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STOCHASTIC SYSTEMS

Counting Triangles in the Generalized Clustering Attachment Model

N. M. Markovich

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: nat.markovich@gmail.com

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Abstract—The clustering attachment model (CAM) was introduced in Bagrow and Brockmann (2013) as an alternative to preferential attachment models. In Markovich and Vaičiulis (2024), a generalized clustering attachment model (GCAM) is proposed that includes an arbitrary attachment function f. This paper investigates the behavior of the mathematical expectation of the number of triangles under weak constraints on the function f. The logarithmic growth of the mathematical expectation of the number of triangles has been proven. More accurate results, including a central limit theorem for the triangle count, are obtained for a constant f. The expectation of the triangle count in the GCAM with an almost constant function f is considered in our simulation study.

Keywords: generalized clustering attachment model, triangle count, clustering coefficient, node weight, random graph, evolution

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1. INTRODUCTION

Let $G_n = (V_n, E_n)$, $n = 1, 2, \dots$ denote the sequence of random graphs, where V_n is the set of nodes and E_n is the set of edges. The sequence G_n, $n = 1, 2, \dots$ is formed by the evolution, i.e., the dynamic of the graph in time and space. The preferential attachment model (PAM) for the graph evolution has been first introduced in [1, 2]. The majority of the evolution models of the random graphs concerns to the PAM's, see [3–8] among others due to application to numerous real-world graphs. Let us provide here the definition of linear preferential attachment model (LPAM), considered in [9].

Definition 1. Fix $m \geq 2$ and $\delta > -m$. Then the LPAM is a sequence of random graphs $G_n = (V_n, E_n)$, $n \geq 1$ defined as follows:

(i) the graph G_1 consists of a single node with no edges;

(ii) the graph G_2 consists of two nodes with m edges between them;

(iii) the graph G_{n+1}, $n \geq 2$ is constructed recursively as follows: conditioning on the graph G_n, we add a node $n+1$ to the graph, with m new edges. Newly attached edges connect node $n+1$ and nodes $i_1, \dots, i_m \in V_n$, which are chosen by using weighted sampling with replacement. All nodes of the graph G_n are equipped with the weights

$$p_{i,n}(\delta) = \frac{D_{i,n} + \delta}{2m(n-1) + \delta n}, \quad i \in V_n, \tag{1}$$

which are recalculated after the choice of each of nodes $i_1, \dots, i_m \in V_n$. Here and below $D_{i,n}$ denotes the degree of node $i \in V_n$.

In [10], the attachment function f is included in the definition of the PAM. Namely, (1) is replaced by

$$p_{i,n}(f,\delta) = \frac{f(D_{i,n})+\delta}{\sum_{j\in V_n}(f(D_{i,n})+\delta)}, \quad i \in V_n, \tag{2}$$

where the preferential attachment function f is deterministic, non-decreasing and non-negative.

The clustering attachment model (CAM) was introduced in [11] as an alternative to the PAM, see [11] for comprehensive argument. This model is less known, even though it is often observed in practice and it can be applicable to local networks, e.g., local social networks, and "systems where nodes are drawn not towards hubs, but towards densely connected groups" [11].

Before providing the definition of so-called generalized CAM, let us recall the definition of the clustering coefficient. The clustering coefficient of node $i \in V_n$ is defined as

$$c_{i,n} \;=\; \begin{cases} 0, & D_{i,n}=0 \text{ or } D_{i,n}=1,\\ 2\Delta_{i,n}/\left(D_{i,n}\left(D_{i,n}-1\right)\right), & D_{i,n}\ge 2, \end{cases}$$

where $\Delta_{i,n}$ is the number of triangles of node i, [12].

The clustering coefficient $c_{i,n}$ was proposed first in [13] as "the propensity for two neighbors of the same vertex also to be neighbors of one another, forming a triangle of connections in the network." It measures a capacity of the ith node to form triangles of the nearest nodes in its neighborhood.

Let $\#A$ denote the cardinality (the number of elements) of arbitrary finite set A. The following definition of the generalized CAM (further the GCAM) is a slight modification of one proposed in [14].

Definition 2. Fix $m \ge 2$ and $\epsilon \ge 0$. Then the GCAM is a sequence of random graphs $G_n = (V_n, E_n)$, $n \ge 1$ defined as follows:

(i) G_1 is a non-random, finite and simple graph, which consists of at least m nodes, i.e.,

$$\#V_1 \ge m. \tag{3}$$

(ii) Each node $i \in V_n$, $n \ge 1$ is equipped with the weight

$$p_{i,n}(f,\epsilon) = \frac{f\left(c_{i,n}\right)+\epsilon}{\sum_{j\in V_n}\left(f\left(c_{j,n}\right)+\epsilon\right)}, \tag{4}$$

where the attachment function f is deterministic, non-decreasing, non-negative on $[0,1]$ and $f(1) < \infty$.

(iii) For $n \ge 1$, G_{n+1} is constructed recursively as follows: conditioning the graph G_n, we add a node $\#V_1 + n$ to the graph, with m new edges. The new edges connect the pairs of nodes

$$\{\#V_1+n, i_1\}, \{\#V_1+n, i_2\}, \ldots, \{\#V_1+n, i_m\},$$

where nodes $i_1, \ldots, i_m \in V_n$ are chosen by applying weighted sampling without replacement.

It is easy to see that the CAM in [11], i.e., GCAM with

$$f(x) = x^{\alpha}, \quad x \ge 0,\ \alpha > 0 \tag{5}$$

is constructed by replacing $D_{i,n}$ with $c_{i,n}$ in the corresponding PA model. The same noting holds by comparing (2) and (4).

Let us provide several remarks related to comparison of PAM and GCAM.

(1) Cardinalities $\#V_n$ and $\#E_n$ are deterministic for both models. In particular, it follows from Definition 2(iii) that for any $n \geq 1$,

$$\#V_n = \#V_1 + n - 1, \quad \#E_n = \#E_1 + m(n-1). \tag{6}$$

As for LPAM, we have $\#V_n = n$ and $\#E_n = m(n-1)$.

(2) From Definition 2(iii) that for any $n \geq 1$, G_n is a simple graph. The situation is different with LPAM. From Definition 1(iii) it follows that multiple edges allowed in LPAM.

(3) It is known that the LPAM with $m \geq 2$ and $\delta > 0$ produces a power law degree distribution with degree exponent $3 + \delta/m$, see, e.g., [9]. For our best knowledge, the degree distribution is not investigated for the GCAM. Using a heuristic argument it is concluded in [11] that the degree distribution for the CAM belongs to the class of light-tailed distributions. The simulation results in [15] confirm this conclusion.

(4) The main difference between LPAM and GCAM is the following. By considering LPAM one can note that it is likely that the node (say i) with large degree in the graph G_n will be connected with newly introduced node $n+1$. This implies $D_{i,n} < D_{i,n+1}$. This means that LPAM exibits a rich-get-richer phenomenon (Matthew effect). The GCAM has not such property. For example, if $c_{i,n} = 1$ for some node $i \in V_n$ and this node is connected with new node $\#V_1 + n$, then $c_{i,n} > c_{i,n+1}$. Indeed, suppose first that the nodes $i \in V_n$ and $j \in V_n$ selected by weighted random choice with replacement are not connected by an edge. Then $D_{i,n+1} = D_{i,n} + 1$ and $\Delta_{i,n+1} = \Delta_{i,n}$ hold. The last equalities together with the equality $2\Delta_{i,n} = D_{i,n}(D_{i,n} - 1)$ allow us to obtain $c_{i,n+1} = (\Delta_{i,n} - 1)/(\Delta_{i,n} + 1) < 1$. It is easy to verify that when nodes $i \in V_n$ and $j \in V_n$ are connected by an edge, then $c_{i,n+1} = (D_{i,n}(D_{i,n} - 1) + 2)/(\Delta_{i,n}(\Delta_{i,n} + 1)) < 1$ due to $D_{i,n} \geq 2$.

The total number of triangles Δ_n in a random graph G_n is defined by

$$\Delta_n = \frac{1}{3} \sum_{i \in V_n} \Delta_{i,n}.$$

Our objective is to investigate the behavior of the sequence $\mathrm{E}(\Delta_n) - \Delta_1$, $n \geq 2$ for the GCAM with $m = 2$ and $\epsilon > 0$. The GCAM with $m = 2$ can be characterized as a basic. Namely, the clustering coefficient of a newly appended node is equal to 1 when a pair of chosen nodes i_1 and i_2 are connected, or 0 otherwise.

The paper is organized as follows. Related results regarding the expected number of triangles are surveyed in Section 2. Main results including Theorems 1 and 2 are presented in Section 3. The algorithm and simulation results of the sequence $\mathrm{E}\Delta_n - \Delta_1$, $n \geq 2$ are considered in Section 4. Conclusions are given in Section 5. Proofs are provided in Appendix.

2. RELATED RESULTS FOR EXPECTED NUMBER OF TRIANGLES

A triangle of connected nodes is the most studied subgraph and it can be considered as the basic community. Perhaps, it was first proposed for study in [13].

In [16] it is shown that $\mathrm{E}(\Delta_n)$ in the Albert-Barabási model (the LPAM with $\delta = 0$, $m \geq 2$) is of order $\ln^3(n)$. The same order is obtained in [17] for the PAM, where all m edges from the next new vertex are drawn simultaneously and independently, and the probabilities of drawing an edge to vertex i depend linearly on the degree of this vertex in G_n, but are not recalculated after drawing the next edge, as in the definition of 1.

$\mathrm{E}(\Delta_n)$ of LPAM with two parameters δ, m was investigated in [18], see also [19]. In [18] it is shown that for $\delta > 0$ and $m \geq 2$,

$$|\mathrm{E}(\Delta_n) - C_1(\beta, m)\ln(n)| \leq C_2, \quad n \geq 2$$

holds, where C_1 and C_2 are some positive constants. More general, Theorem 2.5 in [9] states that for the LPAM parameters $m \in \mathbb{N} \geq 2$ and $\delta > -m$,

$$\mathrm{E}\left(\Delta_n\right) = C_1(m, \delta) g_1(n) + g_2(n), \quad n \to \infty$$

holds, where the function g_2 is such that $|g_2(n)| \to 0$ as $n \to \infty$, while the function g_1 has the form

$$g_1(n) = \begin{cases} n^{-\delta/(2m+\delta)}, & -m < \delta < 0, \\ \ln^3(n), & \delta = 0, \\ \ln(n), & \delta > 0. \end{cases}$$

The explicit form of the asymptotic constant $C_1(m, \delta)$ is given in Theorem 2.5 of in [9].

Let us list articles containing results on the number of triangles Δ_n. The asymptotic behavior of the number of triangles Δ_n in an inhomogeneous random graph with different connection probabilities to form edges is derived in [20, 21], and a short survey can be found in [22]. It was shown that Δ_n converges in distribution to a Poisson distribution. The asymptotic expected number of Δ_n in power-law uniform random graphs is studied in [23]. The survey of the literature regarding the triangle count Δ_n for homogeneous random graphs is given in [21].

3. MAIN RESULTS

The main result of this work is the following theorem.

Theorem 1. *Let $m = 2$ and $\epsilon > 0$ be parameters of the GCAM. Then for $n \geq 2$,*

$$\frac{1}{\mu^2} + g_1(n) \leq \frac{\mathrm{E}(\Delta_n) - \Delta_1}{4\ln(n)} \leq \mu^2 + g_2(n), \tag{7}$$

where

$$\mu = \frac{\epsilon + f(1)}{\epsilon + f(0)} \tag{8}$$

and $g_i(n) = O\left(1/\ln(n)\right)$, $n \to \infty$, $i = 1, 2$.

Without loss of generality we may assume that $f(1) = 1$ in (4). Indeed, if $f(1) \neq 1$, then $p_{i,n}(f, \epsilon) = p_{i,n}(f/f(1), \epsilon/f(1))$. By considering particular case of the attachment function f we have the following result.

Theorem 2. *Let $m = 2$ and $\epsilon > 0$ be parameters of the GCAM and f is the constant function, namely,*

$$f(x) = \lambda, \quad 0 \leq x \leq 1, \quad \lambda \geq 0. \tag{9}$$

Then

$$\lim_{n\to\infty} \frac{\mathrm{E}(\Delta_n) - \Delta_1}{4\ln(n)} = 1 \tag{10}$$

and

$$\frac{\Delta_n - 4\ln(n)}{2\ln^{1/2}(n)} \xrightarrow{\mathrm{d}} \mathcal{N}\left(0, 1\right), \quad n \to \infty, \tag{11}$$

where $\xrightarrow{\mathrm{d}}$ denotes the convergence in distribution and $\mathcal{N}(0,1)$ denotes the standard normal distribution.

The statement (11) is the central limit theorem (CLT). The author has not encountered an analogue for a LMAP. A review of the literature on local CLT applied to Δ_n in Erdös-Rényi random graphs is contained in [24].

4. MONTE-CARLO SIMULATIONS

The classical approach to analyze the total triangle count is to simulate a "long enough" sequence of graphs $G_1, \ldots, G_n$ and then build a sample $\Delta_1, \ldots, \Delta_n$. However, this procedure may require a long simulation time. To overcome the problem, we present an approximation for $\mathrm{E}(\Delta_n) - \Delta_1$, $n \geq 2$, based on theoretical calculations presented in the Appendix, to which several references will be made below.

Let $U_k^{(j)}$, $1 \leq j \leq N$, $1 \leq k \leq n-1$ be independent and uniformly distributed random variables (r.v.) on $(0,1)$, and let the random event B_k be defined in (A.4). Then

$$\overline{D_{n,N}} = \frac{1}{N} \sum_{j=1}^{N} \sum_{k=1}^{n-1} I\left\{U_k^{(j)} < \mathrm{P}(B_k)\right\}, \quad n \geq 2, \tag{12}$$

is an approximation for $\mathrm{E}(\Delta_n) - \Delta_1$. Indeed, using (A.6) and (A.7), we get $\mathrm{E}\left(\overline{D_{n,N}}\right) = \mathrm{E}(\Delta_n) - \Delta_1$. Moreover, for any fixed n the sequence $\overline{D_{n,N}}$, $N = 1, 2, \ldots$ converges almost everywhere to $\mathrm{E}(\Delta_n) - \Delta_1$ as $N \to \infty$. This follows from the strong law of large numbers (see, e.g., Theorem 14 in [28]), since the terms in the sum (12) by j are independent and identically distributed r.v.s with expectation $\mathrm{E}(\Delta_n) - \Delta_1$.

Let $\tilde{n} \geq 2$ denote some natural number. Using (12), one can quickly generate an approximation of the sequence $\mathrm{E}(\Delta_n) - \Delta_1$, $2 \leq n \leq \tilde{n}$. To this end, we need to calculate probabilities $\mathrm{P}(B_k)$, $1 \leq k \leq \tilde{n} - 1$. In the simplest case, when f is a constant function (see, (9)), we have

$$\mathrm{P}(B_k) = \frac{2(\#E_1 + 2(k-1))}{(\#V_1 + k - 1)(\#V_1 + k - 2)}, \quad k \geq 1,$$

see (A.13). Calculating the probabilities $\mathrm{P}(B_k)$, $1 \leq k \leq \tilde{n} - 1$ becomes significantly more complicated when replacing a constant function with an almost everywhere constant (in terms of Lebesgue measure) function

$$f(x) = \begin{cases} 0, & x = 0, \\ 1, & 0 < x \leq 1. \end{cases} \tag{13}$$

For the function f presented in (13), weights (4) have the form

$$p_{i,n}(f, \epsilon) = \frac{1}{(1+\epsilon)V_{n,1} + \epsilon V_{n,2}} \begin{cases} \epsilon, & i \in V_{n,2}, \\ 1 + \epsilon, & i \in V_{n,1}, \end{cases} \tag{14}$$

where $V_{n,1} = \{i : c_{i,n} > 0\}$, $V_{n,2} = V_n \setminus V_{n,1}$. Let us also decompose the set of edges E_n into disjoint subsets. Let $E_{n,1} = \{\{i,j\} : \{i,j\} \in E_n,\ i \in V_{n,1},\ j \in V_{n,1}\}$, $E_{n,3} = \{\{i,j\} : \{i,j\} \in E_n,\ i \in V_{n,2},\ j \in V_{n,2}\}$ and $E_{n,2} = E_n \setminus (E_{n,1} \cup E_{n,3})$ hold. For example, if $G_1 = (V_1, E_1)$ with

$$V_1 = \{1, 2, \ldots, 9\}, \quad E_1 = \{\{1,2\}, \{1,3\}, \{2,3\}, \{2,5\}, \{3,4\}, \{6,7\}\}, \tag{15}$$

then $V_{1,1} = \{1,2,3\}$, $V_{1,2} = \{4,5,\ldots,9\}$, $E_{1,1} = \{\{1,2\},\{1,3\},\{2,3\}\}$, $E_{1,2} = \{\{2,5\},\{3,4\}\}$, $E_{1,3} = \{\{6,7\}\}$.

Taking into account the introduced notations (A.7) can be rewritten as follows:

$$\mathrm{P}(B_k) = q_{k,1} + q_{k,2} + q_{k,3}, \quad k \geq 1,$$

where

$$q_{k,\ell} = \sum_{\{i_1,i_2\}\in E_{k,\ell}} \left(\frac{p_{i_1,k}(f,\epsilon)p_{i_2,k}(f,\epsilon)}{1-p_{i_1,k}(f,\epsilon)} + \frac{p_{i_1,k}(f,\epsilon)p_{i_2,k}(f,\epsilon)}{1-p_{i_2,k}(f,\epsilon)} \right), \quad \ell = 1,2,3.$$

Here and further, we assume that the initial graph G_1 and the parameter ϵ are chosen so that $\mu < \#V_1$. Using (14), we find

$$q_{k,1} = \frac{2(1+\epsilon)^2\#E_{k,1}}{\lambda_k\left(\lambda_k - 1 - \epsilon\right)}, \quad q_{k,2} = \frac{\epsilon(1+\epsilon)\#E_{k,2}}{\lambda_k\left(\lambda_k - 1 - \epsilon\right)} + \frac{\epsilon(1+\epsilon)\#E_{k,2}}{\lambda_k\left(\lambda_k - \epsilon\right)}, \quad q_{k,3} = \frac{2\epsilon^2\#E_{k,3}}{\lambda_k\left(\lambda_k - \epsilon\right)},$$

where $\lambda_k = (1+\epsilon)\#V_{k,1} + \epsilon\#V_{k,2}$. In addition, we introduce the quantities

$$\bar{q}_{k,1} = \frac{2(1+\epsilon)^2\left(\#V_{k,1}(\#V_{k,1}-1)/2 - \#E_{k,1}\right)}{\lambda_k\left(\lambda_k - 1 - \epsilon\right)},$$

$$\bar{q}_{k,2} = \frac{\epsilon(1+\epsilon)\left(\#V_{k,1}\#V_{k,2} - \#E_{k,2}\right)}{\lambda_k\left(\lambda_k - 1 - \epsilon\right)} + \frac{\epsilon(1+\epsilon)\left(\#V_{k,1}\#V_{k,2} - \#E_{k,2}\right)}{\lambda_k\left(\lambda_k - \epsilon\right)},$$

$$\bar{q}_{k,3} = \frac{2\epsilon^2\left(\#V_{k,2}(\#V_{k,2}-1)/2 - \#E_{k,3}\right)}{\lambda_k\left(\lambda_k - \epsilon\right)}.$$

It is easy to see that $q_{k,1}$ is the probability of choosing an unordered pair of nodes connected by an edge from $E_{k,1}$, and $\bar{q}_{k,1}$ is the probability of choosing an unconnected unordered pair of nodes such that both nodes belong to $V_{k,1}$. The quantities $q_{k,\ell}$ and $\bar{q}_{k,\ell}$, $\ell = 2,3$ can be interpreted similarly.

Let us recall that to form the graph G_{k+1}, a new node $\#V_1 + k$ is added to the graph G_k. Depending on the link between the unordered pair of nodes selected using weighted random selection without deletion, the newly introduced node is added to the set $V_{k+1,1}$ or to the set $V_{k+1,2}$. We also add two edges connected to the node $\#V_1 + k$ to one of the disjoint sets $E_{k+1,1}$, $E_{k+1,2}$, or $E_{k+1,3}$. In particular, the cardinalities of the sets $\#V_{k+1,\ell}$, $\ell = 1,2$, $k \geq 1$ and $\#E_{k+1,\ell}$, $\ell = 1,2,3$, $k \geq 1$ are controlled by the following rules:

1. If the event $\{U_k^{(j)} < q_{k,1}\}$ occurs (i.e. both selected nodes belong to $V_{k,1}$ and they are connected by an edge from $E_{k,1}$), then $\#V_{k+1,1} = \#V_{k,1} + 1$ and $\#E_{k+1,1} = \#E_{k,1} + 2$. If $\{q_{k,1} + q_{k,2} + q_{k,3} \leq U_k^{(j)} < q_{k,1} + q_{k,2} + q_{k,3} + \bar{q}_{k,1}\}$ (i.e., both selected nodes belong to $V_{k,1}$ and are not connected), then $\#V_{k+1,2} = \#V_{k,2} + 1$ and $\#E_{k+1,2} = \#E_{k,2} + 2$.
2. If the event $\{q_{k,1} \leq U_k^{(j)} < q_{k,1} + q_{k,2}\}$ occurs (i.e. one of the selected nodes belongs to $V_{k,1}$, and another one belongs to the set $V_{k,2}$ and the nodes are connected by the edge from $E_{k,2}$), then $\#V_{k+1,1} = \#V_{k,1} + 2$, $\#V_{k+1,2} = \#V_{k,2} - 1$, $\#E_{k+1,1} = \#E_{k,1} + 3$ and $\#E_{k+1,2} = \#E_{k,2} - 1$. If the event $\{q_{k,1} + q_{k,2} + q_{k,3} + \bar{q}_{k,1} \leq U_k^{(j)} < q_{k,1} + q_{k,2} + q_{k,3} + \bar{q}_{k,1} + \bar{q}_{k,2}\}$ occurs (i.e. one of the selected nodes belongs to $V_{k,1}$, and another one belongs to the set $V_{k,2}$, and nodes are not connected), then $\#V_{k+1,2} = \#V_{k,2} + 1$, $\#E_{k+1,2} = \#E_{k,2} + 1$ and $\#E_{k+1,3} = \#E_{k,3} + 1$.
3. If the event $\{q_{k,1} + q_{k,2} \leq U_k^{(j)} < q_{k,1} + q_{k,2} + q_{k,3}\}$ occurs (i.e. both of the selected nodes belong to $V_{k,2}$ and they are connected by the edge from $E_{k,3}$), then $\#V_{k+1,1} = \#V_{k,1} + 3$, $\#V_{k+1,2} = \#V_{k,2} - 2$, $\#E_{k+1,1} = \#E_{k,1} + 3$ and $\#E_{k+1,3} = \#E_{k,3} - 1$. If the event $\{U_k^{(j)} \geq 1 - \bar{q}_{k,3}\}$ occurs (i.e. both of selected nodes belong to $V_{k,2}$ and they are disconnected), then $\#V_{k+1,2} = \#V_{k,2} + 1$ and $\#E_{k+1,3} = \#E_{k,3} + 2$.

The following initial graphs were used in the simulation $G_1 = (V_1, E_1)$:

a) a complete graph $G_1^{(1)}$ with three nodes and three edges;

b) the graph $G_1^{(2)} = (V_1, E_1)$, where V_1 and E_1 are determined in (15);

c) a complete graph $G_1^{(3)}$ with 17 nodes and 136 edges.

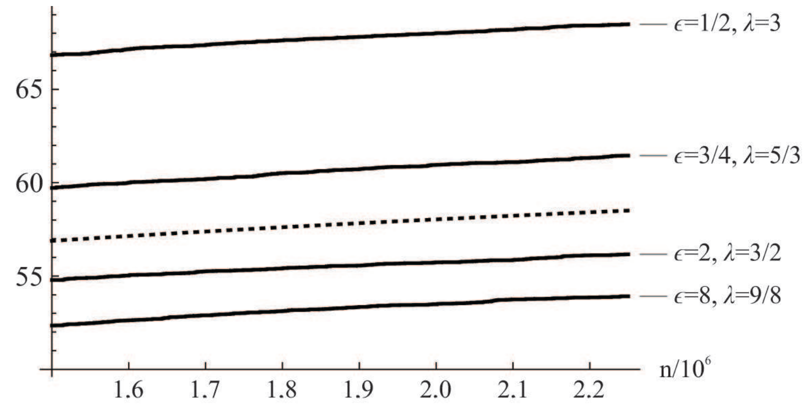

Fig. 1. The graph $g(n) = 4\ln(n)$ (dotted line) and $\overline{D_{n,100}}$, $1.5 \times 10^6 \leq n \leq 2.25 \times 10^6$ for initial graph $G_1^{(1)}$.

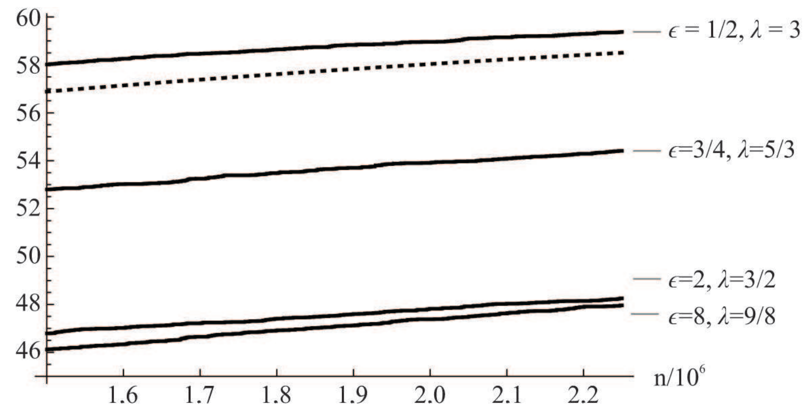

Fig. 2. The graph $g(n) = 4\ln(n)$ (dotted line) and $\overline{D_{n,100}}$, $1.5 \times 10^6 \leq n \leq 2.25 \times 10^6$ for initial graph $G_1^{(2)}$.

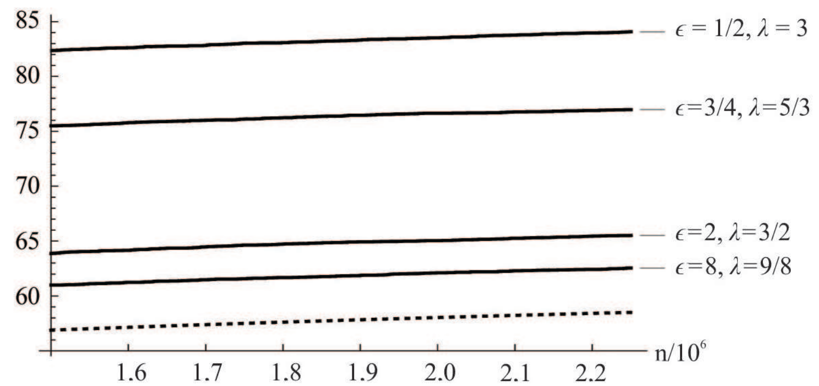

Fig. 3. The graph $g(n) = 4\ln(n)$ (dotted line) and $\overline{D_{n,100}}$, $1.5 \times 10^6 \leq n \leq 2.25 \times 10^6$ for initial graph $G_1^{(3)}$.

In the simulation, the difference $\mathrm{E}(\Delta_n) - \Delta_1$ is approximated by an empirical mean $\overline{D_{N,n}}$. The values $\overline{D_{n,N}}$ are calculated based on (12) using $N = 100$ independent sequences of triangle numbers. The approximation values $\overline{D_{n,100}}$, $2 \leq n \leq 2.25 \times 10^6$ are calculated for $\epsilon = 1/2.3/4.2.8$ and for each initial graph. The simulation results are presented in Figs. 1–3, the smoothness of the curves of which indicates that the chosen value of N is sufficiently large. To show that $\overline{D_{n,100}}$, as a function of n, varies slowly for large n, the simulation results are presented on the interval of natural numbers $[1.5 \times 10^6, 2.25 \times 10^6]$.

Relation (10) states that for $\lambda = 1$, the mathematical expectation of the number of triangles $\mathrm{E}(\Delta_n) - \Delta_1$ is asymptotically equal to $g(n) = 4\ln(n)$ as $n \to \infty$. The graph of the function g is also shown in Figs. 1–3.

We see that the graphs $\overline{D_{n,100}}$ in Figs. 1–3 are approximately parallel to $g(n) = 4\ln(n)$. Therefore, we can assume that for each initial graph G_1 and $\epsilon > 0$, there exists a positive parameter ν, probably depending on G_1 and ϵ, such that the mathematical expectation $\mathrm{E}(\Delta_n) - \Delta_1$ is asymptotically equal to $\nu\ln(n)$ as $n \to \infty$.

The simulation results are as follows:

(1) the graphs $\overline{D_{n,100}}$ in Figs. 1–3 allow us to conclude that ν depends on the parameter ϵ, which is also confirmed by the estimates of the parameter $\hat{\nu}$ obtained by the least squares method and presented in table;

Empirical characteristics $\bar{D} = \overline{D_{2.5\times 10^6,100}}$ and $\hat{\nu}$

	Initial graph											
	$G_1^{(1)}$				$G_1^{(2)}$				$G_1^{(3)}$			
ϵ	1/2	3/4	2	8	1/2	3/4	2	8	1/2	3/4	2	8
$\overline{D}$	68	61	56	54	59	54	48	48	84	77	66	62
$\hat{\nu}$	4.69	4.20	3.85	3.69	4.07	3.71	3.29	3.26	5.77	5.29	4.49	4.28

(2) the table shows that $\bar{D} = \overline{D_{2.5\times 10^6,100}}$ and ν depend on the assignment of the initial graph;

(3) for all approximations $\overline{D_{2.5\times 10^6,100}}$ it holds

$$\lambda^{-2} \leq \frac{\overline{D_{2.5\times 10^6,100}}}{4\ln(2.5\times 10^6)} \leq \lambda^2, \quad \lambda = \frac{1+\epsilon}{\epsilon},$$

that does not contradict the theoretical result (7);

(4) the approximation (12) can be applied for the next attachment function as well:

$$f(x) = \begin{cases} 0, & 0 \leq x < 1, \\ 1, & x = 1. \end{cases}$$

When designing an algorithm to compute $\mathrm{P}(B_k)$, $1 \leq k \leq n-1$, in this case, it is important to keep in mind that $c_{i,n} = 1$ if and only if i is a vertex of the complete graph.

5. CONCLUSIONS

The expected number of node triangles for the GCAM with $m = 2$, any $\epsilon > 0$ and arbitrary attachment function f satisfying Definition 2 is investigated.

The function f does not have to be continuous in Definition 2(ii), and therefore, in Theorem 1. This function may be discontinuous, with one or several removable discontinuities.

Regarding the initial graph $G_1 = (V_1, E_1)$ from which the evolutions begins, Theorem 1 is valid for any set E_1 including the empty one. The technical assumption (3) with $m = 2$ ensures that each graph G_n, $n \geq 1$ contains at least two nodes, which may be chosen by applying the weighted sampling without replacement. The expected number of triangles is proved to be of the rate $\ln(n)$. It is noted in Section 2, the number of triangles in the linear PAM with $m \geq 2$ and $\delta > 0$ has the same rate $\ln(n)$, see [18], [9].

A fast algorithm to generate a long sequence $\mathrm{E}\Delta_n - \Delta_1$, $n \geq 2$ for the GCAM with the attachment function (13) is provided. Our simulation results validate theoretical result in Theorem 1.

The weights $p_{i,n}(f,\epsilon)$, $i \in V_n$ in (4) are uniformly distributed on V_n if the attachment function is constant. This allowed us to find the asymptotic of $\mathrm{E}\Delta_n - \Delta_1$ and prove the asymptotic normality, see (10) and (11), respectively, in Theorem 2.

The case $m > 2$ requires more calculations due to a complicated combinatorics. As is noted in [11], to investigate the growth of the triangle count by using simulations is a challenge. One can suppose that the expected number of triangles has the rate $\ln(n)$ in the case $m > 2$, too.

Having information about the behavior of the sequence $\mathrm{E}\Delta_n$, $n \geq 2$, one can investigate the behavior of the average clustering coefficient

$$\bar{C}_n = \frac{3\Delta_n}{\text{the number of pairs of connected edges in a graph} G_n}.$$

This is a subject for further research.

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APPENDIX

We will provide proofs for the statements here.
The digamma function ψ is defined as

$$\psi(x) = \frac{\mathrm{d}}{\mathrm{d}x} \ln\left(\Gamma(x)\right), \quad x > 0, \tag{A.1}$$

where $\Gamma(x) = \int_0^\infty y^{x-1}\mathrm{e}^{-y}\mathrm{d}y$ is the Euler's gamma function, see, e.g., [25]. Let us define

$$\begin{aligned}\Upsilon_n(\mu, \#V_1, \#E_1) = 4\mu^2\psi(n + \#V_1 - 1 - \mu)& \\ + 2\mu(2\#V_1 - \#E_1)\left(\psi(n + \#V_1 - 1) - \psi(n + \#V_1 - 1 - \mu)\right)&,\end{aligned}$$

where μ is the same as in (8), initial graph $G_1 = (V_1, E_1)$ satisfies Definition 2(i) and natural n is such that $n \geq n_0$, where natural n_0 satisfies $n_0 + \#V_1 - 1 - \mu > 0$.

Lemma 1. *Let $\#V_k$, $k \geq 1$ and $\#E_k$, $k \geq 1$ be the sequences defined in* (6) *and β be a real positive number, while natural numbers $1 \leq n' < n'' < \infty$ are such that $n' + \#V_1 - 1 - \beta > 0$. Then*

$$2\beta^2 \sum_{k=n'}^{n''-1} \frac{\#E_k}{\#V_k(\#V_k - \beta)} = \Upsilon_{n''}(\beta, \#V_1, \#E_1) - \Upsilon_{n'}(\beta, \#V_1, \#E_1). \tag{A.2}$$

Proof of Lemma 1. The left-hand side of (A.2) is equal to the sum

$$2\beta(\#E_1 - 2\#V_1) \sum_{k=n'}^{n''-1} \frac{1}{k + \#V_1 - 1} + 2\beta\left(2\beta - (\#E_1 - 2\#V_1)\right) \sum_{k=n'}^{n''-1} \frac{1}{k + \#V_1 - 1 - \beta}.$$

The equality (A.2) follows by applying the identity

$$\sum_{k=n'}^{n''-1} \frac{1}{k + \gamma} = \psi(n'' + \gamma) - \psi(n' + \gamma), \quad n' + \gamma > 0. \tag{A.3}$$

To prove (A.3) we decompose the left-hand side of (A.3) as following:

$$\sum_{k=n'}^{\infty}\frac{1}{k+\gamma}-\sum_{k=n''}^{\infty}\frac{1}{k+\gamma} = -\left\{-\gamma_E+\sum_{j=0}^{\infty}\frac{1}{j+1}-\sum_{j=0}^{\infty}\frac{1}{j+n'+\gamma}\right\} + \left\{-\gamma_E+\sum_{j=0}^{\infty}\frac{1}{j+1}-\sum_{j=0}^{\infty}\frac{1}{j+n''+\gamma}\right\},$$

where $\gamma_E\approx 0.5772$ is the Euler–Mascheroni's constant, and use the series expansion

$$\psi(x)=-\gamma_E+\sum_{j=0}^{\infty}\frac{1}{j+1}-\sum_{j=0}^{\infty}\frac{1}{j+x},\quad x>0,$$

(see, e.g., [25]).

Lemma 2. *Let β be a real positive number, while natural numbers $1\le n'<n<\infty$ are such that $n'+\#V_1-1-\beta>0$. Then*

$$\Upsilon_n(\beta,\#V_1,\#E_1)-4\beta^2\ln(n)=g(n)$$

holds, where $g(n)=O(1/n)$, as $n\to\infty$.

Proof of Lemma 2. We can write $\Upsilon_n(\beta,\#V_1,\#E_1)-4\beta^2\ln(n)=g(n)$, where

$$\begin{aligned} g(n) &= 2\beta\left(2\beta-(2\#V_1-\#E_1)\right)\left(\psi(n+\#V_1-1-\beta)-\ln(n+\#V_1-1-\beta)\right)\\ &\quad+2\beta(2\#V_1-\#E_1)\left(\psi(n+\#V_1-1)-\ln(n+\#V_1-1)\right)\\ &\quad+4\beta^2\ln\left(1+\frac{\#V_1-1-\beta}{n}\right)+2\beta(2\#V_1-\#E_1)\ln\left(1+\frac{\beta}{n+\#V_1-1-\beta}\right).\end{aligned}$$

By using inequalities $x/(1+x)\le\ln(1+x)\le x$, $x>-1$ and $|\psi(x)-\ln(x)|<1/x$, $x>0$ (see, e.g., [26]), we find that $|g(n)|\le C/n$, where $0<C<\infty$ is some constant.

Proof of Theorem 1. Obviously, that two random events $\{\Delta_{n+1}=\Delta_n+1\}$ and

$$B_n=\{\text{an unordered pair of nodes chosen (by weighted sampling without replacement) from } V_n \text{ is connected by an edge from } E_n\} \tag{A.4}$$

are equivalent. Thus, we get

$$\Delta_n-\Delta_1=\sum_{k=1}^{n-1}I(B_k),\quad n\ge 2, \tag{A.5}$$

where $I(B_k)$ denotes the indicator of the event B_k. Whence, keeping in mind that the graph G_1 is non-random, we get

$$\mathrm{E}\left(\Delta_n\right)-\Delta_1=\sum_{k=1}^{n-1}\mathrm{P}(B_k),\quad n\ge 2. \tag{A.6}$$

By applying the weighted sampling without replacement, the first node of V_k is randomly selected, using weights $p_{i,k}(f,\epsilon)$, $i\in V_k$, $k\ge 1$. The same node, say i', cannot be selected again. The second node is chosen using renormalized weights $p_{i,k}(f,\epsilon)/\left(1-p_{i',k}(f,\epsilon)\right)$, $i\in V_k\setminus\{i'\}$. If $\#E_k=0$, then it immediately follows that $\mathrm{P}(B_k)=0$. As for the case $\#E_k\ne 0$, we have

$$\mathrm{P}(B_k) = \sum_{\{i_1,i_2\}\in E_k}\left(\frac{p_{i_1,k}(f,\epsilon)p_{i_2,k}(f,\epsilon)}{1-p_{i_1,k}(f,\epsilon)}+\frac{p_{i_1,k}(f,\epsilon)p_{i_2,k}(f,\epsilon)}{1-p_{i_2,k}(f,\epsilon)}\right),\ k\ge 1. \tag{A.7}$$

Here, the sum is taken over all unordered pairs of nodes $\{i_1, i_2\}$ belonging to the set of edges E_k. The assumption $\epsilon > 0$ yields $0 < p_{i,k}(f, \epsilon) < 1$ for any $i \in V_k$, $k \geq 1$. Thus, all summands on the right-hand side of (A.7) are positive.

Under the adopted assumptions (see Definition 2 (ii)) the function f does not decrease on $[0, 1]$, i.e., $f(0) \leq f(x) \leq f(1)$ for all $0 \leq x \leq 1$. Due to gross inequality $0 \leq c_{i,k} \leq 1$, $i \in V_k$, $k \geq 1$, this leads to $f(0) \leq f(c_{i,k}) \leq f(1)$, $i \in V_k$, $k \geq 1$. From here follow the inequalities for the weights (4):

$$\frac{1}{\mu \# V_k} \leq p_{i,k}(f, \epsilon) \leq \frac{\mu}{\# V_k}, \quad i \in V_k,\ k \geq 1. \tag{A.8}$$

Let us consider the upper bound of $\mathrm{E}\,(\Delta_n) - \Delta_1$. Assume first that $\# V_1 - \mu > 0$. By combining (6), (A.7), (A.8) we get

$$\mathrm{P}(B_k) \leq 2\mu^2 \frac{\# E_k}{\# V_k(\# V_k - \mu)} \tag{A.9}$$

for $k \geq 1$. This, together with (A.6) gives

$$\mathrm{E}\,(\Delta_n) - \Delta_1 \leq 2\mu^2 \sum_{k=1}^{n-1} \frac{\# E_k}{\# V_k(\# V_k - \mu)}.$$

By applying Lemma 1 we obtain

$$\begin{aligned} \mathrm{E}\,(\Delta_n) - \Delta_1 &\leq \Upsilon_n(\mu, \# V_1, \# E_1) - \Upsilon_1(\mu, \# V_1, \# E_1) \\ &\leq 4\mu^2 \ln(n) + \left| \Upsilon_n(\mu, \# V_1, \# E_1) - 4\mu^2 \ln(n) \right| + \left| \Upsilon_1(\mu, \# V_1, \# E_1) \right|. \end{aligned}$$

Hence, it follows

$$\frac{\mathrm{E}\,(\Delta_n) - \Delta_1}{4\ln(n)} \leq \mu^2 + \frac{\left| \Upsilon_n(\mu, \# V_1, \# E_1) - 4\mu^2 \ln(n) \right|}{4\ln(n)} + \frac{\left| \Upsilon_1(\mu, \# V_1, \# E_1) \right|}{4\ln(n)}.$$

Now, by Lemma 2, it follows

$$\frac{\mathrm{E}\,(\Delta_n) - \Delta_1}{4\ln(n)} \leq \mu^2 + g_2(n), \tag{A.10}$$

where $g_2(n) = O\left(1/\ln(n)\right)$, $n \to \infty$.

Assume now $\# V_1 - \mu \leq 0$. Then there exists natural k_0, such that $\# V_k - \mu \leq 0$ for $1 \leq k \leq k_0$ and $\# V_k - \mu > 0$ for $k > k_0$. We decompose the sum in right-hand side of (A.6) as follows:

$$\mathrm{E}\,(\Delta_n) - \Delta_1 = \sum_{k=1}^{k_0} \mathrm{P}(B_k) + \sum_{k=k_0+1}^{n-1} \mathrm{P}(B_k). \tag{A.11}$$

By using a rough bound $\mathrm{P}(B_k) \leq 1$ we get that first sum in right-hand side of (A.11) does not exceed k_0. To find an upper bound of the second sum, we apply the bound (A.9), which holds for $k > k_0$. Thus, we have

$$\mathrm{E}\,(\Delta_n) - \Delta_1 \leq k_0 + \Upsilon_n(\mu, \# V_1, \# E_1) - \Upsilon_{k_0+1}(\mu, \# V_1, \# E_1).$$

It remains to apply Lemma 2 again, to ensure that (A.10) is satisfied.

A similar reasoning can be applied to obtain the lower bound of $(\mathrm{E}\,(\Delta_n) - \Delta_1) \,/\, (4\ln(n))$.

Proof of Theorem 2. By substituting (9) into (4) we get that for any $\epsilon > 0$,

$$p_{i,k}(f, \epsilon) = 1/\#V_k, \quad i \in V_k,\ k \geq 1. \tag{A.12}$$

By combining (A.6), (A.7) and (A.12) we derive

$$\mathrm{E}\,(\Delta_n) - \Delta_1 = 2\sum_{k=1}^{n-1} \frac{\#E_k}{\#V_k(\#V_k - 1)},\ n \geq 2, \tag{A.13}$$

and hence,

$$\mathrm{E}\,(\Delta_n) - \Delta_1 = \Upsilon_n(1, \#V_1, \#E_1) - \Upsilon_1(1, \#V_1, \#E_1),\ n \geq 2. \tag{A.14}$$

Now (10) follows by applying Lemma 2 with $\beta = 1$.

Let us prove (11). We have

$$\frac{\Delta_n - 4\ln(n)}{2\ln^{1/2}(n)} = \frac{\mathrm{E}(\Delta_n) - \Delta_1 - 4\ln(n)}{2\ln^{1/2}(n)} + \frac{\Delta_1}{2\ln^{1/2}(n)} + \frac{\Delta_n - \mathrm{E}(\Delta_n)}{2\ln^{1/2}(n)}. \tag{A.15}$$

Using (A.14), it is easy to see that the first term on the right-hand side of (A.15) is equal to the ratio

$$\frac{\{\Upsilon_n(1, \#V_1, \#E_1) - 4\ln(n)\} - \Upsilon_1(1, \#V_1, \#E_1)}{2\ln^{1/2}(n)},$$

which by Lemma 2 tends to 0 as $n \to \infty$. Evidently, the second term in the right-hand side of (A.15) tends to 0 as $n \to \infty$, too. It remains to prove that

$$\frac{\Delta_n - \mathrm{E}(\Delta_n)}{2\ln^{1/2}(n)} \xrightarrow{\mathrm{d}} \mathcal{N}\,(0,1)\,, \quad n \to \infty. \tag{A.16}$$

Applying (A.5) and (A.6), we get

$$\Delta_n - \mathrm{E}\,(\Delta_n) = \sum_{k=1}^{n-1} \left(I(B_k) - \mathrm{P}\,(B_k)\right).$$

The summands in the sum over k are independent since the occurrence of $B_1, \ldots, B_{k-1}$ does not affect the probability $\mathrm{P}(B_k)$. Also, $(I(B_k) - \mathrm{P}\,(B_k))$, $k \geq 1$ are non-identically distributed random variables with zero mean.

Let us define

$$\sigma_n^2 = \sum_{k=1}^{n-1} \mathrm{E}\left(I(B_k) - \mathrm{P}\,(B_k)\right)^2.$$

By Lyapunov's theorem (see, e.g., Theorem 27.3 in [27]), the relation (A.16) will be proven if we show that it holds

$$\lim_{n\to\infty} \frac{\sigma_n^2}{4\ln(n)} = 1 \tag{A.17}$$

and

$$\lim_{n\to\infty} \frac{1}{\sigma_n^{2+\eta}} \sum_{k=1}^{n-1} \mathrm{E}|I(B_k) - \mathrm{P}\,(B_k)\,|^{2+\eta} = 0 \tag{A.18}$$

for some positive η.

Let us prove (A.17) first. It is easy to find that

$$\sigma_n^2 = \sum_{k=1}^{n-1} \mathrm{P}(B_k) - \sum_{k=1}^{n-1} \mathrm{P}^2(B_k). \tag{A.19}$$

From (A.13) it immediately follows that $\mathrm{P}(B_k) \leq C_1/(k + \#V_1 - 2)$, $k \geq 1$, where $C_1 = 4 + 2|\# E_1 - 2\#V_1 + 2|$. This implies

$$\sum_{k=1}^{n-1} \mathrm{P}^2(B_k) \leq C_1^2 \sum_{k=1}^{n-1} \frac{1}{(k + \#V_1 - 2)^2} \leq C_1^2 \sum_{k=1}^{\infty} \frac{1}{k^2} = \frac{C_1^2 \pi^2}{6}.$$

Whence, it follows that $\ln^{-1}(n) \sum_{k=1}^{n-1} \mathrm{P}^2(B_k) \to 0$, $n \to \infty$. The latter relation with (10), (A.6) and (A.19) yields (A.17).

Lyapunov's condition (A.18) is satisfied with $\eta = 1$. To verify this we write

$$\sum_{k=1}^{n-1} \mathrm{E}|I(B_k) - \mathrm{P}\,(B_k)\,|^3 = \sum_{k=1}^{n-1} \Big\{\mathrm{P}(B_k) - \mathrm{P}^2(B_k)\Big\} \Big\{(1 - \mathrm{P}(B_k))^2 + \mathrm{P}^2(B_k)\Big\} \leq 2\sigma_n^2.$$

Thus, the left-hand side of (A.18) (with $\eta = 1$) does not exceed $\lim_{n\to\infty} 2/\sigma_n = 0$.

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