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LINEAR SYSTEMS

Tuning Non-Fragile PI Controllers: Analysis

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Abstract—This paper is devoted to the fragility analysis of PI controllers. Two different approaches are proposed to estimate the fragility of a given PI controller; they can be used independently of each other. The first approach is based on the ideas of ellipsoidal estimation and the technique of linear matrix inequalities. Within the second approach, involving the so-called *parameter loop opening*, the parameter under study is "extracted" from the plant–controller system, forming a fictitious control loop; after opening, this loop is analyzed by classical frequency-domain methods (the Nyquist criterion and D-partition). The features of both fragility analysis methods are described, and numerical examples are provided.

Keywords: linear control system, PI controller, fragility, linear matrix inequalities (LMIs), Lyapunov inequality, Nyquist criterion, D-partition

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1. INTRODUCTION

PI and PID controllers are the most common types of automatic controllers; according to various estimates (for example, see [1, 2]), they account for over 90% of currently used controllers. Based on medium-term forecasts, this share will even remain exceptionally high, nearly 80%. The extensive application of PI and PID controllers in industry is due to several circumstances. In addition to their suitability for solving most practical problems and low cost, a high demand for such controllers is associated with their simplicity: one needs to choose correctly only two (three) coefficients—gains—to tune a PI (PID, respectively) controller.

At the same time, the procedures for their practical tuning remain much heuristic: in real plants, PI and PID controllers are often tuned manually, based on an intuitive understanding of the industrial process and the influence of separate PID/PI control components on the latter. Accordingly, the problem of developing regular approaches to their tuning still retains its meaningfulness and topicality.

The general trend of recent decades is the transition to numerical PI/PID controller design methods (including their modifications, called low-order controllers) based on solving optimization problems with a certain performance criterion: the LQR approach [3, 4], H_∞ optimization [5, 6], and others [7, 8].

In practical problems, uncertainty is inevitably introduced into a control system due to the inaccurate technical implementation of the controller itself or the need to tune its parameters during operation. In this case, an optimal controller (in one sense or another) may turn out to be "fragile," i.e., small variations in its parameters can lead to a significant deterioration in control performance or even instability of the closed-loop system. Therefore, the problem of designing non-fragile controllers, originating from the work [9], particularly PI/PID controllers, has obvious practical significance.

In this paper, the problem is to estimate the non-fragility of *given* (or *nominal*) PI controllers. We propose two approaches to solve the problem as follows.

The first approach utilizes a special Lyapunov inequality; when satisfied, it means the existence of a common quadratic Lyapunov function for a set of parameter perturbations of a given PI controller lying in some ellipse. The approach ensures stability for any (including simultaneous and time-varying) parameter perturbations and, moreover, yields ellipsoidal estimates of the system output under bounded exogenous disturbances. The technique of linear matrix inequalities (LMIs) [10, 11] is used as a technical tool here.

The second approach is based on the so-called *parameter loop opening* [12], commonly used in robust analysis problems. The core of this method is to represent a closed-loop plant–controller system in a special form: the parameter under study is "extracted" from the system, forming a *fictitious control loop*. Then this loop is investigated using a classical stability analysis method, i.e., the Nyquist criterion [13]. The approach allows analyzing static perturbations in each PI controller parameter (i.e., finding the set of stability intervals for separate parameters).

These approaches involve different ideas and can be applied both independently and jointly, complementing each other. For example, if a stabilizing PI controller is a priori unknown for a plant, then the second approach can be used to "grope" a region of stabilizing controllers, and the first approach will be naturally applied inside it.

The remainder of this paper is organized as follows. Section 2 is devoted to the formal problem statement. Sections 3 and 4 contain the main results, and Section 5 provides numerical examples.

From now on, $\|\cdot\|$ is the Euclidean norm of a vector and the spectral norm of a matrix; $^{\mathrm{T}}$ is the transpose symbol; I denotes an identity matrix of appropriate dimensions; $\lambda_i(A)$ are the eigenvalues of a matrix A; $\sigma(A) \doteq -\max_i \operatorname{Re}\left(\lambda_i(A)\right) > 0$ means the stability degree of a Hurwitz matrix A. Note that all matrix inequalities are understood in the sense of positive/negative definiteness of corresponding matrices. Finally, s is the Laplace transform symbol.

2. PROBLEM STATEMENT

Consider a linear continuous-time control system described by

$$\begin{aligned} \dot{x} &= Ax + bu + Dw, \quad x(0) = x_0, \\ y &= c^{\mathrm{T}} x, \\ z &= Cx, \end{aligned} \tag{1}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $D \in \mathbb{R}^{n\times m}$, $c \in \mathbb{R}^n$, $C \in \mathbb{R}^{r\times n}$, with the state vector $x(t) \in \mathbb{R}^n$, the measured output $y(t) \in \mathbb{R}$, the controlled output $z(t) \in \mathbb{R}^r$, an exogenous disturbance $w(t) \in \mathbb{R}^m$ satisfying the constraint

$$\|w(t)\| \leqslant 1 \quad \text{for all } t \geqslant 0, \tag{2}$$

and the control input $u(t) \in \mathbb{R}$ in the form of a PI controller

$$u(t) = -k_1 y(t) - k_2 \int\limits_0^t y(\tau) d\tau. \tag{3}$$

Let the gains k_1 and k_2, called the *parameters* of the PI controller (and sometimes identified with the controller itself), be combined into the vector

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{R}^2.$$

Following [14] and introducing the auxiliary scalar variable ξ by

$$\dot{\xi} = y, \quad \xi(0) = 0,$$

we arrive at the equations of the closed-loop system

$$\dot{g} = A_{\rm cl}(k)g + D_{\rm cl}w = \begin{pmatrix} A - k_1 bc^{\rm T} & -k_2 b \\ c^{\rm T} & 0 \end{pmatrix} g + \begin{pmatrix} D \\ 0 \end{pmatrix} w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$
$$z = C_{\rm cl} g = \begin{pmatrix} C & 0 \end{pmatrix} g$$

with respect to the augmented state vector

$$g = \begin{pmatrix} x \\ \xi \end{pmatrix} \in \mathbb{R}^{n+1}.$$

Let a given (nominal) PI controller (3) with numerical parameters

$$k = k^0 = \begin{pmatrix} k_1^0 \\ k_2^0 \end{pmatrix},$$

chosen by some considerations, stabilize the closed-loop system for (1). Two problems are of interest here.

Problem 1. What are the admissible limits to vary the parameters of the nominal PI controller with k^0 without violating the stability of the closed-loop system?

This problem is a special case of the one below.

Problem 2. It is required to find the set of stabilizing PI controllers for system (1), i.e., construct a set $\mathcal{K} \subset \mathbb{R}^2$ such that

$$\max_i \operatorname{Re} \lambda_i \big(A_{\rm cl}(k)\big) < 0 \quad \forall k \in \mathcal{K}.$$

The exact solution of Problem 2 is obtained by the D-partition method [15], which constructs the boundary $\partial\mathcal{K}$ of the set $\mathcal{K}$ analytically. However, in practice, D-partition is not as convenient for solving the same problem for a PID controller, which already includes three parameters.

Below, we propose two approaches to the approximate solution of Problem 1, which provide simple inner estimates of the desired set. In Section 3, the problem is studied from the standpoint of quadratic stability, and a simple and effective approach is presented to obtain ellipsoidal estimates of the set $\mathcal{K}$; also, a non-fragile bounding ellipsoid is constructed. The second approach (Section 4) yields the set $\mathcal{K}$ in the form of stability intervals, separately for each PI controller parameter; the application of this approach as a boundary oracle is considered.

3. FRAGILITY ANALYSIS OF A PI CONTROLLER BASED ON ELLIPSOIDAL ESTIMATION

3.1. Estimation of the Non-Fragility Radius of a PI Controller

We introduce the notation

$$A_0 = \begin{pmatrix} A & 0 \\ c^{\rm T} & 0 \end{pmatrix}, \quad F = \begin{pmatrix} -b \\ 0 \end{pmatrix}, \quad H = \begin{pmatrix} c^{\rm T} & 0 \\ 0 & 1 \end{pmatrix},$$

with

$$A_{\rm cl}(k) = A_0 + Fk^{\rm T}H.$$

Perturbing the parameters k^0 of the stabilizing PI controller (3) in the form

$$k_1^0 + \delta_1, \quad k_2^0 + \delta_2,$$

we obtain the vector $k^0 + \delta$ of the perturbed controller parameters. By assumption, the perturbations belong to some ellipse

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \in \mathcal{E}(R) = \{\delta \in \mathbb{R}^2 : \quad \delta^{\mathrm{T}} R^{-1} \delta \leqslant 1\}, \quad R \succ 0. \tag{4}$$

It is necessary to find the largest ellipse (4) (by some criterion) such that system (1) with the stabilizing PI controller with the parameters k^0 will remain stable under all perturbations

$$\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix} \in \operatorname{int} \mathcal{E}(R)$$

of its parameters. In particular, an important issue is to calculate the largest *non-fragility radius* $\rho(k)$ such that system (1) will remain stable under all perturbations $\delta \colon \|\delta\| < \rho$ of the PI controller parameters.

Consider the matrix of the so-called *perturbed* system, i.e., the closed-loop one with the controller with the "perturbed" parameters:

$$A_{\mathrm{cl}}(k^0 + \delta) = A_0 + F(k^0 + \delta)^{\mathrm{T}} H = A_{\mathrm{cl}}(k^0) + F\delta^{\mathrm{T}} H. \tag{5}$$

As is well known (e.g., see [11]), if the Lyapunov inequality

$$\left(A_{\mathrm{cl}}(k^0) + F\delta^{\mathrm{T}} H\right)^{\mathrm{T}} Q + Q\left(A_{\mathrm{cl}}(k^0) + F\delta^{\mathrm{T}} H\right) \prec 0$$

with some matrix $0 \prec Q \in \mathbb{S}^{n+1}$ holds under all *admissible* uncertainties δ, then the family (5) has a common quadratic Lyapunov function $V(x) = x^{\mathrm{T}} Q x$ and is robustly quadratically stable.

The last inequality can be written as

$$A_{\mathrm{cl}}^{\mathrm{T}}(k^0) Q + Q A_{\mathrm{cl}}(k^0) + H^{\mathrm{T}} \delta F^{\mathrm{T}} Q + Q F \delta^{\mathrm{T}} H \prec 0$$

or

$$A_{\mathrm{cl}}^{\mathrm{T}}(k^0) Q + Q A_{\mathrm{cl}}(k^0) + H^{\mathrm{T}} R^{1/2} \bar{\delta} F^{\mathrm{T}} Q + Q F \bar{\delta}^{\mathrm{T}} R^{1/2} H \prec 0$$

with respect to the uncertainty $\bar{\delta} \doteq R^{-1/2}\delta$, $\|\bar{\delta}\| \leqslant 1$.

Using Petersen's lemma [16], we arrive at the matrix inequality

$$\begin{pmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(k^0) Q + Q A_{\mathrm{cl}}(k^0) + \varepsilon H^{\mathrm{T}} R H & QF \\ F^{\mathrm{T}} Q & -\varepsilon I \end{pmatrix} \prec 0$$

with respect to the scalar variable ε and the matrix variables $Q, R \succ 0$. Due to the homogeneity of this LMI in the aggregate of variables, we set $\varepsilon = 1$. Thus, the following result is true.

Lemma 1. *The feasibility of the LMI*

$$\begin{pmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(k^0) Q + Q A_{\mathrm{cl}}(k^0) + H^{\mathrm{T}} R H & QF \\ F^{\mathrm{T}} Q & -I \end{pmatrix} \prec 0$$

with respect to the matrix variables $0 \prec Q \in \mathbb{S}^{n+1}$ *and* $0 \prec R \in \mathbb{S}^2$ *is equivalent to the quadratic stability of the family* $A_{\mathrm{cl}}(k^0 + \delta)$ *under all admissible uncertainties* δ.

Now, simple inner estimates can be proposed for the admissibility region of the uncertainty δ by minimizing a certain criterion under the LMI constraint established in Lemma 1.

As is known (for more details, see [11]), an ellipsoid of the form

$$\mathcal{E} = \{x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \leqslant 1\}, \quad P \succ 0, \tag{6}$$

has volume $c_n \sqrt{\det P}$, where c_n is the volume of the unit ball in the n-dimensional space; in addition, the function $f(X) = -\operatorname{logdet} X$ of a matrix argument $X \succ 0$ is convex. Accordingly, the volume of an ellipsoid (in the two-dimensional case under consideration, the area of an ellipse) is maximized as follows.

Lemma 2. *Let* $\widehat{R}_{\mathrm{area}}$ *be the solution of the problem*

$$\min(-\operatorname{logdet} R)$$

subject to the constraints

$$\begin{pmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(k^0)Q + QA_{\mathrm{cl}}(k^0) + H^{\mathrm{T}}RH & QF \\ F^{\mathrm{T}}Q & -I \end{pmatrix} \prec 0, \quad Q \succ 0, \quad R \succ 0,$$

where optimization is performed with respect to the matrix variables $Q \in \mathbb{S}^{n+1}$ *and* $R \in \mathbb{S}^2$. *Then system* (1) *closed by the stabilizing PI controller* (3) *with the parameters* k^0 *remains stable under all perturbations* $\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$ *of its parameters such that*

$$\delta^{\mathrm{T}} \widehat{R}_{\mathrm{area}}^{-1} \delta < 1.$$

The uncertainty ellipse can be optimized using other criteria, e.g., by maximizing the radius of the circle contained in it. (This also gives the maximum magnitude of the uncertainty δ.) As is known, if

$$P \succcurlyeq \gamma I,$$

the ellipsoid (6) will contain a ball of radius $\sqrt{\delta}$ centered at the origin (details can be found in [11, Remark 2.2.3]). Therefore, we arrive at the following result.

Lemma 3. *Let* $\widehat{\gamma}$, $\widehat{R}_{\mathrm{axis}}$ *be the solution of the problem*

$$\max \gamma$$

subject to the constraints

$$\begin{pmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(k^0)Q + QA_{\mathrm{cl}}(k^0) + H^{\mathrm{T}}RH & QF \\ F^{\mathrm{T}}Q & -I \end{pmatrix} \prec 0, \quad R \succcurlyeq \gamma I, \quad Q \succ 0,$$

where optimization is performed with respect to the matrix variables $Q \in \mathbb{S}^{n+1}$ *and* $R \in \mathbb{S}^2$ *and the scalar variable* γ. *Then system* (1) *closed by the stabilizing PI controller* (3) *with the parameters* k^0 *remains stable under all perturbations* $\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$ *of its parameters such that*

$$\delta^{\mathrm{T}} \widehat{R}_{\mathrm{axis}}^{-1} \delta < 1,$$

and, in particular, those satisfying

$$\|\delta\| < \sqrt{\widehat{\gamma}}.$$

Thus, the non-fragility radius $\rho(k^0)$ of a given PI controller with the parameters k^0 is defined by the value $\sqrt{\widehat{\gamma}}$.

Remark 1. The last assertion of Lemma 3 can be obtained more simply: it suffices to set $R = rI$, which leads to the following result. Let $\widehat{r}$ be the solution of the problem

$$\max r \ \text{ subject to the constraints } \begin{pmatrix} A_{\mathrm{cl}}^{\mathrm{T}}(k^0)Q + QA_{\mathrm{cl}}(k^0) + rH^{\mathrm{T}}H & QF \\ F^{\mathrm{T}}Q & -I \end{pmatrix} \prec 0, \quad Q \succ 0,$$

where optimization is performed with respect to the matrix variable $Q \in \mathbb{S}^{n+1}$ and the scalar variable r. Then system (1) closed by the stabilizing PI controller (3) with the parameters k^0 remains stable under all perturbations $\delta = \begin{pmatrix} \delta_1 \\ \delta_2 \end{pmatrix}$ of its parameters such that

$$\|\delta\| < \sqrt{\widehat{r}}.$$

3.2. Non-Fragile Bounding Ellipsoid

Now it is natural to estimate the influence of uncertainties in the controller parameters on the performance criterion in a certain control problem associated with system (1). In this paper, we consider the classical problem of suppressing nonrandom bounded exogenous disturbances (2). An effective solution approach is to find the minimal bounding ellipsoid containing the controlled output of system (1); for details, see [11].

Within the analysis problem, we still assume a given stabilizing PI controller (3) with parameters k^0. It is required to find the minimal bounding ellipsoid for the output z under all perturbations δ of the controller parameters such that $\|\delta\| \leqslant \gamma < \rho$. It will be called the *non-fragile bounding ellipsoid* corresponding to the non-fragility level γ.

According to [11], the minimal bounding ellipsoid for the controlled output of system (1) closed by a PI controller with parameters k^0 is given by the matrix $C_{\mathrm{cl}}\widetilde{P}C_{\mathrm{cl}}^{\mathrm{T}}$, where $\widetilde{P}$ is the solution of the parameterized semidefinite programming problem

$$\min \operatorname{tr} C_{\mathrm{cl}}PC_{\mathrm{cl}}^{\mathrm{T}}$$

subject to the constraint

$$A_{\mathrm{cl}}(k^0)P + PA_{\mathrm{cl}}^{\mathrm{T}}(k^0) + \alpha P + \frac{1}{\alpha}D_{\mathrm{cl}}D_{\mathrm{cl}}^{\mathrm{T}} \preccurlyeq 0,$$

where optimization is performed with respect to the matrix variable $0 \prec P \in \mathbb{S}^{n+1}$ and the scalar parameter $\alpha > 0$. (The minimal ellipsoid is understood in terms of the so-called trace criterion, i.e., the one obtained by minimizing the sum of its squared semi-axes.)

By utilizing this result and passing to the matrix (5) of the perturbed system, we have the condition

$$\left(A_{\mathrm{cl}}(k^0) + F\delta^{\mathrm{T}}H\right)P + P\left(A_{\mathrm{cl}}(k^0) + F\delta^{\mathrm{T}}H\right)^{\mathrm{T}} + \alpha(\delta)P + \frac{1}{\alpha(\delta)}D_{\mathrm{cl}}D_{\mathrm{cl}}^{\mathrm{T}} \preccurlyeq 0,$$

holding for all $\delta\colon \|\delta\| \leqslant \gamma$. Suppose the existence of a number α such that the above matrix inequality is valid under all admissible values of δ. We represent it in the form

$$A_{\mathrm{cl}}(k^0)P + PA_{\mathrm{cl}}^{\mathrm{T}}(k^0) + \alpha P + \frac{1}{\alpha}D_{\mathrm{cl}}D_{\mathrm{cl}}^{\mathrm{T}} + PH^{\mathrm{T}}\delta F^{\mathrm{T}} + F\delta^{\mathrm{T}}HP \preccurlyeq 0$$

and use Petersen's lemma [16], taking the non-fragility level γ into account by appropriately scaling the matrix F. Consequently,

$$A(k^0)P + PA^{\mathrm{T}}(k^0) + \alpha P + \frac{1}{\alpha}D_{\mathrm{cl}}D_{\mathrm{cl}}^{\mathrm{T}} + \varepsilon\gamma^2 FF^{\mathrm{T}} + \frac{1}{\varepsilon}PH^{\mathrm{T}}HP \preccurlyeq 0$$

which is equivalent to

$$\begin{pmatrix} A_{\mathrm{cl}}(k^0)P + PA_{\mathrm{cl}}^{\mathrm{T}}(k^0) + \alpha P + \frac{1}{\alpha}D_{\mathrm{cl}}D_{\mathrm{cl}}^{\mathrm{T}} + \varepsilon\gamma^2 FF^{\mathrm{T}} & PH^{\mathrm{T}} \\ HP & -\varepsilon I \end{pmatrix} \preccurlyeq 0,$$

holding for some scalar $\varepsilon > 0$. Thus, the following fact has been established.

Theorem 1. *Let $\widehat{P}$ be the solution of the problem*

$$\min \operatorname{tr} C_{\mathrm{cl}}PC_{\mathrm{cl}}^{\mathrm{T}}$$

subject to the constraints

$$\begin{pmatrix} A_{\mathrm{cl}}(k^0)P + PA_{\mathrm{cl}}^{\mathrm{T}}(k^0) + \alpha P + \frac{1}{\alpha}D_{\mathrm{cl}}D_{\mathrm{cl}}^{\mathrm{T}} + \varepsilon\gamma^2 FF^{\mathrm{T}} & PH^{\mathrm{T}} \\ HP & -\varepsilon I \end{pmatrix} \preccurlyeq 0, \quad P \succ 0,$$

where optimization is performed with respect to the matrix variable $P \in \mathbb{S}^{n+1}$, the scalar variable ε, and the scalar parameter $\alpha > 0$.

Then the ellipsoid with the matrix $C_{\mathrm{cl}}\widehat{P}C_{\mathrm{cl}}^{\mathrm{T}}$ is a non-fragile bounding ellipsoid for the output z of system (1) with $x_0 = 0$ closed by the stabilizing PI controller (3) with the parameters k^0, corresponding to the admissible non-fragility level γ.

Remark 2. Within the approach discussed, the case of a nonzero initial condition can also be considered easily. In this case, the LMI

$$\begin{pmatrix} 1 & x_0^{\mathrm{T}} \\ x_0 & P \end{pmatrix} \succcurlyeq 0$$

is added to the constraints of the theorem, meaning that the point x_0 belongs to the invariant ellipsoid with the matrix P; see details in [11].

4. FRAGILITY ANALYSIS OF A PI CONTROLLER BY PARAMETER LOOP OPENING

4.1. Idea of the Approach

This section describes a frequency-domain approach to the fragility analysis of a PI controller based on the so-called *parameter loop opening* [12]. The approach estimates the set $\mathcal{K}$ of Problem 2 as follows: for a known stabilizing controller with parameters k^0, with one fixed parameter $k_i^0 = \mathrm{const}$, $i = 1, 2$, the stability interval (or union of intervals) is determined for the second parameter k_j, $j = 1, 2$, $i \neq j$.

The plant (1) is written in the "input–output" form as

$$a(s)y = b(s)u, \tag{7}$$

where $b(s)$ and $a(s)$ denote the numerator and denominator polynomials of its transfer function $b(s)/a(s) = c^{\mathrm{T}}(sI - A)^{-1}b$.

We apply the Laplace transform to the PI controller equation (3):

$$u = -\frac{k_1 s + k_2}{s} y. \tag{8}$$

In view of the negative feedback in the Nyquist criterion, this yields the open-loop transfer function

$$w_o(s) = \frac{b(s)}{a(s)} \frac{k_1 s + k_2}{s},$$

which is associated with the characteristic polynomial

$$D(s) = a(s)s + b(s)(k_1 s + k_2)$$

of the closed-loop system.

The Nyquist criterion [13] can be used to analyze the stability of the polynomial $D(s)$ from the Nyquist plot of the frequency response $w_o(j\omega)$. In classical automatic control theory, the robustness measure of a control system is the gain and phase margins (the frequency-domain segment stability criteria in [13]).

These classical tools for stability and robustness analysis are used in parameter loop opening [12]. Let us demonstrate the use of this method for analyzing the fragility of a nominal stabilizing PI controller k^0 with respect to the parameter k_1.

We introduce the variables

$$\tilde{u} \doteq -k_1^0 \tilde{y}, \quad \tilde{y} \doteq y,$$

where $\tilde{u}$ and $\tilde{y}$ are the fictitious control and output of the plant, respectively; then the plant equations (7) can be written as

$$a(s)y = b(s)\Big(\tilde{u} - \frac{k_2^0}{s} y\Big) \quad \Rightarrow \quad \Big(a(s) + \frac{k_2^0 b(s)}{s}\Big)\tilde{y} = b(s)\tilde{u}$$
$$\Rightarrow \quad \tilde{y} = \frac{s b(s)}{s a(s) + k_2^0 b(s)} \tilde{u} \quad \Rightarrow \quad \tilde{y} = \tilde{w}_1(s)\tilde{u}.$$

The resulting fictitious control loop consists of the transfer function $\tilde{w}_1(s)$ of a fictitious plant closed by the static controller $-k_1^0$. Performing a similar opening with respect to the parameter k_2 (setting $\tilde{u} = -k_2^0 \tilde{y}$), we get the following transfer function of the fictitious plant:

$$\tilde{w}_2(s) = \frac{b(s)}{s\big(a(s) + k_1^0 b(s)\big)}.$$

The next result is easily proved.

Lemma 4. *The characteristic polynomials obtained by closing the transfer functions of the open fictitious loops coincide with the characteristic polynomial $D(s)$ of the original closed-loop system* (7), (8).

Indeed, these open-loop transfer functions are written as

$$\tilde{w}_{oi}(s) = k_i^0 \tilde{w}_i(s), \quad i = 1, 2, \tag{9}$$

yielding $D(s)$ after closing, where the sign of the feedback in the Nyquist criterion is taken into account similarly to $w_o(s)$. In other words, the functions (9) can be employed to judge the stability of system (7), (8).

Note another important fact: both transfer functions (9) contain the PI controller parameters as cofactors. Then stability intervals of the closed-loop system with respect to the parameters k_i, $i = 1, 2$, are found by studying the intersection points of the real axis with the corresponding Nyquist plots $\mathrm{Im}\{\tilde{w}_{oi}(j\omega)\} = 0$, $i = 1, 2$. As shown in [15], such an approach is equivalent to the one-dimensional D-partition of the polynomial $D(s) = \tilde{a}_i(s) + k_i\tilde{b}_i(s)$, where $\tilde{b}_i(s)$ and $\tilde{a}_i(s)$ are the numerator and denominator of the corresponding transfer function $\tilde{w}_{oi}(s)$.

4.2. Fragility Analysis Procedure

Using the parameter k_1 as an example, we describe the procedure for finding stability intervals with respect to $\tilde{w}_{o1}(s)$ provided that all roots of $b(s)$ are nonzero and there are no zero or pure imaginary numbers among the roots of $\tilde{a}(s)$.

1. The Nyquist plot $\tilde{w}_{o1}(j\omega)$ starts and ends at the point $(0,\ j0)$ when varying ω from 0 to ∞, i.e., it forms a closed contour. Solve the polynomial equation $\mathrm{Im}\{\tilde{w}_{o1}(j\omega)\} = 0$, $\omega \in [0\ \infty)$, denoting its solutions by ω_i, $i = 1, \dots, r$.

Compute $g_i = \mathrm{Re}\{\tilde{w}_{o1}(j\omega_i)\}$, $i = 1, \dots, r$, and associate with each number g_i an index $f_i \in \{0,\ \pm 1/2,\ \pm 1\}$, $i = 1, \dots, r$, defined by the following rule [13, p. 141]:

$$f_i = \begin{cases} 1 & \text{if } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i - \epsilon))\} < 0 \text{ and } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i + \epsilon))\} > 0, \\ 1/2 & \text{if } \tilde{w}_{o1}(\omega_i) \text{ is the start of the plot and } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i + \epsilon))\} > 0, \\ 0 & \text{if } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i - \epsilon))\} \cdot \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i + \epsilon))\} > 0,\ i = 1, \dots, r, \\ -1/2 & \text{if } \tilde{w}_{o1}(\omega_i) \text{ is the start of the plot and } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i + \epsilon))\} < 0, \\ -1 & \text{if } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i - \epsilon))\} > 0 \text{ and } \mathrm{Im}\{\tilde{w}_{o1}(j(\omega_i + \epsilon))\} < 0, \end{cases}$$

where $\epsilon > 0$ is a sufficiently small number such that

$$\epsilon \ll \min_i |\omega_i - \omega_{i-1}|, \quad i = 2, 3, \dots, r.$$

2. According to the Nyquist criterion [13], the closed-loop system is stable if and only if the frequency response $\tilde{w}_{o1}(j\omega)$ has $s_{\tilde{a}}/2$ counterclockwise encirclements of the critical point $-1/k_1$, where $s_{\tilde{a}}$ is the number of unstable roots of the polynomial $\tilde{a}(s)$. For a stable $\tilde{w}_{o1}(j\omega)$, one obtains $s_{\tilde{a}} = 0$; therefore, for closed-loop stability, its frequency response shall not encircle the critical point.

The number of encirclements of the critical point can change only at the real axis points g_i, $i = 1, \dots, r$. An encirclement occurs clockwise for $f_i > 0$ and counterclockwise for $f_i < 0$; if $f_i = 0$, the number of encirclements remains invariable.

Sort the numbers g_i, $i = 1, \dots, r$, in ascending order and rearrange the subscripts of g_i and f_i accordingly.

3. First consider the case where $g_i < 0$, $i = 1, \dots, r-1$, and $g_r = 0$.

Split the real axis into the open intervals

$$(-\infty,\ g_1), \quad (g_1,\ g_2), \quad \dots, \quad (g_{r-1},\ g_r), \quad (g_r,\ \infty).$$

For the first interval $(-\infty,\ g_1)$, the number of encirclements of the critical point by the frequency response is zero, and the closed-loop system will be stable only if $\tilde{w}_1(s)$ is stable; then the parameter k_1 varies within the interval

$$k_1^{[-\infty]} \in \left[0,\ -\frac{k_1^0}{g_1}\right).$$

The left bound is included in this interval since the open-loop system (7) is stable.

For the remaining intervals, starting from the second, check the equality

$$\sum_{i=1}^{q} f_i = -\frac{s_{\tilde{a}}}{2}, \quad q = 1, \ldots, r-1, \tag{10}$$

where q is the subscript of the left bound of the interval $(g_q,\ g_{q+1})$.

For all intervals where condition (10) holds, the closed-loop system is stable; then for a particular qth interval, one has the following interval of the parameter k_1 :

$$k_1^{[-q]} \in \left(-\frac{k_1^0}{g_q},\ -\frac{k_1^0}{g_{q+1}}\right).$$

The complete stability region is the union of all stability intervals: $\bigcup k_1^{[-q]}$.

4. In the general case, g_i, $i = 1, \ldots, r$, can have different signs or be zero.

For all $g_i < 0$, perform the actions of Step 3. For $g_i = 0$, the stability of the closed-loop system is determined by the open-loop transfer function (9).

For the remaining $g_i > 0$, apply the following technique: simultaneously change the signs of $g_i > 0$ and their corresponding f_i, which means changing the sign of the feedback in the closed-loop system, and then perform the actions of Step 3, albeit with calculating the bounds of the qth stability interval as

$$k_1^{[+q]} \in \left(\frac{k_1^0}{g_q},\ \frac{k_1^0}{g_{q+1}}\right);$$

for the interval $(g_r,\ \infty)$, denote the corresponding stability interval by

$$k_1^{[+\infty]} \in \left(\frac{k_1^0}{g_r},\ 0\right).$$

The complete stability region is determined by analogy with Step 3.

Note that if stability holds in the intervals $k_1^{[-\infty]}$ and $k_1^{[+\infty]}$ simultaneously, they can be united into one common interval $k_1^{[\infty]}$. However, for the points g_i with $f_i = 0$ (the number of encirclements remains invariable!), such a union cannot be performed for stable intervals $k_1^{[-(i-1)]}$ and $k_1^{[-i]}$.

Remark 3. When analyzing a general-form transfer function $\tilde{w}_{oi}$ containing zero or pure imaginary poles, the above procedure is preserved, but it becomes more complicated to count encirclements. (See the analysis of the transfer function $\tilde{w}_{o2}$ with one zero pole in Example 2.)

The approach developed gives a complete description of stability intervals, even if the controller k^0 is not stabilizing. In this case, the approach also detects stability intervals (if they exist) with respect to one controller parameter with the second fixed.

4.3. Fragility Analysis Procedure as a Boundary Oracle

The fragility analysis procedure proposed in the previous section can be effectively used as a *boundary oracle* for an approximate description of the region of stabilizing PI (and also PID) controllers. It is based on a modification of *Hit-and-Run* [15, 17], a variant of the Monte Carlo method to generate points with the uniform distribution inside some region $\mathcal{D} \subset \mathbb{R}^m$, generally neither convex nor connected.

Hit-and-Run includes three sequential steps:

1) Through a current point $x_i \in \mathcal{D}$ in a random direction $d \in \mathbb{R}^m$, a line $x_i + \tau d$, $\tau \in (-\infty,\ \infty)$, is drawn.

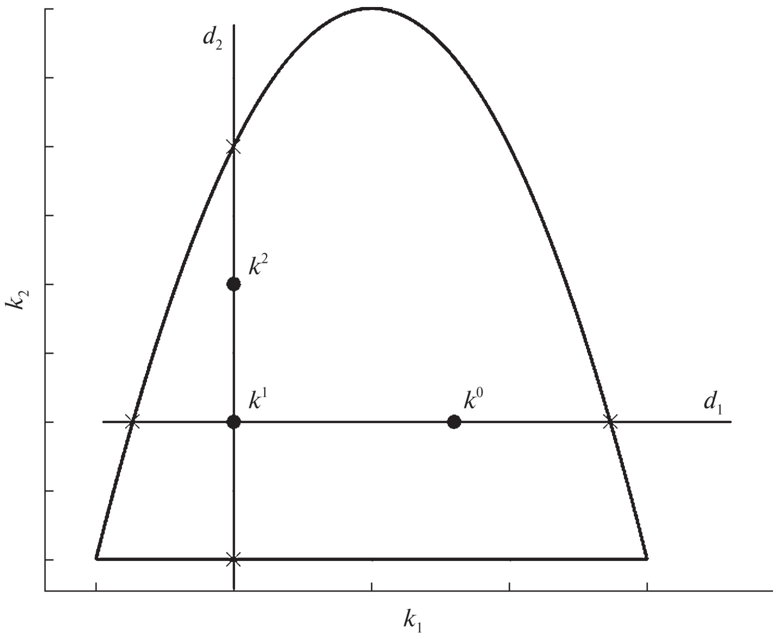

Fig. 1. The general scheme of the coordinate-wise modification of Hit-and-Run.

2) The so-called boundary oracle is used to find the intersection points of this line with the boundary of the region $\mathcal{D}$.

3) The next point x_{i+1} is chosen randomly on the resulting segment (or segments in the multiply connected case), and the process is repeated.

According to the results of Section 4.2, we implement the boundary oracle by providing the intersection points of the boundary of the region $\mathcal{K}$ (all stabilizing PI controllers) with the line $k + \tau d$, where k is some stabilizing PI controller and $d \in \mathbb{R}^2$ is one of the two directions

$$\tilde{d}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \tilde{d}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix},$$

coinciding with those of the coordinate axes in the plane (k_1, k_2). Therefore, in this (coordinate-wise) modification of the Hit-and-Run method, the direction $d^i \in \mathbb{R}^2$ at the current point k^i is chosen not randomly but as

$$d^i = \begin{cases} \tilde{d}_1 & \text{if } i \bmod 2 = 1, \\ \tilde{d}_2 & \text{if } i \bmod 2 = 0. \end{cases}$$

In other words, the directions $\tilde{d}_1$ and $\tilde{d}_2$ are alternated.

Figure 1 shows the first two iterations of this modification of the Hit-and-Run method in the parameter plane $(k_1,\ k_2)$. Since the region $\mathcal{K}$ of all stabilizing PI controllers has dimension 2 (for PID controllers, dimension 3), this simplification is not overly restrictive; the examples below confirm the effectiveness of the corresponding procedure.

5. EXAMPLES

Example 1. Consider the linearized model of a first-order plant with delay, widely used in practice [18]:

$$\frac{\kappa}{Ts+1} e^{-\tau s} \to \frac{\kappa(-\tau s/2 + 1)}{(Ts+1)(\tau s/2 + 1)}, \tag{11}$$

where κ is the gain, T is the time constant, and τ is the delay.

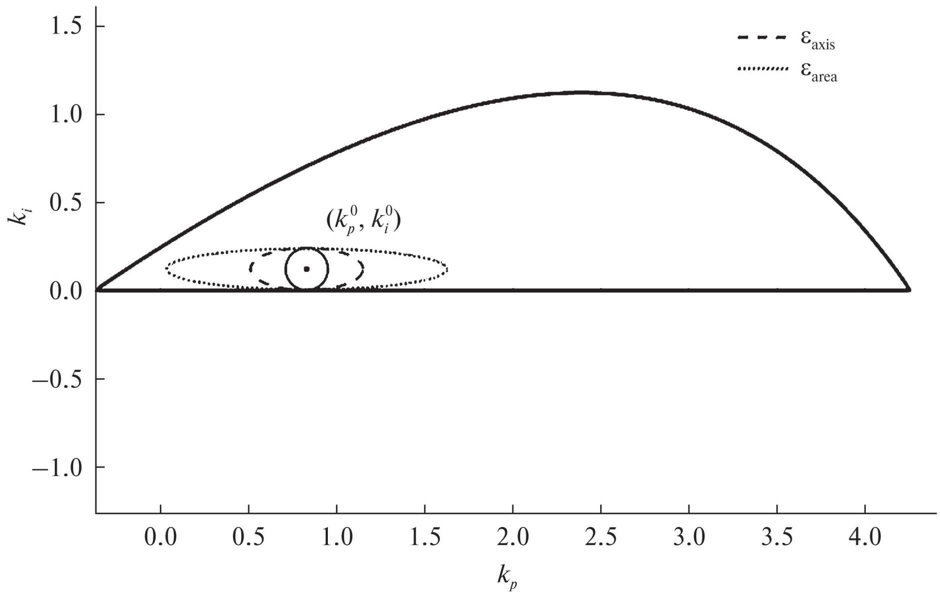

Fig. 2. The ellipses of admissible perturbations.

In the state space, this model is described by a system of the form (1) with the matrices

$$A = \begin{pmatrix} -a_1 & 1 \\ -a_2 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} b_1 \\ b_2 \end{pmatrix}, \quad c = \begin{pmatrix} 1 \\ 0 \end{pmatrix},$$

where

$$a_1 = \frac{2T + \tau}{T\tau}, \quad a_2 = \frac{2}{T\tau}, \quad b_1 = -\frac{\kappa}{T}, \quad b_2 = \frac{2\kappa}{T\tau}. \tag{12}$$

Taking

$$\kappa = 2.7, \quad T = 8.4, \quad \tau = 1.6 \tag{13}$$

as the parameters of the original model, we have

$$A = \begin{pmatrix} -1.3690 & 1 \\ -0.1488 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} -0.3214 \\ 0.4018 \end{pmatrix}.$$

Let a stabilizing controller have the parameters [19]

$$k^0 = \begin{pmatrix} 0.832 \\ 0.120 \end{pmatrix}.$$

Lemmas 2 and 3 give the ellipses of admissible perturbations with the matrices

$$\widehat{R}_{\text{area}} = \begin{pmatrix} 0.6343 & 0 \\ 0 & 0.0135 \end{pmatrix} \quad \text{and} \quad \widehat{R}_{\text{axis}} = \begin{pmatrix} 0.1021 & 0 \\ 0 & 0.0144 \end{pmatrix},$$

respectively; and the non-fragility radius is $\rho(k^0) = 0.12$.

In Fig. 2, the bold lines show the set $\mathcal{K}$ of all stabilizing PI controllers yielded by D-partition (for details, see [15]), as well as the ellipses of admissible perturbations: the dotted line corresponds to the ellipse $\mathcal{E}_{\text{area}}$, obtained by maximizing the area; the dashed line, to the ellipse $\mathcal{E}_{\text{axis}}$, obtained by

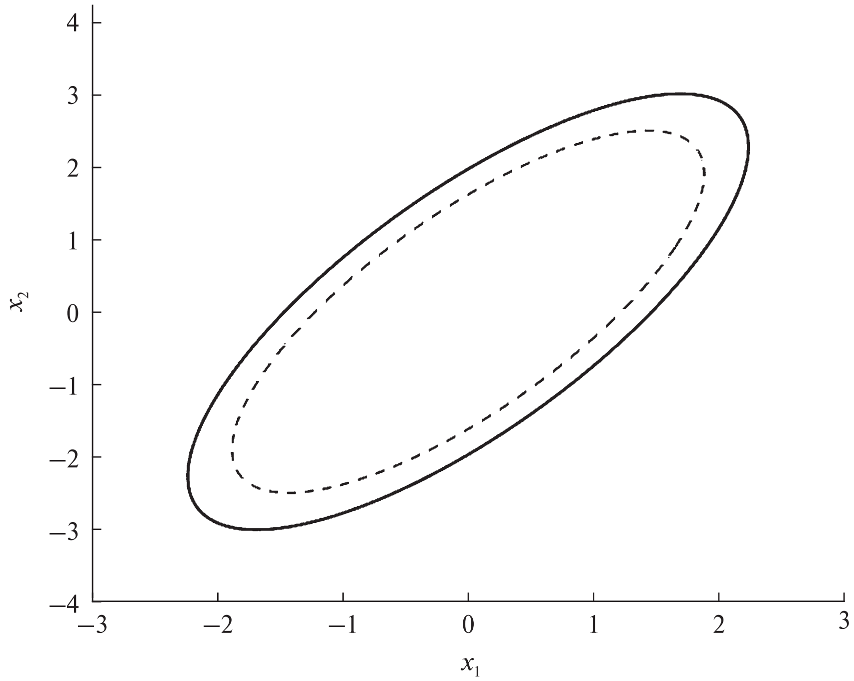

Fig. 3. The non-fragile bounding ellipsoid.

maximizing the length of the smallest semi-axis. The thin solid line is a circle with the non-fragility radius $\rho(k^0)$. The point indicates the PI controller with the parameters k^0.

We choose the controlled output $z = x$ (the full state of the system), i.e., $C = I$, and set $D = b$. The minimal bounding ellipsoid for the system closed by the PI controller with the parameters k^0 is given by the matrix

$$C_{\rm cl}\widetilde{P}C_{\rm cl}^{\rm T} = \begin{pmatrix} 3.5473 & 3.5892 \\ 3.5892 & 6.2562 \end{pmatrix}.$$

Let the non-fragility level be $\gamma = 0.02$. Theorem 1 provides the matrix

$$C_{\rm cl}\widehat{P}C_{\rm cl}^{\rm T} = \begin{pmatrix} 5.0026 & 5.0881 \\ 5.0881 & 9.0666 \end{pmatrix}$$

of the non-fragile bounding ellipsoid corresponding to this non-fragility level.

In Fig. 3, the dotted line shows the minimal bounding ellipsoid constructed without the non-fragility requirement, and the solid line is the non-fragile bounding ellipsoid found; it exceeds the minimal one by about 40% in terms of the trace criterion.

In all examples, numerical simulation was carried out in MATLAB using the `cvx` package [20].

Example 2. We return to the plant from Example 1. The transfer function (11), written in the form (7), is

$$(s^2 + a_1 s + a_2)y = (b_1 s + b_2)u;$$

in view of (12)–(13), it becomes

$$(s^2 + 1.369s + 0.149)y = (-0.321s + 0.402)u.$$

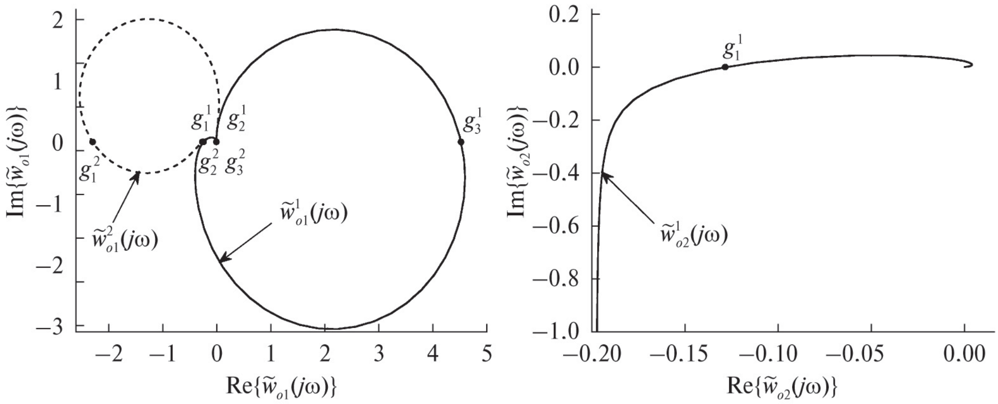

Fig. 4. The frequency response of the open-loop system: $\tilde{w}^i_{o1}(j\omega)$, $i = 1, 2$ (left) and $\tilde{w}^1_{o2}(j\omega)$ (right).

Using the procedure described in Section 4, we analyze the fragility of two stabilizing PI controllers with the parameters

$$k^1 = \begin{pmatrix} k_1^1 \\ k_2^1 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.1 \end{pmatrix}, \quad k^2 = \begin{pmatrix} k_1^2 \\ k_2^2 \end{pmatrix} = \begin{pmatrix} 1 \\ 0.5 \end{pmatrix}.$$

We start with the parameter k_1. In the corresponding transfer functions (9), the superscript indicates which of the PI controllers, k^1 or k^2, is being analyzed. (Similar notation will be employed for all parameters of the fragility analysis procedure.) As a result,

$$\tilde{w}^1_{o1}(s) = \frac{-0.321s^2 + 0.402s}{s^3 + 1.369s^2 + 0.117s + 0.0402},$$
$$\tilde{w}^2_{o1}(s) = \frac{-0.321s^2 + 0.402s}{s^3 + 1.369s^2 - 0.012s + 0.201}.$$

Their frequency responses are demonstrated in Fig. 4 on the left, namely, $\tilde{w}^1_{o1}(j\omega)$ (the solid line) and $\tilde{w}^2_{o1}(j\omega)$ (the dotted line).

The polynomial equations $\mathrm{Im}\{\tilde{w}^i_{o1}(j\omega)\} = 0$, $i = 1, 2$, have the solutions

$$\omega_1^1 = 0, \quad \omega_2^1 = 0.1670, \quad \omega_3^1 = 1.3417,$$
$$\omega_1^2 = 0, \quad \omega_2^2 = 0.4043, \quad \omega_3^2 = 1.2393,$$

and the corresponding pairs of g_i^j and f_i^j, $j = 1, 2$, $i = 1, 2, 3$ (sorted in ascending order of g_i^j) are

$$\{g_1^1 = -0.2387,\ f_1^1 = 1\}, \quad \{g_2^1 = 0,\ f_2^1 = 1/2\}, \quad \{g_3^1 = 4.5263,\ f_3^1 = -1\},$$
$$\{g_1^2 = -2.2907,\ f_1^2 = -1\}, \quad \{g_2^2 = -0.2596,\ f_2^2 = 1\}, \quad \{g_3^2 = 0,\ f_3^1 = 1/2\}.$$

They are presented in Fig. 4 on the left, with the values g_i^1 above the real axis and g_i^2 below it.

We begin with the frequency response $\tilde{w}^2_{o1}(j\omega)$ since the corresponding arrangement of the points g_i^2, $i = 1, 2, 3$, coincides with the assumption of Step 3 of the fragility analysis procedure. The transfer function $\tilde{w}^2_{o1}(s)$ has a pair of unstable complex conjugate poles; therefore, $s^2_{\tilde{a}} = 1$, and condition (10) holds only for $q = 1$ on the interval $(g_1^2,\ g_2^2)$. As a result, we arrive at the following interval:

$$k_1^{[-1]} \in (0.4366,\ 3.8523).$$

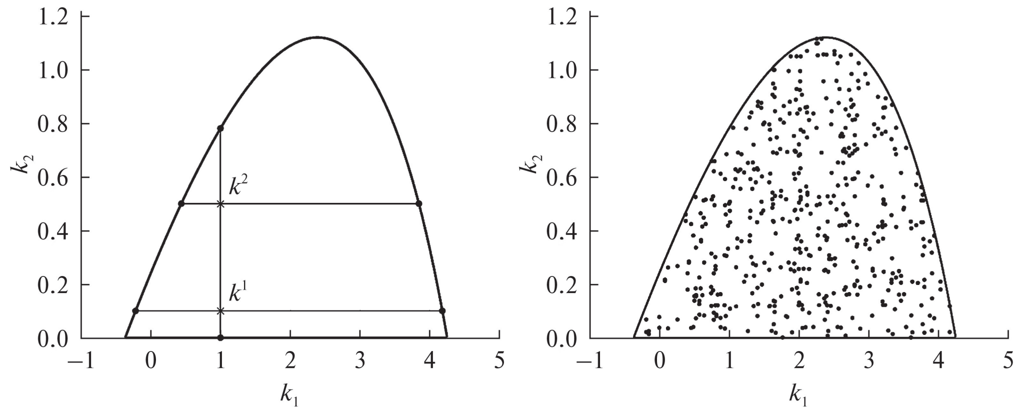

Fig. 5. Comparison with D-partition: fragility analysis of k^1 and k^2 (left) and Hit-and-Run implementation (right).

The transfer function $\tilde{w}^1_{o1}(s)$ is stable ($s^1_{\tilde{a}} = 0$), but g^1_i, $i = 1, 2, 3$, have different signs; hence, it is necessary to use Step 4 of the fragility analysis procedure of PI controllers. Considering g^1_i shows that $k_1^{[-\infty]}$ and $k_1^{[+\infty]}$ are stable and can be united, finally yielding

$$k_1^{[\infty]} \in (-0.2209,\ 4.1898).$$

When analyzing the parameter k_2, any of the controllers k^1 and k^2 can be chosen because $k^1_1 = k^2_1 = 1$. We will use k^1, the corresponding transfer function (9) is

$$\tilde{w}^1_{o2}(s) = \frac{-0.0321s + 0.0402}{s(s^2 + 1.048s + 0.551)}.$$

This function is neutral: it has one zero pole and the others are stable ($s^1_{\tilde{a}} = 0$).

The frequency response $\tilde{w}^1_{o2}(j\omega)$ is shown in Fig. 4 on the right. The polynomial equation $\mathrm{Im}\{\tilde{w}^1_{o2}(j\omega)\} = 0$ has the solution $\omega_1 = 0.547$, giving the pair

$$\{g^2_1 = -0.128,\ f^2_1 = 1\}.$$

Stability holds on the interval $(-\infty,\ g^2_1)$ or, recalculated into the parameter k_2,

$$k_2^{[-\infty]} \in (0,\ 0.781).$$

Here, the left bound does not include zero since $\tilde{w}^1_{o2}(s)$ is not asymptotically stable.

In Fig. 5 on the left, in the parameter plane $(k_1,\ k_2)$, the bold lines present the set of stabilizing PI controllers found by D-partition [15], the crosses indicate the controllers with the parameters k^1 and k^2, and the thin lines demonstrate the stability intervals for each parameter.

In Fig. 5 on the right, the results of the coordinate-wise modification of the Hit-and-Run method are shown (see Section 4.3). A PI controller with the parameters $k^2 = (1,\ 0.5)^{\mathrm{T}}$ was chosen as the starting point, and 500 points were generated.

Example 3. Consider a plant (7) of the form

$$(s^3 + s^2 + s + 1)y = (s^2 + 2s + 5)u,$$

which is regulated by a PI controller with the parameters $k = (2,\ 0.01)^{\mathrm{T}}$.

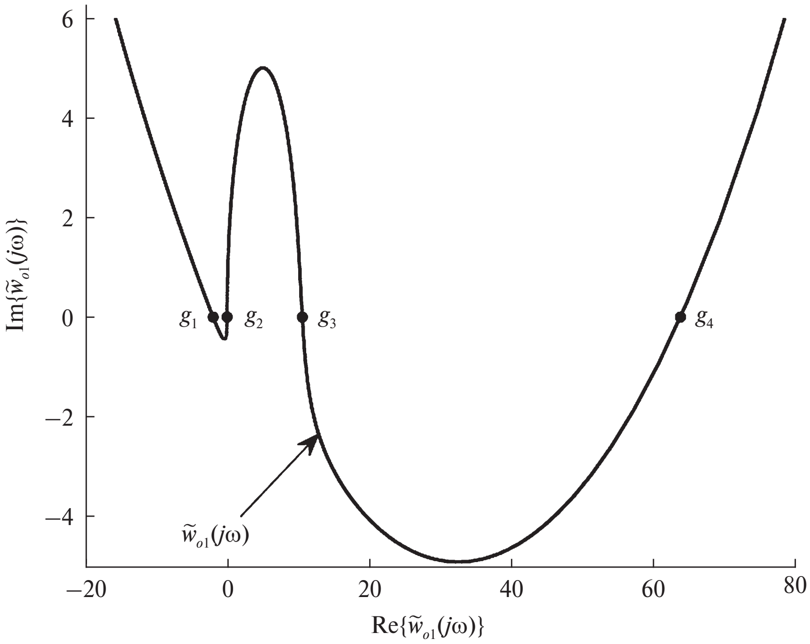

Fig. 6. The frequency response $\tilde{w}_{o1}(j\omega)$ of the open-loop system.

We analyze fragility with respect to the parameter $k_1 = 2$ using the procedure from Section 4. The transfer function (9) is written as

$$\tilde{w}_{o1}(s) = \frac{2s^3 + 4s^2 + 10s}{s^4 + s^3 + 1.01s^2 + 1.02s + 0.05};$$

it has one pair of unstable complex conjugate poles and $s_{\tilde{a}} = 1$ for it. The frequency response $\tilde{w}_{o1}(j\omega)$ is shown in Fig. 6.

The polynomial equation $\mathrm{Im}\{\tilde{w}_{o1}(j\omega)\} = 0$ has the solution

$$\omega_1 = 0, \quad \omega_2 = 0.3045, \quad \omega_3 = 0.944, \quad \omega_4 = 1.7396,$$

yielding the corresponding pairs of g_i and f_i, $i = 1, 2, 3, 4:$

$$\{g_1 = -1.9679,\ f_1 = -1\}, \quad \{g_2 = 0,\ f_2 = 1/2\},$$
$$\{g_3 = 10.5841,\ f_3 = -1\}, \quad \{g_4 = 63.8024,\ f_4 = 1\}$$

(see Fig. 6).

Condition (10) holds on the intervals $(g_1,\ g_2)$ and $(g_3,\ g_4)$; therefore, for $k_2 = 0.01$, the closed-loop system will be stable on the intervals

$$\bigcup_q k_1^{[q]}, \quad q \in \{-1,\ +3\},$$

where

$$k_1^{[-1]} \in (1.0163,\ \infty), \quad k_1^{[+3]} \in (-0.189,\ -0.0313).$$

6. CONCLUSIONS

This paper has proposed two approaches to the fragility analysis of PI controllers. The first one is based on the technique of linear matrix inequalities; for a nominal stabilizing PI controller, it effectively constructs ellipsoidal estimates for admissible uncertainties in the PI controller parameters under which the system remains quadratically stable. Also, a method for constructing a non-fragile bounding ellipsoid has been presented.

The second approach, involving the Nyquist criterion, provides intervals of static variations for each PI controller parameter under which the closed-loop system is still stable. The use of this approach as a boundary oracle in a modification of the Hit-and-Run method has been described.

Both approaches are simply implementable and employ standard functions available in most modern software tools for engineers.

A natural further development is to extend the above results to the fragility analysis of PID controllers, as well as to apply these approaches to design non-fragile PI and PID controllers.

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