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CONTROL IN SOCIAL ECONOMIC SYSTEMS

Aggregate Indicators of Oligopoly Game Dynamics

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Abstract—This paper addresses the problem of designing aggregate indicators of reflexive collective decision-making dynamics in oligopoly models. Competitive markets with an arbitrary number of Cournot-reflexive agents are considered. The indicators proposed below are used to estimate the dynamics analytically, establish conditions for its convergence to an equilibrium, and identify the moments of dynamics reversal towards convergence. According to a comparative analysis provided, the numerical simulation results are consistent with the analytical estimation counterparts obtained using the aggregate indicators. The perfect consistency of the results is proved for stabilizing sequences of indicators.

Keywords: Cournot oligopoly, reflexive collective behavior, aggregate description, equilibrium, stabilizing sequences, aggregate indicators

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1. INTRODUCTION

This work is devoted to solving the game of agents in an oligopoly market. The state-of-the-art of this problem, as one of the most important in game theory, was reviewed in [1–3].

Under incomplete awareness and the interdependence of behavior, the dynamics of agent's decision-making in a competitive market are built on the basis of reflexive considerations regarding the best individual choice, taking the responses of competitors into account [4–16]. Below, we develop an approach to describing reflexive collective behavior and solving the problem of achieving equilibrium [17].

In contrast to conventional approaches, where the emphasis is on the actions of each agent in each period (time step) of decision-making, the convergence conditions within this approach are formulated for an aggregate description of agent behavior. Aggregation over agent actions and aggregation over time are used. The former aggregation is implemented for each period by summing the residuals of agent actions. And the latter aggregation is implemented over a set of sequential periods by summing the values of dynamics elements [17].

The problem under consideration is to design aggregate indicators that can be used to monitor overall market dynamics and estimate the movement of each agent's dynamics towards an equilibrium analytically. The Cournot oligopoly model is taken as a basic model, where reflexive agents follow a collective behavior procedure under incomplete awareness.

The remainder of this paper is organized as follows. Sections 2 and 3 present the theoretical background of the research under assumptions regarding the awareness of agents that are traditional for indicator-based collective behavior models, as well as a thorough justification of the indicators proposed. For several special cases, the numerical simulation results on the asymptotic convergence of reflexive dynamics are compared with the analytical estimation counterparts obtained using the new aggregate indicators. In Section 4, we consider the applicability of these indicators for the agents themselves. Additional assumptions regarding the awareness of agents are introduced, allowing them to estimate uniformly, via the indicators, the market dynamics independently and on par with the competitors.

2. STARTING POINTS FOR THE RESEARCH

This section presents the theoretical background (previous and new results) underlying the design of aggregate indicators for the convergence of agent dynamics.

The Cournot oligopoly model is a noncooperative game with n players (rational agents) competing in the outputs of a homogeneous product. The classical assumptions of this model (adopted here as well) include a linear inverse demand function $p\left(\sum_{j\in N} q_j\right) = a - b\sum_{j\in N} q_j$ and linear cost functions $\varphi_i(q_i) = c_i q_i + d_i$ of agents, with the following notation: q_i is the output of agent i ($i \in N = \{1,\dots,n\}$); c_i and d_i are the marginal and fixed costs of agent i, respectively; $p\left(\sum_{j\in N} q_j\right)$ is the uniform market price; the parameter a characterizes the maximum possible price of the product under which the demand will tend to zero; and the parameter b characterizes the slope of the demand curve. The payoff (goal) function of agent i depends on his strategy q_i and the set of strategies of all players and has the form [18, 19]

$$\Pi_i\left(q_i, \sum_{j\in N} q_j\right) = \left(a - b\sum_{j\in N} q_j\right) q_i - c_i q_i - d_i \to \max_{q_i}, \quad i \in N. \tag{1}$$

In the case of linear demand and cost functions, a unique solution $q^* = (q_1^*, \dots, q_i^*, \dots, q_n^*)$, $q_i^* > 0\ \forall i \in N$, of the oligopoly game, understood as a static Nash equilibrium in the normal form game [20], exists under the well-known conditions; for example, see [17].

Under game uncertainty (about the actions chosen by the competitive environment) and incomplete knowledge (of the costs, goal functions, etc., of competitors), the game solution is found as the outcome of the reflexive collective behavior dynamics [3, 4, 9, 17, 19, 21]:

$$q_i^{t+1} = q_i^t + \gamma_i^{t+1}(x_i^t - q_i^t), \quad i \in N. \tag{2}$$

Here:

t ($t = 0, 1, 2, \dots$) is the time instant (period) number;

$x_i^t = x_i\left(\sum_{j\in N\setminus\{i\}} q_j^t\right)$ is the current position of the target of agent i (the action ensuring the maximum value of this agent's goal function in period t); to determine it, the agent does not need to know the goal functions of the competitors since the actions of all agents observed by him in period t are supposed to be the same in the current period;

$\gamma_i^{t+1} \in [0, 1]$ is a parameter independently chosen by each agent in period $(t + 1)$ (his "step" from the current strategy to the current position of his target);

$q^0 = (q_1^0, \dots, q_i^0, \dots, q_n^0)$ is the vector of initial strategies of all agents. For the linear Cournot model [15, 17, 19], we have

$$x_i^t = \frac{(a - c_i)/b - \sum_{j\in N\setminus\{i\}} q_j^t}{2}. \tag{3}$$

With the change of variables

$$\varepsilon_i^t = q_i^* - q_i^t \quad (i \in N;\ t = 0, 1, 2, \dots), \tag{4}$$

the profit model (1) can be transformed to

$$\Pi_i^t = -\left(\sum_{j\in N} \varepsilon_j^t\right) \varepsilon_i^t \to \max_{\varepsilon_i^t} \quad (i \in N), \tag{5}$$

and the dynamics (2) to

$$\varepsilon_i^{t+1} = \varepsilon_i^t + \gamma_i^{t+1}\left(-\frac{\sum_{j\in N\setminus\{i\}} \varepsilon_j^t}{2} - \varepsilon_i^t\right). \tag{6}$$

In the general case, problem (5) in the static statement may have multiple solutions. For example, for $n = 2$, any vector $\varepsilon = (\varepsilon_1, \varepsilon_2)$ will formally be a solution if $\varepsilon_1 = -\varepsilon_2$, or, more generally, if $\sum_{j\in N} \varepsilon_j = 0$. However, possible nonzero solutions (e.g., $\varepsilon = (5, -5)$ in the case $n = 2$) are not Nash equilibria. Only equilibrium solutions are of interest.

The starting points for this research, explaining the transition to the model with residuals, are presented in the three propositions below.

Proposition 1. *In model* (5)*, there exists the unique static Nash equilibrium* $\varepsilon^* = (\varepsilon_1^*, \ldots, \varepsilon_i^*, \ldots, \varepsilon_n^*)$ *with* $\varepsilon_i^* = 0 \quad \forall i \in N$, *and the convergence of process* (6) *means that* $\varepsilon_i^t \to \varepsilon_i^* = 0$ *as* $t \to \infty$.

The proof of Proposition 1 is given in the Appendix for the first time.

The other two were proved in [17].

Proposition 2. *Process* (6) *converges if and only if process* (2), (3) *converges for model* (1).

Proposition 3. *If* $\sum_{j\in N} \varepsilon_j^t \to 0$ *as* $t \to \infty$, *then* $\varepsilon_i^t \to \varepsilon_i^* = 0 \ \forall i \in N$.

The following relations are also important for the subsequent analysis:

$$\sum_{j\in N} \varepsilon_j^{t+1} = \left(1 - 2\tilde{\gamma}^{t+1}\right) \sum_{j\in N} \varepsilon_j^t, \tag{7}$$

$$\left|\sum_{j\in N} \varepsilon_j^{t_0+\tau}\right| \leqslant (1 - 2\bar{\tilde{\gamma}}_+^\tau)^{t_+} \left|1 - 2\bar{\tilde{\gamma}}_-^\tau\right|^{t_-} \left|\sum_{j\in N} \varepsilon_j^{t_0}\right| \quad (\tau = t_+ + t_-). \tag{8}$$

(Here, the expression $\frac{\tilde{\gamma}^t(1+n)}{4}$, used in [17], is redenoted by $\tilde{\gamma}^t$.)

In (8) and the Appendix, the superscripts t_+ and t_- are the powers, whereas the superscripts $(t_0 + \tau)$ and t_0 are not.

In (8), we also employ the following designations: t_0 is an arbitrary period and $\tau > 0$; $t_+(t_-)$ is the number of periods in the set T_+ (T_-, respectively), $T_+ = \{t \in T | 1 - 2\tilde{\gamma}^t > 0\}$, $T_- = \{t \in T | 1 - 2\tilde{\gamma}^t < 0\}$, $T = \{t_0 + 1, \ldots, t_0 + \tau\}$, and $\bar{\tilde{\gamma}}_+^\tau = \frac{1}{t_+} \sum_{t\in T_+} \tilde{\gamma}^t$ and $\bar{\tilde{\gamma}}_-^\tau = \frac{1}{t_-} \sum_{t\in T_-} \tilde{\gamma}^t$ are the average values of the parameter

$$\tilde{\gamma}^t = \left(\sum_{j\in N} \gamma_j^t + \sum_{j\in N} \gamma_j^t \varepsilon_j^{t-1} \Big/ \sum_{j\in N} \varepsilon_j^{t-1}\right) \Big/ 4 \tag{9}$$

over τ periods and the sets T_+ and T_-, respectively.

In the special case where $\tilde{\gamma}^t = \frac{1}{2}$ (and therefore, $\sum_{j\in N} \varepsilon_j^t = 0$), the market dynamics improve in the sense that the residuals of each agent in period $(t + 1)$ are smaller, by absolute value, than his residuals in period t [17].

In dynamics series, the ratio of two sequential levels of the series is called the chain growth rate. In the case under consideration, for the series of aggregate residuals, this rate is expressed via the aggregate indicator $\tilde{\gamma}^t$ of agent actions as $\frac{\sum_{j\in N} \varepsilon_j^t}{\sum_{j\in N} \varepsilon_j^{t-1}} = 1 - 2\tilde{\gamma}^t$, see (7).

3. AGGREGATE INDICATORS IN NEW STUDIES OF DYNAMICS CONVERGENCE

First, let us dwell on an important special case of the sequence $\{\tilde{\gamma}^t\}$, as it has a quite natural explanation in terms of the steps γ_i^t chosen by each agent i and can be simply verified experimentally; for example, see [4, 15, 19]). For instance, an agent does not change his steps over time or repeats them in a known way.

Assume that the sequence $\{\tilde{\gamma}^t\}$ stabilizes to $\tilde{\gamma}^{t_s}$ starting from period t_s if t_s is the smallest natural number such that $\tilde{\gamma}^t = \tilde{\gamma}^{t_s}$ $\forall t \geqslant t_s$ [22].

Proposition 4. *Let the sequence* $\{\tilde{\gamma}^t\}$ *stabilize to* $\tilde{\gamma}^{t_s}$. *The dynamics* (6) *converge if and only if* $0 < \tilde{\gamma}^{t_s} < 1$.

The proof of this proposition is given in the Appendix.

Example 1. Consider $n = 4$ agents in a market and compile the vector of agents' steps $\gamma^t = (\gamma_1^t, \gamma_2^t, \gamma_3^t, \gamma_4^t)$. First, we analyze the case where $\gamma^t = (0.2; 1; 1; 1)$ in all periods, i.e., the agents do not change their steps. According to a numerical experiment, the sequence $\{\tilde{\gamma}^t\}$ stabilizes, within the third decimal place, to the level $\tilde{\gamma}^{t_s} = 1.039$ starting from period $t_s = 35$, and the dynamics (6) diverge. By Proposition 4, the dynamics (6) also diverge since, in this case, $\tilde{\gamma}^{t_s} > 1$. Now let $\gamma^1 = \gamma^5 = \gamma^9 = \ldots = (0.2; 1; 1; 1)$, $\gamma^2 = \gamma^6 = \gamma^{10} = \ldots = (1; 0.2; 1; 1)$, $\gamma^3 = \gamma^7 = \gamma^{11} = \ldots = (1; 1; 0.2; 1)$, and $\gamma^4 = \gamma^8 = \gamma^{12} = \ldots = (1; 1; 1; 0.2)$. In this case, the sets of parameters repeat every 3 periods. According to a numerical experiment, starting from period $t_s = 21$, the sequence $\{\tilde{\gamma}^t\}$ stabilizes to $\tilde{\gamma}^{t_s} = 0.955$, and the dynamics (6) converge. By Proposition 4, the dynamics (6) also converge since $\tilde{\gamma}^{t_s} < 1$. Note that in both cases described, three agents take full steps (equal to one) and one takes a partial step (equal to 0.2), the average values of the steps coincide (equal to 0.8), and the sequences $\{\tilde{\gamma}^t\}$ stabilize. However, the results turn out to be directly opposite: in the first case, the dynamics (6) diverge, converging in the second, which may be somewhat non-obvious and unexpected.

For stabilizing sequences, the numerical simulation results are always perfectly consistent with the analytical estimation counterparts obtained using $\tilde{\gamma}^t$; see Example 1 above. Also, this example illustrates situations where the aggregate description makes it possible to formulate the convergence conditions of decision-making dynamics quite simply and uniformly, which may be not the case with a detailed description.

Let us proceed to general cases of the dynamics (6), where the sequences $\{\tilde{\gamma}^t\}$ may be non-stabilizing.

We introduce the following aggregate indicator for process (6):

$$z_{t_0,t_1} = \sum_{t=t_0}^{t_1} z^t \quad (z_{t_0,t_0} = z^{t_0}), \tag{10}$$

where

$$z^t = \begin{cases} \tilde{\gamma}^t, & \tilde{\gamma}^t < \dfrac{1}{2}(t \in T_+), \\ 1 - \tilde{\gamma}^t, & \tilde{\gamma}^t > \dfrac{1}{2}(t \in T_-), \\ \dfrac{1}{2}, & \tilde{\gamma}^t = \dfrac{1}{2}. \end{cases} \tag{11}$$

The properties of $z_{t_0,t}$:

(1) For $t_0 < t_1 < t_2$, $z_{t_0,t_2} = z_{t_0,t_1} + z_{t_1+1,t_2}$.

(2) For $t^* \in M(\tau) = \left\{t^* | z_{1,t^*} = \min_{t=1,\ldots,\tau} z_{1,t}\right\}$, $z_{t^*+1,t} \geqslant 0 \quad \forall t = t^* + 1, \ldots, \tau$.

(3) For $t^* = \max_{t \in M(\tau)} t$, $z_{t^*+1,t} > 0 \quad \forall t = t^* + 1, \ldots, \tau$.

Proposition 5. *Let the parameters* $\tilde{\gamma}^t$, z^t, *and* $z_{1,t}$ *of the Cournot oligopoly be given by* (9), (10), *and* (11), *respectively. If* $M(\tau) = \left\{t^* | z_{1,t^*} = \min_{t=1,\ldots,\tau} z_{1,t}\right\} \neq \oslash$ *as* $\tau \to \infty$, *then process* (6) *converges to an equilibrium.*

The proof of this (main) result of the paper is provided in the Appendix.

By the proposition, if the minimum of $z_{1,t}$ with respect to t exists (i.e., $z_{1,t}$ does not go to minus infinity) and t^* is the point of minimum, then $z_{t^*+1,t} \geqslant 0$ $(\forall t \geqslant t^* + 1)$ and the dynamics become convergent.

In the context of Proposition 5, the formation of $z_{1,t}$ plays a key role. In this regard, we discuss the possible changes of its value in the (next) period $(t+1)$.

The case $\tilde{\gamma}^{t+1} < \frac{1}{2}$ corresponds to the fact that, due to (10), $z_{1,t+1} = z_{1,t} + \tilde{\gamma}^{t+1}$. If $0 < \tilde{\gamma}^{t+1}$, then $z_{1,t+1} > z_{1,t}$. The value of z increases, which favorably affects the convergence of the process. If $\tilde{\gamma}^{t+1} < 0$, then $z_{1,t+1} < z_{1,t}$. The value of z decreases, which may deteriorate the convergence of the process. If $\tilde{\gamma}^{t+1} = 0$, then $z_{1,t+1} = z_{1,t}$. The value of z remains the same.

The case $\frac{1}{2} < \tilde{\gamma}^{t+1}$ corresponds to the fact that, due to (10) and (11), $z_{1,t+1} = z_{1,t} + 1 - \tilde{\gamma}^{t+1}$. If $\tilde{\gamma}^{t+1} < 1$, then $z_{1,t+1} > z_{1,t}$. The value of z increases, which favorably affects the convergence of the process. If $1 < \tilde{\gamma}^{t+1}$, then $z_{1,t+1} < z_{1,t}$. The value of z decreases, which may deteriorate the convergence of the process. If $\tilde{\gamma}^{t+1} = 1$, then $z_{1,t+1} = z_{1,t}$. The value of z remains the same.

The case $\tilde{\gamma}^t = \frac{1}{2}$ favors the convergence of the process and $z_{1,t+1} = z_{1,t} + \frac{1}{2}$.

Proposition 5 and the above discussion give grounds for treating $z_{t_0,t}$ as an aggregate indicator for estimating the convergence of the dynamics (6). A positive current value of the indicator points to an improvement in the overall market dynamics relative to the reference period (in the sense of (8)), which (see Proposition 3) favors the convergence of each agent's dynamics. A negative current value of the aggregate indicator is insufficient for asserting a relative improvement in the dynamics. Generally speaking, a positive contribution to the indicator and to the convergence of the process is made by periods where $0 < \tilde{\gamma}^t < 1$.

In dynamics series, the base growth rate is used as an intensity indicator; in the case under consideration, for the series of aggregate residuals and the time interval $[t_0, t_0 + \tau]$, it has the form $\frac{\sum_{j \in N} \varepsilon_j^{t_0+\tau}}{\sum_{j \in N} \varepsilon_j^{t_0}}$. The indicator $z_{t_0,t_0+\tau}$, aggregated over the agents' actions and the interval, estimates the dynamics of this rate and points the periods of its reversal towards convergence to zero.

Example 2. A numerical calculation fragment. This example illustrates Proposition 5. As before, consider $n = 4$ agents. Figure 1 shows the dynamics of the indicator $\tilde{\gamma}^t$, which is calculated by (9) based on some initial conditions ε_i^0 and agents' steps γ_i^t $(i = \overline{1,4})$. On an interval of thirty periods, the dynamics of $\tilde{\gamma}^t$ do not stabilize. This figure also shows the corresponding dynamics of the indicator $z_{1,t}$, which is calculated by (10), (11). The influence of the dynamics of the indicator $z_{1,t}$ on the dynamics of the residuals ε_i^t $(i = \overline{1,4})$ can be traced in Fig. 2. Up to the twentieth period, no convergence trend of the residuals of all agents is observed since on the interval of thirty periods, the indicator $z_{1,t}$ achieves minimum only in the twentieth period (see Fig. 1). Then the indicator $z_{21,t}$, which can be calculated by the formula $z_{21,t} = z_{1,t} - z_{1,20}$ (see Property 1) above), has a positive sign for $t \geqslant 21$ and increases up to $t = 29$ inclusive. Therefore, the dynamics of agents' residuals move towards the theoretical static Cournot–Nash equilibrium $\varepsilon_i^* = 0$ $(i = \overline{1,4})$.

Let us return to Example 1 with stabilizing indicator sequences $\{\tilde{\gamma}^t\}$ to demonstrate their application. For the case of convergent dynamics of ε_i^t, in view of (11), the aggregate indicator starting from $t_s = 21$ will be calculated by the formula $z_{t_s,\tau} = \sum_{t=t_s}^{\tau} (1 - \tilde{\gamma}^t) = (\tau - t_s + 1)(1 - \tilde{\gamma}^{t_s}) = (\tau - t_s + 1)\, 0.445$. Its value will be bounded from below as $\tau \to \infty$; therefore, based on Proposi-

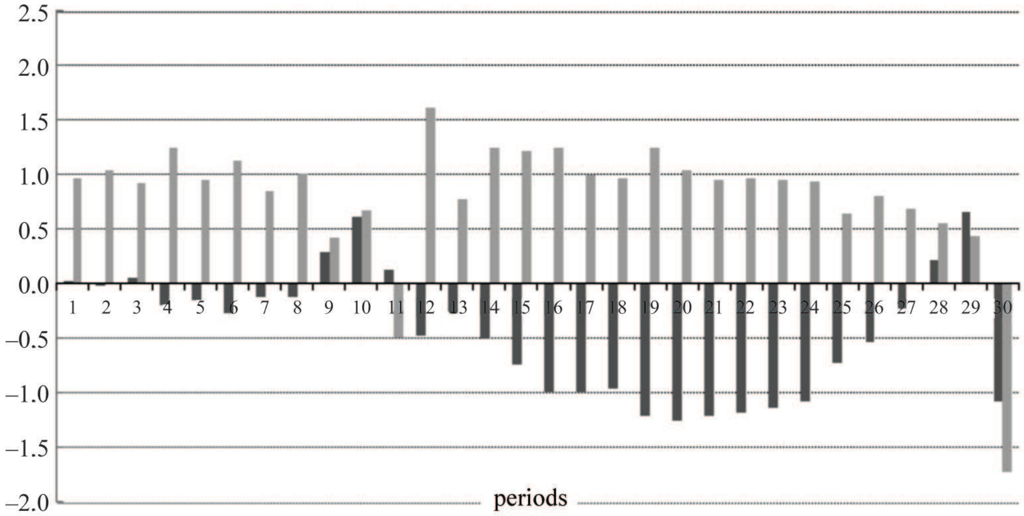

Fig. 1. The dynamics of the indicators $\tilde{\gamma}^t$ (light bars) and $z_{1,t}$ (dark bars).

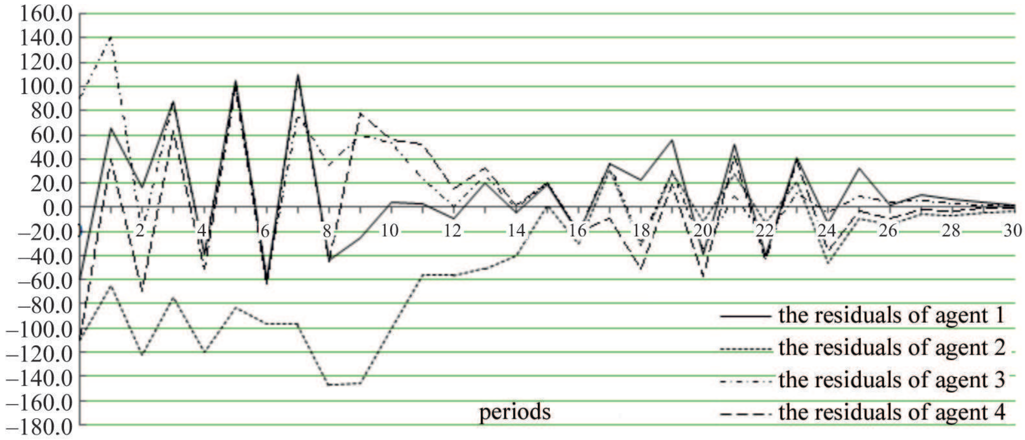

Fig. 2. The dynamics of the residuals of individual agent actions.

tion 5, the convergence of (6) according to this aggregate indicator is confirmed. For the case of divergent dynamics of ε_i^t, the aggregate indicator starting from $t_s = 35$ will be calculated by the same formula $z_{t_s,\tau} = (\tau - t_s + 1)(1 - \tilde{\gamma}^{t_s}) = (\tau - t_s + 1)(-0.039)$. Its value will not be bounded from below as $\tau \to \infty$. Hence, $M(\tau) = \varnothing$ as $\tau \to \infty$, and by Proposition 5, the convergence of (6) according to the z-indicator is not confirmed.

4. APPLICATION OF INDICATORS BY AGENTS

The above indicators may be of not only theoretical but also practical interest. Here is a possible scenario in which agents apply aggregate indicators in order to estimate their dynamics and the overall market dynamics as well.

Traditionally, in the collective behavior procedure (2), an agent decides on his current output without knowledge of the equilibrium output and without the ability to judge how his previous actions and the actions of other agents have contributed to achieving the equilibrium. Obviously, such an ability could be useful for an agent to make his current choice.

An agent will acquire this ability by using the aggregate indicators $\tilde{\gamma}^t$ and $z_{t_0,t}$.

If $0 < \tilde{\gamma}^t < 1$, the steps chosen in the previous period t by all agents have positively contributed to the convergence of the process. And if $z_{t_0,t} > 0$, the steps chosen on the time interval $t_0, t_0+1, \ldots, t$ have improved the overall market dynamics in period t relative to period t_0.

The main challenge regarding the applicability of the indicators for the agents themselves is that the collective behavior procedures (2) and (6) are described in different "coordinate frames": in one frame, the current outputs of agents are calculated by (2); in the other, the residuals of current outputs are calculated by (6), and the indicators are formed by (9)–(10).

To link these coordinate frames, we present the following considerations. With $Q^t = \sum_{j\in N} q_j^t$, from (4) and $\sum_{j\in N} \varepsilon_j^{t+1} = (1 - 2\tilde{\gamma}^{t+1}) \sum_{j\in N} \varepsilon_j^t$ it follows that $\sum_{j\in N} \varepsilon_j^t = \frac{Q^{t+1}-Q^t}{2\tilde{\gamma}^{t+1}}$. Then $\sum_{j\in N} \varepsilon_j^t - \sum_{j\in N} \varepsilon_j^{t-1} = \frac{Q^{t+1}-Q^t}{2\tilde{\gamma}^{t+1}} - \frac{Q^t-Q^{t-1}}{2\tilde{\gamma}^t} = -(Q^t - Q^{t-1})$, and we arrive at the recurrence relation

$$\tilde{\gamma}^{t+1} = \frac{Q^{t+1} - Q^t}{Q^t - Q^{t-1}} \cdot \frac{\tilde{\gamma}^t}{1 - 2\tilde{\gamma}^t}. \tag{12}$$

Let us adopt several assumptions so that each agent, while staying within the coordinate frame of process (2), will objectively estimate the market dynamics on par with other competitors.

(1) All agents are "intellectual" to the level of knowing formula (12) for calculating the indicator $\tilde{\gamma}^t$ recursively and formulas (10), (11) for calculating the indicator $z_{t_0,t}$.

(2) In a certain period t_0, all agents choose the same step γ^{t_0}. (Starting from t_0, the beliefs of all agents about the process will be synchronized. The agents choose it by agreement. According to (9), $\tilde{\gamma}^{t_0} = \gamma^{t_0}(1+n)/4$. The value $\tilde{\gamma}^{t_0}$ is supposed to be known to all agents.)

The first assumption is quite natural for an agent using indicators. It means that in period $(t+1)$, each agent will know the calculated indicators $\tilde{\gamma}^t$ and the aggregate outputs Q^t $(t = 1, 2, \ldots)$ of the previous periods. The indicators $\tilde{\gamma}^t$ are calculated by the agents themselves using the recurrence formula (12), and the agents know the aggregate outputs based on the conditions of the collective behavior procedure (2), (3). In turn, in period $(t+1)$, each agent will also know the calculated indicators $z_{t_0,t}$, which (see Property 1) above) are calculated by the agent using the recurrence formula $z_{t_0,t} = z_{t_0,t-1} + z^t$, with z^t given by (11).

The second assumption is intended to eliminate the influence of equilibrium outputs on the indicator $\tilde{\gamma}^t$ and process (6).

Under these assumptions, by observing the previous aggregate outputs Q^t according to model (2), each agent can obtain, independently of the others, the uniform estimates $\tilde{\gamma}^t$ and $z_{t_0,t}$ of the market dynamics synchronized with the estimates of his competitors starting from period t_0.

5. CONCLUSIONS

The practical and scientific significance, as well as the prospects for future studies of collective behavior dynamics, is determined by an analytical confirmation of the intuitive idea that their aggregate description over significant time intervals can be no less accurate than a detailed one. The application of aggregate indicators expands our understanding of the set of situations that, under incomplete awareness and the absence of common knowledge, can lead oligopoly models to reflexive equilibria (stable outcomes of agent interaction). It seems promising to design and apply aggregate indicators for analytically solving reflexive decision-making problems within other oligopoly and behavior models, different from those considered in this work, e.g., with the Stackelberg responses of agents, as well as taking into account the mental and behavioral components of agent activity [9].

APPENDIX

Proof of Assertion 1. Based on (5), the optimal strategies of agents can be determined by solving the system of n homogeneous linear equations $\partial\Pi_i/\partial\varepsilon_i = -\left(\sum_{j\in N}\varepsilon_j\right) - \varepsilon_i = 0$ $(i \in N)$ with n unknowns. The determinant of the coefficient matrix of the unknowns is nonzero. Therefore, the system has only the trivial solution $\varepsilon_i^* = 0$ $\forall i \in N$. This solution will be a Nash equilibrium. Suppose that only agent i has a finite deviation $\Delta\varepsilon_i \neq 0$ from the trivial solution. Due to (5), we have $\Pi_i(\varepsilon^*) = 0 > -(\Delta\varepsilon_i)^2$: by deviating, the agent reduces his profit. In other words, the trivial solution is an equilibrium.

The proof of Proposition 1 is complete.

Proof of Assertion 4. Let $0 < \tilde{\gamma}^{t_s} < 1$. According to (7), we have $\sum_{j\in N}\varepsilon_j^{t_0+\tau} = (1-2\tilde{\gamma}^{t_s})^\tau\sum_{j\in N}\varepsilon_j^{t_0}$. Hence, $\sum_{j\in N}\varepsilon_j^{t_0+\tau}\to 0$ as $\tau\to\infty$ and, by Proposition 3, $\varepsilon_i^{t_0+\tau}\to\varepsilon_i^* = 0$ $\forall i\in N$.

If $\tilde{\gamma}^{t_s} < 0$ or $1 < \tilde{\gamma}^{t_s}$, then $\left|\sum_{j\in N}\varepsilon_j^{t_0+\tau}\right| \to \infty$, and therefore the dynamics cannot converge.

The proof of Proposition 4 is complete.

Proof of Assertion 5. Let $t^* = \max_{t\in M(\tau)} t$ as $\tau\to\infty$. Then, by Property 3) for $z_{t^*+1,t}$, we obtain $z_{t^*+1,t^*+\tau} = \sum_{t\in T_+}\tilde{\gamma}^t + t_- - \sum_{t\in T_-}\tilde{\gamma}^t > 0$ with $\tau \geqslant 1$. In view of the the average value expressions $\bar{\tilde{\gamma}}_+^t = \frac{1}{t_+}\sum_{t\in T_+}\tilde{\gamma}^t$ and $\bar{\tilde{\gamma}}_-^t = \frac{1}{t_-}\sum_{t\in T_-}\tilde{\gamma}^t$, it follows that $t_+\bar{\tilde{\gamma}}_+^\tau - t_-(\bar{\tilde{\gamma}}_-^\tau - 1) > 0$. This inequality implies $\frac{t_+(1-2\bar{\tilde{\gamma}}_+^\tau)+t_-(2\bar{\tilde{\gamma}}_-^\tau-1)}{t_++t_-} < 1$.

Due to inequality (8), we have $\left|\sum_{j\in N}\varepsilon_j^{t^*+\tau}\right| \leqslant (1-2\bar{\tilde{\gamma}}_+^\tau)^{t_+}|1-2\bar{\tilde{\gamma}}_-^\tau|^{t_-}\left|\sum_{j\in N}\varepsilon_j^{t^*}\right|$ $(\tau = t_+ + t_-)$.

The expression $(1-2\bar{\tilde{\gamma}}_+^\tau)^{t_+}|1-2\bar{\tilde{\gamma}}_-^\tau|^{t_-}$ on the right-hand side of this inequality can be estimated by using a well-known result: for $x_1, x_2 \geqslant 0$ and $1 > \alpha > 0$, $\alpha x_1 + (1-\alpha)x_2 \geqslant x_1^\alpha x_2^\alpha$, and the strict equality holds only if $x_1 = x_2$.

Consequently,

$$(1-2\bar{\tilde{\gamma}}_+^\tau)^{t_+}|1-2\bar{\tilde{\gamma}}_-^\tau|^{t_-} = (1-2\bar{\tilde{\gamma}}_+^\tau)^{t_+}(2\bar{\tilde{\gamma}}_-^\tau-1)^{t_-} \leqslant \left[\frac{t_+(1-2\bar{\tilde{\gamma}}_+^\tau)+t_-(2\bar{\tilde{\gamma}}_-^\tau-1)}{t_++t_-}\right]^{t_++t_-}.$$

It has been shown that the expression in square brackets is below 1. Therefore, $\left[\frac{t_+(1-2\bar{\tilde{\gamma}}_+^\tau)+t_-(2\bar{\tilde{\gamma}}_-^\tau-1)}{t_++t_-}\right]^{t_++t_-} \to 0$ as $\tau\to\infty$, and based on Proposition 3, process (6) converges to an equilibrium.

The proof of Proposition 5 is complete.

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