

Analytical Optimal Solution to the Minimum Time and Energy Problem of Programmed Control for Spacecraft Spatial Motion in the Class of Generalized Helical Conic Motions

Yu. N. Chelnokov^{*,a}, A. V. Molodenkov^{*,b}, and I. A. Pankratov^{*,**,c}

^{*}*Institute of Precision Mechanics and Control, Russian Academy of Sciences, Saratov, Russia*

^{**}*Saratov National Research State University named after N.G. Chernyshevsky, Saratov, Russia*
e-mail: ^achelnokovyun@gmail.com, ^bmolalxei@yandex.ru, ^cpankratovia.mechanic@gmail.com

Received May 8, 2025

Revised July 28, 2025

Accepted July 29, 2025

Abstract—The study examines the problem of optimal programmed control for the spatial motion (particularly, maneuvering) of a spacecraft, treated as a free rigid body of arbitrary dynamic configuration, with a combined cost functional incorporating both time and energy expended during the controlled motion of the spacecraft. In the class of generalized helical conic motions, an optimal analytical solution to the problem is obtained for arbitrary boundary conditions on the angular and linear positions as well as the angular and linear velocities of the spacecraft, which is then formulated into an algorithm. To describe the spatial motion, four-dimensional dual Euler parameters (Rodrigues–Hamilton parameters) are used, which are components of the Clifford dual quaternion (parabolic biquaternion) of finite displacement. The solution is derived using Chasles’ theorem on the motion of a free rigid body and the Kotelnikov–Study transfer principle. Numerical examples demonstrating the effectiveness of the proposed solution are provided.

Keywords: spacecraft, optimal programmed control, spatial motion, Clifford dual quaternion (biquaternion), dual integral combined cost functional, analytical solution, algorithm

DOI: 10.7868/S1608303226020042

1. INTRODUCTION

It is well known that the general spatial displacement of a free rigid body is equivalent to its helical displacement (Chasles’ theorem in mechanics: any displacement of a free rigid body can be accomplished by a single helical displacement along some axis, called the axis of finite helical displacement). Therefore, the motion of a free rigid body represents a continuous sequence of instantaneous helical motions. Here, the orientation of the axis of instantaneous helical motion coincides with the orientation of the instantaneous rotation axis of the rigid body, and the body’s rotation around this axis and its translational motion along this axis form (using Clifford’s duality) the so-called dual rotation angle: a complex composition of the body’s rotational and translational displacements.

Previously, the literature considered the kinematic problem of constructing a time-optimal helical displacement of a free rigid body equivalent to its general spatial displacement [1, 2]. Here, the vectors of angular and linear velocities of the body served as programmed controls. To obtain an exact (analytical) solution to the problem, the dual quaternion (biquaternion) kinematic equation of helical motion of a free rigid body, first proposed by Yu.N. Chelnokov in 1980 [3–5], was used.

Also, in [6–8], dynamic problems of analytical construction of control for the spatial motion of a rigid body (using the dual quaternion models of free rigid body motion proposed in these works) were solved within the framework of the feedback control concept.

In [6], a method for the analytical construction of control for the spatial motion of a rigid body, equivalent to the composition of angular (rotational) and translational (orbital) motions, was developed in a nonlinear dynamic formulation using Clifford dual quaternions and dual matrices. The controls ensure asymptotic stability “in the large” of any chosen programmed spatial motion in the inertial coordinate system and the desired dynamics of the controlled spatial motion of the rigid body. To construct the control laws, dual quaternion and matrix models of the spatial motion of a rigid body, the concept of solving inverse problems of dynamics, the principle of feedback control, and an approach based on reducing the nonlinear differential equations of perturbed spatial motion of the rigid body constructed in the paper to reference linear stationary differential forms of a chosen structure through the use of proposed nonlinear feedbacks in the control laws were employed. Various dual matrix (screw) control laws for the spatial motion of a rigid body are proposed, for which the nonlinear non-stationary differential equations of perturbed spatial motion of the rigid body take the form of linear stationary dual matrix differential equations of the second order (with respect to the screw part of the dual quaternion of the body’s position error), invariant with respect to any chosen programmed spatial motion of the rigid body. The constant coefficients (scalar dual or matrix dual) of these equations are the gain coefficients of the nonlinear feedbacks in the proposed dual control laws, ensuring the desired quality of control transients. The determination of the gain coefficients of the nonlinear feedbacks and the properties of such controlled motion of the rigid body are discussed.

In [7], another method for the analytical construction of control for the spatial motion of a rigid body (in particular, a spacecraft considered as a free rigid body) was developed in a nonlinear dynamic formulation using dual quaternions (Clifford parabolic biquaternions). The control ensures asymptotic stability “in the whole” of any chosen programmed motion in the inertial coordinate system and the desired dynamics of the controlled body motion. To construct the control laws, new dual quaternion differential equations of perturbed spatial motion of a rigid body are proposed, which, unlike [6], use non-normalized dual quaternions of finite displacements, dual quaternions of angular and linear velocities and accelerations of the body with non-zero dual scalar parts (i.e., four-dimensional, rather than three-dimensional, dual angular and linear velocities and accelerations). This allowed for effectively solving the synthesis problem of control ensuring asymptotic stability “in the whole” of any chosen programmed motion. To construct the control laws, the concept of solving inverse problems of dynamics, the principle of feedback control, and an approach based on reducing the constructed equations of perturbed body motion to linear stationary differential forms of a chosen structure, invariant with respect to any chosen programmed motion, through the appropriate choice of dual nonlinear feedbacks in the proposed dual quaternion control laws were also used. This approach is similar to that used in [6], but is applied to more general dual quaternion differential equations of perturbed spatial motion of a rigid body. Analytical solutions of the dual quaternion differential equations describing the dynamics of the spatial motion control process using the proposed dual quaternion control laws are constructed. The properties and regularities of such control are analyzed.

This article addresses the problem of constructing interrelated time-optimal and energy-optimal programmed control for the spatial motion of a spacecraft, considered as a free rigid body of arbitrary dynamic configuration, performing an optimal helical conic motion in an inertial coordinate system, equivalent to the composition of the translational (orbital) motion of the spacecraft’s center of mass and its angular motion (rotation) around the center of mass. The spacecraft is subject to the principal vector and principal moment of external forces, including the control force vector and

the control torque vector. Four-dimensional dual Euler parameters (Rodrigues–Hamilton parameters) and three-dimensional dual velocities of the spacecraft, which are components of the Clifford dual quaternion of finite displacement and the dual quaternion of angular and linear velocities, respectively, i.e., dual quaternions with complex (dual) components, are used to describe the general spatial motion of the spacecraft. The Clifford duality s used to describe this motion possesses the property that its square is zero: $s^2 = 0$.

The problem formulation (for arbitrary boundary conditions on the angular and linear positions of the spacecraft in inertial space and on its absolute angular and linear velocities, with the dual control vector being unconstrained) is based on using a dynamic model of the general spatial motion of the spacecraft, equivalent to its helical motion. To solve the problem, the dual quaternion (biquaternion) kinematic equation of motion of a free rigid body [3–5] (where the dual variable is the dual quaternion (biquaternion) of the body's finite displacement in inertial space) was used, as well as analytical expressions obtained by the authors in the article as explicit functions of time for the four-dimensional dual quaternion of finite displacement and for the three-dimensional dual quaternions of angular and linear velocities and accelerations of the spacecraft, describing the generalized helical conic motion of the spacecraft in inertial space. To obtain these analytical expressions, a new analytical solution to the dual quaternion kinematic equation of spatial motion of a free rigid body for its generalized helical conic motion is constructed in the article. Using these analytical expressions (after extracting their principal and moment parts), the vectors of programmed control force and programmed control torque applied to the spacecraft are formed in accordance with the concept of solving inverse problems of dynamics.

Using the Kotelnikov–Study transfer principle, which allows extending quaternion formulas describing angular motion control to dual quaternion (biquaternion) formulas describing the control of general spatial motion of a rigid body, an algorithm for programmed control of spacecraft spatial motion, optimal in terms of time and minimum energy costs in the class of generalized helical conic motions, is obtained. This algorithm is constructed based on a generalization of the previously obtained approximate analytical solution of the quaternion problem of time-optimal and energy-optimal spacecraft (as a rigid body) reorientation under arbitrary boundary conditions on the angular position and angular velocity of the spacecraft. This solution was obtained within the framework of the classical Poincaré concept, interpreting arbitrary angular motion of a rigid body in terms of precession cones, or otherwise, generalized conic angular motion [9–11].

To find the optimal parameters of the generalized helical conic motion and the corresponding optimal angular and linear velocities and accelerations of the spacecraft, the L.S. Pontryagin maximum principle was used. The found analytical solution of the problem can be considered as a solution to the problem of time-optimal and energy-optimal programmed control of spacecraft spatial motion under arbitrary boundary conditions in the class of generalized dual (helical) conic motions. The results of modeling optimal spatial maneuvering of a spacecraft in an inertial coordinate system using the proposed dual quaternion models and dual control laws are presented, demonstrating the effectiveness of the proposed method for controlling spacecraft motion.

The proposed optimal analytical solution to the problem of programmed control of spacecraft spatial motion has not only theoretical but also applied significance, as it allows using in the spacecraft motion control system the obtained analytical expressions (as explicit functions of time) for the dual quaternion of the spacecraft's finite displacement and for the dual quaternion of its absolute angular and linear velocities, describing the optimal programmed helical conic motion of the spacecraft in the inertial coordinate system. It also allows using optimal analytical time-varying laws for the programmed control force and programmed control torque applied to the spacecraft. This is important, in particular, for spacecraft spatial maneuvering at exceptionally high rates of displacement, when the time for calculating the optimal spatial trajectory of the spacecraft

maneuver and for calculating the programmed control laws enabling this trajectory is extremely limited.

The article continues the research initiated in [12].

2. INITIAL EQUATIONS OF SPATIAL MOTION OF A FREE RIGID BODY

Consider a free rigid body, e.g., a spacecraft, capable of performing arbitrary spatial motion relative to the main (inertial) coordinate system, equivalent to a spatial helical motion, and also equivalent to the composition of the translational motion of the body together with an arbitrarily chosen point of the body and the rotation of the body around that point. The body is subject to an arbitrary principal vector and principal moment of external forces. They include the control force vector and the control torque vector. We will consider the controlled spatial motion of the body relative to the inertial coordinate system $O_1\xi_1\xi_2\xi_3$ (ξ) (its origin O_1 is the Earth's center of mass, the axis $O_1\xi_3$ is directed along the Earth's rotation axis, and the axes $O_1\xi_1$ and $O_1\xi_2$ lie in the equatorial plane and do not participate in the Earth's diurnal rotation). We rigidly attach to the rigid body a coordinate system $CX_1X_2X_3$ (X) with origin at the body's center of mass C .

Let us denote: \mathbf{r} and \mathbf{v} —the radius vector and velocity vector of the center of mass of the rigid body (spacecraft) in the inertial coordinate system; $\boldsymbol{\lambda}$ —the quaternion of orientation of the body in this coordinate system, its components are the Rodrigues–Hamilton (Euler) parameters λ_j , $j = \overline{0,3}$, identical in bases ξ and X (it is assumed that in the initial position (before body rotation), the orientations of coordinate systems ξ and X coincide); $\boldsymbol{\omega}$ and $\boldsymbol{\varepsilon}$ —the vectors of absolute angular velocity and absolute angular acceleration of the body; \mathbf{F}_c and \mathbf{M}_c —the vectors of control force and control torque applied to the body; $\mathbf{F} = \mathbf{F}(t, \mathbf{r}, \mathbf{v})$ and $\mathbf{M} = \mathbf{M}(t, \boldsymbol{\lambda}, \boldsymbol{\omega})$ —the principal vector of other external forces acting on the rigid body (gravity, drag, and other forces of interaction of the body with the external environment), and the principal moment of these forces, calculated relative to the body's center of mass, assumed to be known functions of time t and the variables \mathbf{r} , \mathbf{v} and $\boldsymbol{\lambda}$, $\boldsymbol{\omega}$.

The initial differential equations of motion of a rigid body, written in the body-fixed coordinate system X , have the form:

$$m [\dot{\mathbf{v}}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{v}_x] = \mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x) + \mathbf{F}_{cx}, \quad \dot{\mathbf{r}}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{r}_x = \mathbf{v}_x; \quad (2.1)$$

$$\dot{\boldsymbol{\omega}}_x = \boldsymbol{\varepsilon}_x = \mathbf{J}^{-1} [\mathbf{M}_x(t, \boldsymbol{\lambda}, \boldsymbol{\omega}_x) - \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{J}\boldsymbol{\omega}_x + \mathbf{M}_{cx}]; \quad (2.2)$$

$$2\dot{\boldsymbol{\lambda}} = \boldsymbol{\lambda} \circ \boldsymbol{\omega}_x; \quad (2.3)$$

$$\mathbf{r}_x = x_1\mathbf{i} + x_2\mathbf{j} + x_3\mathbf{k}, \quad \dot{\mathbf{r}}_x = \dot{x}_1\mathbf{i} + \dot{x}_2\mathbf{j} + \dot{x}_3\mathbf{k},$$

$$\mathbf{v}_x = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}, \quad \dot{\mathbf{v}}_x = \dot{v}_1\mathbf{i} + \dot{v}_2\mathbf{j} + \dot{v}_3\mathbf{k};$$

$$\boldsymbol{\lambda} = \lambda_0 + \boldsymbol{\lambda}_v = \lambda_0 + \lambda_1\mathbf{i} + \lambda_2\mathbf{j} + \lambda_3\mathbf{k}, \quad \dot{\boldsymbol{\lambda}} = \dot{\lambda}_0 + \dot{\boldsymbol{\lambda}}_v = \dot{\lambda}_0 + \dot{\lambda}_1\mathbf{i} + \dot{\lambda}_2\mathbf{j} + \dot{\lambda}_3\mathbf{k};$$

$$\boldsymbol{\omega}_x = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}, \quad \dot{\boldsymbol{\omega}}_x = \dot{\omega}_1\mathbf{i} + \dot{\omega}_2\mathbf{j} + \dot{\omega}_3\mathbf{k}, \quad \boldsymbol{\varepsilon}_x = \varepsilon_1\mathbf{i} + \varepsilon_2\mathbf{j} + \varepsilon_3\mathbf{k};$$

$$\mathbf{J} = \begin{pmatrix} J_{11} & -J_{12} & -J_{13} \\ -J_{21} & J_{22} & -J_{23} \\ -J_{31} & -J_{32} & J_{33} \end{pmatrix}, \quad \mathbf{K}(\boldsymbol{\omega}_x) = \begin{pmatrix} 0 & -\omega_3 & \omega_2 \\ \omega_3 & 0 & -\omega_1 \\ -\omega_2 & \omega_1 & 0 \end{pmatrix}.$$

Here in equations (2.1), (2.2) \mathbf{r}_x , \mathbf{v}_x , $\boldsymbol{\omega}_x$, $\boldsymbol{\varepsilon}_x$, \mathbf{F}_x , \mathbf{M}_x , \mathbf{F}_{cx} , \mathbf{M}_{cx} —column vectors of size 3×1 or, further, quaternions with zero scalar parts, composed of the projections x_k , v_k , ω_k , ε_k , F_k , M_k , F_{ck} , M_{ck} , $k = \overline{1,3}$, of vectors \mathbf{r} , \mathbf{v} , $\boldsymbol{\omega}$, $\boldsymbol{\varepsilon}$, \mathbf{F} , \mathbf{M} , \mathbf{F}_c , \mathbf{M}_c onto the axes of the body-fixed coordinate system X ; m —body mass, \mathbf{J} —constant inertia matrix of the rigid body; $\mathbf{K}(\boldsymbol{\omega}_x)$ —skew-symmetric matrix of angular velocities of the body, associated with the vector $\boldsymbol{\omega}$; \mathbf{i} , \mathbf{j} , \mathbf{k} —orts of the hypercomplex space (Hamilton's vector imaginary units); \mathbf{a}_y —mapping of vector \mathbf{a} onto basis Y

($Y = \xi, X$), defined as a quaternion $\mathbf{a}_y = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$, whose components are the projections a_k of vector \mathbf{a} onto basis Y ; the superscript dot denotes the time derivative t (when calculating the derivative of a quaternion, the orts $\mathbf{i}, \mathbf{j}, \mathbf{k}$ are considered constant), the sign “ \circ ” denotes quaternion multiplication. The first matrix equation (2.1) and the matrix equation (2.2) are dynamic, while the second matrix equation (2.1) and the quaternion equation (2.3)—are kinematic equations of spatial motion of a rigid body, representing the composition of translational (trajectory) and angular (rotational) motions. These equations constitute a system of nonlinear, non-stationary differential equations of the thirteenth order with respect to the variables x_k, v_k and λ_j, ω_k . In the case where the coordinate system $OX_1X_2X_3$ (X) with origin at another arbitrarily chosen point O of the body is rigidly attached to the rigid body, the principal vector \mathbf{F} and principal moment \mathbf{M} of external forces include the translational inertial force and its moment relative to the point O of the rigid body.

3. PROBLEM STATEMENT FOR SPACECRAFT CONTROL USING DUAL QUATERNIONS AND THE CONCEPT OF SOLVING THE INVERSE PROBLEM OF DYNAMICS

In accordance with the concept of solving inverse problems of dynamics, the laws for forming the column vector \mathbf{F}_{cx} of the control force \mathbf{F}_c and the column vector \mathbf{M}_{cx} of the control torque \mathbf{M}_c , composed of the projections F_{ck} and M_{ck} , $k = \overline{1,3}$, of the vectors \mathbf{F}_c and \mathbf{M}_c onto the axes of the coordinate system X associated with the spacecraft, are obtained based on the equations of motion (2.1), (2.2) and have the form:

$$\mathbf{F}_{cx} = m [\dot{\mathbf{v}}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{v}_x] - \mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x), \quad (3.1)$$

$$\mathbf{M}_{cx} = \mathbf{J}\boldsymbol{\varepsilon}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{J}\boldsymbol{\omega}_x - \mathbf{M}_x(t, \boldsymbol{\lambda}, \boldsymbol{\omega}_x). \quad (3.2)$$

The vector-matrix control law (3.1) includes the column vector $\dot{\mathbf{v}}_x = \mathbf{w}_x$, composed of the time derivatives of the projections v_k of the vector \mathbf{v} of the velocity of the body's center of mass C in the inertial coordinate system ξ (the vector of absolute velocity of the spacecraft's center of mass) onto the axes of the rotating coordinate system X , and the vector-matrix control law (3.2) includes the column vector $\boldsymbol{\varepsilon}_x$, composed of the projections ε_k of the vector $\boldsymbol{\varepsilon}$ of angular acceleration of the spacecraft in the inertial coordinate system (the vector of absolute angular acceleration of the body) onto the axes of the coordinate system X .

These column vectors $\dot{\mathbf{v}}_x = \mathbf{w}_x$ and $\boldsymbol{\varepsilon}_x$ will be considered as new controls. They can be constructed, for example, using the L.S. Pontryagin maximum principle based on the vector-matrix and quaternion equations:

$$\dot{\mathbf{v}}_x = \mathbf{w}_x, \quad \dot{\mathbf{r}}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{r}_x = \mathbf{v}_x, \quad (3.3)$$

$$\dot{\boldsymbol{\omega}}_x = \boldsymbol{\varepsilon}_x, \quad 2\dot{\boldsymbol{\lambda}} = \boldsymbol{\lambda} \circ \boldsymbol{\omega}_x, \quad (3.4)$$

derived from the equations of motion of a rigid body (2.1)–(2.3).

We emphasize that the new controls \mathbf{w}_x and $\boldsymbol{\varepsilon}_x$ have a clear mechanical meaning. \mathbf{w}_x —is the column vector whose components are the projections onto the axes of the body-fixed coordinate system X of the component $\dot{\mathbf{v}} = (d\mathbf{v}/dt)_{\text{loc}}$ of the vector \mathbf{W} of absolute linear acceleration of the spacecraft's center of mass C (projections of the local derivative $(d\mathbf{v}/dt)_{\text{loc}}$ of the vector \mathbf{v} of absolute velocity of point C , computed in this coordinate system). The total vector \mathbf{W} of absolute linear acceleration of the spacecraft's center of mass is equal to the sum of the vectors $\dot{\mathbf{v}}$ and $\boldsymbol{\omega} \times \mathbf{v}$, defined by their projections in the coordinate system X : $\mathbf{W}_x = \dot{\mathbf{v}}_x + \boldsymbol{\omega}_x \times \mathbf{v}_x = \mathbf{w}_x + \boldsymbol{\omega}_x \times \mathbf{v}_x$ (here “ \times ” denotes the cross product).

$\boldsymbol{\varepsilon}$ —is the vector of absolute angular acceleration of the rigid body (one of the main characteristics of absolute angular (rotational) motion of a rigid body), $\boldsymbol{\varepsilon}_x$ —is the column vector whose components are the projections of the vector $\boldsymbol{\varepsilon}$ onto the axes of the coordinate system X .

Introducing the component $\dot{\mathbf{v}} = (d\mathbf{v}/dt)_{\text{loc}}$ of the vector of absolute linear acceleration of the spacecraft's center of mass as a control allows proposing, for describing the spatial motion of the spacecraft, a biquaternion differential model of spacecraft spatial motion, convenient for solving the problem of controlling spacecraft spatial motion using dual quaternions and the maximum principle.

After finding the new controls \mathbf{w}_x and $\boldsymbol{\varepsilon}_x$ based on the differential equations (3.3) and (3.4), the phase variables \mathbf{r}_x , \mathbf{v}_x , $\boldsymbol{\lambda}$, $\boldsymbol{\omega}_x$, appearing in these equations, corresponding to these controls will also be found, and then the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} will be found in accordance with the algebraic relations (3.1), (3.2).

Thus, the problem of constructing the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} using the concept of solving inverse problems of dynamics is reduced by the authors to constructing the required component \mathbf{w}_x of absolute linear acceleration and the required absolute angular acceleration $\boldsymbol{\varepsilon}_x$, which act as new controls in the differential equations (3.3), (3.4).

Obviously, equations (3.3), (3.4) are valid for any moving object (i.e., a free rigid body). Therefore, the controls \mathbf{w}_x and $\boldsymbol{\varepsilon}_x$ for all such bodies are constructed in the same way. The specifics of the object (its mass-inertial and other characteristics, the acting external perturbing forces and their moments) are taken into account when constructing the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} based on the final relations (3.1), (3.2); their right-hand sides include the principal vector $\mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x)$ of other external forces acting on the rigid body (gravity, drag, and other forces of interaction of the body with the external environment) and the principal moment $\mathbf{M} = \mathbf{M}(t, \boldsymbol{\lambda}, \boldsymbol{\omega})$ of these forces, calculated relative to the body's center of mass. The principal vector and principal moment are assumed to be known functions of time t and the variables \mathbf{r} , \mathbf{v} and $\boldsymbol{\lambda}$, $\boldsymbol{\omega}$.

Let us set the problem of finding the new controls \mathbf{w}_x and $\boldsymbol{\varepsilon}_x$ in terms of dual quaternions. Let us introduce the kinematic screw \mathbf{U} of the spacecraft, whose mapping \mathbf{U}_x onto the body-fixed (spacecraft) basis X is defined by the dual quaternion:

$$\mathbf{U}_x = U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k} = \boldsymbol{\omega}_x + s\mathbf{v}_x, \quad U_k = \omega_k + sv_k, \quad (3.5)$$

where $\boldsymbol{\omega}_x$ and \mathbf{v}_x are quaternions: $\boldsymbol{\omega}_x = \omega_1\mathbf{i} + \omega_2\mathbf{j} + \omega_3\mathbf{k}$, $\mathbf{v}_x = v_1\mathbf{i} + v_2\mathbf{j} + v_3\mathbf{k}$; s —is the Clifford symbol (duality) possessing the property $s^2 = 0$; $U_k = \omega_k + sv_k$, $k = \overline{1, 3}$ —are the dual orthogonal projections of the kinematic screw \mathbf{U} onto basis X .

Then the vector-matrix (3.3) and quaternion (3.4) differential equations can be replaced by the following two dual quaternion (biquaternion) differential equations [6–8]:

$$\dot{\mathbf{U}}_x = \boldsymbol{\varepsilon}_x + s\mathbf{w}_x = \mathbf{H}_x; \quad (3.6)$$

$$2\dot{\boldsymbol{\Lambda}} = \boldsymbol{\Lambda} \circ \mathbf{U}_x; \quad (3.7)$$

$$\begin{aligned} \mathbf{U}_x &= U_1\mathbf{i} + U_2\mathbf{j} + U_3\mathbf{k}, & U_k &= \omega_k + sv_k, & \boldsymbol{\varepsilon}_x &= \varepsilon_1\mathbf{i} + \varepsilon_2\mathbf{j} + \varepsilon_3\mathbf{k}, \\ \mathbf{w}_x &= w_1\mathbf{i} + w_2\mathbf{j} + w_3\mathbf{k}, & \mathbf{H}_x &= H_1\mathbf{i} + H_2\mathbf{j} + H_3\mathbf{k}, & H_k &= \varepsilon_k + sw_k; \end{aligned} \quad (3.8)$$

$$\boldsymbol{\Lambda} = \Lambda_0 + \boldsymbol{\Lambda}_v = \Lambda_0 + \Lambda_1\mathbf{i} + \Lambda_2\mathbf{j} + \Lambda_3\mathbf{k} = \boldsymbol{\lambda} + s\boldsymbol{\lambda}^0, \quad \Lambda_j = \lambda_j + s\lambda_j^0. \quad (3.9)$$

Here, the sought variables are the dual quaternion $\mathbf{U}_x = \boldsymbol{\omega}_x + s\mathbf{v}_x$ (mapping of the kinematic screw \mathbf{U} of the spacecraft onto the spacecraft-fixed basis X) and the dual quaternion $\boldsymbol{\Lambda} = \boldsymbol{\lambda} + s\boldsymbol{\lambda}^0$ of the spacecraft's displacement in inertial space. Its principal part (the quaternion $\boldsymbol{\lambda}$) describes the orientation of the spacecraft in the inertial coordinate system, while the moment part (the quaternion $\boldsymbol{\lambda}^0$) describes the location of the spacecraft in this coordinate system. The Cartesian coordinates x_k , $k = \overline{1, 3}$, of the spacecraft's center of mass in this coordinate system can be determined by knowing the components of the quaternions $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^0$. The dual control in this problem is the dual quaternion $\mathbf{H}_x = \boldsymbol{\varepsilon}_x + s\mathbf{w}_x$, i.e., the dual composition of the required absolute angular

acceleration ε_x and the required component \mathbf{w}_x of absolute linear acceleration of the spacecraft (the dual composition of the new controls \mathbf{w}_x and ε_x).

Introducing the component of the vector of absolute linear acceleration of the spacecraft's center of mass (the local derivative of the vector of absolute velocity of the center of mass) as a control allows proposing, for describing the spatial motion of the spacecraft, the differential dual quaternion equation (3.6) for the kinematic screw, which, together with the kinematic dual quaternion equation (3.7) of spatial motion of the spacecraft, forms a complete mathematical model of its spatial motion, convenient for solving the problem of controlling spacecraft spatial motion using dual quaternions and the maximum principle.

The coordinates ξ_k of the spacecraft's center of mass in the inertial coordinate system ξ (i.e., the projections of its radius vector \mathbf{r}) and the projections x_k of this vector onto the axes of the spacecraft-fixed coordinate system X are related to the components of the quaternions $\boldsymbol{\lambda}$ and $\boldsymbol{\lambda}^0$ by the relations [3, 5]:

$$\mathbf{r}_\xi = \xi_1 \mathbf{i} + \xi_2 \mathbf{j} + \xi_3 \mathbf{k} = 2\boldsymbol{\lambda}^0 \circ \tilde{\boldsymbol{\lambda}}, \quad \mathbf{r}_x = x_1 \mathbf{i} + x_2 \mathbf{j} + x_3 \mathbf{k} = \tilde{\boldsymbol{\lambda}} \circ \boldsymbol{\lambda}^0, \quad (3.10)$$

where the upper tilde denotes quaternion conjugation.

Let us formulate the following problem for finding the new controls \mathbf{w}_x and ε_x using dual quaternions: it is necessary to construct, in the class of optimal generalized helical conic motions, a dual quaternion programmed control $\mathbf{H}_x = \varepsilon_x + s\mathbf{w}_x$ (the dual composition of the component of absolute linear acceleration of the spacecraft's center of mass and the vector of absolute angular acceleration of the rigid body), ensuring a programmed transfer of the spacecraft, whose motion is described by equations (3.6), (3.7), from its arbitrary given initial state

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(0) = \boldsymbol{\lambda}(0) + s\boldsymbol{\lambda}^0(0), \quad \mathbf{U}_x = \mathbf{U}_x(0) = \boldsymbol{\omega}_x(0) + s\mathbf{v}_x(0) \quad (3.11)$$

to its given final state

$$\boldsymbol{\Lambda} = \boldsymbol{\Lambda}(t_1) = \boldsymbol{\Lambda}_k = \boldsymbol{\lambda}_k + s\boldsymbol{\lambda}_k^0, \quad \mathbf{U}_x = \mathbf{U}_x(t_1) = \mathbf{U}_x^k = \boldsymbol{\omega}_x^k + s\mathbf{v}_x^k \quad (3.12)$$

in time t_1 , which is to be determined.

The boundary conditions $\boldsymbol{\lambda}^0(0)$ and $\boldsymbol{\lambda}_k^0$ are found through the given initial and final values of the quaternions $\boldsymbol{\lambda}$ and \mathbf{r}_ξ using the formula:

$$\boldsymbol{\lambda}^0 = (1/2) \mathbf{r}_\xi \circ \boldsymbol{\lambda} = (1/2) (\xi_1 \mathbf{i} + \xi_2 \mathbf{j} + \xi_3 \mathbf{k}) \circ (\lambda_0 + \lambda_1 \mathbf{i} + \lambda_2 \mathbf{j} + \lambda_3 \mathbf{k}), \quad (3.13)$$

where ξ_k , $k = \overline{1, 3}$ are the coordinates of the object in the inertial coordinate system. After solving the stated problem, it is necessary to extract the principal part ε_x and the moment part $\mathbf{w}_x = \dot{\mathbf{v}}_x$ from the constructed control $\mathbf{H}_x = \varepsilon_x + s\mathbf{w}_x$. After this extraction, the laws for forming the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} are obtained in accordance with the concept of solving inverse problems of dynamics using formulas (3.1) and (3.2).

4. ANALYTICAL SOLUTION OF THE OPTIMAL PROGRAMMED CONTROL PROBLEM

Using the Kotelnikov–Study transfer principle, which allows extending quaternion formulas describing angular motion control to dual quaternion formulas describing the control of general spatial motion of a rigid body, we obtain an algorithm for programmed control of spacecraft spatial motion, optimal in the class of generalized helical conic motions, for the problem of Section 3 of the article. This algorithm is constructed using Clifford dual quaternions and the Kotelnikov–Study transfer principle by generalizing the analytical solution of the quaternion problem of time-optimal and energy-optimal spacecraft reorientation in the class of generalized conic angular motions of a

rigid body under arbitrary boundary conditions on the angular position and angular velocity of the spacecraft [9–11], which, in turn, was obtained based on the exact solution of a modified problem of optimal rotation of a rigid body introduced by Ya.G. Sapunkov [13]. We present this biquaternion solution.

In the case where a free rigid body performs a spatial generalized helical conic motion, for which the kinematic screw \mathbf{U}_x of the body has the form

$$\mathbf{U}_x(t) = U_1 \mathbf{i} + U_2 \mathbf{j} + U_3 \mathbf{k} = \boldsymbol{\omega}_x + s \mathbf{v}_x = \left(\dot{F}(t) \sin G(t) \right) \mathbf{i} + \left(\dot{F}(t) \cos G(t) \right) \mathbf{j} + \dot{G}(t) \mathbf{k}, \quad (4.1)$$

where $F(t)$, $G(t)$, $\dot{F}(t)$, $\dot{G}(t)$ —are arbitrary differentiable dual functions of time:

$$\begin{aligned} F(t) &= f(t) + sf^0(t), & \dot{F}(t) &= \dot{f}(t) + s\dot{f}^0(t), \\ G(t) &= g(t) + sg^0(t), & \dot{G}(t) &= \dot{g}(t) + s\dot{g}^0(t), \end{aligned}$$

the dual quaternion kinematic equation (3.7) of the general spatial motion of a rigid body has an analytical solution constructed by the authors:

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}(0) \circ \mathbf{exp} \{ - (G(0)/2) \mathbf{k} \} \circ \mathbf{exp} \{ ((F(t) - F(0))/2) \mathbf{j} \} \circ \mathbf{exp} \{ (G(t)/2) \mathbf{k} \} \quad (4.2)$$

—a dual analogue of the known quaternion solution [9–11] of the kinematic equation of rotational motion of a body, where “ $\mathbf{exp}\{\cdot\}$ ” denotes the dual quaternion exponential [5].

The helical motion of a free rigid body (spacecraft) described by relations (4.1), (4.2) can be generalized by adding an arbitrary dual rotation in the inertial coordinate system by a dual constant angle around some axis. Such a dual rotation is given by an arbitrary constant dual quaternion $\mathbf{K} = \boldsymbol{\kappa} + s\boldsymbol{\kappa}^0$, $\|\mathbf{K}\| = 1$ (multiplications on the left and right by the dual quaternions $\tilde{\mathbf{K}}$ and \mathbf{K} respectively, where $\tilde{\mathbf{K}} \circ \mathbf{K} = \mathbf{K} \circ \tilde{\mathbf{K}} = 1$, are added to the right-hand sides of formulas (4.1), (4.2)):

$$\mathbf{U}_x(t) = \tilde{\mathbf{K}} \circ \left[\left(\dot{F}(t) \sin G(t) \right) \mathbf{i} + \left(\dot{F}(t) \cos G(t) \right) \mathbf{j} + \dot{G}(t) \mathbf{k} \right] \circ \mathbf{K}, \quad (4.3)$$

$$\boldsymbol{\Lambda}(t) = \boldsymbol{\Lambda}(0) \circ \tilde{\mathbf{K}} \circ \mathbf{exp} \{ - (G(0)/2) \mathbf{k} \} \circ \mathbf{exp} \{ ((F(t) - F(0))/2) \mathbf{j} \} \circ \mathbf{exp} \{ (G(t)/2) \mathbf{k} \} \circ \mathbf{K}. \quad (4.4)$$

Formulas (4.3), (4.4) contain (for $s = 0$) all previously found exact quaternion analytical solutions of the classical problem of optimal rotation of a dynamically symmetric rigid body in the case where the angular velocity vector maintains a constant direction or describes a circular cone in space throughout its motion [14, 15], and their dual analogues (for $s \neq 0$), constructed here in this article. The proposed structure of the kinematic screw (4.1) or (4.3), in the quaternion case—the angular velocity vector of the spacecraft, correlates well with the Poincot concept: any arbitrary angular motion of a rigid body around a fixed point can be considered as some generalized conic motion of the rigid body.

We restrict ourselves to the case where the constant dual quaternion \mathbf{K} is represented as a product of two constant dual quaternions:

$$\mathbf{K} = \mathbf{K}_2 \circ \mathbf{K}_1, \quad \mathbf{K}_1 = \mathbf{exp} \{ (A_1/2) \mathbf{i} \}, \quad \mathbf{K}_2 = \mathbf{exp} \{ (A_2/2) \mathbf{j} \}, \quad (4.5)$$

where $A_1 = \alpha_1 + s\alpha_1^0$, $A_2 = \alpha_2 + s\alpha_2^0$ —are some dual scalar constants. Note that the dual quaternions \mathbf{K}_1 and \mathbf{K}_2 define dual rotations of the screw \mathbf{U}_x around the axes \mathbf{i} and \mathbf{j} . The dual rotation around the axis \mathbf{k} is already included in (4.4), as an additive constant is included in the dual function $G(t)$.

The conjugate dual quaternion $\tilde{\mathbf{K}}$ will be represented as:

$$\tilde{\mathbf{K}} = \tilde{\mathbf{K}}_1 \circ \tilde{\mathbf{K}}_2, \quad \tilde{\mathbf{K}}_1 = \mathbf{exp} \{ - (A_1/2) \mathbf{i} \}, \quad \tilde{\mathbf{K}}_2 = \mathbf{exp} \{ - (A_2/2) \mathbf{j} \}. \quad (4.6)$$

Then the kinematic screw \mathbf{U}_x of the free rigid body (spacecraft) and the analytical solution of the dual quaternion kinematic equation (3.7) for the considered spatial helical motion of the spacecraft take the form:

$$\begin{aligned} \mathbf{U}_x(t) = & \mathbf{exp}\{- (A_1/2) \mathbf{i}\} \circ \mathbf{exp}\{- (A_2/2) \mathbf{j}\} \\ & \circ \left[\left(\dot{F}(t) \sin G(t) \right) \mathbf{i} + \left(\dot{F}(t) \cos G(t) \right) \mathbf{j} + \dot{G}(t) \mathbf{k} \right] \\ & \circ \mathbf{exp}\{(A_2/2) \mathbf{j}\} \circ \mathbf{exp}\{(A_1/2) \mathbf{i}\}, \end{aligned} \quad (4.7)$$

$$\begin{aligned} \Lambda(t) = & \Lambda(0) \circ \mathbf{exp}\{- (A_1/2) \mathbf{i}\} \circ \mathbf{exp}\{- (A_2/2) \mathbf{j}\} \\ & \circ \mathbf{exp}\{- (G(0)/2) \mathbf{k}\} \circ \mathbf{exp}\{((F(t) - F(0))/2) \mathbf{j}\} \\ & \circ \mathbf{exp}\{(G(t)/2) \mathbf{k}\} \circ \mathbf{exp}\{(A_2/2) \mathbf{j}\} \circ \mathbf{exp}\{(A_1/2) \mathbf{i}\}. \end{aligned} \quad (4.8)$$

The Cartesian coordinates ξ_k of the body's center of mass in the inertial coordinate system and the projections x_k of the radius vector of the center of mass onto the axes of the body-fixed coordinate system X are found through the components of the quaternions λ and λ^0 using formulas (3.10), and the projections of the velocity vector of the spacecraft's center of mass in the inertial coordinate system onto its own coordinate axes are found using the quaternion formula

$$\begin{aligned} \mathbf{v}_\xi = \dot{\mathbf{r}}_\xi = & \dot{\xi}_1 \mathbf{i} + \dot{\xi}_2 \mathbf{j} + \dot{\xi}_3 \mathbf{k} = 2 \left(\dot{\lambda}^0 \circ \tilde{\lambda} + \lambda^0 \circ \tilde{\dot{\lambda}} \right) \\ = & 2 \dot{\lambda}^0 \circ \tilde{\lambda} - \lambda^0 \circ \omega_x \circ \tilde{\lambda} = \left(2 \dot{\lambda}^0 - \lambda^0 \circ \omega_x \right) \circ \tilde{\lambda}. \end{aligned} \quad (4.9)$$

Let us set the problem of finding the optimal parameters of the helical conic motion of the spacecraft (generalizing the problem of finding the optimal parameters of the generalized angular conic motion of the spacecraft [9–11]). We will consider the second derivatives of the dual functions $F(t)$ and $G(t)$ (of the dual parameters of the helical conic motion) as control parameters. Let $\dot{F} = F_1$, $\dot{G} = G_1$ —be the dual phase coordinates, and U_{c1} , U_{c2} —be the dual control parameters. Then the controlled system will be described by the following system of scalar dual differential equations:

$$\dot{F} = F_1, \quad \dot{G} = G_1, \quad \dot{F}_1 = U_{c1}, \quad \dot{G}_1 = U_{c2}, \quad (4.10)$$

where $F = f + sf^0$, $G = g + sg^0$, $F_1 = f_1 + sf_1^0$, $G_1 = g_1 + sg_1^0$ —are the dual phase coordinates, $U_{c1} = u_{c1} + su_{c1}^0$, $U_{c2} = u_{c2} + su_{c2}^0$ —are the dual controls.

System (4.10) of four scalar dual differential equations is equivalent to a system of eight scalar real differential equations:

$$\dot{f} = f_1, \quad \dot{g} = g_1, \quad \dot{f}_1 = u_{c1}, \quad \dot{g}_1 = u_{c2}; \quad (4.11)$$

$$\dot{f}^0 = f_1^0, \quad \dot{g}^0 = g_1^0, \quad \dot{f}_1^0 = u_{c1}^0, \quad \dot{g}_1^0 = u_{c2}^0. \quad (4.12)$$

The subsystems of real differential equations (4.11) and (4.12) are independent, since the original system (4.10) of dual differential equations is linear.

Then, for the dual controlled system (4.10) or the real controlled system (4.11), (4.12), the following problem can be formulated: it is required to find the dual optimal controls $U_{c1}(t) = u_{c1}(t) + su_{c1}^0(t)$, $U_{c2}(t) = u_{c2}(t) + su_{c2}^0(t)$ or the real optimal controls $u_{c1}(t)$, $u_{c2}(t)$ and $u_{c1}^0(t)$, $u_{c2}^0(t)$, which transfer the controlled system (4.10) ((4.11), (4.12)) from the initial state

$$\begin{aligned} & \left. \begin{aligned} F = F(0) = f(0) + sf^0(0), \quad G = G(0) = g(0) + sg^0(0), \\ F_1 = F_1(0) = f_1(0) + sf_1^0(0), \quad G_1 = G_1(0) = g_1(0) + sg_1^0(0) \end{aligned} \right\} \\ & \sim \left\{ \begin{aligned} f = f(0), \quad f^0 = f^0(0), \quad g = g(0), \quad g^0 = g^0(0), \\ f_1 = f_1(0), \quad f_1^0 = f_1^0(0), \quad g_1 = g_1(0), \quad g_1^0 = g_1^0(0) \end{aligned} \right. \end{aligned} \quad (4.13)$$

to the final state

$$\left. \begin{aligned} F &= F(t_1) = f(t_1) + sf^0(t_1), & G &= G(t_1) = g(t_1) + sg^0(t_1), \\ F_1 &= F_1(t_1) = f_1(t_1) + sf_1^0(t_1), & G_1 &= G_1(t_1) = g_1(t_1) + sg_1^0(t_1) \end{aligned} \right\} \quad (4.14)$$

$$\sim \begin{cases} f = f(t_1), & f^0 = f^0(t_1), & g = g(t_1), & g^0 = g^0(t_1), \\ f_1 = f_1(t_1), & f_1^0 = f_1^0(t_1), & g_1 = g_1(t_1), & g_1^0 = g_1^0(t_1) \end{cases}$$

and minimize the dual combined functional:

$$J = \int_0^{t_1} (1 + U_{c1}^2 + U_{c2}^2) dt \quad (4.15)$$

and satisfy relations (4.7), (4.8) at the initial and final instants of time (the final time instant t_1 is not fixed and is subject to determination):

$$\begin{aligned} &[(F_1(0) \sin G(0)) \mathbf{i} + (F_1(0) \cos G(0)) \mathbf{j} + G_1(0) \mathbf{k}] \\ &= \mathbf{exp} \{(A_2/2) \mathbf{j}\} \circ \mathbf{exp} \{(A_1/2) \mathbf{i}\} \circ \mathbf{U}_x(0) \\ &\quad \circ \mathbf{exp} \{-(A_1/2) \mathbf{i}\} \circ \mathbf{exp} \{-(A_2/2) \mathbf{j}\}, \end{aligned} \quad (4.16)$$

$$\begin{aligned} &[(F_1(t_1) \sin G(t_1)) \mathbf{i} + (F_1(t_1) \cos G(t_1)) \mathbf{j} + G_1(t_1) \mathbf{k}] \\ &= \mathbf{exp} \{(A_2/2) \mathbf{j}\} \circ \mathbf{exp} \{(A_1/2) \mathbf{i}\} \circ \mathbf{U}_x(t_1) \\ &\quad \circ \mathbf{exp} \{-(A_1/2) \mathbf{i}\} \circ \mathbf{exp} \{-(A_2/2) \mathbf{j}\}; \end{aligned} \quad (4.17)$$

$$\begin{aligned} &\Lambda(0) \circ \mathbf{exp} \{-(A_1/2) \mathbf{i}\} \circ \mathbf{exp} \{-(A_2/2) \mathbf{j}\} \\ \circ &[\mathbf{exp} \{-(G(0)/2) \mathbf{k}\} \circ \mathbf{exp} \{((F(t_1) - F(0))/2) \mathbf{j}\} \circ \mathbf{exp} \{(G(t_1)/2) \mathbf{k}\}] \\ &\quad \circ \mathbf{exp} \{(A_2/2) \mathbf{j}\} \circ \mathbf{exp} \{(A_1/2) \mathbf{i}\} = \Lambda(t_1) \end{aligned} \quad (4.18)$$

or

$$\begin{aligned} &\mathbf{exp} \{-(G(0)/2) \mathbf{k}\} \circ \mathbf{exp} \{((F(t_1) - F(0))/2) \mathbf{j}\} \circ \mathbf{exp} \{(G(t_1)/2) \mathbf{k}\} \\ &= \mathbf{exp} \{(A_2/2) \mathbf{j}\} \circ \mathbf{exp} \{(A_1/2) \mathbf{i}\} \circ \tilde{\Lambda}(0) \circ \Lambda(t_1) \\ &\quad \circ \mathbf{exp} \{-(A_1/2) \mathbf{i}\} \circ \mathbf{exp} \{-(A_2/2) \mathbf{j}\}. \end{aligned} \quad (4.19)$$

The laws for forming the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} are obtained in accordance with the concept of solving inverse problems of dynamics using the formulas:

$$\begin{aligned} \mathbf{F}_{cx} &= F_{c1} \mathbf{i} + F_{c2} \mathbf{j} + F_{c3} \mathbf{k} = m [\dot{\mathbf{v}}_x + \mathbf{K}(\boldsymbol{\omega}_x) \mathbf{v}_x] - \mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x) \\ &= m [\mathbf{w}_x + \mathbf{K}(\boldsymbol{\omega}_x) \mathbf{v}_x] - \mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x), \end{aligned} \quad (4.20)$$

$$\mathbf{M}_{cx} = M_{c1} \mathbf{i} + M_{c2} \mathbf{j} + M_{c3} \mathbf{k} = \mathbf{J} \boldsymbol{\varepsilon}_x + \mathbf{K}(\boldsymbol{\omega}_x) \mathbf{J} \boldsymbol{\omega}_x - \mathbf{M}_x(t, \boldsymbol{\lambda}, \boldsymbol{\omega}_x), \quad (4.21)$$

where the quaternions $\mathbf{v}_x = v_1 \mathbf{i} + v_2 \mathbf{j} + v_3 \mathbf{k}$, $\boldsymbol{\omega}_x = \omega_1 \mathbf{i} + \omega_2 \mathbf{j} + \omega_3 \mathbf{k}$, $\dot{\mathbf{v}}_x = \mathbf{w}_x = \dot{v}_1 \mathbf{i} + \dot{v}_2 \mathbf{j} + \dot{v}_3 \mathbf{k}$, $\dot{\boldsymbol{\omega}}_x = \dot{\omega}_1 \mathbf{i} + \dot{\omega}_2 \mathbf{j} + \dot{\omega}_3 \mathbf{k}$, $\boldsymbol{\varepsilon}_x = \dot{\boldsymbol{\omega}}_x = \varepsilon_1 \mathbf{i} + \varepsilon_2 \mathbf{j} + \varepsilon_3 \mathbf{k}$ are obtained by extracting the principal and moment parts from the dual relation (4.7):

$$\begin{aligned} \omega_1 &= f_1 \sin g \cos \alpha_2 - g_1 \sin \alpha_2, \\ \omega_2 &= f_1 (\sin g \sin \alpha_1 \sin \alpha_2 + \cos g \cos \alpha_1) + g_1 \sin \alpha_1 \cos \alpha_2, \\ \omega_3 &= f_1 (\sin g \cos \alpha_1 \sin \alpha_2 - \cos g \sin \alpha_1) + g_1 \cos \alpha_1 \cos \alpha_2; \end{aligned} \quad (4.22)$$

$$\begin{aligned}
\dot{\omega}_1 &= (u_{c1} \sin g + f_1 g_1 \cos g) \cos \alpha_2 - u_{c2} \sin \alpha_2, \\
\dot{\omega}_2 &= u_{c1} (\sin g \sin \alpha_1 \sin \alpha_2 + \cos g \cos \alpha_1) \\
&\quad + f_1 g_1 (\cos g \sin \alpha_1 \sin \alpha_2 - \sin g \cos \alpha_1) + u_{c2} \sin \alpha_1 \cos \alpha_2, \\
\dot{\omega}_3 &= u_{c1} (\sin g \cos \alpha_1 \sin \alpha_2 - \cos g \sin \alpha_1) \\
&\quad + f_1 g_1 (\cos g \cos \alpha_1 \sin \alpha_2 + \sin g \sin \alpha_1) + u_{c2} \cos \alpha_1 \cos \alpha_2;
\end{aligned} \tag{4.23}$$

$$\begin{aligned}
\mathbf{v}_x &= e^{-\frac{\alpha_1}{2}\mathbf{i}} \circ \left[\boldsymbol{\beta} \times (\alpha_1^0 \mathbf{i} + \alpha_2^0 \mathbf{j}) + e^{-\frac{\alpha_2}{2}\mathbf{j}} \circ ((f_1^0 \sin g + f_1 g^0 \cos g) \mathbf{i} \right. \\
&\quad \left. + (f_1^0 \cos g - f_1 g^0 \sin g) \mathbf{j} + g_1^0 \mathbf{k}) \circ e^{\frac{\alpha_2}{2}\mathbf{j}} \right] \circ e^{\frac{\alpha_1}{2}\mathbf{i}}, \\
\boldsymbol{\beta} &= e^{-\frac{\alpha_2}{2}\mathbf{j}} \circ (f_1 \sin g \mathbf{i} + f_1 \cos g \mathbf{j} + g_1 \mathbf{k}) \circ e^{\frac{\alpha_2}{2}\mathbf{j}};
\end{aligned} \tag{4.24}$$

$$\begin{aligned}
\dot{\mathbf{v}}_x &= e^{-\frac{\alpha_1}{2}\mathbf{i}} \circ \left[\dot{\boldsymbol{\beta}} \times (\alpha_1^0 \mathbf{i} + \alpha_2^0 \mathbf{j}) + e^{-\frac{\alpha_2}{2}\mathbf{j}} \circ \left((u_{c1}^0 - f_1 g^0 g_1) \sin g \right. \right. \\
&\quad \left. \left. + (f_1^0 g_1 + u_{c1} g^0 + f_1 g_1^0) \cos g \right) \mathbf{i} + ((u_{c1}^0 - f_1 g^0 g_1) \cos g \right. \\
&\quad \left. - (f_1^0 g_1 + u_{c1} g^0 + f_1 g_1^0) \sin g \right) \mathbf{j} + u_{c2} \mathbf{k} \right] \circ e^{\frac{\alpha_1}{2}\mathbf{i}},
\end{aligned} \tag{4.25}$$

$$\dot{\boldsymbol{\beta}} = e^{-\frac{\alpha_2}{2}\mathbf{j}} \circ ((u_{c1} \sin g + f_1 g_1 \cos g) \mathbf{i} + (u_{c1} \cos g - f_1 g_1 \sin g) \mathbf{j} + u_{c2} \mathbf{k}) \circ e^{\frac{\alpha_2}{2}\mathbf{j}}.$$

The problem (4.3)–(4.25) is solved by the authors using the maximum principle. The dual Hamilton–Pontryagin function for the controlled system (4.10) or (4.11) and (4.12) and the combined functional (4.15) has the form:

$$H = - \left(1 + U_{c1}^2 + U_{c2}^2 \right) + \Psi_1 F_1 + \Psi_2 G_1 + \Psi_3 U_{c1} + \Psi_4 U_{c2} = 0, \quad \forall t \in [0, t_1], \tag{4.26}$$

where $\Psi_1 = \psi_1 + s\psi_1^0$, $\Psi_2 = \psi_2 + s\psi_2^0$, $\Psi_3 = \psi_3 + s\psi_3^0$, $\Psi_4 = \psi_4 + s\psi_4^0$ —dual adjoint variables satisfying the dual system of equations:

$$\dot{\Psi}_1 = 0, \quad \dot{\Psi}_2 = 0, \quad \dot{\Psi}_3 = -\Psi_1, \quad \dot{\Psi}_4 = -\Psi_2, \tag{4.27}$$

which is equivalent to two systems of real equations:

$$\dot{\psi}_1 = 0, \quad \dot{\psi}_2 = 0, \quad \dot{\psi}_3 = -\psi_1, \quad \dot{\psi}_4 = -\psi_2; \tag{4.28}$$

$$\dot{\psi}_1^0 = 0, \quad \dot{\psi}_2^0 = 0, \quad \dot{\psi}_3^0 = -\psi_1^0, \quad \dot{\psi}_4^0 = -\psi_2^0. \tag{4.29}$$

The general solutions of equations (4.27) or (4.28), (4.29), containing dual arbitrary integration constants $C_i = c_i + sc_i^0$, $i = \overline{0, 4}$, or real arbitrary integration constants c_1, \dots, c_4 and c_1^0, \dots, c_4^0 , have the form:

$$\Psi_1 = C_1, \quad \Psi_2 = C_2, \quad \Psi_3 = -C_1 t + C_3, \quad \Psi_4 = -C_2 t + C_4; \tag{4.30}$$

$$\psi_1 = c_1, \quad \psi_2 = c_2, \quad \psi_3 = -c_1 t + c_3, \quad \psi_4 = -c_2 t + c_4, \tag{4.31}$$

$$\psi_1^0 = c_1^0, \quad \psi_2^0 = c_2^0, \quad \psi_3^0 = -c_1^0 t + c_3^0, \quad \psi_4^0 = -c_2^0 t + c_4^0. \tag{4.32}$$

The sought optimal dual controls

$$U_{c1} = (-C_1 t + C_3)/2, \quad U_{c2} = (-C_2 t + C_4)/2 \tag{4.33}$$

maximize the Hamilton–Pontryagin function (4.26).

Consider the dual controls (4.33), which contain the dual quantities

$$\begin{aligned} U_{c1} &= u_{c1} + su_{c1}^0, & U_{c2} &= u_{c2} + su_{c2}^0, \\ \Psi_3 &= \psi_3 + s\psi_3^0, & \Psi_4 &= \psi_4 + s\psi_4^0, & C_i &= c_i + sc_i^0, \quad i = \overline{0,4}. \end{aligned}$$

From these controls, we extract the principal and moment parts, i.e., the scalar real controls u_{c1} , u_{c2} and u_{c1}^0 , u_{c2}^0 :

$$\begin{aligned} U_{c1} &= u_{c1} + su_{c1}^0 = (\psi_3 + s\psi_3^0) / 2 = [(-c_1t + c_3) + s(-c_1^0t + c_3^0)] / 2, \\ U_{c2} &= u_{c2} + su_{c2}^0 = (\psi_4 + s\psi_4^0) / 2 = [(-c_2t + c_4) + s(-c_2^0t + c_4^0)] / 2. \end{aligned}$$

Substituting (4.33) into the system of dual differential equations (4.10), we find the general solution for the phase coordinates F , G , F_1 , G_1 , containing eight dual arbitrary constants $C_1 = c_1 + sc_1^0, \dots, C_8 = c_8 + sc_8^0$:

$$\begin{aligned} F(t) &= -C_1t^3/12 + C_3t^2/4 + C_5t + C_6, \\ G(t) &= -C_2t^3/12 + C_4t^2/4 + C_7t + C_8, \\ F_1(t) &= -C_1t^2/4 + C_3t/2 + C_5, \\ G_1(t) &= -C_2t^2/4 + C_4t/2 + C_7. \end{aligned} \tag{4.34}$$

The unknown dual constants C_1, \dots, C_8 are subject to determination. Since the constant C_6 enters the function F as an additive constant, from formula (4.8), containing the expression $\mathbf{exp}\{((F(t) - F(0))/2)\mathbf{j}\}$, it can be seen that this dual constant has no effect. For this reason, the constant C_6 can be set to zero. Thus, to determine the nine unknown dual constants of the problem $C_1, \dots, C_5, C_7, C_8, A_1, A_2$ and the transient time t_1 , we use nine dual equations from the system (4.16), (4.17), (4.19) and the scalar part of condition (4.26) at the time instant t_1 (note that in the dual quaternion equation (4.19) only three equations in dual scalar form are independent due to the normalization of the dual quaternion $\mathbf{\Lambda}$). Substituting formulas (4.33), (4.34) into (4.7), (4.8), we obtain analytical dual formulas for finding the sought optimal laws of variation of the kinematic screw \mathbf{U}_x of the free rigid body and the dual quaternion $\mathbf{\Lambda}$, describing the optimal trajectories of the angular (rotational) and translational (orbital) motions of the free rigid body. These formulas will describe the optimal, in the sense of the combined functional, displacement of the body in the class of generalized helical conic motions. The laws for forming the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} are found in accordance with the concept of solving inverse problems of dynamics using formulas (4.20), (4.21).

The problem of optimal, in the sense of a combined functional incorporating time and energy, displacement of a free rigid body (spacecraft) in the class of generalized helical conic motions is thus completely solved.

The optimal algorithm for the spatial motion of a spacecraft of arbitrary dynamic configuration under arbitrary boundary conditions has the form:

1) from the given dual quantities $\mathbf{\Lambda}(0)$, $\mathbf{\Lambda}(t_1)$, $\mathbf{U}_x(0)$, $\mathbf{U}_x(t_1)$ (formulas (3.11), (3.12) for the boundary conditions of the problem) by solving the system of algebraic dual equations (4.5), (4.6), (4.16), (4.17), (4.19), supplemented by condition (4.26) for the scalar part of the Hamilton–Pontryagin function, the nine dual undetermined constants $C_1, \dots, C_5, C_7, C_8, A_1, A_2$ and the time t_1 are calculated; then the dual functions F , G , F_1 , G_1 are found;

2) using (4.5), the constant dual quaternion \mathbf{K} is calculated;

3) using formula (4.7), the kinematic screw of the spacecraft $\mathbf{U}_x(t)$ is determined;

4) using formula (4.8), the dual quaternion of the spatial helical motion of the spacecraft $\mathbf{\Lambda}(t)$ is determined;

5) using formulas (4.20), (4.21), the laws for forming the control force \mathbf{F}_{cx} and control torque \mathbf{M}_{cx} of the spacecraft are constructed;

6) from expression (4.15), the value of the optimization criterion for the problem of optimal spatial motion (maneuvering) of the spacecraft is found.

It should be noted that in the cases of quaternion problems of optimal reorientation of a spherically symmetric spacecraft in the classes of planar Euler rotations and regular angular conic motions, the known optimal solutions of the classical problem and the analytical solutions of the problem of controlling angular motion of the spacecraft [9–11], optimal in the class of generalized angular conic motions, completely coincide. Based on the Kotelnikov–Study transfer principle, the same can be stated regarding the dual spatial analogues of these controlled motions of the spacecraft.

5. NUMERICAL EXAMPLES

This section considers examples of calculations of optimal, in the class of generalized helical conic motions, spatial motion (maneuvering) of a spacecraft according to the proposed algorithm, demonstrating the effectiveness of the obtained analytical solution of the problem for various dynamic configurations of the spacecraft.

Let us transition from dimensional variables of the problem to dimensionless ones using the formulas (the quaternion $\boldsymbol{\lambda}$ is dimensionless) [16]:

$$\begin{aligned} J^{\text{scale}} &= \left((J_1^2 + J_2^2 + J_3^2) / 3 \right)^{1/2}, & J_k^{\text{dimless}} &= J_k / J^{\text{scale}}, \\ \boldsymbol{\omega}_x^{\text{dimless}} &= \left(J^{\text{scale}} \right)^{1/2} a^{1/4} \boldsymbol{\omega}_x, & t^{\text{dimless}} &= \left(J^{\text{scale}} \right)^{-1/2} a^{-1/4} t, \\ \mathbf{M}^{\text{dimless}} &= a^{1/2} \mathbf{M}, \end{aligned}$$

and also by the formulas:

$$\boldsymbol{\lambda}^{\text{dimless}} = \boldsymbol{\lambda}^0 / L, \quad \boldsymbol{\varepsilon}_x^{\text{dimless}} = J^{\text{scale}} a^{1/2} \boldsymbol{\varepsilon}_x, \quad \mathbf{w}_x^{\text{dimless}} = L \left(J^{\text{scale}} \right)^{-1} a^{-1/2} \mathbf{w}_x.$$

Here $a = 1(\text{N} \times \text{m})^{-2} = 1 \text{ kg}^{-2} \times \text{m}^{-4} \times \text{s}^4$, L —is the scale factor for distance.

In this case, the phase equations of the problem and the main formulas of the proposed algorithm will not change.

We assume that the principal vector of other external forces $\mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x)$, appearing in formula (4.20) for calculating the control force, is caused solely by the central force of gravitational attraction of the spacecraft to the Earth, equal to

$$\mathbf{F}_x(t, \mathbf{r}_x, \mathbf{v}_x) = \mathbf{F}_x(\mathbf{r}_x) = -G \frac{M_0 m}{r^3} \mathbf{r}_x,$$

where G is the gravitational constant, and M_0 —is the mass of the attracting body (the Earth), $r = |\mathbf{r}_x|$.

The gravitational moment in this central force field is determined by the following relation [17]:

$$\mathbf{M}_{\text{grav}} = 3\mu \frac{\boldsymbol{\gamma} \times \mathbf{J}\boldsymbol{\gamma}}{r^3}, \quad \boldsymbol{\gamma} = \frac{\mathbf{r}}{r},$$

where $\mu = GM_0$ is the gravitational parameter of the attracting body (Earth).

Then the expressions for the control force and torque (4.20), (4.21) in dimensionless form will be written as follows:

$$\begin{aligned} \mathbf{F}_x &= \mathbf{w}_x + \mathbf{K}(\boldsymbol{\omega}_x) \mathbf{v}_x + N_G \mathbf{r}_x / |\mathbf{r}_x|^3, \\ \mathbf{M}_{cx} &= \mathbf{J}\boldsymbol{\varepsilon}_x + \mathbf{K}(\boldsymbol{\omega}_x) \mathbf{J}\boldsymbol{\omega}_x - 3N_G \mathbf{r}_x \times \mathbf{J}\mathbf{r}_x / |\mathbf{r}_x|^5. \end{aligned} \quad (5.1)$$

Here $N_G = GM_0T^2/L^3$ —is the dimensionless parameter of the problem, $T = (J^{\text{scale}})^{1/2} \times a^{1/4}$ — the scale factor for time.

Taking into account that $\|\boldsymbol{\lambda}\| = 1$, formulas (5.1) for finding the control force and control torque in dimensionless form take the form

$$\begin{aligned} \mathbf{F}_{cx} &= \mathbf{w}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{v}_x + N_G \frac{\tilde{\boldsymbol{\lambda}} \circ \boldsymbol{\lambda}^0}{\|\boldsymbol{\lambda}^0\|^{3/2}}, \\ \|\boldsymbol{\lambda}^0\| &= (\lambda_0^0)^2 + (\lambda_1^0)^2 + (\lambda_2^0)^2 + (\lambda_3^0)^2. \\ \mathbf{M}_{cx} &= \mathbf{J}\boldsymbol{\varepsilon}_x + \mathbf{K}(\boldsymbol{\omega}_x)\mathbf{J}\boldsymbol{\omega}_x - 3N_G \frac{(\tilde{\boldsymbol{\lambda}} \circ \boldsymbol{\lambda}^0) \times \mathbf{J}(\tilde{\boldsymbol{\lambda}} \circ \boldsymbol{\lambda}^0)}{\|\boldsymbol{\lambda}^0\|^{5/2}}. \end{aligned} \quad (5.2)$$

Note that expressions (5.1) and (5.2) for the control force do not contain the inertia tensor \mathbf{J} , i.e., the law of variation of the control force does not depend on the type of mass distribution of the spacecraft.

The boundary conditions for the angular position in space and angular velocity of the spacecraft have the form ([16, p. 137],) (reference [16] considered the problem of optimal reorientation of a spacecraft as a rigid body of arbitrary dynamic configuration under arbitrary boundary conditions on the angular position and angular velocity of the spacecraft's reorientation in the absence of translational displacement and gravitational torque):

$$\begin{aligned} \boldsymbol{\lambda}(0) &= (0.7951; 0.2981; -0.3975; 0.3478), \\ \boldsymbol{\omega}(0) &= (0.2739; -0.2388; -0.3), \end{aligned} \quad (5.3)$$

$$\begin{aligned} \boldsymbol{\lambda}_k &= (0.8443; 0.3984; -0.3260; 0.1485), \\ \boldsymbol{\omega}_k &= (0.0; 0.0; -0.59). \end{aligned} \quad (5.4)$$

SC 1. Spherically symmetric rigid body: $J_1 = J_2 = J_3 = J^* \text{ kg} \times \text{m}^2$. Here $J^{\text{scale}} = J^* \text{ kg} \times \text{m}^2$, then the dimensionless moments of inertia are $J_1 = J_2 = J_3 = J^*/J^{\text{scale}} = 1.0$.

SC 2. International Space Station (ISS) [18]:

$J_1 = 4\,853\,000 \text{ kg} \times \text{m}^2$, $J_2 = 23\,601\,000 \text{ kg} \times \text{m}^2$, $J_3 = 26\,278\,000 \text{ kg} \times \text{m}^2$ (in dimensional form) or $J_1 = 0.2358$, $J_2 = 1.1466$, $J_3 = 1.2766$ (in dimensionless form).

SC 3. Space Shuttle spacecraft (almost axisymmetric rigid body): $J_1 = 3\,400\,648 \text{ kg} \times \text{m}^2$, $J_2 \approx J_3 = 21\,041\,672 \text{ kg} \times \text{m}^2$ or $J_1 = 0.1967$, $J_2 \approx J_3 = 1.2168$.

In order to compare the calculation results with those obtained in [16] in the absence of translational displacement of the spacecraft, the initial and final values of the spacecraft's linear velocity vector must be set to the zero vector. The values of the real constants $\alpha_1, \alpha_2, c_1, \dots, c_8$, appearing in the analytical solution of the problem obtained in [16] for angular motion with boundary conditions (5.3), (5.4) are as follows ($\alpha_1^0 = \alpha_2^0 = 0$, $c_1^0 = c_2^0 = \dots = c_8^0 = 0$):

$$\begin{aligned} \alpha_1 &= -0.0421; & \alpha_2 &= -0.2226; & c_1 &= 3.4020; & c_2 &= -2.0123; & c_3 &= 2.2293; \\ c_4 &= -1.7026; & c_5 &= -0.4156; & c_6 &= 0; & c_7 &= -0.2220; & c_8 &= -0.9216. \end{aligned}$$

Furthermore, the values of the components of the radius vector of the spacecraft's center of mass (in the inertial coordinate system) at the initial and final instants of time must be the same. In the authors' calculations, the coordinates of the dimensionless radius vector were calculated from

the spacecraft orbit parameters given in [19, p. 95]:

$$\begin{aligned}\xi_1^0 &= \xi_1^k = 23\,399\,727.8 \text{ m}, \\ \xi_2^0 &= \xi_2^k = 23\,962\,416.6 \text{ m}, \\ \xi_3^0 &= \xi_3^k = -18\,801\,552.4 \text{ m}.\end{aligned}$$

In [19], the effectiveness of numerical prediction of transformations regularizing and stabilizing the equations of motion was investigated using this orbit as an example.

In this problem formulation, the mass-inertial characteristics of the spacecraft appear only in the formulas for finding the control force and control torque and do not affect the laws of variation of the optimal controls (optimal angular and linear accelerations). Therefore, the calculation results in dimensionless variables correspond to the calculations presented in [16] when solving the spacecraft reorientation problem.

In the case where the initial and final positions of the spacecraft coincide, the values of the components of the dimensionless control torque vector \mathbf{M} found by the authors at the beginning, middle, and end of the motion in the absence of gravitational torque are close to those given in Tables 1, 4, 5 of reference [16, pp. 176–177] for a spherically symmetric spacecraft, the ISS, and the Space Shuttle spacecraft, respectively.

The value of the integral

$$J^{\text{modif}} = \int_0^{t_1} (1 + \mathbf{M}^2) dt \quad (5.5)$$

(the dimensionless minimized functional in [16]) in the case of spherical symmetry of the spacecraft turned out to be 1.4746, which is close to the number 1.4749 indicated at the end of p. 176 of reference [16] for the optimal control problem of angular motion of a spacecraft in the class of angular conic motions. The final time of the controlled process turned out to be 0.9657, which is close to the number 0.966 given in Table 1.

At the same time, the value of the optimization criterion (4.15) for the problem of optimal spatial motion (maneuvering) of the spacecraft turned out to be $J = 1.4708 + 0 \cdot s$, i.e., the functional value is a scalar. Let us further consider the case where the scale factors correspond to the work [20]: $L = 37\,000\,000.0 \text{ m}$, $T = 11\,272.855470 \text{ s}$. Then with $M_0 = 5.9722 \times 10^{24} \text{ kg}$ and $G = 6.67408 \times 10^{-11} \text{ N} \cdot \text{m}^2 \cdot \text{kg}^{-2}$, the dimensionless parameter of the problem is $N_G = 0.99997 \approx 1.0$.

Note that in the case where the initial and final positions of the spacecraft coincide and do not change during its angular motion, the specific values of the components of the vector \mathbf{r}_x do not affect the values of the sought constants $C_1, \dots, C_5, C_7, C_8, A_1, A_2$. Therefore, the laws of variation of the phase variables and optimal control also do not depend on them. Moreover, the projections of the spacecraft's radius vector onto the axes of the inertial coordinate system ξ will be constant. Also, in the coordinate system ξ , the control force will be a constant vector. The control force vector will be anti-collinear with respect to the gravitational force vector (with the magnitudes of these vectors being equal). Over time, the projections of the gravitational force onto the axes of the spacecraft-fixed coordinate system X will change (the projections of this force onto the axes of the ξ coordinate system will not change).

Figure 1 shows the time-varying laws of the components of the control torque and control force vectors (they do not depend on the spacecraft's inertia tensor) in the body-fixed coordinate system, for $N_G = 0.99997 \approx 1.0$ for a spherically symmetric spacecraft (in this case, the gravitational torque is absent). In the coordinate system ξ , the control force vector is a constant vector.

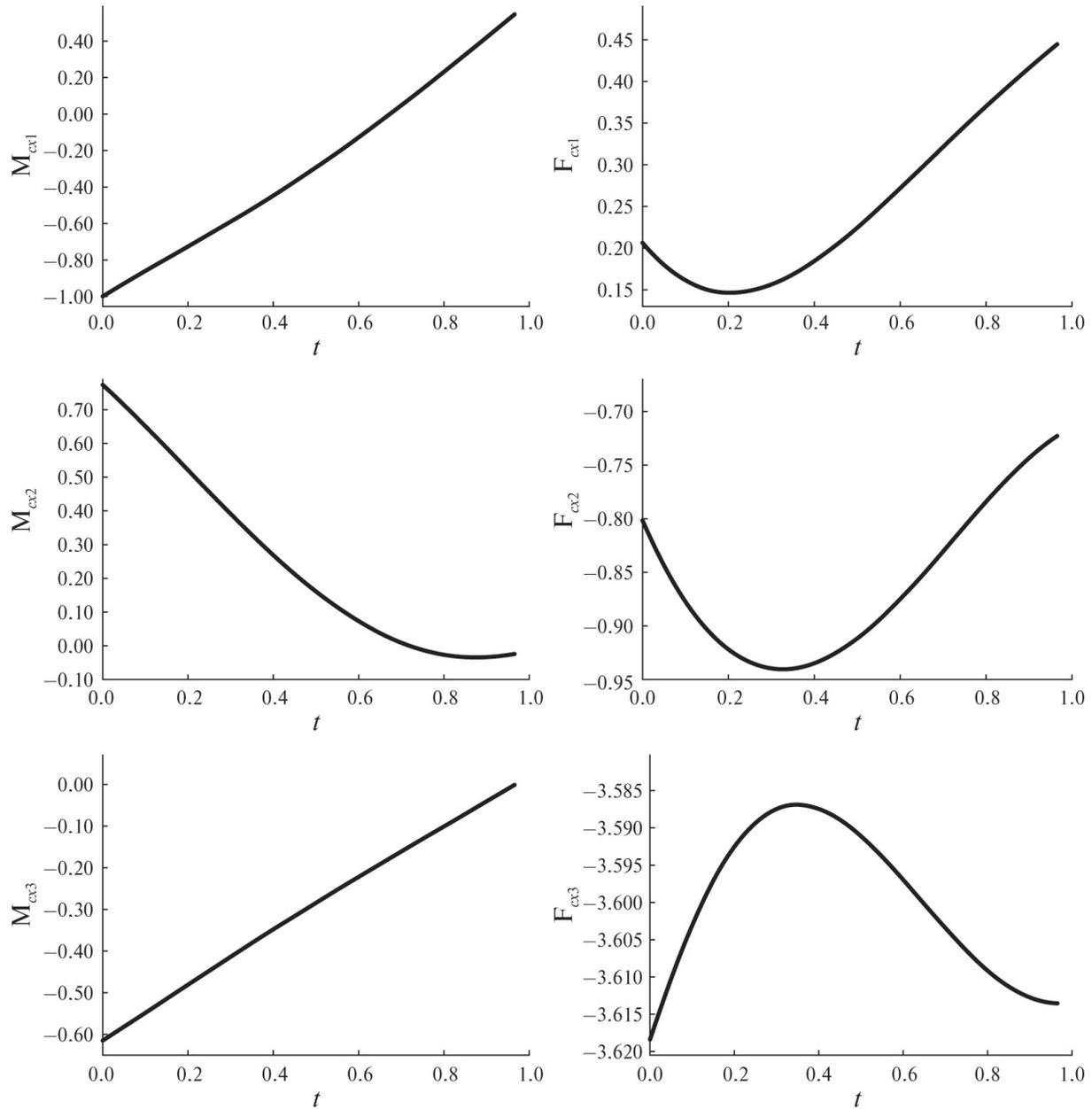


Fig. 1. Components of the control torque and control force vectors for SC 1 in the absence of translational displacement.

Laws of variation of the control torque vector components were also found for the ISS and the Space Shuttle spacecraft. In this case, the values of the integral (5.5) in the absence of gravitational torque turned out to be 1.3671 (ISS) and 1.3680 (Space Shuttle spacecraft), which is close to the numbers 1.3674 and 1.3683, respectively, indicated on p. 177 of reference [16] for the modified optimal control problem of angular motion.

Figures 2 and 3 show the dimensionless time-varying laws of the control torque vector components in the body-fixed coordinate system for the ISS and the Space Shuttle spacecraft (left column—in the absence of gravitational torque, right column—in its presence) when the coordinates of the initial and final positions of the spacecraft coincide.

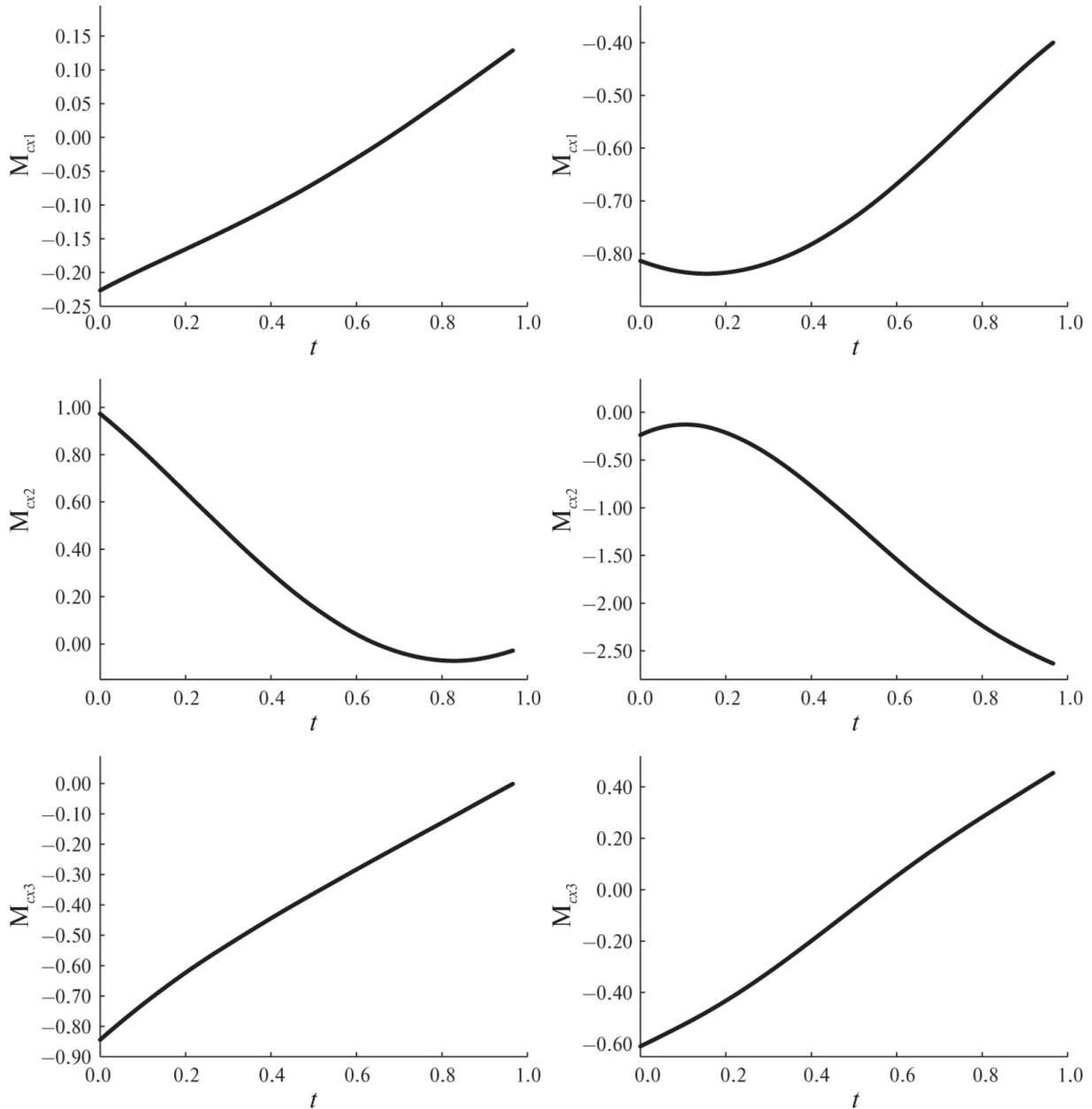


Fig. 2. Control torque vector components for SC 2 in the absence of translational displacement.

From the calculation results, it follows that for all spacecraft, the range of variation of the third component of the control force vector is an order of magnitude smaller than the ranges of variation of the other two components.

Next, calculations are presented for the case where, in addition to angular motion, the spacecraft also performs controlled translational motion. Let the position and velocity of the spacecraft in dimensionless variables correspond to the following orbit parameters [21]:

initial position of the spacecraft

$$\xi_1^0 = -12\,194\,795.0 \text{ m}, \quad \xi_2^0 = 21\,779\,195.0 \text{ m}, \quad \xi_3^0 = 8\,278\,547.0 \text{ m}.$$

$$\dot{\xi}_1^0 = -1080.750 \text{ m/s}, \quad \dot{\xi}_2^0 = -1849.256 \text{ m/s}, \quad \dot{\xi}_3^0 = 3274.225 \text{ m/s},$$

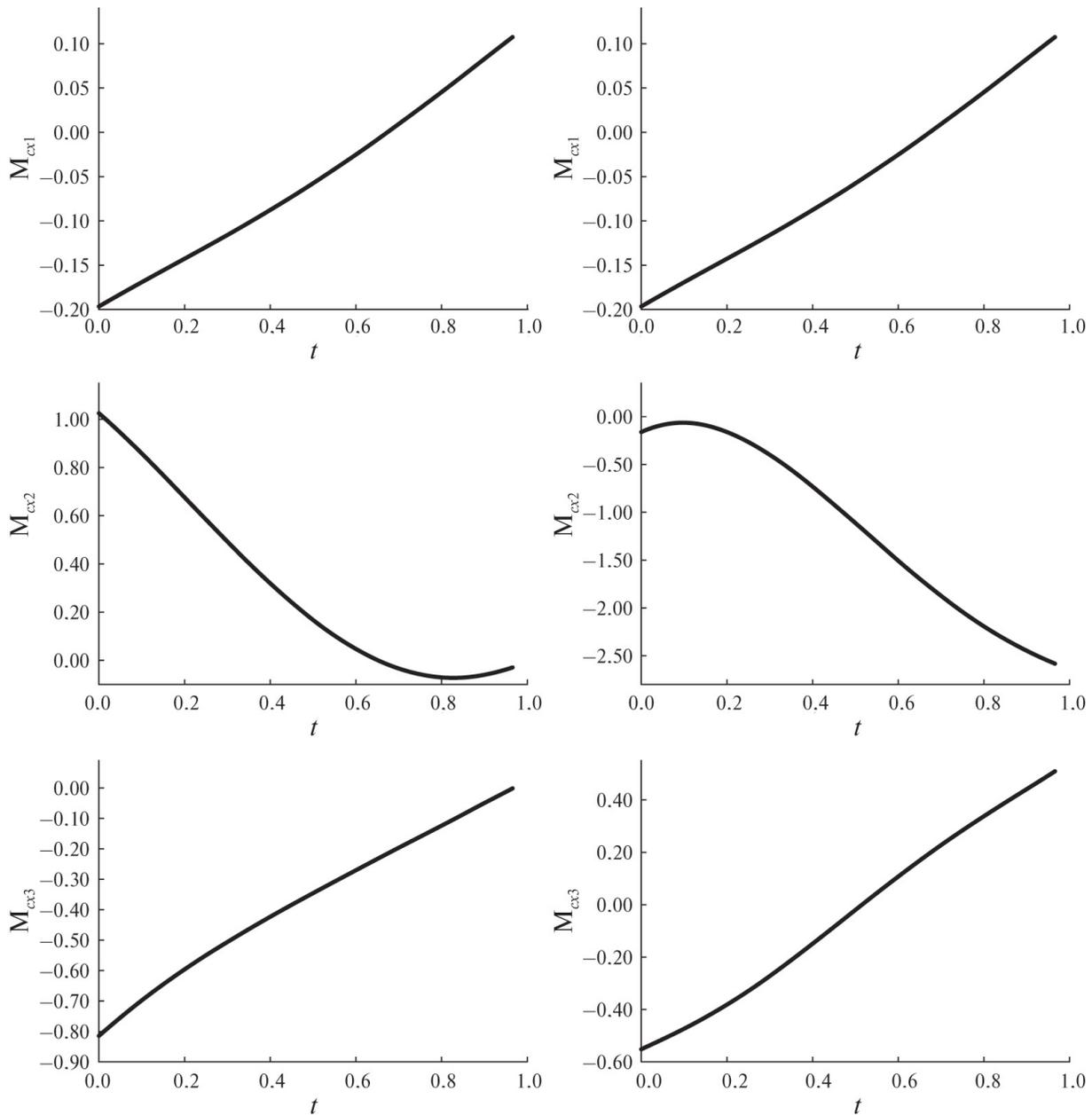


Fig. 3. Control torque vector components for SC 3 in the absence of translational displacement.

final position of the spacecraft

$$\xi_1^k = -12\,110\,249.0 \text{ m}, \quad \xi_2^k = 21\,643\,438.0 \text{ m}, \quad \xi_3^k = 8\,744\,787.0 \text{ m}.$$

$$\dot{\xi}_1^k = -1063.392 \text{ m/s}, \quad \dot{\xi}_2^k = -1905.728 \text{ m/s}, \quad \dot{\xi}_3^k = 3247.462 \text{ m/s}.$$

As an initial approximation for the undetermined dual constants $C_1, \dots, C_5, C_7, C_8, A_1, A_2$ in the presence of translational displacement, the real values of these constants corresponding to a pure rotation of the spacecraft were taken. In this case, the principal parts of these dual constants (which correspond to the pure rotation of the spacecraft) and the final time of the controlled process remained practically unchanged as a result of solving the system of nonlinear algebraic equations.

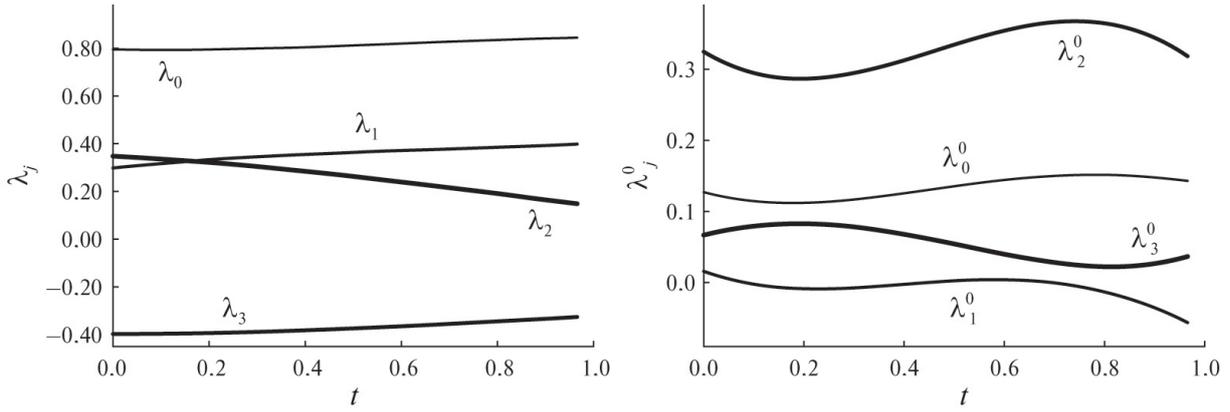


Fig. 4. Components of the dual quaternion of finite displacement for SC 1 in the presence of translational displacement.

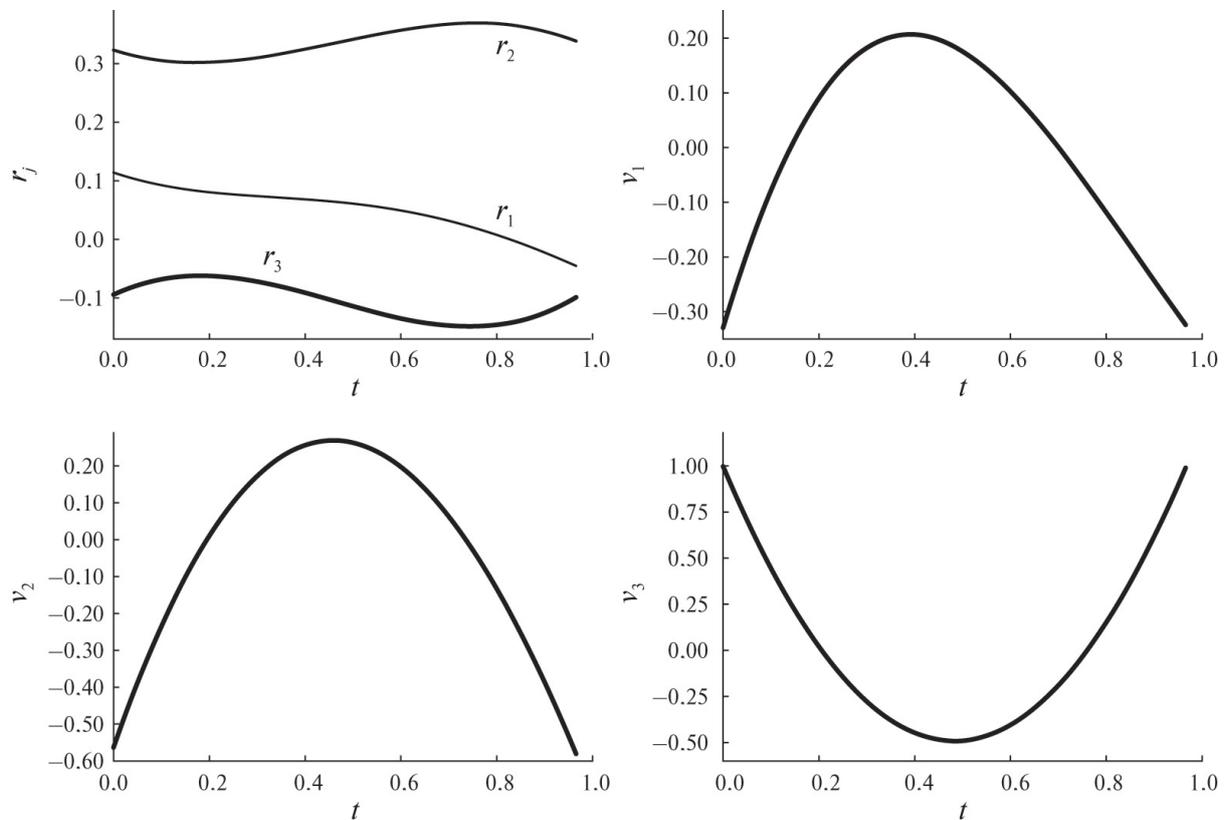


Fig. 5. Components of the radius vector and linear velocity vector for SC 1 in the presence of translational displacement.

The value of the optimization criterion (4.15) for the problem of optimal spatial motion (maneuvering) of the spacecraft turned out to be $J = 1.4708 + 1.2897 \cdot s$ (its principal part is the same as in the case of pure rotation of the spacecraft [16]).

Below are the calculation results in the presence of translational displacement of the spacecraft's center of mass. Note that the error in determining the components of the dual quaternion of the spacecraft's finite displacement at the final instant of time, both in the absence and in the presence of translational displacement, was about 10^{-16} dimensionless units. In this case, the error in

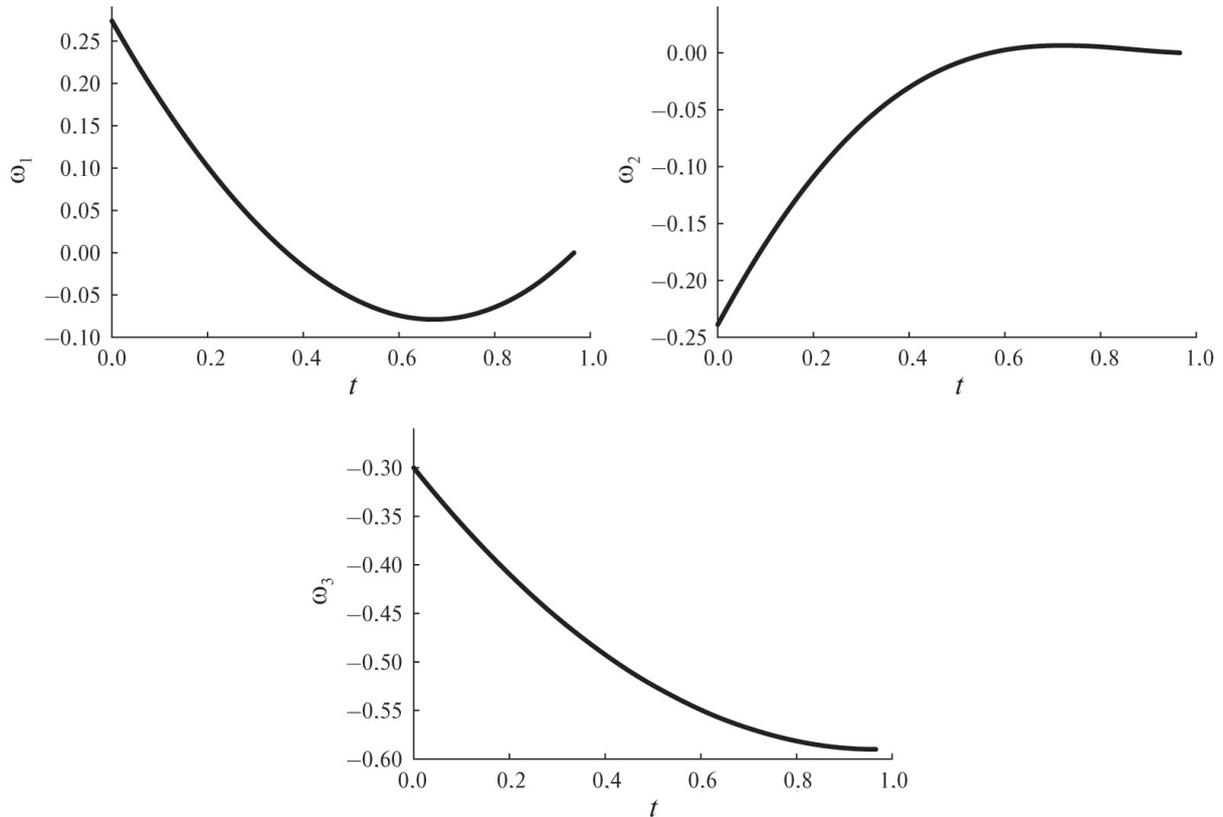


Fig. 6. Angular velocity vector components for SC 1 in the presence of translational displacement.

determining the components of the linear and angular velocity vectors was about 10^{-15} and 10^{-17} dimensionless units, respectively. The order of these values is due to the accuracy of solving the system of algebraic dual equations (4.5), (4.6), (4.16), (4.17), (4.19) for finding the undetermined dual constants $C_1, \dots, C_5, C_7, C_8, A_1, A_2$, which was approximately 10^{-15} dimensionless units.

Figures 4 and 5 show the time-varying laws of the components of the dual quaternion of the spacecraft's finite displacement in inertial space, as well as the projections r_j ($j = \overline{1, 3}$) of the radius vector and the projections v_j ($j = \overline{1, 3}$) of the velocity vector of its center of mass onto the axes of the spacecraft-fixed coordinate system X for the case of a spherically symmetric spacecraft. The functions $r_j(t)$ and $\lambda_j^0(t)$ are harmonic functions of time (here and below, the laws of variation of dimensionless quantities are discussed). The laws of variation of the components of the vector \mathbf{v}_x can be approximated by parabolas. One of the components is a convex downward function, while the other two are convex upward. Each of the components of the vector \mathbf{v}_x changes sign twice during the spacecraft's motion (the components of this vector change sign almost simultaneously). The extremum points of the components of the vector \mathbf{r}_x coincide with the time instants at which the corresponding component of the vector \mathbf{v}_x is zero.

Figure 6 shows the time-varying laws of the angular velocity vector components for the case of a spherically symmetric spacecraft.

Figures 7 and 8 show the time-varying laws of the optimal angular and linear accelerations, control torque, and control force for the case of a spherically symmetric spacecraft. In the case of a spherically symmetric spacecraft, the optimal laws of variation of the angular acceleration vector components completely coincide with the laws of variation of the optimal control torque vector components. The second and third components of the optimal angular acceleration vector are close to zero at the end of the motion. The second and third components of the optimal linear

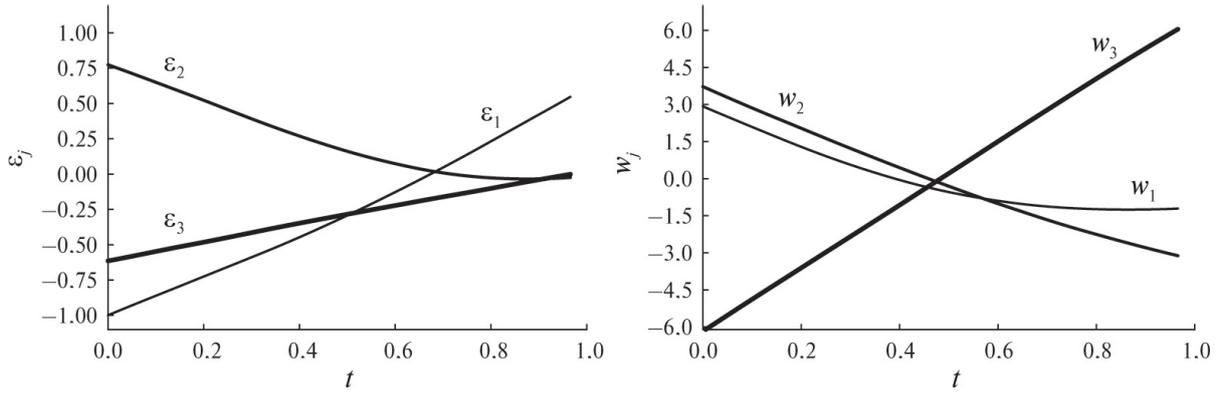


Fig. 7. Optimal control for SC 1 in the presence of translational displacement.

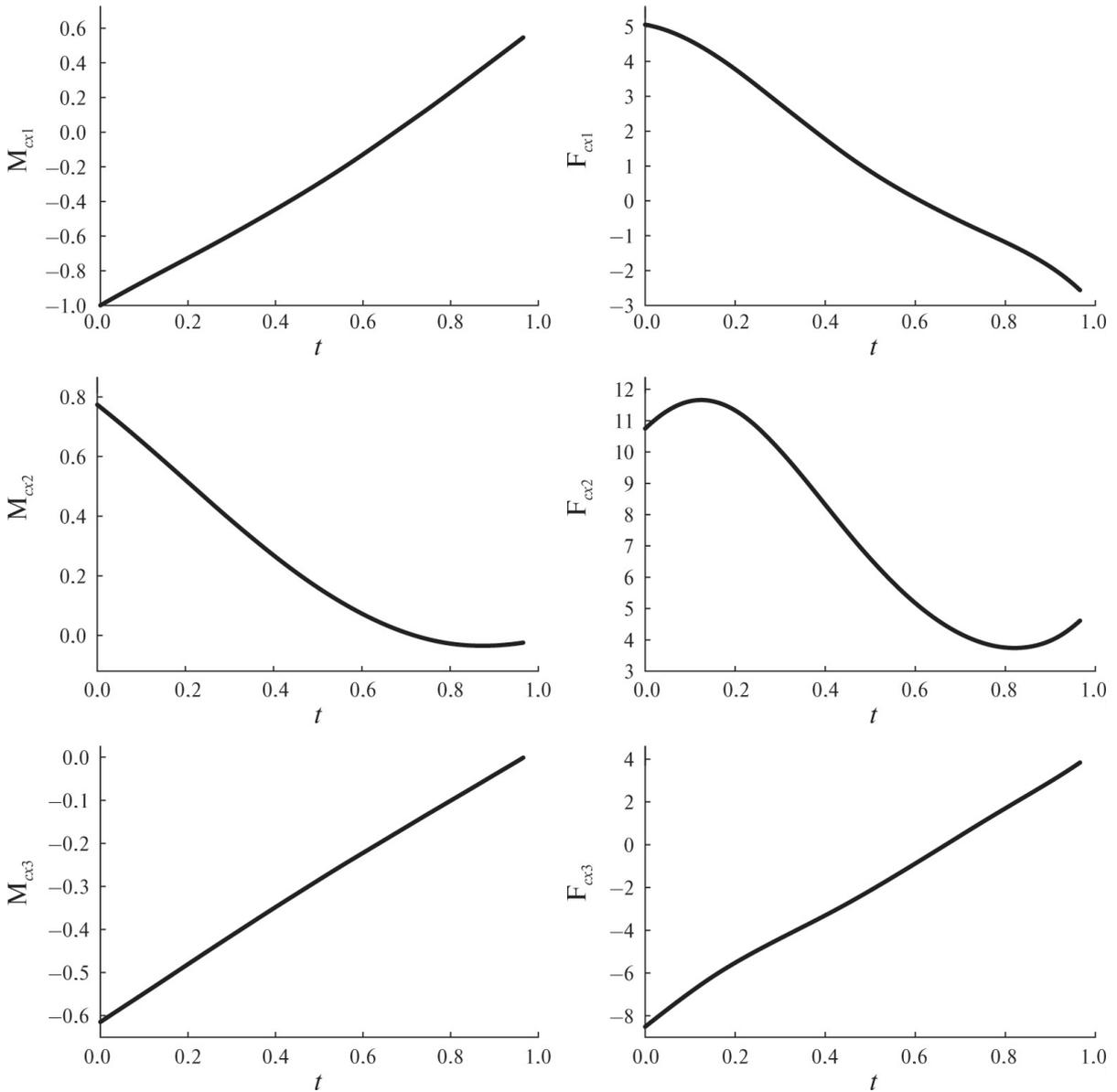


Fig. 8. Components of the control torque and control force vectors for SC 1 in the presence of translational displacement.

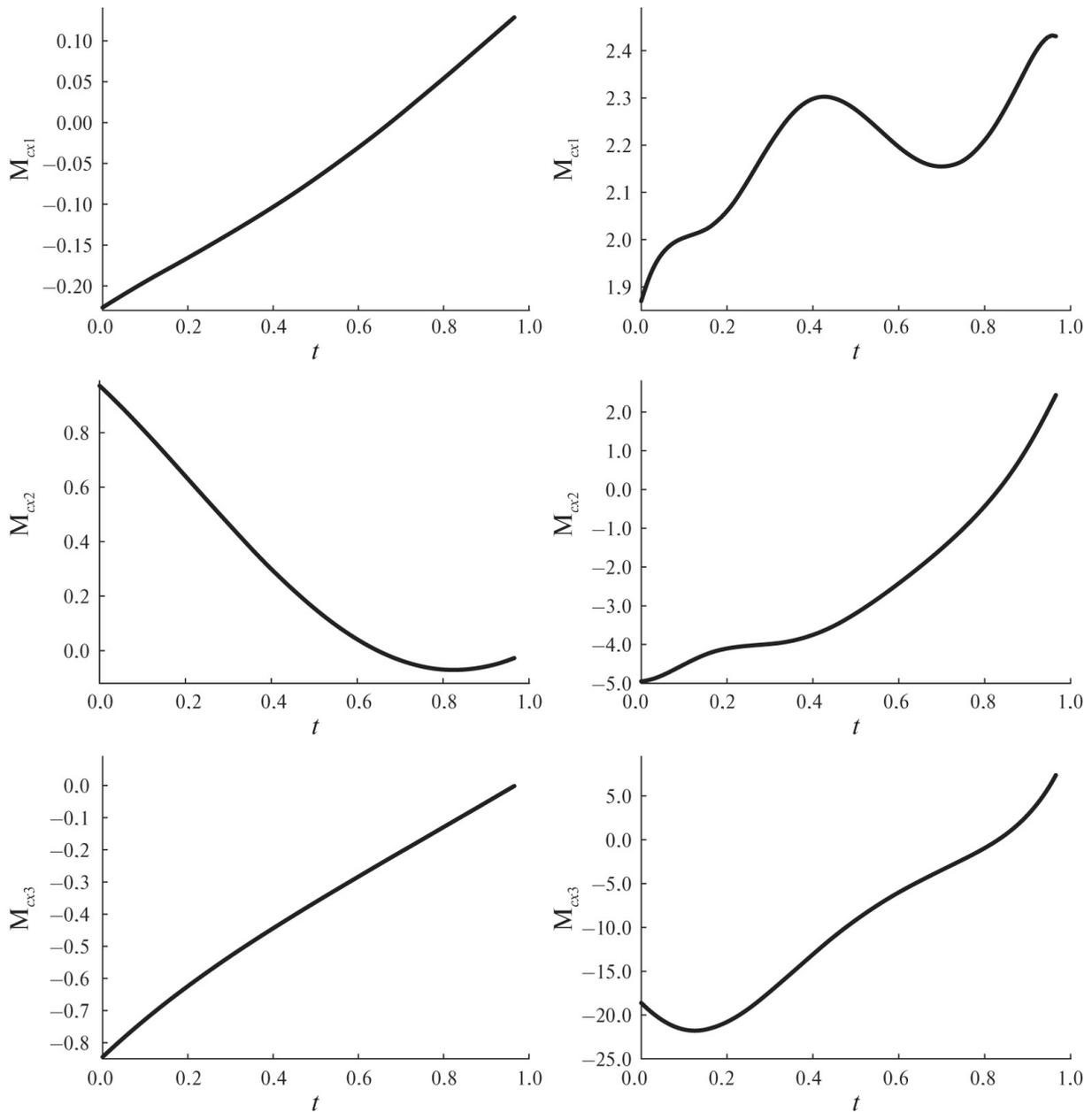


Fig. 9. Control torque vector components for SC 2 in the presence of translational displacement.

acceleration vector are close to linear functions of time. The sign change point of each component of this vector corresponds to the extremum point of the corresponding component of the spacecraft's linear velocity vector. The first component of the control force vector is a decreasing function, and the third is an increasing function. The second component of this vector varies according to a harmonic law.

Next, Figs. 9 and 10 show the time-varying laws of the control torque for the case where the mass distribution of the spacecraft corresponds to the ISS and the Space Shuttle spacecraft (left column—in the absence of gravitational torque, right column—in its presence). Note that since the inertia tensor \mathbf{J} does not appear in expressions (5.1) and (5.2) for the control force, the laws of variation of the control force for the ISS and the Space Shuttle spacecraft coincide with the law shown in the second column of Fig. 8 for the case of a spherically symmetric spacecraft.

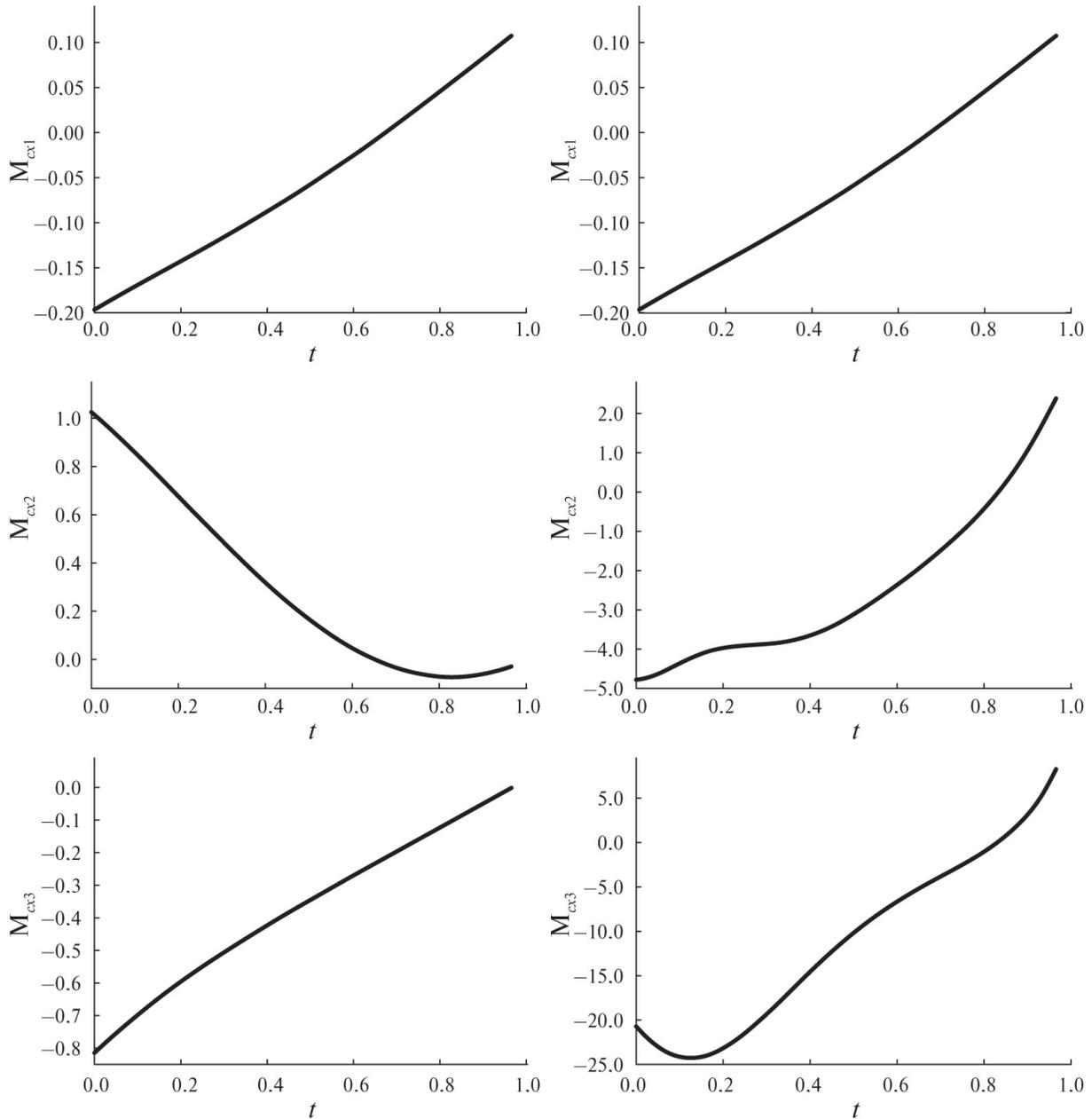


Fig. 10. Control torque vector components for SC 3 in the presence of translational displacement.

Note that the laws of variation of the other sought quantities qualitatively remained the same as in the previous case.

6. CONCLUSION

A dual quaternion (biquaternion) theory of optimal control of the spatial motion of a free rigid body (spacecraft) as an interrelated control of its spatial angular (rotational) and translational (orbital) motions has been constructed. The second derivatives of the parameters of the generalized helical conic motion of the spacecraft, equivalent to its general spatial motion, are used as optimized controls. The vectors of programmed control force and programmed control torque are found in accordance with the concept of solving inverse problems of dynamics.

The analytical algorithm for controlling the spatial motion (maneuvering) of a spacecraft as a free rigid body of arbitrary dynamic configuration with arbitrary boundary conditions, obtained on this basis, is optimal in the class of generalized helical conic motions and is applicable in spacecraft control systems. Moreover, it does not require a numerical solution of a complex high-dimensional differential boundary value optimization problem or any other complex numerical solution. This is important, in particular, for spacecraft spatial maneuvering at exceptionally high rates of displacement, when the time for calculating the optimal spatial trajectory of the spacecraft maneuver and for calculating the control laws enabling this trajectory is extremely limited.

REFERENCES

1. Strelkova, N.A., Time-optimal kinematic control of the helical displacement of a rigid body, *Izv. AN SSSR. MTT*, 1982, no. 4, pp. 73–76.
2. Malanin, V.V. and Strelkova, N.A., *Optimal'noe upravlenie orientatsiei i vintovym dvizheniem tverdogo tela (Optimal Control of Orientation and Helical Motion of a Rigid Body)*, Moscow–Izhevsk: R&C Dynamics, 2004.
3. Chelnokov, Yu.N., On integration of the kinematic equations of helical motion of a rigid body, *PMM*, 1980, vol. 44, no. 1, pp. 32–39.
4. Chelnokov, Yu.N., On a form of inertial navigation equations, *Izv. AN SSSR. MTT*, 1981, no. 5, pp. 20–28.
5. Chelnokov, Yu.N., *Kvaternionnyye i bikvaternionnyye modeli i metody mekhaniki tverdogo tela i ikh prilozheniya: Geometriya i kinematika dvizheniya (Quaternion and Biquaternion Models and Methods of Rigid Body Mechanics and Their Applications: Geometry and Kinematics of Motion)*, Moscow: Fizmatlit, 2006.
6. Chelnokov, Yu.N., Control of spatial motion of a rigid body using biquaternions and dual matrices, *Izv. RAN. MTT*, 2021, no. 1, pp. 17–43.
7. Chelnokov, Yu.N., Synthesis of control of spatial motion of a rigid body using dual quaternions, *PMM*, 2019, vol. 83, no. 5–6, pp. 704–733.
8. Chelnokov, Yu.N., Control of spatial motion of a rigid body using dual quaternions, *Proc. XII All-Russ. Congr. Fundam. Probl. Theor. Appl. Mech.*, 2019, pp. 288–290.
9. Molodenkov, A.V. and Sapunkov, Ya.G., Analytical quasi-optimal solution of the problem of rotating an arbitrary rigid body under arbitrary boundary conditions, *Izv. RAN. MTT*, 2019, no. 2, pp. 140–154.
10. Molodenkov, A.V. and Sapunkov, Ya.G., Analytical quasi-optimal algorithm for programmed control of spacecraft angular motion, *Izv. RAN. TiSU*, 2023, no. 4, pp. 125–136.
11. Sapunkov, Ya.G. and Molodenkov, A.V., Quasioptimal spacecraft attitude control constructed according to the Poincaré concept, *Aerospace*, 2023, vol. 10, no. 5, pp. 402–417.
12. Chelnokov, Yu.N., Molodenkov, A.V., and Loginov, M.Yu., Biquaternion quasi-optimal analytical solution of the programmed control problem for spacecraft spatial motion, *Proc. XXX St. Petersburg Int. Conf. Integr. Navig. Syst.*, 2023, pp. 411–414.
13. Molodenkov, A.V. and Sapunkov, Ya.G., Solution of the optimal rotation problem for a spherically symmetric rigid body under arbitrary boundary conditions in the class of generalized conic motions, *Izv. RAN. MTT*, 2014, no. 5, pp. 22–34.
14. Branets, V.N. and Shmyglevsky, I.P., *Primenenie kvaternionov v zadachakh orientatsii tverdogo tela (Application of Quaternions in Problems of Rigid Body Orientation)*, Moscow, Nauka, 1973.
15. Molodenkov, A.V. and Sapunkov, Ya.G., Analytical solution of the optimal rotation problem for a spherically symmetric spacecraft in the class of conic motions, *Izv. RAN. TiSU*, 2013, no. 3, pp. 167–176.
16. Molodenkov, A.V. and Sapunkov, Ya.G., Analytical approximate solution of the optimal spacecraft rotation problem under arbitrary boundary conditions, *Izv. RAN. TiSU*, 2015, no. 3, pp. 131–141.
17. Beletsky, V.V., *Dvizhenie iskusstvennogo sputnika otnositel'no tsentra mass (Motion of an Artificial Satellite about Its Center of Mass)*, Moscow: Nauka, 1965.

18. Banit, Yu.R., Belyaev, M.Yu., Dobrinskaya, T.A., et al., Determination of the inertia tensor of the International Space Station from telemetric information, *Preprint No. 57*, Moscow: Keldysh Inst. Appl. Math. RAS, 2002.
19. Bordovitsyna, T.V., *Sovremennye chislennye metody v zadachakh nebesnoi mekhaniki* (Modern Numerical Methods in Problems of Celestial Mechanics), Moscow: Nauka, 1984.
20. Pankratov, I.A., Sapunkov, Ya.G., and Chelnokov, Yu.N., Quaternion models and algorithms for solving the general problem of optimal reorientation of a spacecraft orbit, *Izv. Saratov Univ. (N.S.), Ser. Math. Mech. Inform.*, 2020, vol. 20, no. 1, pp. 93–104.
21. Bordovitsyna, T.V. and Avdyushev, V.A., *Teoriya dvizheniya iskusstvennykh sputnikov Zemli. Analiticheskie i chislennye metody* (Theory of Motion of Artificial Earth Satellites. Analytical and Numerical Methods), Tomsk: Tomsk State Univ. Publ., 2016.

This paper was recommended for publication by A.I. Matasov, a member of the Editorial Board