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STOCHASTIC SYSTEMS

Invariance of Stationary Distributions of Exponential Networks with Prohibitions

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Abstract—We consider queuing networks with prohibitions on transitions between network nodes that determine the protocol of their operation. In the graph of transient network intensities, a set of base vertices is allocated (proportional to the number of edges) and the question is raised about deleting some subset of it so that the stationary distribution of the Markov process describing the functioning of the network is preserved. In order for this condition to be fulfilled, it is sufficient that the set of vertices of the graph of transient intensities, after removing a subset of the base vertices, coincide with the set of states of the Markov process and this graph is connected. It is proved that the ratio of the number of remaining base vertices to their total number n converges to $1/2$ for $n \to \infty$.

Keywords: queuing network, transient intensity graph, base vertex of the graph

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1. INTRODUCTION

The transition intensity graph of a queuing network determines its operating protocol, and removing its elements reduces the possible transitions between network states. Such a change in the transition intensity graph signifies a change in the queuing network operating protocol and is associated, for example, with solving problems in the design of transportation or computer networks (see [1–6]). One way to change the network operating protocol is to introduce blocking probabilities of transitions between network states [7–9]. However, this method does not imply the introduction of transition restrictions, which is a relevant applied problem and requires the connectivity of the transition intensity graph.

The authors have their own work in this area [10, 11], and this paper is a continuation of that work. These studies consider exponential networks with various operating protocols, including protocols with random variation, and construct a transition intensity graph during their simulation. The stationary distributions of the networks under consideration are calculated under conditions of connectivity of the transition intensity graph.

In this paper, we define conditions under which the removal of some edges of the transition intensity graph that determines the functioning of a queuing network does not change the stationary distribution in it. We are talking about an open Jackson network, including one with a total limited number of requests, and the closed Gordon–Newell network [12, 13]. The search for the required conditions is based on the representation of the transition intensity graph as a union of base graphs. Each base graph consists of a base vertex and is complete; the intersection of the set of edges of any two base graphs is empty. This representation (decomposition) enables to determine a set of base vertices and the graphs corresponding to them, whose removal preserves the total set of graph vertices and the connectivity of the transition intensity graph. Graph decomposition leads to decomposition in the multiplicative theorem.

The main result of the work is the constructive removal of a subset of basic vertices that preserves the stationary distribution of the Markov process describing the network's operation. Removing any vertex from the set of remaining basic vertices violates the condition for preserving this distribution, leading to an optimal solution in a certain sense. The set of vertices in the transition intensity graph after removing the subset of basic vertices coincides with the state set of the Markov process, and this graph is connected. It has been proven that the ratio of the number of remaining basic vertices to the total number of vertices n converges to $1/2$ as $n \to \infty$.

2. DECOMPOSITION IN AN OPEN NETWORK

Consider an open Jackson network G with a Poisson input flow of intensity λ, consisting of m single-channel queuing systems with exponential service times of intensity μ_i, $i = 1, \ldots, m$. The dynamics of a request's movement through the network is defined by the routing matrix $\Theta = ||\theta_{ij}||_{i,j=0}^m$, where θ_{ij} is the transition probability after service from the ith node to the jth node, $\theta_{00} = 0$, and the node number 0 is an external source. It is assumed that the routing matrix is indecomposable:

$$\forall i, j \in \{0, 1, \ldots, m\} \; \exists \, i_1, i_2, \ldots, i_r \in \{1, \ldots, m\} : \; \theta_{ii_1} > 0, \; \theta_{i_1 i_2} > 0, \; \ldots, \; \theta_{i_r j} > 0.$$

Then the vector $\Lambda = (\lambda, \lambda_1, \lambda_2, \ldots, \lambda_m)$ is the only solution to the system of balance relations

$$\Lambda = \Lambda\Theta. \tag{1}$$

The network functioning (the number of requests in the nodes) is described by a discrete Markov process $y(t)$ with a set of states $\mathbf{N} = \{\mathbf{n} = (n_1, \ldots, n_m) : \; n_1, \ldots, n_m \geq 0\}$ and transition intensities:

$$\begin{gathered} L(\mathbf{n}, \mathbf{n} + \mathbf{e}_k) = \lambda\theta_{0k}, \quad L(\mathbf{n} + \mathbf{e}_k, \mathbf{n}) = \mu_k\theta_{k0}, \\ L(\mathbf{n} + \mathbf{e}_k, \mathbf{n} + \mathbf{e}_i) = \mu_k\theta_{ki}, \quad 1 \leq k \neq i \leq m, \quad \mathbf{n} \in \mathbf{N}. \end{gathered} \tag{2}$$

Here the element $\mathbf{e}_k \in \mathbf{N}$, has the kth coordinate equal to 1, the rest are 0.

Consider a graph $\Gamma(\mathbf{n})$ with vertex set $V(\mathbf{n}) = \{\mathbf{n}, \; \mathbf{n} + \mathbf{e}_k, \; k = \overline{1, m}\}$ and set of edges connecting them $\{(\mathbf{n}, \mathbf{n} + \mathbf{e}_k), \; (\mathbf{n} + \mathbf{e}_i, \mathbf{n} + \mathbf{e}_j), \; i \neq j, \; 1 \leq k, i, j \leq m\}$, $\mathbf{n} \in \mathbf{N}$ (if $\mathbf{n}_1 \neq \mathbf{n}_2$, then $\Gamma(\mathbf{n}_1, \; \mathbf{n}_2)$ have no common edges). We call the vertex $\mathbf{n}$ defining the graph $\Gamma(\mathbf{n})$ a base vertex, the corresponding graph $\Gamma(\mathbf{n})$ a base graph, and the set of all such vertices $\mathbf{N}_0$ a base set. Note that $\mathbf{N}_0 = \mathbf{N}$ in the case of an open network G.

For some $\mathbf{N}_* \subseteq \mathbf{N}_0$ we set

$$\Gamma(\mathbf{N}_*) = \bigcup_{\mathbf{n} \in \mathbf{N}_*} \Gamma(\mathbf{n}), \quad V(\mathbf{N}_*) = \bigcup_{\mathbf{n} \in \mathbf{N}_*} V(\mathbf{n}).$$

Let us describe the network G_* with restrictions by associating it with the graph $\Gamma(\mathbf{N}_*)$. The absence of an edge $(\mathbf{n}, \mathbf{n} + \mathbf{e}_k)$ in the graph $\Gamma(\mathbf{N}_*)$ means that no requests arrive from outside the node k and do not leave after being serviced there. The absence of an edge $(\mathbf{n} + \mathbf{e}_k, \mathbf{n} + \mathbf{e}_i)$ in the graph $\Gamma(\mathbf{N}_*)$ means that after being serviced at node k, no requests arrive at node i, $k \neq i$. Such permissions and restrictions on transitions between network nodes define its operational protocol.

Theorem 1. *If $\rho_i = \frac{\lambda_i}{\mu_i} < 1$, $i = 1, \ldots, m$, $V(\mathbf{N}_*) = \mathbf{N}$, and the graph $\Gamma(\mathbf{N}_*)$ is connected, then the Markov process $y(t)$, describing the functioning of the network G_*, is ergodic, and its stationary distribution $\pi(\mathbf{n})$ where $\mathbf{n} \in \mathbf{N}$, is calculated by the formula*

$$\pi(\mathbf{n}) = C^{-1} \prod_{i=1}^{m} \rho_i^{n_i}, \quad C = \sum_{\mathbf{n} \in N} \prod_{i=1}^{m} \rho_i^{n_i}, \quad \mathbf{n} \in \mathbf{N}. \tag{3}$$

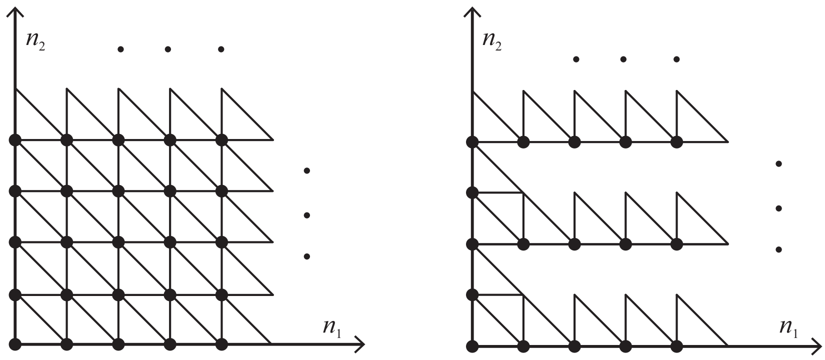

Fig. 1. Graph $\Gamma(\mathbf{N})$ (left) and graph $\Gamma(\mathbf{N}_*)$ (right), sets $\mathbf{N}_0$ (left) and $\mathbf{N}_*$ (right) are highlighted with bold dots.

Connectivity of an undirected graph $\Gamma(\mathbf{N}_*)$ means the existence of a path between any two of its vertices.

Remark 1. The stationary distribution (3) of the Markov process $y(t)$, describing the functioning of network G, remains unchanged for network G_*.

Remark 2. A similar theorem can be formulated and proven for the case of multi-channel nodes.

Remark 3. Theorem 1 can be extended to the case of a finite set of states $\mathbf{N}$ of the Markov process $y(t)$ without the constraints $\rho_i < 1$, $i = 1, \ldots, m$, where $\mathbf{N}_* \subseteq \mathbf{N}_0 \subseteq \mathbf{N}$, $\mathbf{N}_0$ is the set of base nodes that also satisfies the conditions $V(\mathbf{N}_0) = \mathbf{N}$, and the graph $\Gamma(\mathbf{N}_0)$ is connected.

The objective of this work is to construct a subset $\mathbf{N}_*$ of the set of base vertices $\mathbf{N}_0$ in such a way that, on the one hand, the stationary distribution of the Markov process is preserved, and on the other hand, the ratio of the number of vertices of set $\mathbf{N}_*$ to the number of vertices of set $\mathbf{N}_0$ tends to 1/2 with an increase in the number of requests in the network. To preserve the stationary distribution in an open network, it is sufficient to require that the equality $V(\mathbf{N}_*) = \mathbf{N}$ and the graph connectivity $\Gamma(\mathbf{N}_*)$ be satisfied.

Unlimited number of requests. Consider an open network whose operation is described by a discrete $y(t)$ with a set of states $\mathbf{N} = \{\mathbf{n} = (n_1, \ldots, n_m) :\ n_1, \ldots, n_m \geq 0\}$ and transition intensities (2). We construct a network with restrictions. We introduce a subset of the set of base vertices $\mathbf{N}_0 = \mathbf{N}$:

$$\mathbf{N}_* = \bigcup_{k=0}^{\infty} \Big(\mathbf{N}_*(2k) \bigcup (2k+1)\mathbf{e}_m\Big) \subset \mathbf{N}_0, \quad \mathbf{N}_*(k) = \{\mathbf{n} \in \mathbf{N} :\ n_1, \ldots, n_{m-1} \geq 0,\ n_m = k\}. \tag{4}$$

As an example, the constructed graphs $\Gamma(\mathbf{N})$ and $\Gamma(\mathbf{N}_*)$, for $m = 2$ are shown on Fig. 1. Bold dots indicate the set of base vertices $\mathbf{N}_0$, as well as its subset $\mathbf{N}_*$.

Theorem 2. *For the constructed open network with restrictions* (*the set of base vertices is determined by formula* (4)), *the equality* $V(\mathbf{N}_*) = \mathbf{N}$ *holds, and the graph* $\Gamma(\mathbf{N}_*)$ *is connected. Removing a vertex from the set* $\mathbf{N}_*$ *violates either the equality* $V(\mathbf{N}_*) = \mathbf{N}$, *or the connectivity of the graph* $\Gamma(\mathbf{N}_*)$.

We define the set $\Pi(2k) = \{\mathbf{n} \in \mathbf{N} :\ 0 \leq n_1, \ldots, n_m \leq 2k\}$ and denote by $|D|$ the number of elements of the set D.

Theorem 3. *For the constructed open network with restrictions* (*the set of base vertices is determined by formula* (4)) *the following relation is valid*

$$\frac{|\mathbf{N}_* \bigcap \Pi(2k)|}{|\mathbf{N}_0 \bigcap \Pi(2k)|} \to \frac{1}{2}, \quad k \to \infty, \tag{5}$$

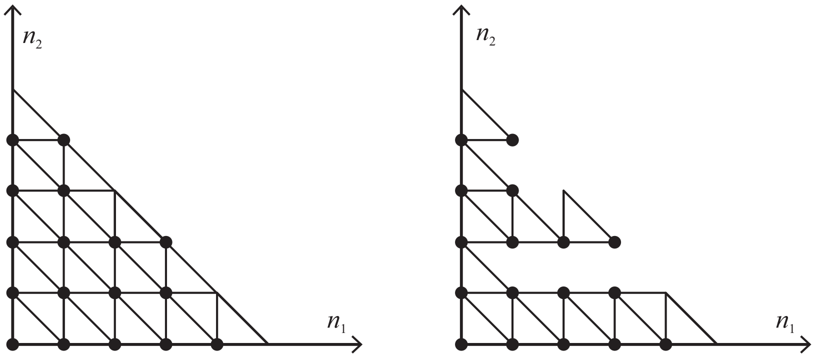

Fig. 2. Graph $\Gamma(\mathbf{N})$ (left) and graph $\Gamma(\mathbf{N}_*)$ (right), sets $\mathbf{N}_0$ (left) and $\mathbf{N}_*$ (right) are highlighted with bold dots.

where by

$$\frac{|\mathbf{N}_* \bigcap \Pi(2k)|}{|\mathbf{N}_0 \bigcap \Pi(2k)|} - \frac{1}{2} = \frac{1}{2(2k+1)} + \frac{k}{(2k+1)^m} \to 0, \quad k \to \infty. \tag{6}$$

The calculation results of $\dfrac{|\mathbf{N}_* \bigcap \Pi(2k)|}{|\mathbf{N}_0 \bigcap \Pi(2k)|} - \dfrac{1}{2}$ for different values of m and k are given in Table.

Estimate of the convergence rate

k	$m = 2$	$m = 3$
5	0.08677686	0.049211119
10	0.046485261	0.024889321
20	0.024092802	0.012485309
50	0.009851975	0.004999025

Limited total number of requests. Consider an open network whose operation is described by a discrete Markov process $y(t)$ with a finite set of states $\mathbf{N} = \{\mathbf{n} = (n_1, \ldots, n_m) : \ n_1, \ldots, n_m \geq 0, \sum_{k=1}^m n_k \leq 2K+1\}$ and transition intensities (2). We construct a network with restrictions by defining a set of base nodes $\mathbf{N}_0 = \{\mathbf{n} \in \mathbf{N} : \ \sum_{k=1}^m n_k \leq 2K\}$. The set $\mathbf{N}_0$ is maximal in the sense that adding a node $\mathbf{n} \notin \mathbf{N}_0$ to it leads to the relation $V(\mathbf{N}_0 \bigcup \mathbf{n}) \neq \mathbf{N}$. We introduce a subset of the set of base nodes

$$\mathbf{N}_* = \bigcup_{k=0}^{K} \mathbf{N}_*(2k) \bigcup \{(2k+1)\mathbf{e}_m, \ k = 0, \ldots, K-1\} \in \mathbf{N}_0, \tag{7}$$

$$\mathbf{N}_*(k) = \left\{ \mathbf{n} \in \mathbf{N} : \ \sum_{k=1}^{m} n_k \leq 2K - k, \ n_m = k \right\}.$$

As an example, the constructed graphs $\Gamma(\mathbf{N})$ and $\Gamma(\mathbf{N}_*)$ for $m = 2$ are shown on Fig. 2. The set of base vertices is highlighted with bold dots $\mathbf{N}_0$, and as well as its subset $\mathbf{N}_*$.

Theorem 4. *For the constructed open network with restrictions* (*the set of basic vertices is determined by formula* (7)), *the equality* $V(\mathbf{N}_*) = \mathbf{N}$ *holds and the graph* $\Gamma(\mathbf{N}_*)$ *is connected. Removing a vertex from the set* $\mathbf{N}_*$ *violates either the equality* $V(\mathbf{N}_*) = \mathbf{N}$ *or the connectivity of the graph* $\Gamma(\mathbf{N}_*)$.

Theorem 5. *For the constructed open network with restrictions* (*the set of base nodes is determined by formula* (7)) *the following relation is valid:*

$$\frac{|\mathbf{N}_*|}{|\mathbf{N_0}|} \to \frac{1}{2}, \quad K \to \infty. \tag{8}$$

3. DECOMPOSITION IN A CLOSED NETWORK

Consider a closed network (Gordon–Newell network) G with a constant number of requests $2K+1$ and a number of service nodes $m>2$. Request movements in the network are described by the routing matrix $\Theta=||\theta_{ij}||_{i,j=1}^m$, which is assumed to be indecomposable. Then, for any $B>0$, a solution $\Lambda=(\lambda_1,\dots,\lambda_m)$ to the system $\Lambda=\Lambda\Theta$ of balance relations exists and is unique, provided $B=\sum_{i=1}^m\lambda_i>0$. The network's operation (the number of requests at nodes) is described by a discrete Markov process $y(t)$ with a finite set of states

$$\mathbf{N}=\left\{\mathbf{n}=(n_1,\dots,n_m):\ n_1,\dots,n_m\ge 0,\ \sum_{k=1}^m n_k=2K+1\right\}$$

and transition intensities:

$$L(\mathbf{n}+\mathbf{e}_i,\mathbf{n}+\mathbf{e}_k)=\mu_i\theta_{ik},\quad 1\le i\ne k\le m,\quad \mathbf{n}+\mathbf{e}_i,\mathbf{n}+\mathbf{e}_k\in\mathbf{N}. \tag{9}$$

We define the set of all base vertices $\mathbf{N}_0$ by the equality

$$\mathbf{N}_0=\{\mathbf{n}\in\mathbf{N}:\ n_m>0\}.$$

We rewrite equalities (9) for $\mathbf{n}\in\mathbf{N}_0$:

$$L(\mathbf{n},\mathbf{n}-\mathbf{e}_m+\mathbf{e}_i)=\mu_m\theta_{mi},\quad L(\mathbf{n}-\mathbf{e}_m+\mathbf{e}_i,\mathbf{n}-\mathbf{e}_m+\mathbf{e}_k)=\mu_i\theta_{ik},\quad 1\le i\ne k<m. \tag{10}$$

Using equalities (10), we define the set

$$V(\mathbf{n})=\{\mathbf{n},\mathbf{n}-\mathbf{e}_m+\mathbf{e}_i,\ 1\le i<m\}.$$

Consider a graph $\Gamma(\mathbf{n})$, $\mathbf{n}\in\mathbf{N}_0$, with the set of vertices $V(\mathbf{n})$ and the set of edges connecting them ($\Gamma(\mathbf{n}_1)$, $\Gamma(\mathbf{n}_2)$, $\mathbf{n}_1\ne\mathbf{n}_2$, have no common edges). Obviously, $V(\mathbf{N}_0)=\mathbf{N}$.

Let $\mathbf{N}_*\subset\mathbf{N}_0$. Similar to the open network, we can describe a closed network G with restrictions by associating it with the graph $\Gamma(\mathbf{N}_*)$, and denote it by G^*.

Theorem 6. *If $V(\mathbf{N}_*)=\mathbf{N}$, and the graph $\Gamma(\mathbf{N}_*)$ is connected, then the Markov process $y(t)$ describing the functioning of the network G^* is ergodic and its stationary distribution $\pi(\mathbf{n})$, $\mathbf{n}\in\mathbf{N}$, is calculated by the formula*

$$\pi(\mathbf{n})=C^{-1}\prod_{i=1}^m\left(\frac{\lambda_i}{\mu_i}\right)^{n_i},\ C=\sum_{\mathbf{n}\in\mathbf{N}}\prod_{i=1}^m\left(\frac{\lambda_i}{\mu_i}\right)^{n_i},\quad 1\le i\le m.$$

Let us construct $\mathbf{N}_*\subset\mathbf{N}_0$ such that the stationary distribution of the Markov process is preserved and the relation $\frac{|\mathbf{N}_*|}{|\mathbf{N}_0|\to\frac{1}{2}}$ as $K\to\infty$. Due to Theorem 6, to preserve the stationary distribution in a closed network, it suffices to require that the equality $V(\mathbf{N}_*)=\mathbf{N}$ be satisfied and that the graph $\Gamma(\mathbf{N}_*)$ is connected.

To construct the set of base vertices $\mathbf{N}_*$ we associate with each point of the set $\mathbf{n}=(n_1,\dots,n_m)\in\mathbf{N}$ a point $\mathbf{n}'=(n_1,\dots,n_{m-1},0)$ and denote the set of such points $\mathbf{N}'=\{\mathbf{n}':\sum_{k=1}^{m-1}n_k\le 2K+1\}$. Then the set $\mathbf{N}_0$ will correspond to the set $\mathbf{N}'_0=\{\mathbf{n}'\in\mathbf{N}':\ \sum_{k=1}^{m-1}n_k\le 2K\}$. We now define

$$\mathbf{N}'_*=\bigcup_{k=0}^K\mathbf{N}'_*(2k)\bigcup\{(2k+1)\mathbf{e}_{m-1},\ k=0,\dots,K-1\},$$

$$\mathbf{N}'_*(k)=\left\{\mathbf{n}'\in\mathbf{N}':\ \sum_{k=1}^{m-1}n_k\le 2K-k,\ n_{m-1}=k\right\}.$$

Then we construct the set

$$\mathbf{N}_* = \{\mathbf{n} \in \mathbf{N}_0 : \ \mathbf{n}' \in \mathbf{N}'_*\}. \tag{11}$$

Theorem 7. *For the constructed closed network with restrictions (the set of base vertices is determined by formula (11)) the equality $V(\mathbf{N}_*) = \mathbf{N}$ holds and the graph $\Gamma(\mathbf{N}_*)$ is connected. Removing a vertex from the set $\mathbf{N}_*$ violates either the equality $V(\mathbf{N}_*) = \mathbf{N}$ or the connectivity of the graph $\Gamma(\mathbf{N}_*)$.*

Theorem 8. *For the constructed closed network with restrictions (the set of base vertices is determined by formula (11)) the relation*

$$\frac{|\mathbf{N}_*|}{|\mathbf{N}_0|} \to \frac{1}{2}, \quad K \to \infty \tag{12}$$

holds.

4. CONCLUSION

In all the queuing models considered, the set of base nodes $\mathbf{N}_0$, in terms of the number of nodes and, consequently, the number of edges in the transition intensity graph, is approximately half the subset of base nodes $\mathbf{N}_* \subset \mathbf{N}_0$. Exponential queuing systems with restrictions, while maintaining a stationary distribution, were constructed by transitioning to a special type of transition intensity graph. This work began with a special case, namely, with $m = 2$ for an open network and $m = 3$ for a closed one. The variants of the set $\mathbf{N}_*$ proposed in this work are not unique. We plan to continue this work, generalizing the obtained results and extending them to queuing systems whose operation is described by Markov processes.

The authors are currently collaborating with maritime logistics specialists and plan to jointly apply the obtained results to the design of a logistics network for container transportation from one mode of transport to another (road, rail and sea) to ensure uninterrupted delivery of goods.

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APPENDIX

A.1. PROOF OF THEOREM 1

The proof of formula (3) is based on the ergodicity theorem formulated in [14] for a discrete Markov process. Let us verify its sufficient conditions. 1) To verify that the distribution (3) satisfies the Kolmogorov–Chapman equations

$$\sum_{\mathbf{n}' \in V(\mathbf{N}_0)} \pi(\mathbf{n}) L(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \in V(\mathbf{N}_0)} \pi(\mathbf{n}') L(\mathbf{n}', \mathbf{n}), \quad \mathbf{n} \in \mathbf{N}, \tag{A.1}$$

it is enough to verify that it satisfies for $\mathbf{n}_0 \in \mathbf{N}_0$ the equalities

$$\sum_{\mathbf{n}' \in V(\mathbf{n}_0)} \pi(\mathbf{n}) L(\mathbf{n}, \mathbf{n}') = \sum_{\mathbf{n}' \in V(\mathbf{n}_0)} \pi(\mathbf{n}') L(\mathbf{n}', \mathbf{n}), \quad \mathbf{n} \in \mathbf{V}(\mathbf{n_0}). \tag{A.2}$$

Substituting the distribution (3) into formula (A.2) leads to the balance relations (1), with $C < \infty$. 2) The states of the process $y(t)$ are communicating, which follows from the assumption that the graph $\Gamma(\mathbf{N}_*)$ is connected. 3) The regularity condition is also satisfied, since for any $\mathbf{n} \in \mathbf{N}$ the inequality $\sum_{\mathbf{n}' \in \mathbf{N}} L(\mathbf{n}, \mathbf{n}') <$ const. holds. Theorem 1 is proved.

A.2. PROOF OF THEOREM 2

It is obvious that

$$V(\mathbf{N}_*) = \bigcup_{k=0}^{\infty} (\mathbf{N}_*(2k) \cup \mathbf{N}_*(2k+1)) = \mathbf{N}.$$

Since the graph $\Gamma(\mathbf{N}_*(2k))$ is connected and the edge $((2k+1)\mathbf{e}_m, (2k+2)\mathbf{e}_m)$ connects the graphs $\Gamma(\mathbf{N}_*(2k))$, $\Gamma(\mathbf{N}_*(2k+2))$, $k = 0, 1, \ldots$, then the graph $\Gamma(\mathbf{N}_*)$ is also connected.

Let's check the second assertion of the theorem. First, consider the case $m = 2$. The following options are possible.

1. If $\mathbf{n} = (0, 2k)$ and $k > 0$, then the graph $\Gamma(\mathbf{N}_*\backslash\mathbf{n})$ does not contain a path between vertices $\mathbf{n} = (0, 2k)$, $(0, 2k+1)$ and, therefore, is disconnected. If $k = 0$, then $\mathbf{n} \notin V(\mathbf{N}_*\backslash\mathbf{n})$ and, therefore, $V(\mathbf{N}_*\backslash\mathbf{n}) \neq V(\mathbf{N}_*)$.

2. If $\mathbf{n} = (0, 2k+1)$, $k \geq 0$, then in the graph $\Gamma(\mathbf{N}_*\backslash\mathbf{n})$ there is no path between the vertices $(0, 2k+1)$, $(0, 2k+2)$ and, therefore, the graph $\Gamma(\mathbf{N}_*\backslash\mathbf{n})$ is disconnected.

3. If $\mathbf{n} = (n_1, 2k)$, $k \geq 0$, and $n_1 > 1$, then $(n_1, 2k+1) \notin V(\mathbf{N}_*\backslash\mathbf{n})$ and, consequently, $V(\mathbf{N}_*\backslash\mathbf{n}) \neq V(\mathbf{N}_*)$. If $n_1 = 1$, then the vertex $\mathbf{n} = (1, 2k)$ is not connected with the vertex $(2, 2k)$ in the graph $\Gamma(\mathbf{N}_*\backslash\mathbf{n})$ and, therefore, the graph $\Gamma(\mathbf{N}_*\backslash\mathbf{n})$ is disconnected.

Let us now move on to the case $m > 2$. The verification of conditions **1**, **2** when replacing $(0, 2k)$ with $(0, \ldots, 0, 2k)$, and $(0, 2k+1)$ with $(0, \ldots, 0, 2k+1)$ is similar. When checking condition **3**, we set $\mathbf{n} = (n_1, \ldots, n_{m-1}, 2k)$. If $n_1 + \ldots + n_{m-1} > 1$, then it is easy to establish that $(n_1, \ldots, n_{m-1}, 2k+1) \notin V(\mathbf{N}_*\backslash\mathbf{n})$. If $n_1 + \ldots + n_{m-1} = 1$, then in the graph $\Gamma(\mathbf{N}_*\backslash\mathbf{n})$ there is no connection between the vertices $\mathbf{n} = \mathbf{e}_i + 2k\mathbf{e}_m$ and $2\mathbf{e}_i + 2k\mathbf{e}_m$. The Theorem 2 has been proven.

A.3. PROOF OF THEOREM 3

The number of vertices $|\mathbf{N}_0 \bigcap \Pi_m(2k)| = (2k+1)^m$. In turn, the number of vertices $|\mathbf{N}_* \bigcap \Pi_m(2k)| = (2k+1)^{m-1}(k+1) + k$. From here follow formulas (5), (6). Theorem 3 is proved.

A.4. PROOF OF THEOREM 4

The proof of Theorem 4 almost verbatim repeats the proof of Theorem 2.

A.5. PROOF OF THEOREM 5

From [15] it follows that the number of solutions of the Diaphantine equation $n_1 + \ldots + n_m = n$ in non-negative integers is equal to C_{n+m-1}^{n}. Then

$$|\mathbf{N}_*(k)| = \sum_{i=0}^{2K-k} C_{i+m-2}^{m-2} = C_{2K-k+m-1}^{m-1}, \quad |\mathbf{N}_*| = \sum_{k=0}^{K} |\mathbf{N}_*(2k)| + K. \tag{A.3}$$

From (A.3) we obtain the relations $1 = \mathbf{N}_*(2K) < \mathbf{N}_*(2K-1) < \ldots < \mathbf{N}_*(0)$. Consequently, the inequalities hold

$$\sum_{k=0}^{K} |\mathbf{N}_*(2k)| > \sum_{k=1}^{K} |\mathbf{N}_*(2k-1)| > \sum_{k=0}^{K} |\mathbf{N}_*(2k)| - |\mathbf{N}_*(0)|. \tag{A.4}$$

From (A.4) we have

$$2\sum_{k=0}^{K}|\mathbf{N}_*(2k)| - |\mathbf{N}_*(0)| < |\mathbf{N}_0| = \sum_{k=0}^{K}|\mathbf{N}_*(2k)| + \sum_{k=1}^{K}|\mathbf{N}_*(2k-1)| < 2\sum_{k=0}^{K}|\mathbf{N}_*(2k)|,$$

$$\frac{|\mathbf{N}_0|}{2} < \sum_{k=0}^{K}|\mathbf{N}_*(2k)| < \frac{|\mathbf{N}_0| + |\mathbf{N}_*(0)|}{2},$$

$$\frac{|\mathbf{N}_0|}{2} + K < |\mathbf{N}_*| < \frac{|\mathbf{N}_0| + |\mathbf{N}_*(0)|}{2} + K. \tag{A.5}$$

In turn, for $K \to \infty$ the asymptotic estimates are valid

$$|\mathbf{N}_0| = \sum_{i=0}^{2K} C_{i+m-1}^{m-1} = C_{2K+m}^{m} \sim \frac{(2K)^m}{m!}, \quad |\mathbf{N}_*(0)| = C_{2K+m-1}^{m-1} \sim \frac{(2K)^{m-1}}{(m-1)!},$$

$$\frac{K}{|\mathbf{N}_0|} \to 0, \quad \frac{|\mathbf{N}_*(0)|}{|\mathbf{N}_0|} \to 0.$$

Thus, relation (8) is satisfied by inequality (A.5). Theorem 5 is proven.

A.6. PROOF OF THEOREM 6

The proof of this theorem is similar to the proof of Theorem 1.

A.7. PROOF OF THEOREM 7

By construction, there is a one-to-one correspondence between the sets $\mathbf{N}'_*$ and $\mathbf{N}_*$, and also between the sets $\mathbf{N}'$ and $\mathbf{N}$, so the graphs $\Gamma(\mathbf{N}'_*)$ and $\Gamma(\mathbf{N}_*)$ are isomorphic. In turn, it follows from Theorem 4 that $V(\mathbf{N}'_*) = \mathbf{N}'$ and the graph $\Gamma(\mathbf{N}'_*)$ is connected, and deleting a vertex from the set $\mathbf{N}'_*$ violates at least one of these statements. The statement of the theorem is a consequence of the listed facts. Theorem 7 has been proven.

A.8. PROOF OF THEOREM 8

Since

$$|\mathbf{N}'_*| = |\mathbf{N}_*|, \ |\mathbf{N}'_0| = |\mathbf{N}_0|,$$

then the statement of Theorem 5 implies the statement of Theorem 8.

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