ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 4, pp. 314–335.

SPECIAL ISSUE

Energy Stability Metrics of Linear Continuous Systems

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Received July 8, 2025
Revised October 14, 2025
Accepted November 20, 2025

Abstract—The concept of basic systems as systems with a transfer function in the form of an inverse characteristic polynomial of the system is introduced. For these basic systems, we obtain spectral decompositions of energy-based stability metrics from their state-space representations in canonical forms, including Jordan, modal, controllability, and observability. We present explicit formulas for the spectral decomposition of the squared H_2-norm, specifically for the case where the transfer function possesses repeated poles. Furthermore, we develop efficient recursive algorithms to compute the spectral decompositions of the inverse matrices of gramians of continuous dynamical systems based on the repeated spectra of the original dynamics matrices. Spectral decompositions of energy stability metrics are obtained in the form of spectral decomposition of the square H_2-norm of the transfer function of the base system for the case of its simple and multiple roots. A new approach to analyzing the stability of linear systems based on an enhanced Routh–Hurwitz stability criterion is proposed, which includes forming Routh tables and Xiao matrices, computing the spectrum of the dynamics matrix, computing spectral decompositions of energy-based metrics, and jointly applying the Routh–Hurwitz stability criterion and the criterion of boundedness of energy-based stability metrics of basic systems. Finally, we conclude with practical recommendations for applying these theoretical and algorithmic results in systems analysis and design.

Keywords: spectral decompositions, canonical forms, basis systems, gramians, Lyapunov and Sylvester differential equations, Cauchy matrices, Xiao matrices, energy metrics

DOI: 10.7868/S1608303226040034

1. INTRODUCTION

It is well known that gramians are solutions to Sylvester and Lyapunov equations, which are the subject of a vast number of scientific publications [1–4]. These algebraic and differential matrix equations also play a fundamental role in control theory. In recent years, interest has grown in developing methods for computing various energy-related metrics to analyze the stability and degree of controllability, reachability, and observability of such systems. The initial spectral decompositions of gramians for linear continuous and discrete systems with simple spectra were derived in [1]. This work employed the spectral decomposition of the integral representation of solutions to Lyapunov or Sylvester equations to achieve this result. Furthermore, it pioneered the analysis of the temporal characteristics of gramians in controllability canonical forms, examining the role of eigenvectors and the kernels of Cauchy matrices in synthesizing optimal low-order dynamical models. Subsequently, a series of studies [5–12] proposed energy-based metrics for assessing the stability and reachability of both stable and unstable linear systems. In control theory practice, the gramian method as an approach to solving Lyapunov and Sylvester differential equations is applied to a range of practical problems, including:

- Optimal placement of sensors and actuators [7, 10–13];
- Simplification of linear and bilinear control system models using energy metrics [1–3];

- Analysis and synthesis of modal control systems [13–15, 17];
- Analysis of the accuracy of control systems under random disturbances [2, 4];
- Condition monitoring and technical diagnostics based on the identification of energy balance anomalies [13, 16, 18].

The gramian method provides both qualitative and quantitative evaluation of the energy localized in weakly damped oscillatory modes. Furthermore, the resonant interaction between closely spaced vibrational modes within a dynamical system can be significantly enhanced. By transforming the system's state equations into controllability and observability canonical forms, it becomes possible to extract invariant energy metrics that independent of coordinate transformations. The structural aspects of controllability and observability in relation to gramian computation and analysis are explored in studies [19, 20]. An important problem of optimal sensor and actuator placement based on various energy functionals, including invariant ellipsoids, is considered in [4, 7].

2. PROBLEM STATEMENT

Consider a stationary MIMO LTI dynamic system of the form

$$\dot{x}(t) = Ax(t) + Bu(t), x(0) = 0, y(t) = Cx(t), \tag{1}$$

where $x(t) \in \mathbb{R}^n, u(t) \in \mathbb{R}^m, y(t) \in \mathbb{R}^m$. In case of SISO LTI systems $u(t) \in \mathbb{R}^1, y(t) \in \mathbb{R}^1$. The Lyapunov and Sylvester equations play an important role in control theory and the design of control systems based on integral performance indicators, as well as in the synthesis of optimal systems using state observers. Two types of Lyapunov equations are used to solve these problems

$$AP + PA^T = -I_n. \tag{2}$$

$$AP + PA^T = -\sum_{j,\eta} e_j e_\eta^T. \tag{3}$$

Equations of the first type arise in MIMO LTI systems defined by equations of state of the general form (1). Equations of the second type arise in SISO LTI systems defined by equations of state in modal canonical form and in canonical forms of controllability and observability. In modal canonical form, we have the following expressions for the matrix A_d and vectors b, c

$$A_d = diag\begin{bmatrix} s_1 & \dots & s_n \end{bmatrix}, b = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T, c = \begin{bmatrix} r_1 & r_2 & \dots & r_n \end{bmatrix}^T, \tag{4}$$

where r_i is residue of the transfer function at the pole "i." For the base system we have

$$c = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T. \tag{5}$$

In the canonical forms of controllability in this case the equations of state have the form

$$\dot{x}_c(t) = A_c^F x_c(t) + b^F u(t), x_c(0) = 0, \tag{6}$$
$$y_c(t) = c^F x_c(t),$$

$$A_c^F = \begin{bmatrix} 0 & 1 & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ 0 & 0 & 0 & \dots & 0 \\ 0 & 0 & 0 & \dots & 1 \\ -a_0 & -a_1 & -a_2 & \dots & -a_{n-1} \end{bmatrix}, b^F = \begin{bmatrix} 0 & 0 & \dots & 0 & 1 \end{bmatrix}^T, \tag{7}$$
$$c^F = \begin{bmatrix} \xi_0 & \xi_1 & \dots & \xi_{n-2} & \xi_{n-1} \end{bmatrix},$$

where ξ_i is coefficient of the numerator polynomial of the transfer function

$$W(s) = \frac{\xi_{n-1}s^{n-1} + \dots + \xi_1 s + \xi_0}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0}.$$

For the base system we have

$$c^F = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}. \tag{8}$$

The classical criterion for the stability of continuous linear systems is based on solving equations of the form (2), (3) with a modified right-hand side of the form $-bb^T$. These matrices depend on the similarity transformation matrices, so the controllability gramians and the expressions for the squares of the H_2-norm of the transfer function obtained using the gramians are not invariant under various transformations of the state space. This drawback can be avoided by using right-hand sides of the form (2) or (3). We call these equations of the second type the basic Lyapunov equations, and the corresponding state equations, in which the vector c^F is chosen in the form $c^F = \begin{bmatrix} 1 & 0 & \dots & 0 & 0 \end{bmatrix}$ or $c = \begin{bmatrix} 1 & 1 & \dots & 1 \end{bmatrix}^T$, the basic equations of state. These equations of state correspond to a transfer function (TF)

$$W_{base}(s) = \frac{1}{s^n + a_{n-1}s^{n-1} + \dots + a_1 s + a_0} = \frac{1}{N(s)}, \tag{9}$$

where $N(s)$ is characteristic polynomial of the system.

Definition 1 [5, 6]. Let us call a Xiao matrix a square matrix that has a zero-plaid structure of the form

$$Y = \begin{bmatrix} y_1 & 0 & -y_2 & 0 & y_3 & \dots \\ 0 & y_2 & 0 & -y_3 & 0 & \dots \\ -y_2 & 0 & y_3 & 0 & \dots & \dots \\ 0 & -y_3 & 0 & \dots & \dots & \dots \\ y_3 & 0 & \dots & \dots & \dots & 0 \\ \dots & \dots & \dots & \dots & 0 & y_n \end{bmatrix} y_i \in \mathbb{C}, i = \overline{1,n}. \tag{10}$$

The elements of this matrix are calculated using the formulas

$$y_{j\eta} = \begin{cases} 0, \quad \text{if } \ j+\eta = 2k+1, \quad k = 1,\dots,n, \\ y_n = \frac{1}{2Y_{n,1}}, \\ y_{n-l} = \frac{-\sum_{i=1}^{m-1}(-1)^i Y_{n-l,i+1} y_{n-l+i}}{Y_{n-l,1}}, \text{if } j+\eta = 2k, \ k = 1,\dots,n, l = 1,\dots,n-1, \end{cases}$$

It is known that basic equations play an important role in studying the stability of linear systems. Let us consider a differential equation of state of the form

$$\begin{gathered} y^{(n)}(t) + a_{n-1}y^{(n-1)}(t) + \dots + a_0 y(t) = \delta(t), \\ y^{(n-1)}(0) = 1, \quad y^{(n-2)}(0) = 0, \quad \dots, \quad y(0) = 0, \end{gathered} \tag{11}$$

where $\delta(t) - \delta$ is Dirac function. This equation corresponds to the transfer function of the base system. It is proved that the controllability gramian for the system (11) has the form of a Xiao matrix (10), in which the elements y_i are calculated using the measured quantities $y^{(i)}(t)$ [6]

$$y_i = \int_0^\infty \left[y^{(i)}(t)\right]^2 dt. \tag{12}$$

These estimates have the physical meaning of measured energies of the states of the system, and they can be calculated in various ways, including using state observers [19].

Definition 2. The energy metric of the stability of the system (1) is the square of the H_2-norm of its transfer function [10–12].

We will show that basic systems can be completely controllable if additional conditions are met. The controllability matrix for the modal base system (4), (5) is a Vandermonde matrix, which is a full rank matrix only if all eigenvalues of the dynamics matrix are distinct. The controllability matrix for the base system (7), (8) is not singular. Therefore, the basic system (7), (8) is completely controllable. The gramian controllability matrix for the base system (7), (8) in Xiao form has the form [5]

$$P^{cF} = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j \left(-s_k\right)^{\eta}}{\prod_{\lambda=1,\lambda\neq k}^{n} \left(s_k - s_\lambda\right) \prod_{\lambda=1,}^{n} \left(-s_k - s_\lambda\right)} 1_{j+1\eta+1}, \tag{13}$$

$$P^{cF} = \sum_{k=1}^{n} \sum_{\eta=0}^{n-1} \sum_{j=0}^{n-1} \frac{s_k^j \left(-s_k\right)^{\eta}}{\dot{N}\left(s_k\right) N\left(-s_k\right)} 1_{j+1\eta+1}. \tag{14}$$

It follows that the matrices of the gramians P^{cF} are positive definite.

Remark 1. It follows from (14) that the Xiao matrix depends on the quadratic forms $s_k^j(-s_k)^{\eta}$ formed by the eigenvalues of the original base and antistable base systems, on the residues of the transfer function of the base system, and on the values of the characteristic polynomial of the antistable system in its eigenvalues. Quadratic forms define the property of the zero blanket of Xiao matrices. The denominator of the expression (14), which depends on the characteristic polynomial $N(s)$, determines the value of the square of the H_2-norm of the transfer function, which is the energy metric of the system (1). The important role of the base system in the formation of energy sustainability metrics is obvious. Complete controllability of basic systems is one of the conditions for the positive definiteness of their controllability gramians. This simplifies the calculation of controllability gramians for the base system (7), (8), simplifies the transformation of gramians into Hadamard form and the calculation of the squared H_2-norm of the transfer function (9). Moreover, this guarantees the normality of the gramian matrix, which implies that the diagonal elements and traces of the corresponding gramians are positive. Their elements depend only on the eigenvalues of the dynamics matrix and its characteristic polynomial, which do not depend on similarity transformations and, therefore, are invariants under these transformations. As for the basic system (4), (5), the controllability gramian in it takes the simple form of Cauchy matrices [1, 21]

$$P_c = \frac{-1}{s_j + s_\eta^*}, \forall j, \eta = \overline{1,n}, \forall s_j, s_\eta \in \mathbb{C}^-. \tag{15}$$

The symmetric Cauchy matrix is defined as

$$C = \frac{-1}{x_j + x_{\eta,}}, \forall j, \eta = \overline{1,n}, \forall x_j, x_\eta \in \mathbb{R}^-. \tag{16}$$

Theorem 1 [21]. *A symmetric Cauchy matrix of the form* (16) *is positive definite if and only if the numbers x_j, x_η are positive and mutually distinct*

$$0 < x_1 < x_2 < \ldots < x_n, \quad 0 < x_n < x_{n-1} < \ldots < x_1.$$

Corollary 1. *Let us consider the Cauchy matrix of the form* (16)*, of the stable base system* (4), (5)*, or for the system* (7), (8)*. Let us assume that the systems are stable and the eigenvalues of their dynamics matrices are real and mutually distinct. Then a symmetric Cauchy matrix of the form* (16) *is positive definite if and only if the conditions are satisfied*

$$0 < -s_1 < -s_2 < \ldots < -s_n, \quad 0 < -s_n < -s_{n-1} < \cdots < -s_1.$$

In this case, the role of positive numbers $x_j, x_{\eta,}$ is played by the real parts of the eigenvalues of the dynamics matrix $-s_j, -s_\eta$, which may turn out to be complex-valued. This requires representing gramians as stable Hermitian matrices

$$[P_c]_{Herm} = [P_o]_{Herm}, \quad p_{j\eta Herm} = \frac{1}{2}\left(\frac{-1}{s_j + s_\eta^*} + \frac{-1}{s_j^* + s_{\eta,}}\right) = Re\left(\frac{-1}{s_j + s_\eta^*}\right).$$

In [1, Lemma 9.4] it is proved that the Cauchy matrix (15) of a stable diagonalized base system is positive definite. Note that for basic systems their controllability and observability gramians coincide

$$P_c = P_o.$$

In this case, we obtain an important property of the basic system

$$\sigma_k = (-2Re(s_k))^{-1}, k = \overline{1,n},$$

where σ_k are Hankel singular values of gramian matrices. In this case, we obtain a simple formula for the square of the H_2-norm of the transfer function of the base system

$$||W(s)||_2^2 = trP_C = \sum_{k=1}^{n} \sigma_k = \sum_{k=1}^{n} \frac{1}{-2Re(s_k)}.$$

An important role in the study of spectral decompositions of solutions of equations of the form (2), (3), including the gramians of the basic systems, is played by multiple eigenvalues of the dynamics matrices, which is associated with the consideration of equations of state in Jordan form.

The objectives of this study are as follows. The first goal of the paper is to obtain spectral decompositions of the gramian matrices of the underlying continuous dynamical systems from the multiple spectra of their dynamics matrices. Another objective of the paper is to develop recurrent algorithms for calculating the controllability gramian and its inverse gramian for MIMO LTI continuous systems. The third objective is to analyze the energy metrics of stability of linear stationary systems using the algebraic Routh–Hurwitz stability criterion.

3. MAIN RESULTS

Let us consider the problem of calculating spectral decompositions of MIMO LTI and SISO LTI systems using solutions of differential and algebraic Lyapunov equations in the frequency domain. As is well known, if a linear system is stable, its transfer function is strictly proper, but all poles are distinct, then the following formulas are valid for calculating the square of the H_2-norm $W(s)$[1]

$$\|W(s)\|_2^2 = \sum_{k=1}^{n} r_k W(-s_k), r_k = Res\left[N^{-1}(s), s_k\right]. \tag{17}$$

In the case of the base system (4), (5) energy stability metric is separable across the poles of the transfer function, with each spectral component being a positive number. The energy metrics of stability J_1, J_2 have the form [19]

$$J_1 = \sum_{k=1}^{n} \frac{r_k}{\dot{N}(s_k) N(-s_k)}, \tag{18}$$

$$J_2 = \sum_{k=1}^{n}\sum_{\rho=1}^{n} \frac{-1}{s_k + s_\rho} \frac{1}{\dot{N}(s_k)\dot{N}(s_\rho)}. \tag{19}$$

(18) can be represented in an equivalent form

$$J_1 = \sum_{i=1}^{n}\sum_{k=1}^{n} \frac{r_k}{\prod_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod_{\lambda=1,}^{n}(-s_k - s_\lambda)}.$$

Since the matrix of a stable diagonalized base system is positive definite, (19) can be represented as the trace of the Cauchy matrix

$$J_2 = tr\sum_{k=1}^{n}\sum_{\rho=1}^{n} Re\left(\frac{-1}{s_k + s_\rho^*}\right) e_k e_\rho^T .$$

The spectral decomposition of the transfer function TF by multiple poles of the basic SISO LTI system in general has the form [24]

$$W(s) = \sum_{k=1}^{n}\sum_{\nu=1}^{m_k} \frac{L_{k\nu}}{(s - s_k)^{m_k - \nu + 1}}, \tag{20}$$

where s_k is the pole of the transfer function, $L_{k\nu}$ is the residue of the transfer function of order "ν" at the pole s_k

$$L_{k\nu} = \frac{1}{(\nu - 1)!}\left[\frac{d^{\nu-1}}{ds^{\nu-1}}\left(\frac{(s - s_k)^\nu}{N(s)}\right)\right]_{s=s_k}. \tag{21}$$

For basic systems, calculating the residue of the transfer function TF over multiple poles is greatly simplified. Let the strictly eigentransfer function TF of the general form of a SISO LTI system have the form

$$W(s) = \frac{1}{N(s)} = \frac{1}{\prod_{i=1}^{n}(s - s_i)^{m_i}}.$$

Let us introduce the notation

$$N_{k1}(s) = (s - 1)^{m_1} \times \ldots \times (s - s_{k-1})^{m_{k-1}} \times (s - s_{k+1})^{m_{k+1}} \times \ldots (s - s_n)^{m_n},$$

$$N_{k1}(s_k) = (s_k - s_1)^{m_1} \times \ldots \times (s_k - s_{k-1})^{m_{k-1}} \times (s_k - s_{k+1})^{m_{k+1}} \times \ldots (s_k - s_n)^{m_n}.$$

Then the residue $L_{k\nu}$ of the transfer function $W(s)$ can be calculated using the following recurrent algorithm.

The first step of the algorithm $\nu = 1$

$$L_{k1}(s_k,\ s_\mu) = N_k^{-1}(s_k,\ s_\mu) \quad \forall k, \mu = \overline{1, n}.$$

The second step of the algorithm $\nu = 2$

$$L_{k2}(s_k,\ s_\mu) = -N_k^{-1}(s_k,\ s_\mu)\left(\frac{m_1}{s_k - s_1} + \ldots + \frac{m_{k+1}}{s_k - s_{k+1}} + \ldots + \frac{m_n}{(s_k - s_n)}\right), \quad \forall k, \mu = \overline{1, n},$$

or

$$L_{k2}(s_k,\ s_\mu) = -L_{k1}(s_k,\ s_\mu)\sum_{\mu=1,\mu\neq k}^{m} \frac{m_\mu}{(s_k - s_\mu)}, \quad \forall k, \mu = \overline{1, n},$$

$$L_{k3} = \frac{1}{2!}\left[L_{k2} - \frac{1}{N_{k1}(s)}\sum_{\mu=1,\mu\neq k}^{n} \frac{m_\mu}{(s - s_\mu)^2}\right]\Bigg|_{s=s_k}, \quad \forall k = \overline{1, n},\ \forall \nu = \overline{1, m_k}.$$

The limiting relations follow from the residue formulas

$$\lim_{s_k \to s_i} |L_{k\nu}| = \infty.$$

Theorem 2. *Spectral decompositions of energy metrics of stability of basic systems by multiple poles of the TF.*

Consider a basic general-purpose SISO LTI system with multiple poles and a transfer function of the form (18)*. We assume that the system is fully controllable, strictly eigenvalue-free, and stable.*

Then the following statements are true:

1) *The spectral decomposition of the energy metric of stability of the base system by multiple poles of the TF has the form*

$$J_1 = J_2 = \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} \frac{(-1)^{m_k-\nu}}{(m_k-\nu)!} \left\{ L_{k\nu} \left[\frac{d^{m_k-\nu}}{ds^{m_k-\nu}} \left(\frac{1}{N(-s)} \right) \right]_{s=s_k} \right\}_{Herm}, \tag{22}$$

$$L_{k\nu} = \frac{1}{(\nu-1)!} \left[\frac{d^{\nu-1}}{ds^{\nu-1}} \left(\frac{(s-s_k)^{\nu}}{N(s)} \right) \right]_{s=s_k}. \tag{23}$$

2) *The spectral decomposition of the energy metric of the base system by combinational multiple poles of the TF has the form*

$$J_2 = \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} \sum_{\rho=1}^{n} \sum_{\mu=1}^{m_\rho} \frac{(m_k-\nu+m_\rho-\mu+1)!}{(m_k-\nu)!} \left[L_{k\nu} L_{\rho\mu} \frac{1}{(-s_k-s_\rho^*)^{m_k-\nu+m_\rho-\mu+1}} \right]_{Herm}, \tag{24}$$

where

$$L_{\rho\mu} = \frac{1}{(\mu-1)!} \left[\frac{d^{\mu-1}}{ds^{\mu-1}} \left(\frac{(s-s_\rho)^{\mu}}{N(s)} \right) \right]_{s=s_\rho}. \tag{25}$$

The energy metrics of the base system (4), (5) *or the system* (7), (8) J_1, J_2 *are positive numbers. They are invariants under various similarity transformations.*

Proof. Let us consider the following expansions of the transfer functions of the original system (1) for the case of its multiple poles and its antistable subsystem

$$W(-s) = \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} \frac{L_{k\nu}}{(-s-s_k)^{m_k-\nu+1}}, \tag{26}$$

where $L_{k\nu}$ is residue of order "ν" at the pole s_k of the TF

$$L_{k\nu} = \frac{1}{(\nu-1)!} \left[\frac{d^{\nu-1}}{ds^{\nu-1}} \left(\frac{(-s-s_k)^{\nu}}{N(-s)} \right) \right]_{s=s_k}.$$

Let us take advantage of the fact that in this case the transfer function, like its inverse Laplace transform, are scalar functions

$$J = \left\| N^{-1}(s) \right\|_2^2 = \int_0^{\infty} L^{-1} \left[N^{-1}(s) \right] L^{-1} \left[N^{-1}(-s^*) \right] dt. \tag{27}$$

We calculate the integral (27) as the limit at $T \to \infty$ of another integral

$$J = \lim_{T \to \infty} \int_0^T L^{-1} \left[N^{-1}(s) \right] L^{-1} \left[N^{-1}(-s) \right] dt.$$

Let us introduce the antiderivative function

$$H(t) = L^{-1} \left[N^{-1}(s) \right] L^{-1} \left[N^{-1}(-s) \right],$$

where

$$\left[N^{-1}(s)\right] = \sum_{k=1}^{n}\sum_{\nu=1}^{m_k} L_{k\nu}\frac{1}{(s-s_k)^{m_k-\nu+1}},$$

$$L_{k\nu} = \frac{1}{(\nu-1)!}\left[\frac{d^{\nu-1}}{ds^{\nu-1}}\left(\frac{(s-s_k)^{\nu}}{N(s)}\right)\right]_{s=s_k},$$

$$L^{-1}\left[N^{-1}(s)\right] = \sum_{k=1}^{n}\sum_{\nu=1}^{m_k}\frac{L_{k\nu}}{(m_k-\nu)!}t^{m_k-\nu}e^{s_k t},$$

$$\left[N^{-1}(-s)\right] = \sum_{\rho=1}^{n}\sum_{\mu=1}^{\rho} L_{\rho\mu}\frac{1}{(-s-s_k)^{m_\rho-\mu+1}},$$

$$L_{\rho\mu} = \frac{1}{(\mu-1)!}\left[\frac{d^{\mu-1}}{ds^{\mu-1}}\left(\frac{(-s-s_\rho)^{\mu}}{N(-s)}\right)\right]_{s=s_\rho},$$

$$L^{-1}\left[N^{-1}(-s)\right] = \sum_{\rho=1}^{n}\sum_{\mu=1}^{m_\rho}\frac{L_{\rho\mu}}{(m_\rho-\mu)!}t^{m_\rho-\mu}e^{-s_\rho^* t}.$$

The first statement of the theorem is related to the decomposition of the square of the H_2-norm of $N^{-1}(s)$, when in the primitive function only the function is subject to spectral decomposition $L^{-1}\left[N^{-1}(s)\right]$. In this case, the theorem on the Laplace transform of the product of complex functions of time, the image of which is a fractional rational fraction, takes the form

$$L\left[L^{-1}\left[N^{-1}(s)\right]\times N^{-1}(-s)\right] = L\left[H(t)\right]$$
$$= H(s) = \sum_{k=1}^{n}\sum_{\nu=1}^{m_k}\frac{(-1)^{m_k-\nu}L_{k\nu}}{(m_k-\nu)!}\left[\frac{d^{m_k-\nu}}{ds^{m_k-\nu}}\left(\frac{1}{N(-s)}\right)\right]_{s=s-s_k}.$$

Thus

$$J_1 = \left\|N^{-1}(s)\right\|_2^2 = -H(0) = J_2$$
$$= \sum_{k=1}^{n}\sum_{\nu=1}^{m_k}\frac{(-1)^{m_k-\nu}}{(m_k-\nu)!}\left\{L_{k\nu}\left[\frac{d^{m_k-\nu}}{ds^{m_k-\nu}}\left(\frac{1}{N(-s)}\right)\right]_{s=s_k}\right\}_{Herm}.$$

This proves the first statement of the theorem. If we add to this scheme the spectral decomposition of the second function $L^{-1}\left[N^{-1}(s)\right]$ and we use the theorem on the Laplace transform of the product of complex functions of time, the image of which is a fractional rational fraction, then we obtain

$$L\left[L^{-1}\left[N^{-1}(s)\right]\times N^{-1}(-s)\right] = L\left[H(t)\right]$$
$$= \sum_{k=1}^{n}\sum_{\nu=1}^{m_k}\frac{(-1)^{m_k-\nu}}{(m_k-\nu)!}L_{k\nu}\left\{\frac{d^{m_k-\nu}}{ds^{m_k-\nu}}\left[\sum_{\rho=1}^{n}\sum_{\mu=1}^{m_\rho}L_{\rho\mu}\frac{1}{(-s-s_\rho^*)^{m_\rho-\mu+1}}\right]_{s=s_k}\right\}$$
$$= \sum_{k=1}^{n}\sum_{\nu=1}^{m_k}\sum_{\rho=1}^{n}\sum_{\mu=1}^{m_\rho}\frac{(m_k-\nu+m_\rho-\mu+1)!}{(m_k-\nu)!}\times\left[L_{k\nu}L_{\rho\mu}\frac{1}{(-s_k-s_\rho^*)^{m_k-\nu+m_\rho-\mu+1}}\right]_{Herm}.$$

Since $H(t)$ is a primitive function, then from (20), taking into account the stability of the system, it follows

$$J_2 = \left\| N^{-1}(s) \right\|_2^2 = H(t)]_0^\infty = -H(0)$$
$$= \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} \sum_{\rho=1}^{n} \sum_{\mu=1}^{m_\rho} \frac{(m_k - \nu + m_\rho - \mu + 1)!}{(m_k - \nu)!} \left[L_{k\nu} L_{\rho\mu} \frac{1}{(-s_k - s_\rho^*)^{m_k - \nu + m_\rho - \mu + 1}} \right]_{Herm}.$$

This proves the second assertion of the theorem. Note that the dynamics matrices of the underlying system were not used in the proof of the theorem. It follows that its statements are valid both for the base system (4), (5) and for the system (7), (8). Moreover, the energy metrics J_1, J_2 are equal to each other, they are positive numbers, since the H_2-norm of the transfer function is a positive number. They are invariants under various similarity transformations, since they are functions of the eigenvalues of the system dynamics matrix.

Let us further consider the differential and algebraic Lyapunov equations of the form

$$\frac{dP(t)}{dt} = A_d P(t) + P(t) A_d^T + b_d b_d^T, \quad P(0) = 0_{nxn}, \quad t \in [0, T], \tag{28}$$
$$A_d P(t) + P(t) A_d^T = -b_d b_d^T. \tag{29}$$

Corollary 2. *Let the conditions of Theorem 2 be satisfied, but all poles of the TF be distinct. Then the following statement is true:*

$$J_1(s_1, \ldots, s_n) = \sum_{k=1}^{n} \frac{1}{\dot{N}(s_k)} \frac{1}{(-s_k)^n + \ldots + a_1(-s_k) + a_0} > 0,$$
$$J_2(s_1, \ldots, s_n) = \sum_{k=1}^{n} \sum_{\rho=1}^{n} J_{2,k\rho}(s_1, \ldots, s_n) > 0, \tag{30}$$
$$J_{2,k\rho}(s_1, \ldots, s_n) = 2Re \frac{r_k r_\rho}{(-s_k - s_\rho)} = 2\alpha_{k\rho} Re \frac{(r_k r_\rho)}{\alpha_{k\rho}^2 + \beta_{k\rho}^2},$$
$$\alpha_{k\rho} = -Res_k - Res_\rho, \beta_{k\rho} = -\Im s_k - \Im s_\rho.$$

Proof. Let us consider the basic system (6), (7) and formulas (11), (12). Since for the basic system the input and output vectors have the form (7), (8), we obtain

$$J_2(s_1, \ldots, s_n) = c^F P_c^F (c^F)^T = y_{n-1} = \int_0^\infty k^2(\tau)\, d\tau,$$

where $k(\tau)$ is the impulse response of the system. This is where the formulas for the statements follow.

Theorem 3. *Suppose that the SISO LTI system* (1) *is fully controllable, there exists a non-degenerate coordinate transformation $x_d = Tx$ such that*

$$A_d = TAT^{-1}.$$

Let us assume that all eigenvalues of the matrix A are different, do not belong to the imaginary axis, and the conditions are satisfied

$$s_i + s_j \neq 0, \qquad \forall i, j = \overline{1, n}.$$

Then the following statements are true:

1) *The solution of the equation* (28) *in the complex domain exists, is unique*

$$L\left[P\left(t\right)\right]=\frac{1}{s}\left\{\sum_{i,j=1}^{n}\frac{-1}{\left(s_i+s_j^*\right)}\left[Res\left(Is-A_d\right)^{-1},s_i\right]b_{di}b_{dj}\left[Res\left(Is-A_d^*\right)^{-1},s_j^*\right]\right\}_{Herm}, \quad (31)$$

where b_{di} *is the "i" element of the vector* b_d, b_{dj} *is the "j" element of the vector* b_d.

2) *The solution of the equation* (28) *in the real domain is unique and has the form*

$$P\left(0,T\right)=\int_0^T\left\{\sum_{i,j=1}^{n}\frac{-1}{\left(s_i+s_j^*\right)}b_d b_d^T e^{(s_i+s_j^*)\tau}e_j e_\eta^T d\tau\right\}_{Herm}. \quad (32)$$

3) *The solution of the algebraic equation* (29) *in the real domain exists, is unique, and has the form*

$$P\left(0,\infty\right)=\left\{\sum_{i,j=1}^{n}\frac{-b_{di}b_{dj}}{\left(s_i+s_j^*\right)}e_j e_\eta^T\right\}_{Herm}. \quad (33)$$

4) *The matrix* $P^{-1}\left(0,\infty\right)$ *under the conditions of the theorem exists and is unique*

$$P^{-1}\left(0,\infty\right)=\frac{-P_0}{p_0}, \quad (34)$$

where P_0 *is the free term of the Faddeev matrix polynomial,* p_0 *is the free term of the characteristic polynomial of the matrix* $P(0,T)$*, determined using the Faddeev–Leverrier recurrent algorithm* [22, 23].

Proof. Applying the Laplace transform to matrix functions of time has its own peculiarities. Let's consider diagonal matrices of size $n\times n$

$$F_1\left(t\right)=diag\left[..\ \ \dots\ \ f_{1,\nu}(t)\ \ \dots\ \ \dots\right], F_2\left(t\right)=diag\left[\dots\ \ \dots\ \ f_{2,\eta}(t)\ \ \dots\ \ \dots\right],$$

where the functions $f_{1,\nu}\left(t\right), f_{2,\eta}(t)$ are Laplace transformable, and their images $f_{1,\nu}\left(s\right), f_{2,\eta}\left(s\right)$ are rational fractions that have only n poles and no zeros

$$f_{1,i}\left(s\right)=\frac{1}{\left(s-s_{1,i}\right)},\quad f_{2,j}\left(s\right)=\frac{1}{\left(s-s_{2,j}\right)}.$$

Let us show that for each element "ij" of the Laplace image of the product of matrices $F_1\left(t\right)b_{di}b_{dj}F_2\left(t\right)$ the following formulas are valid

$$L\left[e_i F_1\left(t\right)b_{di}b_{dj}F_2\left(t\right)e_j^T\right]=\sum_{i,j=1}^{n}1_{ii}b_{di}b_{dj}\frac{1}{\left(s-s_{1,i}-s_{2,j}\right)}1_{jj},\quad 1_{ii}=e_i e_i^T, \quad (35)$$

$$f_{1,i}\left(t\right)=e^{s_i t},\quad f_{2,j}\left(t\right)=e^{s_j^* t}.$$

The identities for the residues of resolvents are also valid

$$\left[Res\left(Is-A_d\right)^{-1},s_i\right]=1_{ii},\quad \left[Res\left(Is-A_d^*\right)^{-1},s_j^*\right]=1_{jj}.$$

Using the theorem on the Laplace transform of a product of real functions of time whose image is a fractional rational algebraic function, we obtain (35). As is known, the solution to the Lyapunov differential equation with zero initial conditions is the Lyapunov integral of the form

$$P\left(t\right)=\int_0^T e^{At}b_d b_d^T e^{A^*t}dt.$$

Applying (35), we obtain

$$P(0,T)=\int_0^T\left\{\sum_{i,j=1}^{n}\frac{-1}{\left(s_i+s_j^*\right)}b_{di}b_{dj}e^{\left(s_i+s_j^*\right)\tau}e_je_\eta^Td\tau\right\}_{Herm}. \tag{36}$$

As shown in [13]

$$P(t)=\sum_{i,j=1}^{n}P_{ij}(t)\,,P_{ij}(t)=\left\{\frac{e^{(s_i+s_j^*)t}}{s_i+s_j^*}1_{ii}b_{di}b_{dj}1_{jj}\right\}_{Herm}.$$

The formula in statement 3) is proved by passing to the limit in (36). The existence and uniqueness of a solution to equation (29) in the time domain has been proven in many works, for example [3]. Let us prove the validity of this statement in the complex domain. Since the Lyapunov integral is a solution of the equation (29), it is sufficient to prove this statement for its Laplace transform. Note that the matrix $e^{At}b_db_d^Te^{A^*t}$ is a smooth, non-zero function of time. For any "ij" element, the inequality

$$\lim_{t\to\infty,\varepsilon\leftrightarrow 0}\int_0^T e_j\left|e^{At}b_db_d^Te^{A^*t}\right|e_\eta^Tdt<\infty,\quad \forall t\in[0,T)\,,\quad \forall i,j=\overline{1,n}.$$

It follows that for all time functions $e_j\left|e^{At}b_db_d^Te^{A^*t}\right|e_\eta^T$ satisfying the conditions of the theorem, there exists an abscissa of absolute convergence σ, which proves the existence and uniqueness of the direct Laplace transform for the solution (32). On the other hand, if there exists a direct Laplace transform of the solution, then there also exists an inverse transform

$$\frac{1}{2\pi j}\int_{c-j\infty}^{c+j\infty}e_\nu\left|e^{At}b_db_d^Te^{A^*t}\right|e_\mu^Te^{ts}ds,\quad c>\sigma,\quad \forall\nu,\mu=\overline{1,n}.$$

Moreover, the inverse transformation is unique in the Lebesgue sense. The validity of assertion 4) follows from the complete controllability of the system [4]. This implies the existence and uniqueness of the inverse matrix $P^{-1}(0,T)$ for any T. We write the expansion of the resolvent of the matrix $P(0,T)$ in a Faddeev–Leverrier series

$$[Is-P(0,T)]^{-1}=\frac{P_{n-1}s^{n-1}+\ldots+P_1s+P_0}{p_ns^n+p_{n-1}s^{n-1}+\ldots+p_1s+p_0}. \tag{37}$$

Putting in (37) $s=0$, we obtain the formula

$$P^{-1}(0,\infty)=\frac{-P_0}{p_0}, \tag{38}$$

where P_0 is the free term of the matrix Faddeev polynomial, p_0 is the free term of the characteristic polynomial of the matrix $P(0,T)$. They are determined using the Faddeev–Leverrier recurrent algorithm [22, 23]

$$\begin{gathered}P_{n-1}s^{n-1}=Ip_ns^{n-1},p_n=1,\\ P_{n-2}s^{n-2}=\left(Ip_{n-1}+p_{n-2}P(0,T)\right)s^{n-2},\\ \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\\ P_0=\left(Ip_1+p_2P(0,T)+\ldots+p_nP(0,T)^{n-1}\right).\end{gathered} \tag{39}$$

The coefficients p_i are determined by the formulas

$$p_{n-i}=\frac{-1}{i}tr\left(P(0,T)P_{n-i}\right).$$

Remark 2. In [13] it was proved that the results of the theorem can be extended to the case of canonical Jordan forms of the equations of state (1) for the case when the characteristic equation of the dynamics matrix of the SISO LTI system (1) can be represented in the form

$$N(s) = \prod_{k=1}^{n} (s - s_k)^{m_k}, \sum_{k=1}^{n} m_k = q.$$

The solution of the equation (3) in the real domain for multiple eigenvalues s_k of the matrix A_d with multiplicity m_k has the form

$$P(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{j\eta}(t),$$

$$P_{j\eta}(t) = \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} h_{j\eta}(t)\, b_d b_d^T e_j e_\eta^T,$$

$$h_{j\eta}(t) = L^{-1}\left\{ s^{-1} \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} L_{k,\nu j} \frac{(-1)^{m_k-\nu}}{(m_k-\nu)!} \left[\frac{d^{m_k-\nu}}{ds^{m_k-\nu}} \left(\frac{s^\eta}{N(-s)} \right) \right]_{s=s-s_k} \right\},$$

$$L_{k,\nu j} = \frac{1}{(\nu-1)!} \left[\frac{d^{\nu-1}}{ds^{\nu-1}} \left(\frac{(s-s_k)^{m_k} s^j}{N(s)} \right) \right]_{s=k}.$$

Let us consider a simple basic SISO LTI system with dimension $n = 3$, represented by the equations of state in diagonal canonical form of the form

$$\dot{x}_d = A_d x_d + b_d u,$$

$$A_d = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & -3 \end{bmatrix}, b_d = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix},$$

$$N(s) = a_3 s^3 + a_2 s^2 + a_1 s + a_0, a_3 = 1, a_2 = 6, a_1 = 11, a_0 = 6,$$

$$s_1 = -1, s_2 = -2, s_3 = -3.$$

Let's calculate the solution to the algebraic Lyapunov equation. Since in this example we are solving not a differential equation, but an algebraic Lyapunov equation for a stable system, (38) becomes the

$$P(\infty) = [p_{ij}(\infty)], \quad p_{ij}(\infty) = \frac{-b_{di} b_{dj}}{s_i + s_j}. \tag{40}$$

Substituting the parameters of the example into the formula, we obtain

$$P(\infty) = \begin{bmatrix} 0.5 & 0.666 & 0.75 \\ 0.666 & 1 & 1.2 \\ 0.75 & 1.2 & 1.5 \end{bmatrix}.$$

Putting $s = 0$ in (37), we obtain

$$P^{-1} = \frac{-P_0}{p_0}. \tag{41}$$

The first step of the recurrent algorithm. We set $P_2 = I_2, p_3 = 1$,

$$k = 0.$$

Step two. Calculate P_1, p_2

$$k = 1,$$

$$P_1 = p_2 I_3 + PP_2 = \begin{bmatrix} -2.5 & 0.666 & 0.75 \\ 0.666 & -2 & 1.2 \\ 0.75 & 1.2 & -1.5 \end{bmatrix},$$

$$p_2 = \frac{-1}{1} tr\,(PP_2) = -3.$$

Step three. Calculate P_0, p_1

$$k = 2,$$

$$P_0 = p_2 I_3 + PP_2 = \begin{bmatrix} 0.06 & -0.099 & 0.0492 \\ -0.099 & 0.1875 & -0.1005 \\ 0.0492 & -0.1005 & 0.5844 \end{bmatrix},$$

$$p_1 = \frac{-1}{2} tr\,(PP_1) = -0.30394.$$

Step four: Calculate p_0

$$k = 3,$$

$$p_0 = \frac{-1}{3} tr\,(PP_2) = -0.000966, p_0^{-1} = -1035.19.$$

We substitute the expressions obtained in the last and penultimate steps into (37)

$$P^{-1} = \begin{bmatrix} 62.114 & -102.4838 & 50.9313 \\ -102.4838 & -194.0981 & -104.36 \\ 50.9313 & -104.36 & 58.426 \end{bmatrix}.$$

It is not difficult to verify that the resulting matrix is the inverse matrix of the controllability gramian.

Theorem 4. *Let's consider the MIMO LTI system* (1) *in the canonical form of controllability and write the Lyapunov differential equation*

$$\frac{dP(t)}{dt} = A_c^F P(t) + P(t) \left(A_c^F\right)^T + B^F \left(B^F\right)^T, \quad P(0) = 0_{nxn}, \quad t \in [0, T]. \tag{42}$$

Let us assume that the system (1) *is completely controllable, there exists a non-degenerate coordinate transformation* $x_c = Tx$, *such that*

$$A_c^F = TAT^{-1}.$$

Let us assume that all eigenvalues of the matrix A_c^F *are different, do not belong to the imaginary axis, and the conditions* $s_i + s_j \neq 0, \forall i, j = \overline{1, n}$ *are satisfied.*

Then the following statements are true:

1) *The solution of the equation* (42) *in the complex domain exists, is unique, and has the form*

$$[P(s)] = \frac{1}{s} \left\{ \sum_{j=1}^{n} \left[Res\left(Is - A_c^F\right)^{-1}, s_j \right] B^F \left(B^F\right)^T \left\{ \left[\left[-Is - (A_c^F)^T \right]^{-1} \right] \right\} \right\}_{Herm}, \tag{43}$$

where

$$A_{cj}^F = a_{j+1} I + a_{j+2} A_c^F + \ldots + a_n \left(A_c^F\right)^{n-j-1}, \quad j = \overline{0, n-1},$$

$$A_{c\eta}^F = a_{\eta+1} I + a_{\eta+2} A_c^F + \ldots + a_n \left(A_c^F\right)^{n-\eta-1}, \ \eta = \overline{0, n-1}.$$

2) *The solution of the equation* (42) *in the real domain is unique and has the form*

$$P_{j\eta}(t) = \frac{\sum_{k=1}^{n} s_k^j(-s_k)^\eta}{\prod_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod_{\lambda=1}^{n}(-s_\rho - s_\lambda)}(e^{s_k t}-1)A_{cj}^F B^F \left(B^F\right)^T \left(A_{c\eta}^F\right)^T \tag{44}$$
$$= \Omega_{c,j\eta}^F(t) \circ \Psi_{c,j\eta}^F,$$
$$\Omega_{c,j\eta}^F(t) = \frac{\sum_{k=1}^{n} s_k^j(-s_k)^\eta}{\prod_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod_{\lambda=1}^{n}(-s_\rho - s_\lambda)}(e^{s_k t}-1)e_j e_\eta^T,$$
$$\Psi_{c,j\eta}^F = A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T.$$

3) *The infinite controllability gramian has the form*

$$P(0,\infty) = \left\{\sum_{j=0}^{n-1}\sum_{\eta=0}^{n-1}\frac{\sum_{k=1}^{n} s_k^j(-s_k)^\eta}{\prod_{\lambda=1,\lambda\neq k}^{n}(s_k - s_\lambda)\prod_{\lambda=1}^{n}(-s_\rho - s_\lambda)}e_j e_\eta^T\right\}_{Herm} \tag{45}$$
$$\times A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T.$$

4) *The matrix* $P^{-1}(0,\infty)$ *under the conditions of the theorem exists and is unique*

$$P^{-1}(0,\infty) = \frac{-P_0^F}{p_0}, \tag{46}$$

in which P_0^F *is the free term of the matrix Faddeev polynomial for the equations of state in the canonical form of controllability,* p_0 *is the free term of the characteristic polynomial of the gramian, determined using the Faddeev–Leverrier recurrent algorithm* [22, 23].

5) *The results of the theorem can be extended to the case of canonical Jordan forms of the equations of state* (42) *for the case when the characteristic equation of the dynamics matrix has "k" different roots of multiplicity* m_k*. The solution of the equation* (42) *in the real domain for multiple eigenvalues* s_k *of the matrix* A_c^F *with multiplicity* m_k *has the form*

$$P(t) = \sum_{i=1}^{n}\sum_{j=1}^{n} P_{j\eta}(t),$$
$$P_{j\eta}(t) = h_{j\eta}(t)\, A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T,$$
$$h_{j\eta}(t) = L^{-1}\left\{s^{-1}\sum_{k=1}^{n}\sum_{j=1}^{m_k} L_{k,j}\frac{(-1)^{m_k-\nu}}{(m_k-\nu)!}\left[\frac{d^{m_k-\nu}}{ds^{m_k-\nu}}\left(\frac{s^\eta}{N(-s)}\right)\right]_{s=s-s_k}\right\}, \tag{47}$$
$$L_{k,j} = \frac{1}{(j-1)!}\left[\frac{d^{j-1}}{ds^{j-1}}\left(\frac{(s-s_k)^{m_k}s^j}{N(s)}\right)\right]_{s=k}. \tag{48}$$

Proof. The general solution method is based on the spectral decomposition of the solution of the Lyapunov integral. The proof of the existence and uniqueness of solutions repeats the analogous proof of Theorem 3. The validity of Assertions 2–4 and (39) are proved in exactly the same way. The formulas themselves have a different form, determined by the dynamics matrix in controllability form. The general solution for the case of different roots of the characteristic equation in the complex domain has the form [13]

$$L[P(t)] = \frac{1}{s}\left\{\sum_{j=1}^{n}\left[Res\left(Is - A_c^F\right)^{-1}, s_j\right] B^F \left(B^F\right)^T \left\{\left[\left[-Is - (A_c^F)^T\right]^{-1}\right]\right\}\right\}_{Herm}.$$

Let us introduce the matrix antiderivative function $H_1(t)$ and the scalar matrix function $H(t)$ of the form

$$H_1(t) = H(t) \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T,$$

$$H(t) = \sum_{k=1}^{n} \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{s_k^j (-s_k)^\eta}{\dot{N}(s_k) N(-s_k)} 1_{j+1\eta+1},$$

$$A_{cj}^F = a_{j+1} I + a_{j+2} A_c^F + \ldots + a_n \left(A_c^F\right)^{n-j-1}, j = \overline{0, n-1},$$

$$A_{c\eta}^F = a_{\eta+1} I + a_{\eta+2} A_c^F + \ldots + a_n \left(A_c^F\right)^{n-\eta-1}, \eta = \overline{0, n-1}.$$

According to (14), the matrix $H(t)$ is a Xiao matrix. It follows that the matrix $P_{j\eta}(t)$ can be represented as a Hadamard product [13]

$$P_{j\eta}(t) = \frac{\sum_{k=1}^{n} s_k^j (-s_k)^\eta}{\prod_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod_{\lambda=1}^{n} (-s_\rho - s_\lambda)} (e^{s_k t} - 1) A_{cj}^F B^F \left(B^F\right)^T \left(A_{c\eta}^F\right)^T$$
$$= \Omega_{c,j\eta}^F(t) \circ \Psi_{c,j\eta}^F,$$

$$\Omega_{c,j\eta}^F(t) = \frac{\sum_{k=1}^{n} s_k^j (-s_k)^\eta}{\prod_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod_{\lambda=1}^{n} (-s_\rho - s_\lambda)} (e^{s_k t} - 1) e_j e_\eta^T, \Psi_{c,j\eta}^F = A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T.$$

From these formulas it follows that the infinite controllability gramian for the case of different roots of the characteristic equation in the real domain has the form

$$P(0,\infty) = \left\{ \sum_{j=0}^{n-1} \sum_{\eta=0}^{n-1} \frac{\sum_{k=1}^{n} s_k^j (-s_k)^\eta}{\prod_{\lambda=1,\lambda\neq k}^{n} (s_k - s_\lambda) \prod_{\lambda=1}^{n} (-s_\rho - s_\lambda)} e_j e_\eta^T \right\}_{Herm}$$
$$\times A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T.$$

This proves the validity of statements 1–3. The validity of statement 4 is proved in the same way as Theorem 3. To construct an infinite gramian of controllability, the recurrent Faddeev–Leverrier algorithm is used

$$P^{-1}(0,\infty) = \frac{-P_0}{p_0},$$

in which P_0 is the free term of the Faddeev matrix polynomial, p_0 is the free term of the characteristic polynomial of the controllability gramian matrix

$$P(t) = \sum_{i=1}^{n} \sum_{j=1}^{n} P_{j\eta}(t),$$

$$P_{j\eta}(t) = h_{j\eta}(t) A_{cj}^F B^F \left(B^F\right)^T (A_{c\eta}^F)^T,$$

$$h_{j\eta}(t) = L^{-1} \left\{ s^{-1} \sum_{k=1}^{n} \sum_{j=1}^{m_k} L_{k,j} \frac{(-1)^{m_k-\nu}}{(m_k-\nu)!} \left[\frac{d^{m_k-\nu}}{ds^{m_k-\nu}} \left(\frac{s^\eta}{N(-s)} \right) \right]_{s=s-s_k} \right\},$$

$$L_{k,j} = \frac{1}{(j-1)!} \left[\frac{d^{j-1}}{ds^{j-1}} \left(\frac{(s-s_k)^{m_k} s^j}{N(s)} \right) \right]_{s=k}.$$

Corollary 3. *Let us generalize the results of Theorem 4 to continuous linear SISO LTI systems. Assume that all the conditions of Theorem 4 are satisfied with MIMO LTI systems replaced by SISO LTI systems. Then the following spectral decomposition of the gramian of controllability of a SISO LTI system in the real domain holds*

$$P\left(t\right)=\sum_{k=1}^{n}\sum_{j=1}^{n}\sum_{\eta=1}^{n}\frac{s_k^{j-1}(-s_k)^{\eta-1}}{\prod_{\lambda=1,\lambda\neq k}^{n}\left(s_k-s_\lambda\right)\prod_{\lambda=1}^{n}\left(-s_\rho-s_\lambda\right)}(e^{s_kt}-1)e_je_\eta^T.$$

Proof. For a system in canonical controllability form, the decomposition of the resolvents of the dynamics matrices A^F, A^{F*} has a simple form, therefore the formulas are valid

$$(Is-A^F)^{-1}b^F=\begin{bmatrix}0\\ \vdots\\ 0\\ N(s)^{-1}\end{bmatrix}+\begin{bmatrix}0\\ \vdots\\ N(s)^{-1}\\ 0\end{bmatrix}+\ldots+\begin{bmatrix}N(s)^{-1}\\ \vdots\\ 0\\ 0\end{bmatrix},$$

$$b^{FT}(-Is-A^{F*})^{-1}=\begin{bmatrix}0\\ \vdots\\ 0\\ N(-s)^{-1}\end{bmatrix}^T+\begin{bmatrix}0\\ \vdots\\ N(-s)^{-1}\\ 0\end{bmatrix}^T+\ldots+\begin{bmatrix}N\left(-s\right)^{-1}\\ \vdots\\ 0\\ 0\end{bmatrix}^T.$$

Substituting them into (43), using the theorem on the Laplace transform of the product of complex functions of time, the image of which is a fractional rational fraction, and the theorem on displacement in the complex domain, we prove Corollary 3 [13].

Corollary 4 [13, Theorem 1]. *Assume that the conditions of Theorem 4 are satisfied for a SISO LTI system. Furthermore, assume that the system is asymptotically stable. Then, the spectral decomposition of the inverse gramian of the LTI system's controllability in the complex domain is a solution to the systems of equations*

$$\vec{X}=\left(I_n\otimes P_c^F\right)^{-1}\vec{I_n},\tag{49}$$

$$P_c^F=\sum_{k=1}^{n}\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1}\frac{s_k^j\left(-s_k\right)^\eta}{\dot{N}\left(s_k\right)N\left(-s_k\right)}\mathbf{1}_{j+1\eta+1}.\tag{50}$$

Proof. From the conditions of the theorem it follows that the solution of the system (49)–(50) exists and is unique. In [13] the following algorithm for calculating the matrices of inverse gramians in Xiao form was proposed. Let X be the matrix of the inverse gramian of controllability, x_i be the "ith" column of the matrix X. Then the inverse gramian is defined by the formula

$$P_c^Fx_i=e_i,\qquad i=\overline{1,n},$$

where e_i-x_i is the "ith" column of the matrix I_n. These equations can be rewritten as solutions to Sylvester's algebraic equation

$$\left(I_n\otimes P_c^F\right)\vec{X}=\vec{I_n},\quad X=\left(P_c^F\right)^{-1}.$$

Let's rewrite (14) as

$$P_c^F=\sum_{k=1}^{n}P_{ck}^F,\quad P_{ck}^F=\sum_{\eta=0}^{n-1}\sum_{j=0}^{n-1}\frac{s_k^j\left(-s_k\right)^\eta}{\dot{N}\left(s_k\right)N\left(-s_k\right)}\mathbf{1}_{j+1\eta+1},$$

where P_{ck}^F is the controllability subgramian matrix corresponding to the pole s_k

$$\vec{X} = \left(I_n \otimes P_c^F\right)^{-1} \vec{I_n}.$$

Finally, we obtain the desired spectral decomposition of the inverse gramian of controllability over the spectrum of the dynamics matrix in canonical controllability form in the form (49), (50). The inverse gramian has the structure of the zero plaid of the Xiao matrix ([5, 6]). The columns of the matrix P_c^F will have the same structure. Insignificant elements of the zero plaid are determined by the formula

$$x_{ij} = 0, \quad \forall i, j = i + j \neq 2k, \quad k = \overline{1, n}.$$

On the other hand, the final inverse gramian is defined as a solution of the Sylvester differential equation

$$X = -\int_0^t e^{P_c^F t} dt, \quad \text{or} \quad \frac{dX}{dt} = e^{P_c^F t},$$

which can be solved in the complex domain

$$\mathcal{L}\left(\frac{dX}{dt}\right) = (Is - P_c^F)^{-1}.$$

The solution of this equation using spectral decompositions of the gramian resolvent can be obtained based on the application of the Faddeev–Leverrier algorithm.

Remark 3. An analysis of the formulas of Theorems 3 and 4 shows that they are much more complicated for calculating gramians than the formulas of the energy metrics of Theorem 2 J_1, J_2. However, the latter are invariant under any similarity transformations, including those that transform the MIMO LTI system (1) into a new system whose state equations correspond to one of three canonical forms: Jordan, modal, and controllability.

4. ENERGY METRICS OF STABILITY AND THE ROUTH-HURWITZ STABILITY CRITERION

A necessary and sufficient condition for the asymptotic stability of the basic system of the second type has the form of boundedness of the squares of the norms of the TF of the system [1]

$$\left\|N^{-1}(s)\right\|_2^2 < \infty, \quad \left\|N^{-1}(s)\right\|_\infty^2 < \infty, \tag{51}$$

or in accordance with Theorem 2 of the boundedness of the squares of the norms of the PF of the base system for the case of multiple roots

$$J_1 < \infty, \quad J_2 < \infty, \quad \max_{s_i} J_1 < \infty, \quad \max_{s_i} J_2 < \infty. \tag{52}$$

We define the energy metric of stability of the basic system in the form $J_1(s_1, \ldots, s_n)$, $J_2(s_1, \ldots, s_n)$. Spectral decompositions of the energy metrics $J_1(s_1, \ldots, s_n)$, $J_2(s_1, \ldots, s_n)$ by simple roots of the characteristic equation have the form (3.2), (3.3). The mathematical correctness of these metrics follows from the fact that the metrics are equal to the squares of the H_2 or H_∞ norms of the transfer function of the base system, which are invariants under similarity transformations

$$\left\|N^{-1}(s)\right\|_{inf}^2 = \max_{s_k} \left\|N^{-1}(s)\right\|_2^2, \forall k = 1, \ldots, n.$$

Spectral decompositions of the energy metric $J_2(s_1, \ldots, s_n)$ over multiple roots of the characteristic equation have the form

$$J_2(s_1, \ldots, s_n) =$$
$$= \sum_{k=1}^{n} \sum_{\nu=1}^{m_k} \sum_{\rho=1}^{n} \sum_{\mu=1}^{m_\rho} \frac{(m_k - \nu + m_\rho - \mu + 1)!}{(m_k - \nu)!} \times \left[L_{k\nu} L_{\rho\mu} \frac{1}{(-s_k - s_\rho^*)^{m_k - \nu + m_\rho - \mu + 1}} \right]_{Herm}.$$

Note that the metrics of the base systems are independent of the Faddeev matrices in the decomposition of the controllability gramians. These formulas imply that as the system approaches the stability boundary, caused by the roots of the characteristic equation approaching the imaginary axis, the energy stability metrics tend to infinity. Let us define an acceptable value for the energy stability metric as a sufficiently large positive number N_{perm}

$$J_1(s_1, \ldots, s_n) = N_{1perm}, J_2(s_1, \ldots, s_n) = N_{2perm}. \tag{53}$$

We will consider any basic system to be conditionally unstable if all roots of its characteristic equation are in the left half-plane, but the energy stability metric exceeds the established acceptable value.

Weakly stable and resonant roots of the characteristic equation that determine the system's proximity to the stability boundary. Analysis of the formulas in Theorem 2 shows that two groups of roots are weakly stable. The first group consists of roots close to the imaginary axis. The second group consists of complex root pairs that are close to each other in norm (a resonant root group). As the weakly stable eigenvalues of the first group approach each other, the corresponding terms become positive definite, and their values become arbitrarily large positive numbers. Similar limiting relations hold for a resonant root group. In the latter case, the degree of oscillatory instability is much higher, since the absolute values of the residues $L_{kv}, L_{\rho\mu}$ are inversely proportional to the difference between the roots s_k, s_ρ. The energy assessment of the weakly stable components of the expansions of the energy metrics of stability allows us to determine and evaluate the energy reserve of stability in the form

$$R_{1stab} = 20lg \frac{N_{perm}}{J_1(s_1, \ldots, s_n)}, \tag{54}$$

$$R_{2stab} = 20lg \frac{N_{perm}}{J_2(s_1, \ldots, s_n)}. \tag{55}$$

The introduced stability metrics (52), (53) do not contradict the classical stability criterion based on the proximity of the nearest root of the characteristic equation to the imaginary axis, but supplement it by taking into account the energy accumulated in the system's states. The same applies to the algebraic stability criterion of Routh-Hurwitz linear stationary systems [5, 6]. It can be expected that the combined application of the Routh–Hurwitz criterion and spectral decompositions of gramians in the form of energy stability metrics (54), (55) for stability analysis will create a synergistic effect, enhancing the advantages of each approach and mitigating their disadvantages. The hybrid criterion includes the formation of Routh tables and Xiao matrices, the calculation of the spectrum of the dynamics matrix, the calculation of the spectral decompositions of the energy metrics $J_1(s_1, \ldots, s_n), J_2(s_1, \ldots, s_n)$, the formation of the stability criterion (54), (55), and the analysis of stability for various scenarios of the system's operation. It is obvious that the obtained criteria are valid not only for the base system, but also for a continuous stationary linear system with many inputs and many outputs, the characteristic polynomial of which coincides with the characteristic polynomial of the base system. In [9], a method for calculating the gramian of the base system in canonical controllability form is proposed, based on the use of the Routh table and without the need to calculate the spectrum of the dynamics matrix. The diagonal elements of the Xiao matrices are related to the elements of the Routh table by the relations

$$y_{j\eta}=\begin{cases} 0, \text{ if } j+\eta=2k+1, \quad k=1,\ldots,n; \\ y_n=\frac{1}{2Y_{n,1}}, \\ y_{n-l}=\frac{-\sum_{i=1}^{m-1}(-1)^i Y_{n-l,i+1}y_{n-l+i}}{Y_{n-l,1}}, \text{if } j+\eta=2k, \quad k=1,\ldots,n, l=\overline{1,n-1}, \end{cases} \tag{56}$$

where $R_{j\eta}$ are "$j\eta$" elements of the Routh table, m is the number of elements in the j row of the Routh table. Let us assume that, along with the Routh table, all eigenvalues of the dynamics matrix s_k have been pre-calculated and they are different. According to (13), (14) the controllability gramian of the base system has the form of Xiao matrices, the diagonal elements of which have the form

$$y_i=\sum_{k=1}^{n}\frac{1}{\prod_{\lambda=1,\lambda\neq k}^{n}(s_k-s_\lambda)\prod_{\lambda=1,}^{n}(-s_k-s_\lambda)}(-1), \ i=\overline{1,n}. \tag{57}$$

Expressions (57) show that the energy stability metric depends on the distribution density of the real parts of the eigenvalues s_k, s_λ on the real axis: the closer s_λ to s_k, the greater the value of the energy metric [14, 16]. This is the group effect of nonlinear mode interaction, which intensifies as the absolute values of the real parts of the roots of the characteristic equation and their combinations tend to zero. The advantage of the (56) algorithm is that it eliminates the need to calculate the eigenvalues of the dynamics matrix. This is also its disadvantage, since the coefficients of the characteristic equation in most cases have no physical meaning, as they are not associated with specific physical devices. The advantage of the (57) algorithm is that the diagonal elements of the gramian matrices express the energy of the measured physical variable y_i of the underlying system. At the same time, the values of the metric are inversely proportional to the products of the absolute values of the differences of all eigenvalues that are close to each other in norm. In this sense, the metric (57) estimates the energy of the combined interaction in a group of weakly stable oscillatory modes. The Routh–Hurwitz criterion does not have this property. Furthermore, the variables y_i depend on the eigenvalues s_k, each of which is associated with a specific physical device and has its label. These eigenvalues can be used to calculate the squares of the H_2- and H_∞-norms of transfer functions in problems of optimal synthesis of robust control systems based on energy efficiency criteria. For closed control systems, formulas (51) are used, in which $N^{-1}(s)$ is the characteristic polynomial of the closed system. Energy metrics also allow us to evaluate the the degree of achievability, the volume of attraction ellipsoids, the average minimum energy, and the centrality indices of dynamic networks [7, 10–12 , 15]. Joint use of algorithms (56), (57) in the considered hybrid Routh–Hurwitz criterion provides a more in-depth analysis of the aperiodic and oscillatory stability of the system, taking into account the multiplicity of the system's eigenvalues.

5. CONCLUSION

The method of spectral and singular gramian decompositions is an extension of frequency methods for the analysis and synthesis of control systems. The main tools for constructing decompositions are Laplace transformation theorems for spectral and singular decompositions of solutions of Lyapunov and Sylvester differential equations, decomposition of the resolvent of the dynamics matrix and the gramian matrix into a series of Faddeev–Leverier series, and recursive algorithms for calculating Faddeev matrices. These decompositions for canonical forms of controllability and observability have an explicit physical interpretation in terms of energy measures of quadratic forms formed by the state variables of the basic system. The paper shows that the direct and inverse gramians of controllability can be computed using a single general effective recursive procedure. The article proposes a new method for calculating Laplace images of finite gramians for simple and multiple eigenvalues based on numerically stable recursive algorithms for decomposing the image of the matrix resolvent into a Faddeev–Leverrier series [22, 23]. The application of transformations of system equations into canonical forms provides the following advantages when calculating spectral decompositions of direct and inverse gramians:

1) Gramian matrices are calculated as solutions of Lyapunov differential equations in the frequency and time domains, which allows simultaneous calculation of solutions to Lyapunov algebraic equations, both finite and infinite gramians, both direct and inverse gramians for continuous dynamic systems specified by state equations in canonical forms: Jordan, modal, controllability, observability.

2) Gramians in canonical forms of controllability and observability in the form of Xiao matrices are invariant under similarity transformations. To obtain them, it is sufficient to calculate only n diagonal elements of the gramian matrices instead of n^2 elements.

3) The energy metrics of stability of the basic systems J_1 and J_2 allow us to calculate the degree of stability of the system not only for different roots, but also for multiple roots of the characteristic equation. The presence of multiple roots significantly increases the threat of loss of stability.

4) Spectral decompositions of the gramians of controllability of a stable basic system in Cauchy form depend on the real parts of the simple and multiple eigenvalues of the dynamics matrix, on their distribution density on the real axis [25].

5) Recursive algorithms for spectral decomposition of gramians contain only matrix addition and multiplication operations and do not contain inversion and exponentiation operations. They contain only addition, multiplication, and exponentiation operations for scalar functions, and do not contain division operations for functions equal to zero. The subtractions of the transfer functions of the base systems, which are used in the calculation of energy criteria, can also be calculated using recursive algorithms. All this is a consequence of the decomposition of the resolvent matrices of the dynamics of the initial stable and dual unstable systems into the Faddeev–Leverrier series and the application of classical frequency methods of analysis and synthesis of dynamic systems, which makes this approach a new effective tool for calculating energy criteria of stability.

A certain disadvantage of the new methods and algorithms developed in this work is the problem of their computational implementation for high-dimensional systems. However, it is precisely the use of recursive versions of the developed algorithms for calculating gramians and energy metrics that makes their application effective for high-dimensional dynamic systems [26]. The article presents research results that may be useful in the design of digital twins, methods for optimizing energy costs in control systems, analysis of energy metrics of stability and reachability, condition monitoring based on the analysis of energy balance anomalies, the development of energy metrics for systems defined by state equations for dynamic networks in the form of graphs, and in the vibration analysis of technical objects [13,28].

FUNDING

This work was supported by the Russian Foundation for Basic Research, project no. 25-29-20158.

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This paper was recommended for publication by A.A. Galyaev, a member of the Editorial Board