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SPECIAL ISSUE

Cauchy Problem for the Buckley–Leverett Equation of Two-Phase Filtration with Variable Porosity

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Abstract—The problem of constructing a multivalued solution for the Buckley–Leverett equation describing the process of two-phase filtration in a medium with a variable porosity coefficient is considered. Such an equation is used to calculate the evolution of the phase separation surface "oil–water" during the development of oil fields. It is shown how to construct the caustic of multivalued solutions for given initial conditions.

Keywords: filtration, porous media, multi-valued solutions, caustics, jets, contact structure, contact vector fields

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1. INTRODUCTION

During oil field development, reservoir energy is maintained by injecting a displacing agent (water or solutions with active additives) through special wells. The movement of the oil displaced by this agent, and the agent itself, depends on the pressure difference between the injection and production zones. In homogeneous reservoirs, fluid movement between alternating rows of production and injection wells is often considered one-dimensional. This assumption is justified, for example, when injection and extraction are carried out through parallel wells [1].

One-dimensional fluid flow is particularly evident when using horizontal wells with parallel boreholes. The displacing agent is injected through injection wells and directed into horizontally oriented fractures formed by hydraulic fracturing. Under pressure, fluid flows from these fractures toward production wells, from where oil is extracted to the surface through vertical channels.

This production approach is particularly effective for fields with hard-to-recover reserves, where successful oil displacement requires physical and chemical methods. These methods include the use of active component solutions (e.g., polymers, surfactants, carbon dioxide, and others), thermal methods (such as hot water or steam injection), as well as wave stimulation (hydrodynamic, electromagnetic, ultrasonic), or combinations thereof.

2. MODEL DESCRIPTION

The Buckley–Leverett equation [2] describes one-dimensional two-phase filtration in a porous medium. It is based on Darcy's law, according to which the filtration rate is proportional to the difference in the partial pressures of the phases. In the classical Buckley–Leverett model, this difference is assumed to be constant. The permeability of the pore space, which is assumed to be non-deformable, is also assumed to be constant.

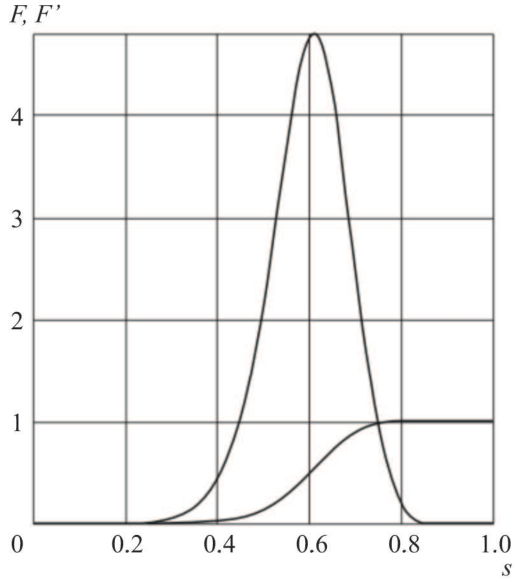

Fig. 1. Graphs of functions F and its derivative.

Here we consider the Buckley–Leverett equation in the following form:

$$m\frac{\partial s}{\partial t}+U\frac{\partial}{\partial x}(F(s))=0. \tag{1}$$

Here t is time, x is the spatial coordinate (the Ox axis is directed in the direction of fluid flow), $s=s(t,x)$ is water saturation (the volume fraction of pores occupied by water), m is the porosity coefficient, and $F(s)$ is the Buckley–Leverett function, indicating the volume fraction of water in the total flow U, which is assumed to be constant. An approximate graph of the Buckley-Leverett function and its derivative is shown in Fig. 1. The prime denotes the derivative of function F with respect to variable s.

It is usually assumed that the porous medium is homogeneous, i.e., in equation (1), the porosity coefficient m is constant. Here, we abandon this assumption and assume that function m depends on the spatial variable, i.e., $m=m(x)$. We assume that the function $m(x)$ is differentiable.

Let's consider the Cauchy problem for equation (1):

$$s(0,x)=S(x), \tag{2}$$

where $S(x)$ is a given function, which we assume to be differentiable.

As is well known, the classical solution to this problem collapses upon reaching a certain critical point in time, regardless of the initial water saturation distribution. The solution develops a discontinuity, and the phase boundary moves like a shock wave.

Therefore, instead of classical solutions, we will consider multivalued solutions [3].

This approach circumvents the difficulties associated with the collapse of solutions to equations and makes it possible to describe the evolution of shock waves.

3. CHARACTERISTICS OF THE BUCKLEY–LEVERETT EQUATION

We use a differential-geometric approach to equation (1). This method, as applied to the problem under consideration, is described in [4]. With this approach, the equation is considered as a hypersurface in the space of 1-jets of functions of two variables.

The space of 1-jets $J^1 = J^1(\mathbb{R}^2)$ of functions of two independent variables t and x is a five-dimensional space with canonical coordinates $t, x, u_{00}, u_{10}, u_{01}$. Let us indicate the coordinates of the 1-jet $\theta = j^1_a(s)$ of the function $s = s(t, x)$ at the point $a = (a_1, a_2) \in \mathbb{R}^2$:

$$t(\theta) = a_1,\ \ x(\theta) = a_2,\ \ u_{00}(\theta) = s(a),\ \ u_{10}(\theta) = \frac{\partial s}{\partial t}(a),\ \ u_{01}(\theta) = \frac{\partial s}{\partial x}(a).$$

The space J^1 is equipped with the four-dimensional contact distribution

$$\mathcal{C} : J^1 \ni \theta \mapsto \mathcal{C} = \ker \varkappa_\theta \subset T_\theta J^1.$$

It is generated by the differential 1-form (the Cartan form)

$$\varkappa = du_{00} - u_{10}dt - u_{01}dx.$$

Equation (1) corresponds to the function $f = m(x)u_{10} + UF'(u_{00})u_{01}$ and a hypersurface in the space of 1-jets is $\mathcal{E} = \{f = 0\}$. For brevity, we introduce the following notation:

$$H(u_{00}) = UF'(u_{00}). \tag{3}$$

Then

$$f = m(x)u_{10} + H(u_{00})u_{01}. \tag{4}$$

The multivalued solution of equation (1) is a two-dimensional surface $L \subset \mathcal{E}$, which is an integral manifold of the contact distribution $\mathcal{C}$, i.e., the restriction of the Cartan form to it is zero: $\varkappa|_L = 0$.

Let us introduce the vector field

$$X = -m(x)\frac{\partial}{\partial t} - H(u_{00})\frac{\partial}{\partial x} \tag{5}$$

on the space of 0-jets J^0. The value of the Cartan form on this field coincides with the function f, and the extension of the field X to the space of 1-jets is a contact vector field with generating function f:

$$X_f = -m(x)\frac{\partial}{\partial t} - H(u_{00})\frac{\partial}{\partial x} + H'(u_{00})u_{01}u_{10}\frac{\partial}{\partial u_{10}} + \Big(m'(x)u_{10} + H'(u_{00})u_{01}^2\Big)\frac{\partial}{\partial u_{01}}.$$

This field is called the *characteristic* vector field for equation (1), and its trajectories are called the *characteristics.*

This vector field is tangent to the hypersurface $\mathcal{E}$ and, moreover, is tangent to every multivalued solution of equation (1) (see [5]). It can be used to construct a multivalued solution to the generalized Cauchy problem. Let us consider this in more detail.

4. CAUCHY PROBLEM

Initial condition (2) generates a Cauchy curve in the space of 1-jets:

$$\mathcal{K} = \{t = 0, u_{00} = S(x), u_{01} = S'(x), m(x)u_{10} + H(u_{00})S'(x) = 0\} \subset \mathcal{E}.$$

Since $\varkappa|_{\mathcal{K}} = 0$, this curve is integral for the contact distribution.

Since the coefficient $m(x) \neq 0$, at each point $a \in \mathcal{K}$, the tangent vector of the field X_f does not lie in the tangent space to this curve: $X_f|_a \notin T_a\mathcal{K}$. By shifting the Cauchy curve along the trajectories of the vector field X_f, we obtain a two-dimensional surface

$$L = \Phi^{(1)}_\tau(\mathcal{K}).$$

Here τ is the shift parameter along the trajectories, and $\Phi_\tau^{(1)}$ is the shift transformation (the flow of the vector field X_f) from $\tau = 0$ to τ. This flow is an extension of the flow Φ_τ of the vector field (5) into the space of 1-jets.

Since the flow $\Phi_\tau^{(1)}$ preserves the contact distribution $\mathcal{C}$, the surface L is a multivalued solution.

Thus, the multivalued solution L is a disjunctive union of the characteristics passing through the curve $\mathcal{K}$. This multivalued solution will be called a *multivalued solution of the Cauchy problem.*

The projections of the characteristics onto the plane of independent variables t, x are integral curves of the following ordinary differential equation:

$$\frac{dx}{dt} = \frac{H(u_{00})}{m(x)}. \tag{6}$$

We also call these curves characteristic curves.

Since $X_f(u_{00}) = 0$, i.e., each characteristic of equation (1) lies entirely in the fiber $u_{00} = \text{const}$. Of course, each characteristic has its own constant. Therefore, the function $H(u_{00})$ is constant on the characteristic.

The integral curves of equation (6) have the form

$$t = \frac{1}{H(u_{00})} \int m(x)dx + C. \tag{7}$$

In this formula, the parameter u_{00} is a constant whose value is determined by the initial point of the characteristic.

Unlike the case of constant porosity, when the projections of the characteristics onto the plane of independent variables are straight lines, here a family of curves is obtained.

5. THE CASE OF LINEAR POROSITY

Let's consider the case of linear porosity:

$$m(x) = \alpha x + \beta.$$

In this case, the condition $m(x) > 0$ must be satisfied. As function (3), we choose the function

$$H(u) = a \exp\left(-\frac{(u-b)^2}{2c^2}\right) + d,$$

which well approximates the derivative F' of the Buckley–Leverett function. Here $\alpha, \beta, a, b, c, d$ are some real numbers, and β, a, c are not equal to zero. As the initial function (2), we choose the function

$$S(x) = \frac{1}{1+x^2},$$

which well approximates the water saturation function near the injection well located at $x = 0$.

For brevity, we denote the coordinate function u_{00} by u, omitting the subscripts. Then, vector field (5) takes the form

$$X = -(\alpha x + \beta)\frac{\partial}{\partial t} - \left(a \exp\Big(-\frac{(u-b)^2}{2c^2}\Big) + d\right)\frac{\partial}{\partial x}. \tag{8}$$

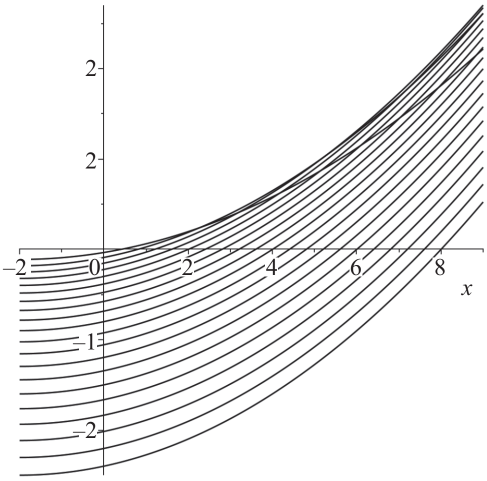

Fig. 2. Characteristics (9) for parameter values (10).

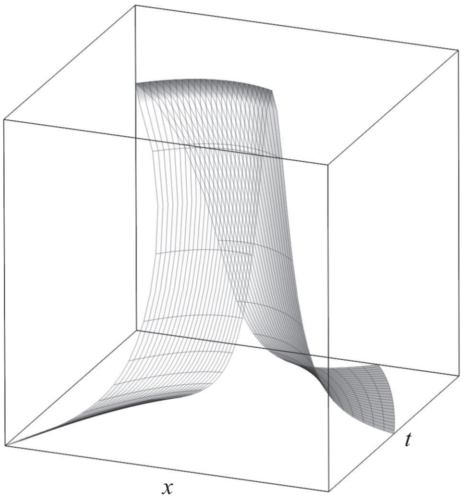

Fig. 3. The caustic Σ on the multivalued solution.

Its flow is found explicitly:

$$\Phi_\tau : \begin{cases} t \mapsto \alpha\left(a\exp\Big(-\dfrac{(u-b)^2}{2c^2}\Big)+d\right)\dfrac{\tau^2}{2}-(\alpha x+\beta)\tau+t, \\ x \mapsto -\left(a\exp\Big(-\dfrac{(u-b)^2}{2c^2}\Big)+d\right)\tau+x, \\ u \mapsto u. \end{cases}$$

Passing through the point (t_0, x_0) characteristic (7) has the form of a parabola (see Fig. 2).

$$\alpha(x^2-x_0^2)+2\beta(x-x_0)-2(t-t_0)\left(a\exp\left(-\frac{(bx_0^2+b-1)^2}{2(x_0^2+1)^2c^2}\right)+d\right)=0. \tag{9}$$

Starting at some point in time $t = T$, the parabolas (9) intersect, and for $t > T$, the solution to equation (1) becomes multivalued (see Fig. 2).

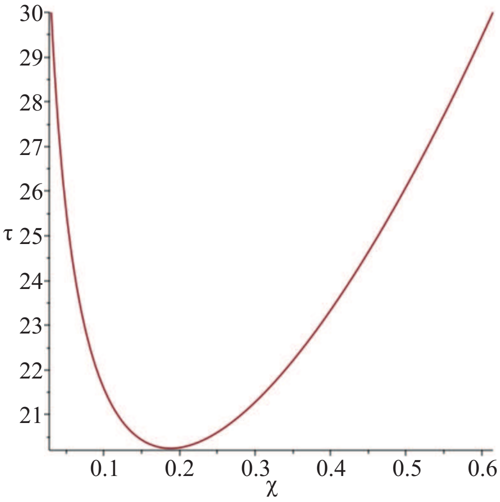

Fig. 4. The graph of multivalued solution.

At $t = T$, a so-called "gradient catastrophe" occurs: the partial derivative with respect to variable x of the classical single-valued solution becomes infinite. A similar phenomenon is observed for the case of constant porosity, however, the characteristics there are not parabolas, but straight lines.

This paper does not address the issue of overcoming this difficulty. The standard method is to construct discontinuous solutions from multivalued solutions using conservation laws and the Hugoniot–Rankine conditions [6].

The envelope of the family of characteristics (9) is the projection of the caustic $\Sigma \subset L$ onto the plane of independent variables. At the caustic points, this projection

$$\pi_{1,0} : L \to \mathbb{R}^2(t, x)$$

has a singularity: the tangent planes to the multivalued solution project onto curves.

Figures 2 and 3 show the characteristics and the caustic Σ for the following parameter values:

$$a = 0.8,\ b = 2,\ c = 0.5,\ d = 0.2,\ \alpha = -0.01,\ \beta = 0.02. \tag{10}$$

The projection of the multivalued solution onto the space $\mathbb{R}^3$ with coordinates t, x, u (see Fig. 4) in parametric form is

$$L : \begin{cases} t = \dfrac{\tau}{2}\left(\tau\alpha a \exp\left(-\dfrac{(b\chi^2+b-1)^2}{2(\chi^2+1)^2c^2)}\right) + (d\tau - 2\chi)\alpha - 2\beta\right), \\ x = -a\tau \exp\left(-\dfrac{(b\chi^2+b-1)^2}{2(\chi^2+1)^2c^2}\right) - d\tau + \chi, \\ u = \dfrac{1}{1+\chi^2}. \end{cases}$$

6. CONCLUSION

It should be noted that other two-phase flow models exist that take capillary forces into account and lead to second-order differential equations. One such example is the Rapoport–Leas equation [7]. However, their study raises higher-order difficulties, overcoming which requires other methods, such as those of finite-dimensional dynamics [8, 9].

Other examples of the application of contact geometry methods to problems in continuum mechanics and thermodynamics are given in [10]. In particular, for certain classes of Buckley–Leverett functions, exact solutions were found for the Barenblatt equations, which describe two-phase flow in porous media in the presence of surface-active reagents dissolved in water and changing the viscosity of the fluids [11].

In [12] the Buckley–Leverett model was constructed and investigated, in which the porosity coefficient in equation (1) is a random variable.

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