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NONLINEAR SYSTEMS

Delay Estimation to Ensure the Maximum Degree of Convergence in Consensus Problems

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Abstract—The Lambert W function is used to study a linear consensus model in multi-agent systems with delay. In particular, the case where all nonzero eigenvalues of the Laplacian matrix are real is considered. An explicit expression is obtained for the delay ensuring the maximum degree of convergence. A formula for the maximum degree of convergence is derived. As proved, the maximum degree of convergence depends only on the maximum and minimum nonzero eigenvalues, while the other eigenvalues have no influence on this characteristic. The results presented are a basis for one still unsolved problem, i.e., the direct estimation of the convergence rate in multi-agent systems with a directed structure.

Keywords: Lambert W function, degree of convergence, degree of stability, control with delay, Laplacian matrix, multi-agent system, consensus

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1. INTRODUCTION

Several results on the theory of multi-agent systems with information links and delay were obtained in [1–9]. However, the dependence of the convergence rate of such systems with delay on their spectral properties remains a relevant topic for research. The degree of convergence conceptualized in this paper and its estimate can serve as a basis for estimating the convergence rate. This is valid for models of multi-agent systems with a Laplacian matrix and, moreover, for systems with an arbitrary stable matrix (when estimating the rate of reaching consensus).

In this paper, we apply the Lambert W function to obtain the main results. Let us recall some key results on this topic.

Over the past 20 years, after the famous publication [10], a series of works on control of systems with delay using the Lambert W function have appeared [11–16]. The *Lambert W function* $W(z)$ is defined by $z = W(z)e^{W(z)}$, i.e., $W(z)$ is the inverse function of $f(z) = ze^z$. The application of this function has proven very effective and, in some cases, allows establishing necessary statements in a shorter way.

The system

$$\dot{y}(t) = Ay(t) + By(t - \tau) \tag{1}$$

with simultaneously triangularizable matrices A and B was studied in [12]; due to this property, the matrix equation can be reduced to a scalar one. A necessary and sufficient condition for robust stability of the corresponding system was presented under certain inequalities with undetermined coefficients. Similar to the results in [12], an expression for the spectrum of system (1) with simultaneously triangularizable or commuting matrices A and B was also derived in [13]. (The overlapping statements in [12] and [13] were obtained simultaneously and independently.)

Consider in detail the method for computing the primary matrix Lambert W function described in [14]. The iterative method (4.4) proposed in [14] was called by the authors a stable variant of the simplified Newton method, namely, that of the iterations $y_{k+1} = ({y_k}^2 + ae^{-y_k})/(y_k + 1)$. Since this formula directly follows from $y_{k+1} = y_k - f(y_k)/f'(y_k)$, it was termed the Newton–Raphson method in [14]. To determine the primary Lambert W function satisfying the equation $We^W - A = 0$, expressed as a polynomial in a matrix A, the Schur decomposition for A is used: the matrix A is replaced by its Schur form $T = Q^{\mathrm{T}}AQ$, where T is a block-diagonal matrix whose diagonal blocks are of order 1 (order 2) if they correspond to a real eigenvalue (to a pair of complex eigenvalues, respectively). Also, the Schur matrix T is represented as a block-triangular matrix with two blocks on the diagonal:

$$T = \begin{pmatrix} T_{11} & T_{12} \\ 0 & T_{22} \end{pmatrix}.$$

For each diagonal block T_{11} and T_{22}, the proposed Newton method with chosen initial matrices leads to the matrices $X_{11} = W_k(T_{11})$ and $X_{22} = W_k(T_{22})$. The missing block X_{12} is found from the Sylvester equation (see [14]). Thus, one obtains

$$W_k(T) = \begin{pmatrix} X_{11} & X_{12} \\ 0 & X_{22} \end{pmatrix}.$$

The inverse transformation $A = QTQ^T$ is applied to compute the primary matrix $W_k(A)$ satisfying the equation $W_k(A)e^{W_k(A)} - A = 0$.

In the scalar case, the stability region of a system with a delay link was also constructed by using the Lambert W function in [16].

The reader can find other applications of the Lambert W function in physical problems in the book [17], where, in particular, results on conformal mappings with the Lambert W function acting on rectangular domains in the z-plane were presented. For a system governed by the equation $x^{(\alpha)}(t) = Bx(t-1)$ with unit delay and the derivative of an integer or fractional order α, a stability analysis using the Lambert W function was described by the same authors in [18].

2. NECESSARY BACKGROUND AND AUXILIARY RESULTS

In this section, when presenting some properties of the Lambert W function, we utilize the results from the famous paper [10] (also, see [19]). The properties of the Lambert W function were also considered in [17].

As in the case of a complex variable, the real Lambert W function $W(x)$ is defined as a solution of the functional equation $W(x)e^{W(x)} = x$.

The function $W(x)$ is defined on the interval $-e^{-1} \leqslant x < +\infty$, where it takes values from $-\infty$ to $+\infty$. For negative arguments, $W(x)$ is negative and two-valued. At the point $(-e^{-1}, -1)$ the graph splits into two branches: the upper one is the principal (or zeroth) branch $W_0(x)$, and the lower one is the minus-first branch $W_{-1}(x)$. In Fig. 1, the zeroth branch is drawn with a solid line and the -1st branch with a dashed line.

In the complex domain, the Lambert W function is multivalued. The complex plane is mapped to a value set (range) $w = W_k(z)$, $k = 0, \pm1, \pm2, \ldots$, $z = x + iy$, $w = u + iv$. The definitional domain of $W_0(z)$ is the entire complex plane with the cut $(-\infty, -e^{-1}]$, i.e., with a cut along the negative real semi-axis: $\{z : -\infty < \mathrm{Re}(z) \leqslant -e^{-1}, \Im(z) = 0\}$. For other branches ($k \neq 0$) the definitional domain is the entire plane with the cut $(-\infty, 0]$, i.e., $\{z : -\infty < \mathrm{Re}(z) \leqslant 0, \Im(z) = 0\}$. Note that the above cuts up to the branching points allow defining single-valued branches of the Lambert

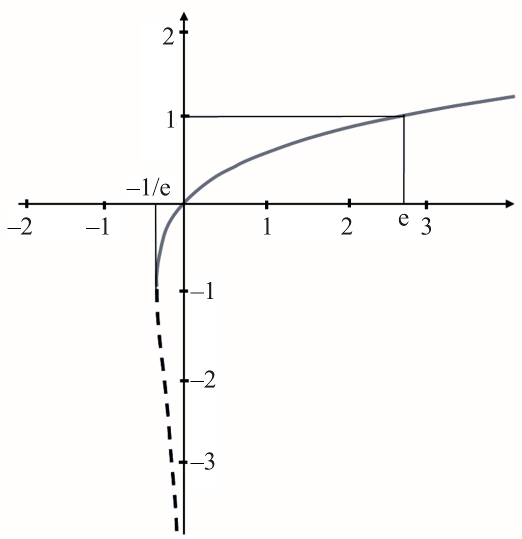

Fig. 1. Two branches of the Lambert function for the real case.

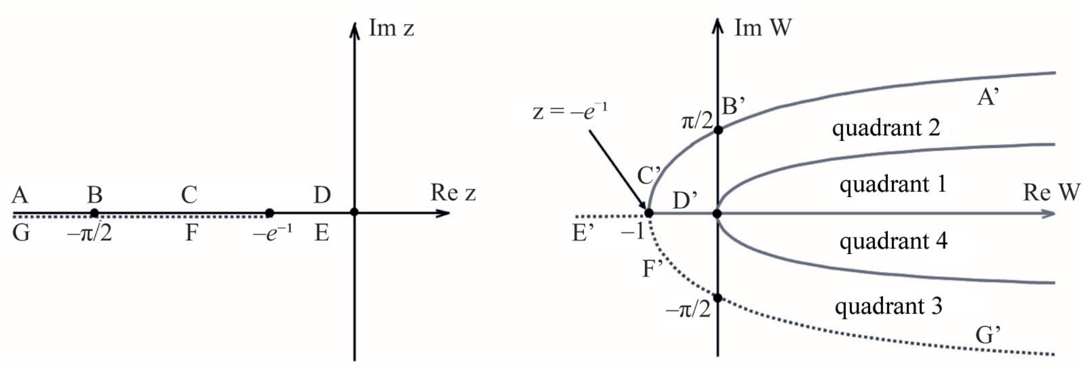

Fig. 2. The solid line shows the mapping of the imaginary and real axes by the function $W_0(z)$. The dashed line, defining the upper boundary of the branch $W_{-1}(z)$, indicates the mappings of the cut and the interval $(-e^{-1}, 0)$ by the function $W_{-1}(z)$.

function that do not transform into each other. Due to $W(z)e^{W(z)} = z$,

$$\begin{cases} x = e^u(u\cos v - v\sin v), \\ y = e^u(v\cos v + u\sin v). \end{cases} \tag{2}$$

From the condition $y = 0$ in (2) we obtain the expression $u = -v\cot v$, $-\pi < v < \pi$, defining the boundary between the range of the branch W_0 and those of the branches W_{-1} and W_1 (see Fig. 2).

Remark 1. For the curve $u = -v\cot v$, the point $v = 0$ is a removable discontinuity point. However, as $W_0(-e^{-1}) = -1$ and $W_{-1}(-e^{-1}) = -1$, the boundary between the ranges will have no discontinuity.

The curves separating the ranges of other branches are defined by the following set of points [10]:

$$\{(u, v) : u = -v\cot v,\ 2\pi k < \pm v < (2k+1)\pi\},\ k = 1, 2, \ldots.$$

Note that for one value of k, two curves are obtained, defining the boundaries between the branches W_k, W_{k+1} and W_{-k}, W_{-k-1}.

To prove the main results, the following fact [12, Lemma 3] will be needed.

Lemma 1. *For any* $z \in \mathbb{C}$,

$$\max\{\mathrm{Re}(W_k(z)) : k = 0, \pm 1, \pm 2, \ldots\} = \mathrm{Re}(W_0(z)).$$

A multi-agent system with a set of agents $V = \{1, \ldots, n\}$ is conveniently represented as a dependency digraph $\Gamma = (V, E)$, where $E \subset V \times V$ is a set of weighted arcs. If agent j influences agent i with a weight a_{ij}, then there exists an arc from vertex j to vertex i with the same weight.

Definition 1. The Laplacian matrix $L = (l_{ij})$ corresponding to a digraph Γ is constructed as follows:

$$l_{ij} = \begin{cases} -a_{ij}, & j \neq i, \\ \sum\limits_{k \neq i} a_{ik}, & j = i. \end{cases}$$

Consider the basic consensus protocol in a first-order multi-agent system with delay

$$\begin{cases} \dot{x}(t) = -Lx(t - \tau), & t > 0, \\ x(\theta) = \phi(\theta), & \theta \in [-\tau, 0], \end{cases} \tag{3}$$

where $x_i(t)$ denotes the characteristic of agent i at a time instant t and $x(t) = (x_1(t), \ldots, x_n(t))^T$ is the column vector of agents' characteristics. The time delay τ in the system can arise from data transmission, measurement, control, etc.

Definition 2. The protocol (3) converges to consensus if there exists the finite limit $\lim_{t \to \infty} x(t) = c\mathbf{1}$, where $x(t)$ is the solution of system (3), $\mathbf{1}$ is the column vector of ones, and c is some constant (the consensus value). In such a case, we say that system (3) reaches asymptotic consensus (or simply, reaches consensus).

For each initial function in Definition 2, only the existence of the constant c is required. However, as proved in [21], for the protocol (3) the value of c depends on the initial function.

For L, zero is always an eigenvalue, and its algebraic multiplicity coincides with the geometric one. This paper considers a system for which L has only a real spectrum with a simple zero eigenvalue and a simple structure. For a graph, the condition of a simple zero eigenvalue is equivalent to connectivity; for a digraph, to the presence of a spanning tree. This condition is necessary for reaching consensus for any initial value vector. (And for $\tau = 0$, it is also sufficient.) The class of Laplacian matrices L satisfying these two conditions is the minimal extension of the class of connected symmetric matrices L. The nonzero eigenvalues of an arbitrary matrix L have positive real parts, which can be verified, e.g., using Gershgorin's circle theorem.

For the protocol to converge, it is necessary that the real parts of all nonzero roots of the characteristic function be negative. The roots of the characteristic function depend on τ. The boundary value τ_0 is a delay value such that consensus is reached in the system for any initial function $\phi(\theta)$ if $\tau < \tau_0$ and is not achieved if $\tau > \tau_0$. The dependence of the boundary delay on the spectral properties of the corresponding matrix has been studied by many researchers. For example, for system (3) with a symmetric matrix, the boundary delay $\tau_0 = \frac{\pi}{2\lambda_{\max}}$ was obtained in [20]. However, a linear system with an arbitrary stable matrix A (and a spectrum possibly containing complex numbers) was considered earlier in [22]; the authors gave a rather complicated proof of a theorem providing an expression for the boundary value τ_0. This result for the consensus problem is presented below as Theorem 1, including a short proof using the Lambert W function.

The characteristic function of system (3) is

$$F(s) = \det(sI + e^{-\tau s}L) = s \prod_{\lambda \in \sigma(L) \setminus \{0\}} (s + \lambda e^{-\tau s}) = s \prod_{\lambda \in \sigma(L) \setminus \{0\}} f_\lambda(s), \tag{4}$$

where $\sigma(L)$ denotes the spectrum of the matrix L.

We write the equation

$$s + \lambda e^{-\tau s} = 0 \tag{5}$$

as $\tau s e^{\tau s} = -\tau\lambda$.

Since $W(z)e^{W(z)} = z$, the number τs (s being the root of equation (5)) is the value of the Lambert W function at $-\tau\lambda$:

$$W(-\tau\lambda) = \tau s. \tag{6}$$

Analogously to the *degree of stability*, a concept introduced for stable matrices, we define the degree of convergence to consensus for system (3).

Definition 3. The number $\zeta_0 = -\max(\mathrm{Re}(s))$, where s is the zero of the function $G(s) = \prod_{\lambda\in\sigma(L)\setminus\{0\}} f_\lambda(s)$, is called the degree of convergence to consensus for system (3).

The aim of this paper is to find the delay value τ^* maximizing the degree of convergence. However, even in the case of a matrix with a real spectrum, the degree of convergence for systems with delay does not always characterize the convergence rate of the corresponding protocol. (For symmetric networks without delay, the degree of convergence coincides with the convergence rate and is called the *Fiedler number*.) Therefore, the concept of the degree of convergence characterizes, to a greater extent, the "margin" of convergence to consensus (like stability margins in control theory). In [23], $\gamma^* = -\zeta_0$ was called the spectral abscissa function, and the corresponding root of the characteristic quasipolynomial was called the dominant root. Based on dominant roots, the authors claimed that for some $\gamma \in \mathbb{R}$, the value $\gamma^* = -\gamma$ approximates the exponential decay rate of the agents' states via the rightmost roots [23].

If consensus is achieved in the system, then $\zeta_0 > 0$. By Lemma 1, to study the degree of convergence, as well as to find the boundary delay, it suffices to consider the root corresponding to the zeroth branch of the Lambert W function.

As mentioned above, for any stable matrix A, the theorem below was established earlier [22, Theorem 3.4]. Here, we provide its short proof for system (3). Note that Theorem 3.4 from [22] can be proved by analogy.

Theorem 1. *If* 0 *is a simple eigenvalue of* L, *then the consensus protocol* (3) *converges for any* $\tau < \tau_0 = \min\limits_{\lambda\in\sigma(L)\setminus\{0\}} \frac{1}{\rho}\left(\frac{\pi}{2} - \varphi\right)$, *where* $\rho = |\lambda|$ *and* $\varphi = |\arg(\lambda)|$.

3. MAIN RESULTS

Consider system (3) in which the matrix L has a real spectrum, i.e., $\sigma(L) = \{0 = \lambda_1 < \lambda_2 \leqslant \lambda_3 \leqslant \ldots \leqslant \lambda_n\}$. For such a system, we pose the following problem: find the value of τ^* ensuring the maximum degree of convergence. For this purpose, under a fixed $\lambda \neq 0$, it is necessary to analyze the behavior of the function $s = \frac{1}{\tau}W_0(-\tau\lambda)$ on the interval $-\pi/2 < -\tau\lambda < 0$. (For $-\tau\lambda < -\pi/2$ the real parts of $W_0(-\tau\lambda)$ are positive.)

Proposition 1. 1) *The real part of the function* $s = \frac{1}{\tau}W_0(-\tau\lambda)$ *increases on the interval* $-\pi/2 < -\tau\lambda < -e^{-1}$. 2) *On the interval* $-e^{-1} < -\tau\lambda < 0$, *this function decreases.*

Proposition 2. *Let* s_0 *be the solution of the equation* $f_\lambda(s) = s + \lambda e^{-\tau s} = 0$ $(\lambda > 0)$ *with the maximum real part. Then* $\arg\min\limits_{\tau} \mathrm{Re}(s_0) = \tau^*_\lambda = \frac{1}{e\lambda}$.

Proposition 3. *For each* $\lambda_i > 0$, $i = 2, \ldots, n$, *let the minimum real part of the solution of equation* (5) *be achieved at* $\tau^*_{\lambda_i}$. *Then*:

1) *If* $\lambda_2 < \lambda_n$, *the maximum degree of convergence of system* (3) *will be achieved at* $\tau^* \in (\tau^*_{\lambda_n}, \tau^*_{\lambda_2})$. 2) *If* $\lambda_2 = \lambda_n = \lambda$, *then* $\tau^* = \tau^*_\lambda$.

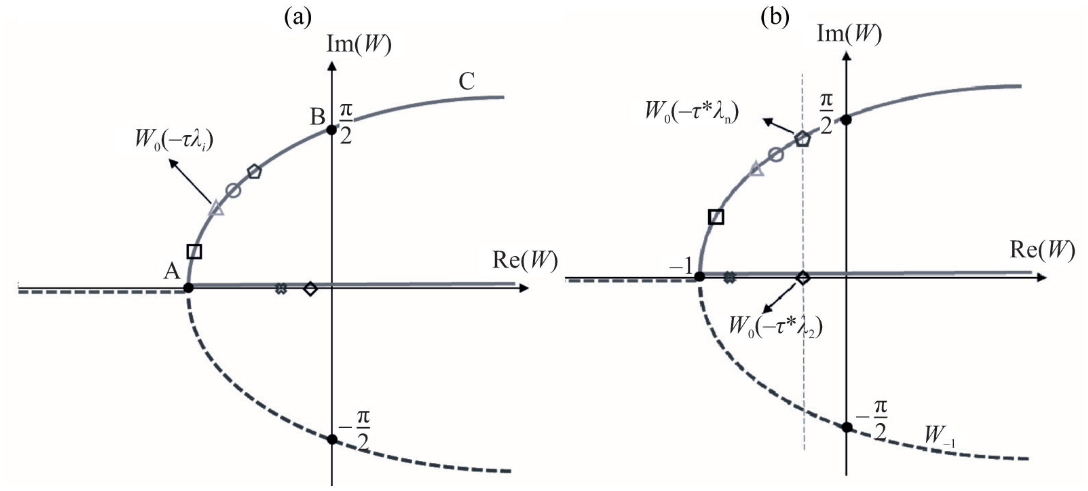

Fig. 3. (a) The values of $W_0(-\tau\lambda)$ and (b) the values of $W_0(-\tau^*\lambda_2)$ and $W_0(-\tau^*\lambda_n)$ for $\tau = \tau^*$. The dashed line indicates the mapping of $W_{-1}(z)$ whereas the solid line the mapping of $W_0(z)$.

Thus, for any $\tau \in (\tau^*_{\lambda_n}, \tau^*_{\lambda_2})$, the value $W_0(-\tau\lambda_n)$ will lie on the arc AC, and $W_0(-\tau\lambda_2)$ will belong to the interval $(-1, 0)$ (see Fig. 3a).

If the real part of $W_0(-\tau\lambda_n)$ is smaller than $W_0(-\tau\lambda_2)$, then by increasing τ, we can achieve the case $\mathrm{Re}(W_0(-\tau\lambda_n)) = W_0(-\tau\lambda_2)$. And if $\mathrm{Re}(W_0(-\tau\lambda_n)) > W_0(-\tau\lambda_2)$, then by decreasing τ, we can achieve the equality here (see Fig. 3b).

The result below provides an expression for τ ensuring the maximum degree of convergence ζ_0.

Theorem 2. *Let* $q = \frac{\lambda_2}{\lambda_n}$ *and* $\xi = -\arccos(q) \cdot \frac{q}{\sqrt{1-q^2}}$. *Then the maximum degree of convergence* ζ_0 *is achieved at* $\tau = \tau^* = \frac{-\xi e^{\xi}}{\lambda_2}$, *and its value is* $\zeta_0 = \frac{\lambda_2}{e^{\xi}}$.

In the paper [24], for an undirected graph and the case of delay, the number ζ_0 was used to characterize the rate of reaching consensus in a second-order consensus protocol with some parameter γ. The authors reduced such a problem to an optimization problem subject to a constraint involving the parameter γ and the first nonzero eigenvalue of the matrix L. Nevertheless, for two systems with the degrees of convergence ζ_0^1 and ζ_0^2 such that $\zeta_0^1 > \zeta_0^2$, the convergence rate of the first system may be less than that of the second. This may be due to, e.g., a Jordan cell of dimension above 1 in the Laplacian matrix. Therefore, for a system with a real spectrum of L, the convergence rate problem shall be investigated considering the convergence rate of the functional series expressing the solution.

4. NUMERICAL EXAMPLE

In this section, we present an example of a multi-agent system with informational influences, delay, and a digraph corresponding to a Laplacian matrix of simple structure with a real spectrum. Figure 4 shows the dependency digraph for five agents. The weights of the agents' influences are indicated on the arcs. The Laplacian matrix of this digraph has the form

$$L = \begin{pmatrix} 1 & -1 & 0 & 0 & 0 \\ 0 & 1.8 & 0 & -1.8 & 0 \\ 0 & -2 & 2.5 & 0 & -0.5 \\ 0 & 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & -2 & 2 \end{pmatrix}.$$

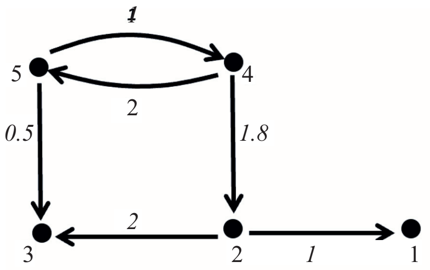

Fig. 4. The dependency digraph for five agents.

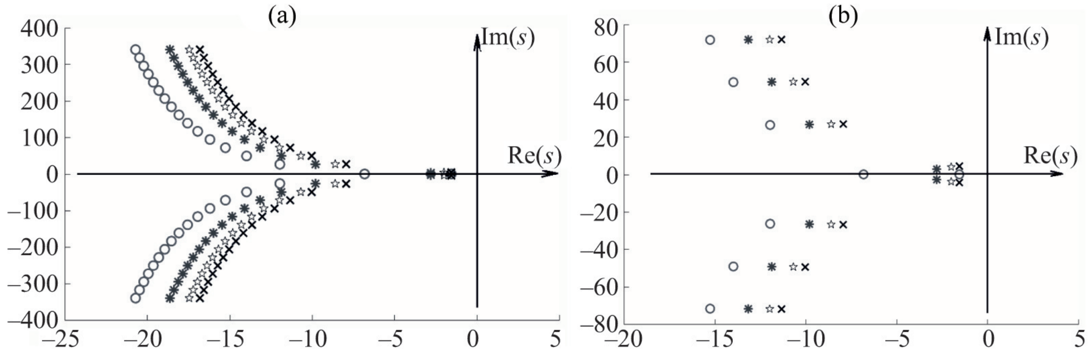

Fig. 5. The localization of roots s of the quasipolynomial computed using the Lambert W function (a) for fourteen branches and (b) for four branches. $\frac{1}{\tau^*}W_0(-\tau^*\lambda_2) = \frac{1}{\tau^*}\mathrm{Re}(W_0(-\tau^*\lambda_n)) = -1.545$.

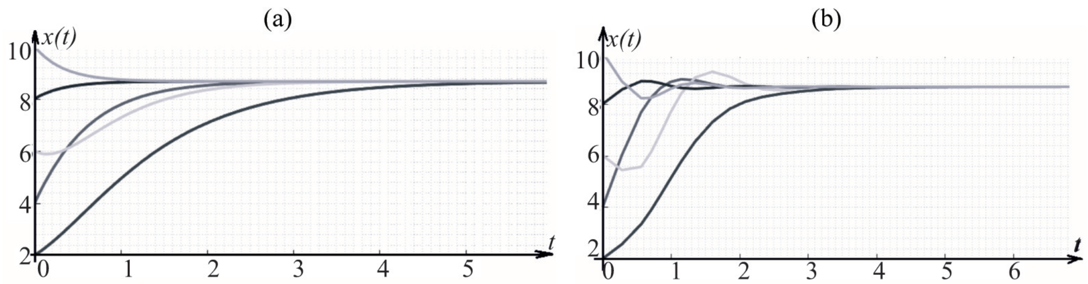

Fig. 6. Reaching consensus (a) without delay and (b) with delay $\tau^* \approx 0.2816$.

Its spectrum is $\sigma(L) = \{0;\ 1;\ 1.8;\ 2.5;\ 3\}$. According to Theorem 2, we find $\lambda_2 = 1;\ \lambda_5 = 3;$ $q = 1/3$, and $-\xi = \arccos(q)\frac{q}{\sqrt{1-q^2}} = \arccos(1/3)\frac{1/3}{\sqrt{8/9}} \approx 0.4352$. Then, for the delay $\tau^* = -\frac{\xi e^{\xi}}{\lambda_2} \approx$ 0.2816, the degree of convergence to consensus achieves maximum, and its value is $\zeta_0 = \frac{\lambda_2}{e^{\xi}} = 1.545$. The roots of the characteristic function (4), except the zero one, are marked in Fig. 5. Note that the boundary delay is $\tau_0 = \frac{\pi}{2\cdot 3} \approx 0.5236$.

Figure 6a demonstrates the graph of reaching consensus without delay with the initial value vector x_0. Clearly, with the delay $\tau^* \approx 0.2816$ and the initial function $\phi(\theta) = x_0$, the agents reach consensus faster than without delay (Fig. 6b). For both cases, the initial value vector is $x_0 = (2, 4, 6, 8, 10)^T$. For this initial condition, the consensus value does not depend on the delay and is given by the expression $px_0 = \frac{26}{3}$, where $p = \left(0, 0, 0, 0, \frac{2}{3}, \frac{1}{3}\right)$ is the row of the eigenprojector of rank 1 for the Laplacian matrix. Note that p is the left eigenvector of the zero eigenvalue for L.

In this example, for a nonsymmetric matrix of simple structure, an increase in the degree of convergence leads to an increase in the convergence rate. However, if a nonsymmetric matrix contains Jordan cells of dimension above 1, such a correlation may be violated. In other words, an increase in the degree of convergence does not always lead to an increase in the convergence rate.

According to the numerical example, an increase in the convergence rate causes oscillations in the transients. Unlike conventional control systems, where filters, damping devices, etc., can be used to reject oscillations, no methods for solving this problem in multi-agent systems are yet known. Indeed, in group motion, this can lead to collisions between agents. This problem is of interest and can be a subject for further study.

5. CONCLUSIONS

In this paper, the Lambert W function has been used to investigate the dependence of the degree of convergence on the delay in a first-order multi-agent system. For a linear system with delay, where the Laplacian matrix has a real spectrum, an expression for the delay τ^* ensuring the maximum degree of convergence has been derived. As has been established, the value τ^* depends only on the minimum nonzero and maximum eigenvalues of the Laplacian matrix with a real spectrum. Owing to the Lambert W function, a new (short) proof for the boundary delay has been obtained. The results of this paper can be applied to an arbitrary system with a nonsymmetric matrix whose spectrum consists of real numbers only.

APPENDIX

Proof of Theorem 1. Consider the characteristic function (4) of system (3), where $f_\lambda(s) = s + \lambda e^{-\tau s}$ and $\lambda = a + ib \in \sigma(L) \setminus \{0\}$. Recall that $a > 0$ for a Laplacian matrix. By (6) and Lemma 1, the boundary delay ensuring the negative real parts of the zeros of the function $f_\lambda(s)$ is given by the equality

$$W_0(-\tau\lambda) = iv,$$

which can be written as

$$ive^{iv} = -\lambda\tau. \tag{A.1}$$

From (A.1) it directly follows that

$$\begin{cases} v \sin v = a\tau, \\ v \cos v = -b\tau. \end{cases} \tag{A.2}$$

Then (A.2) implies $\tan v = -\frac{a}{b}$. Let $\varphi = |\arg(\lambda)|$, $\rho = |\lambda|$ and consider all possible cases of b.

a) $b = 0$. In this case, $-\tau\lambda$ belongs to the negative part of the real axis and $W_0(-\tau\lambda) = W_0(-\frac{\pi}{2})$ (see Fig. 2). Obviously, $-\tau\lambda = -\frac{\pi}{2}$ and $\tau = \frac{\pi}{2\lambda}$.

b) $b < 0$. Here, $-\tau\lambda$ belongs to the second quadrant and $0 < v < \pi/2$, as can be observed in Fig. 2. Therefore, $\tan v = \tan(\frac{\pi}{2} - \varphi) \Rightarrow v = \frac{\pi}{2} - \varphi$.

c) If $b > 0$, then $-\tau\lambda$ belongs to the third quadrant and, according to Fig. 2, $-\pi/2 < v < 0$. Therefore, $\tan v = -\tan(\frac{\pi}{2} - \varphi)$ and consequently, $v = \varphi - \frac{\pi}{2}$.

In case b), substituting $v = \frac{\pi}{2} - \varphi$ into (A.1) yields

$$i\left(\frac{\pi}{2} - \varphi\right) e^{i\left(\frac{\pi}{2} - \varphi\right)} = -\rho\tau e^{-i\varphi}.$$

Multiplying both sides by $-i$, we obtain

$$\left(\frac{\pi}{2} - \varphi\right) e^{i\left(\frac{\pi}{2} - \varphi\right)} = \rho\tau e^{i\left(\frac{\pi}{2} - \varphi\right)} \Rightarrow \frac{\pi}{2} - \varphi = \rho\tau. \tag{A.3}$$

In case c), with $v = \varphi - \frac{\pi}{2}$ substituted into (A.1),

$$i\left(\varphi - \frac{\pi}{2}\right) e^{i\left(\varphi - \frac{\pi}{2}\right)} = -\rho\tau e^{i\varphi},$$

$$\left(\frac{\pi}{2} - \varphi\right) e^{i\left(\varphi - \frac{\pi}{2}\right)} = -i\rho\tau e^{i\varphi} = \rho\tau e^{i\left(\varphi - \frac{\pi}{2}\right)} \Rightarrow \frac{\pi}{2} - \varphi = \rho\tau. \tag{A.4}$$

Thus, from (A.3) and (A.4) it follows that

$$\tau = \frac{1}{\rho}\left(\frac{\pi}{2} - \varphi\right).$$

In view of case a), the boundary delay for system (3) is $\tau_0 = \min\limits_{\lambda \in \sigma(L)\setminus\{0\}} \frac{1}{\rho}\left(\frac{\pi}{2} - \varphi\right)$. □

Proof of Proposition 1. 1) For $-\tau\lambda < -\frac{1}{e}$ (i.e., for $\tau > \frac{1}{e\lambda}$), the function $s = \frac{1}{\tau}W_0(-\tau\lambda)$ takes complex values. Since $W_0(-\frac{\pi}{2}) = i\frac{\pi}{2}$ and $\mathrm{Re}(W_0(x)) < 0$ for $x \in (-\frac{\pi}{2}, 0)$, we have $\mathrm{Re}(s) < 0$ for $-\frac{\pi}{2} < -\tau\lambda < -\frac{1}{e}$ (i.e., for $\frac{1}{e\lambda} < \tau < \frac{\pi}{2\lambda}$). On this interval, as τ increases, the value of $\frac{1}{\tau}$ decreases, like the magnitude of the real part of $W_0(-\tau\lambda)$. Thus, the real part of $s = \frac{1}{\tau}W_0(-\tau\lambda)$ takes a negative value and decreases in magnitude with increasing τ. In other words, it increases on the interval under consideration.

2) Due to

$$W'(x) = \frac{W(x)}{x(1 + W(x))},$$

we have

$$s' = \left(\frac{W_0(-\tau\lambda)}{\tau}\right)'_\tau = -\frac{W_0^2(-\tau\lambda)}{\tau^2(1 + W_0(-\tau\lambda))}.$$

On the other hand, on the interval $-\frac{1}{e} < -\tau\lambda < 0$ (i.e., for $0 < \tau < \frac{1}{e\lambda}$), the values of the function $W_0(-\tau\lambda)$ are negative and greater than -1 (see Fig. 1). Therefore, the function $s = \frac{1}{\tau}W_0(-\tau\lambda)$ decreases on the interval $0 < \tau < \frac{1}{e\lambda}$ with respect to τ. On this interval, $-e\lambda < s < -\lambda$. □

Proof of Proposition 2. If s_0 is the solution of the equation $s + \lambda e^{-\tau s} = 0$, then according to (5) and (6), τs_0 is defined by the multivalued function $W(-\tau\lambda)$. On the other hand, by Lemma 1, the maximum real part of s_0 can be defined by the function $\frac{1}{\tau}W_0(-\tau\lambda)$. Also, based on Proposition 1, the minimum real part of the function $\frac{1}{\tau}W_0(-\tau\lambda)$ is achieved at $-\tau\lambda = -\frac{1}{e}$, i.e., at $\tau = \frac{1}{e\lambda}$. □

Proof of Proposition 3. 1) Obviously, $\tau^*_{\lambda_n} < \ldots < \tau^*_{\lambda_2}$. Let τ^* be the delay value ensuring the maximum degree of convergence for system (3). We show that $\tau^* \in (\tau^*_{\lambda_n}, \tau^*_{\lambda_2})$. Indeed, if $\tau^* \leqslant \tau^*_{\lambda_n}$, then all values $W_0(-\tau^*\lambda_i)$, $i = 2, \ldots, n$, are real and belong to the interval $(-1, 0)$ of the real axis. In this case, the root corresponding to λ_2 will be closest to the imaginary axis. Then, as the value τ^* increases, the root corresponding to λ_2 will move away from the imaginary axis, thereby increasing the degree of convergence of the system. And if $\tau^* \geqslant \tau^*_{\lambda_2}$, then all values $W_0(-\tau^*\lambda_i)$, $i = 2, \ldots, n$, contain an imaginary part and belong to the arc AC (see Fig. 3a). In addition, the root corresponding to λ_n will have the largest real part. In this case, by decreasing τ^*, one can increase the degree of convergence of the system.

Thus, the maximum degree of convergence of system (3) will be achieved at $\tau = \tau^* \in (\tau^*_{\lambda_n}, \tau^*_{\lambda_2})$, for which

$$W_0(-\tau^*\lambda_2) = \mathrm{Re}(W_0(-\tau^*\lambda_n)). \tag{A.5}$$

2) If all nonzero eigenvalues λ of the Laplacian matrix are equal, then the degree of convergence is determined by λ and is equal to $e\lambda$. □

Proof of Theorem 2. According to Proposition 3, the eigenvalues λ_i, $i = 3, \ldots, n-1$, have no influence on τ^*. Therefore, τ^* can be found from equality (A.5).

Let us denote $W_0(-\tau^*\lambda_2) = u$ and $W_0(-\tau^*\lambda_n) = u + iv$.

Note that $-\tau^*\lambda_n$ belongs to the ray $(-\infty, -e^{-1}]$ (see Fig. 2). Then for $W_0(-\tau^*\lambda_n)$ we have $u = -v\mathrm{cot}v$.

Thus, from $W_0(-\tau^*\lambda_2) = \mathrm{Re}(W_0(-\tau^*\lambda_n))$ it follows that

$$\begin{cases} -\tau^*\lambda_2 = ue^u, \\ -\tau^*\lambda_n = e^u(u\cos v - v\sin v), \\ u = -v\mathrm{cot}v. \end{cases} \tag{A.6}$$

We transform the second equation of system (A.6):

$$-\tau^*\lambda_n = ue^u\left(\cos v - \frac{v}{u}\sin v\right).$$

Since $u = -v\mathrm{cot}v$,

$$\cos v - \frac{v}{u}\sin v = \cos v + \frac{\sin^2 v}{\cos v} = \frac{1}{\cos v}.$$

Then system (A.6) becomes

$$\begin{cases} -\tau^*\lambda_2 = ue^u, \\ -\tau^*\lambda_n = ue^u\dfrac{1}{\cos v}, \\ u = -v\mathrm{cot}v. \end{cases}$$

From the first and second equations we obtain

$$v = \arccos(q),$$

where $q = \frac{\lambda_2}{\lambda_n}$. Substituting this expression into the third equation yields

$$u = -\arccos(q)\cot(\arccos(q)).$$

Note that

$$\cot(\arccos(q)) = \frac{\cos(\arccos(q))}{\sin(\arccos(q))} = \frac{q}{\sqrt{1-q^2}}.$$

Hence, $u = -\frac{q}{\sqrt{1-q^2}} \cdot \arccos(q)$, we denote this value by ξ. Then τ^* can be obtained from the first equation of system (A.6):

$$\tau^* = -\frac{\xi e^\xi}{\lambda_2};$$

and the maximum degree of convergence is $\zeta_0 = -\frac{\xi}{\tau^*} = \frac{\lambda_2}{e^\xi}$. □

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