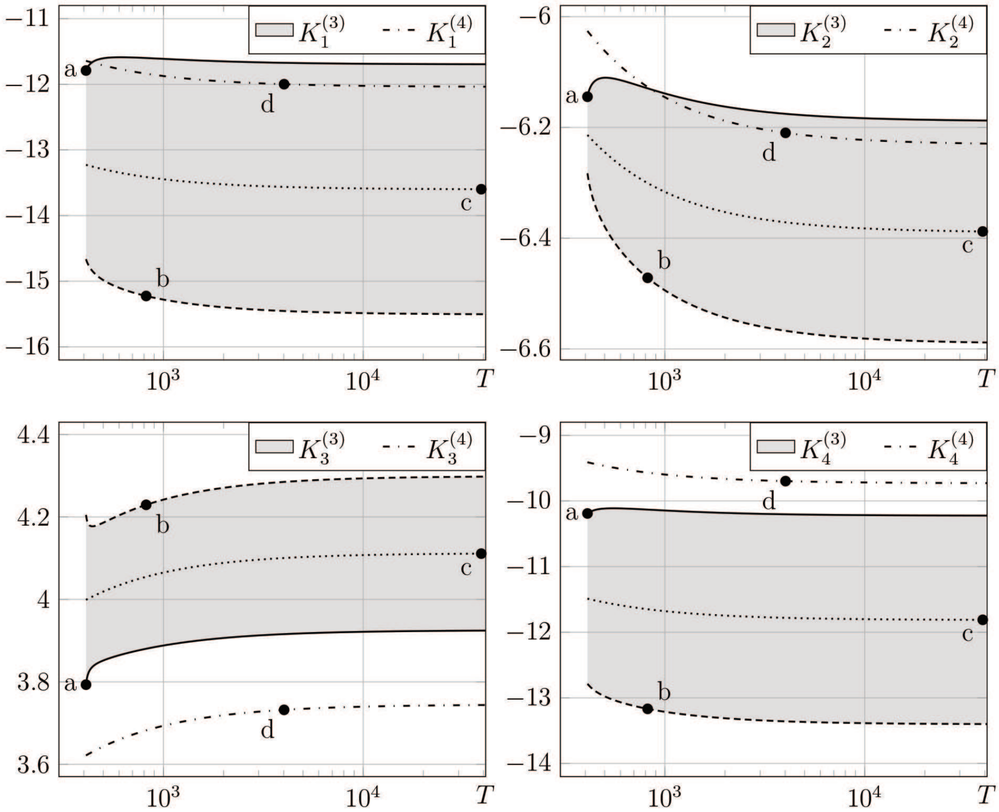
which satisfies the original requirement $|\!|\!| F_{cl} |\!|\!|_{0,1} \leqslant 6$.

Now consider the solution of the parametrization problem for a γ-optimal anisotropic state-feedback controller, as described in Remark 3. For this case, the term $S^{1/2} Y \Phi_{11}^{-1/2}$ from (21) is close to zero, i.e., the controller "loses" dependence on the matrix Y; therefore, only one line is shown in the figure, reflecting the direct dependence of $K^{(4)}$ on T. At the selected point "d", corresponding to the value $T^{(4)} = 4000$, the matrix of the γ-optimal anisotropic controller has the form

$$K^{(4,\mathrm{d})} \approx [-12 \quad -6.21 \quad 3.73 \quad -9.7], \tag{26}$$

and the value of the anisotropic norm of the system closed by this controller is

$$|\!|\!| F_{cl}(K^{(4,\mathrm{d})}) |\!|\!|_{0.1} \approx 5.39.$$



Dependence of the coefficients of suboptimal and γ-optimal anisotropic controllers in the example of Section 5 on the parameter T (logarithmic scale). Solid line corresponds to $Y = Y^{(3)}$; dashed line—to $Y = 0$; shaded area reflects all possible controller coefficients for the intermediate case $Y = (2\lambda{-}1)Y^{(3)}$ for arbitrary $\lambda \in [0; 1]$; dash-dotted line reflects the dependence of the γ-optimal controller $K^{(4)}$ on T. All lines and the shaded area are unbounded to the right (in the direction of increasing T). Points "a", "b", "c", "d" mark the coefficients of the controller matrices considered in the example.

This value is consistent with the previously obtained value for the γ-optimal controller $K^{(2)}$, as do (with some accuracy) the coefficients of the controller matrices themselves, which follows from a comparison of (25) and (26). The only difference is the possibility of choosing controller coefficients from some admissible range of values.

In conclusion of the example, it is worth emphasizing once again that the shaded area in the figure corresponds to all suboptimal anisotropic controllers generated by the matrices $P^{(3)}$, $\Psi^{(3)}$ and the number $\eta^{(3)}$ found at the first step of the solution. For another set of solutions (P, Ψ, η), this set will differ from that shown in the figure, and the union of all such sets will give the entire set of suboptimal anisotropic controllers for the chosen values $a \geqslant 0$ and $\gamma > 0$. This, in particular, explains why the line corresponding to the γ-optimal case (dash-dotted line in the figure) lies outside the shaded region for some values of T, although γ-optimal anisotropic controllers for a fixed $a \geqslant 0$ belong to all possible (but nonempty) sets of suboptimal controllers for the same a.

6. CONCLUSION

The paper proposes an approach to parametrization of suboptimal and γ-optimal anisotropic controllers in the form of static state feedback for linear discrete-time invariant systems with stochastic disturbances of bounded mean anisotropy. The relation of the solution of this problem to the solution of the synthesis problem for suboptimal and γ-optimal anisotropic controllers via convex optimization without parametrization is pointed out.

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APPENDIX

A.1. PROOF OF THEOREM 3

The proof of the theorem is based on "crossing" Theorems 1 and 2 and applying one useful property. In the conditions of Theorem 1, inequality (6) is representable in the form (9), where

$$\Theta = \begin{bmatrix} P & \star & \star & \star \\ 0_{n_w\times n_x} & \eta^2 I_{n_w} & \star & \star \\ AP & B_w & P & \star \\ CP & 0_{n_z\times n_w} & 0_{n_z\times n_x} & I_{n_z} \end{bmatrix}, \quad \Lambda = \begin{bmatrix} 0_{n_x\times n_u} \\ 0_{n_w\times n_u} \\ B_u \\ 0_{n_z\times n_u} \end{bmatrix}, \quad \Pi^{\mathrm{T}} = \begin{bmatrix} I_{n_x} \\ 0_{n_w\times n_x} \\ 0_{n_x\times n_x} \\ 0_{n_z\times n_x} \end{bmatrix}.$$

Since Theorem 2 gives necessary and sufficient conditions for the existence of a solution to inequality (9), its analogue (6) can be replaced by the pair of inequalities corresponding to (10), (11). The first of them, analogous to (10), takes the form (16). The second inequality, analogous to (11), takes the form

$$\begin{bmatrix} \eta^2 I_{n_w} & \star & \star \\ B_w & P & \star \\ 0_{n_z\times n_w} & 0_{n_z\times n_x} & I_{n_z} \end{bmatrix} \succ 0,$$

which can be disregarded in view of the requirement that (7) holds for a positive definite matrix $\Psi \succ 0_{n_w\times n_w}$.

Since adding constraints to an already specified system of conditions can only narrow the corresponding set of solutions, adding inequalities (5), (7), (8) to (10), (11) can only narrow the solution set in terms of the variables P, Ψ, η^2, but does not change the fact that the parametrization of solutions given in the second part of Theorem 2 remains valid. Indeed, all matrices $\widetilde{K}$ given by (12) depend on Θ via (15)—it is this dependence (along with the restrictions on the chosen matrices Y and T) that guarantees the fulfillment of the original inequality (9).

Carrying out the necessary calculations carefully, taking into account that B_u admits the factorization $B_u = B_{uL}B_{uR}$ with full-rank matrices B_{uL} and B_{uR}, we obtain that expressions (20)–(23) are complete analogues of expressions (12)–(15), respectively.

At the end of the proof, the reverse substitution $K = \widetilde{K}P^{-1}$ is motivated by the content of Theorem 1. The theorem is proved.

REFERENCES

1. Semyonov, A.V., Vladimirov, I.G., and Kurdjukov, A.P., Stochastic approach to H_∞-optimization, *Proc. 1994 33rd IEEE Conf. on Decision and Control*, 1994, vol. 3, pp. 2249–2250.

2. Vladimirov, I.G., Kurdjukov, A.P., and Semyonov, A.V., On computing the anisotropic norm of linear discrete-time-invariant systems, *IFAC Proceedings Volumes*, 1996, vol. 29, no. 1, pp. 3057–3062.

3. Vladimirov, I.G., Kurdjukov, A.P., and Semyonov, A.V., State-space solution to anisotropy-based stochastic H_∞-optimization problem, *IFAC Proceedings Volumes*, 1996, vol. 29, no. 1, pp. 3816–3821.

4. Tchaikovsky, M.M., Static output feedback anisotropic controller design by LMI-based approach: General and special cases, *American Control Conference*, 2012, pp. 5208–5213.

5. Timin, V.N., Tchaikovsky, M.M., and Kurdyukov, A.P., A Solution to Anisotropic Suboptimal Filtering Problem by Convex Optimization, *Doklady Mathematics*, 2012, vol. 85, no. 3. pp. 443–445.

6. Timin, N.V., Anisotropy-based suboptimal filtering for the linear discrete time-invariant systems, *Automation and Remote Control*, 2013, vol. 74, no. 11. pp. 1773-1785.

7. Timin, V.N. and Kurdyukov, A.P., Finite horizon anisotropy-based multicriteria filtering, *IFAC-PapersOnLine*, 2015, vol. 48, no. 14, pp. 192–197.

8. Timin, V.N. and Kurdyukov, A.P., Suboptimal anisotropic filtering in a finite horizon, *Automation and Remote Control*, 2016, vol. 77, no. 1, pp. 1-20.

9. Tchaikovsky, M.M. and Timin, V.N., Synthesis of anisotropic suboptimal control for linear time-varying systems on finite time horizon, *Automation and Remote Control*, 2017, vol. 78, no. 7. pp. 1203–1217.

10. Vladimirov, I.G., Diamond, P., and Kloeden, P., Anisotropy-based robust performance analysis of finite horizon linear discrete time varying systems, *Automation and Remote Control*, 2006, vol. 67, pp. 1265–1282.

11. Tchaikovsky, M.M., Suboptimal anisotropic stochastic robust control design via convex optimization, *M.: ICS RAS: PhD thesis*, 2012. (in Russian)

12. Tchaikovsky, M.M., Anisotropic controllers design via convex optimization and semidefinite programming, *UBS*, 2013, no. 42, pp. 100–152. (in Russian)

13. Skelton, R.E., Iwasaki, T., and Grigoriadis, K.M., *A unified algebraic approach to control design*, Routledge, 1998, 304 p.

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