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Global Stabilization of a Second-Order Integrator by a Discontinuous Feedback

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Abstract—The stability of a switching system arising from the use of a discontinuous feedback to stabilize a second-order integrator is analyzed. This is the limiting case of the feedback in the form of nested saturation functions considered previously, where the external saturation function or sigmoid is replaced by the discontinuous function $\operatorname{sgn}(x)$. Such a feedback involves a bounded control resource and ensures constraints on the phase velocity of approaching an equilibrium, which is especially important under large initial deviations. A Lyapunov function for the closed-loop system is proposed, and the global asymptotic stability of the origin is proven using this function and the results of Filippov's theory, provided that the switching curve is a continuously differentiable and monotonic function passing through zero. The global stability of the origin is preserved when relaxing the smoothness requirement for the entire switching function to its continuous differentiability everywhere except for a finite set of points where the derivative does not exist, but continuity holds. In the case of a discontinuous switching curve, the origin is shown to be a sewn center according to Filippov's classification. In this case, the closed-loop system has a semi-stable cycle enclosing a set of discontinuity points on the switching curve. As numerical examples, the level line of the Lyapunov function and phase trajectories on the plane are constructed for different types of switching curves.

Keywords: stabilization of a chain of two integrators, switching system, global asymptotic stability, switching curve, Lyapunov function

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1. INTRODUCTION

In 2024, two papers were written in collaboration with A.V. Pesterev, namely, "Global Stabilization of a Chain of Two Integrators by a Feedback in the Form of Nested *Saturators*"[1] and "Global Stabilization of a Chain of Two Integrators by a Feedback in the Form of Nested *Sigmoids*."[2] The slight distinction in titles (*sigmoids* vs. *saturators*) implies a fundamental difference in the systems under consideration. A system with a smooth feedback was studied in the first paper [1] whereas one with a nonsmooth feedback in the second [2]. Although in both cases global stability was proved using Lyapunov functions with a similar structure (a sum of quadratic and integral terms), the proofs differed significantly, as can be easily seen by comparing the two papers. Chronologically, the authors first succeeded in proving stability for the system with a smooth feedback in the form of nested sigmoids. A natural issue arose about the possibility of generalizing the result to the case of a nonsmooth feedback (in the form of nested saturators) and/or a discontinuous feedback

[1] A saturator is a nonsmooth saturation function $\operatorname{sat}_p(\cdot)$, $p > 0$, defined as follows: $\operatorname{sat}_p(w) = w$ for $|w| \leqslant p$, and $\operatorname{sat}_p(w) = p\,w/|w|$ for $|w| > p$.

[2] A sigmoid is a smooth, strictly increasing, odd, and scalar function $\sigma(x)$ satisfying the following conditions: (a) $\sigma(x) \to \pm 1$ as $x \to \pm\infty$, (b) $\max_x \sigma'(x) = \sigma'(0)$, (c) $\sigma'(0) = 1$, and (d) $\sigma(x) = -\sigma(-x)$, $x \in R$.

(obtained, e.g., by replacing the external saturator or sigmoid with $\text{sgn}(x)$). However, the proof of global stability in [1] turned out to be inapplicable to the nonsmooth system because, unlike the smooth case, the derivative of the corresponding Lyapunov function is not negative definite in the entire space, so the presentation in [1] was limited to the smooth case. Later, after the submission of [1], the authors managed to establish global stability for the nonsmooth case as well, showing that the domains where the Lyapunov function is identically zero cannot contain entire trajectories. The new results were published in [2].

To date, the case of a discontinuous system (see various physical examples of such systems in [3]), as well as smooth saturation functions [4–6], remains unexplored. This type of control functions was first described in [4, 5], but in both cases, only local stabilization was proved. Their main difference from the saturation function $\text{sat}(x)$ is the use of a nonlinear function instead of the linear part, such that the derivative exists and is continuous at the extreme points of the linear segment. This difference does not affect the proof of global stability given in [2], and therefore the results of [2] can be used for functions of this type as well.

This paper is devoted to the second, more complex case, i.e., the global stability analysis of a system closed by a discontinuous feedback. To prove global stability, we will employ a candidate Lyapunov function consisting of the sum of quadratic and integral terms [2]:

$$V(x) = \frac{1}{2}x_2^2 + \int\limits_0^{x_1} \text{sat}(k_3 \text{sat}(k_1\xi))d\xi.$$

As k_3 tends to infinity, the function $V(x)$ takes the form

$$V(x) = \frac{1}{2}x_2^2 + |x_1|. \tag{1}$$

Function (1) is nonsmooth, so Filippov's theorem [7] cannot be directly applied to it. However, it is possible to utilize his proof of global stability for a particular switching system. The idea of the proof is as follows. First, the non-positivity of the derivative outside the switching surface and outside the discontinuity surface of the derivative of the Lyapunov function is shown. Then, the derivatives of the Lyapunov function on the switching curves are computed. Next, trajectories and point singularities where the derivative of the Lyapunov function is zero are considered, and the Barbashin–Krasovskii theorem [8, 9] is applied. An alternative approach was proposed by Matrosov [10, 11] and further developed by Molchanov [12].[3] It consists in generalizing Lyapunov's stability theorems to the case of convex piecewise linear Lyapunov functions, but its application for planar systems with a known nonlinearity is unreasonable.

For the further presentation, the following definitions from [7] will be needed. Consider an equation (or system) in the vector form

$$\dot{x} = f(t, x) \tag{2}$$

with a piecewise continuous function f in a domain G[4]; $x \in R^n$ and $\dot{x} = \frac{dx}{dt}$; finally, $M \in R^{n+1}$ is a null set (a set of measure zero) of discontinuity points of the function f in x.

For each point (t, x) of the domain G, a set $F(t, x)$ in the n-dimensional space is specified. If at a point (t, x) the function f is continuous, then the set $F(t, x)$ consists of a single point coinciding with the value of f at that point. If (t, x) is a discontinuity point of f, then the set $F(t, x)$ is defined in one way or another.

[3] Matrosov's original works were published earlier than [12], but their availability for interested readers is very limited. Therefore, we refer to the modern collections [10, 11] containing various Matrosov's research, which were released significantly later than the original ones.

[4] G is a finite domain of the $(n+1)$-dimensional space of t, x. The domain G consists of a finite number of domains G_i, $i = 1, \ldots, l$, in each of which the function f is continuous up to the boundary.

Definition 1 [7]. A solution of equation (2) is a solution of the differential inclusion

$$\dot{x} \in F(t, x), \tag{3}$$

i.e., an absolutely continuous vector function $x(t)$ defined on an interval or segment I such that $\dot{x}(t) \in F(t, x(t))$ almost everywhere on I.

Definition 2 [7]. The simplest convex extension (completion). For each point $(t, x) \in G$, let $F(t, x)$ be the smallest convex closed set containing all limit values of the vector-function $f(t, x^*)$ when $(t, x^*) \notin M$, $x^* \to x$, $t = \text{const}$. A solution of equation (2) is a solution of inclusion (3) with the just constructed $F(t, x)$. Since M is a null set, for almost all $t \in I$ the measure of the cross-section of M by the plane $t = \text{const}$ is zero. For such t, the set $F(t, x)$ is defined for all $(t, x) \in G$.

At the continuity points of f, the set $F(t, x)$ consists of a single point $f(t, x)$, and the solution satisfies equation (2) in the usual sense. If a point $(t, x) \in M$ lies on the boundaries of the cross-sections of two or more domains $G_1, \ldots, G_k$ by the plane $t = \text{const}$, then the set $F(t, x)$ is a segment, convex polygon, or polyhedron with vertices $f_i(t, x)$, $i \leqslant k$, where $f_i(t, x) = \lim_{(t,x^*) \in G_i, x^* \to x} f(t, x^*)$, and the limit always exists by construction according to Definition 1.

Definition 3 [7]. A solution $x = \chi(t)$ $(t_0 < t < \infty)$ of the differential inclusion (3) with $I = (t_0, \infty)$ is said to be stable if for each $\varepsilon > 0$ it is possible to find $\delta > 0$ such that for each x_0, $|x_0 - \chi(t_0)| < \delta$, each solution $x(t)$ with the initial condition $x(t_0) = x_0$, $t_0 < t < \infty$, exists and satisfies the inequality $|x(t) - \kappa(t)| < \varepsilon$ $(t_0 < t < \infty)$.

The asymptotic stability of (3) is defined by analogy, but with the additional condition $x(t) - \chi(t) \to 0$ as $t \to \infty$.

Definition 4 [7]. Consider a system with control u :

$$\dot{x} = f(t, x, u), \quad u = u(x) \in U(t, x). \tag{4}$$

A solution is a pair of functions, i.e., an absolutely continuous function $x(t)$ and a measurable function $u(x(t))$, that satisfy system (4) almost everywhere on a given interval.

To consider the set of all solutions of system (4), it can be replaced by the differential inclusion (3), where $F(t, x) = f(t, x, U(t, x))$. Then the stability of a solution $x(t) = \chi(t)$ of inclusion (3) means that for $|x(t) - \chi(t)| < \delta$ the solution $x(t)$ of the equation $\dot{x} = f(t, x, U(t, x))$ satisfies equation (4) for all admissible controls $u(t)$.

Definition 5 [7]. A point c is said to be stationary if $x(t) \equiv c$ is a solution of inclusion (3).

Definition 6 [7]. The distance between points or sets is denoted by r :

$$r(a, b) = \|a - b\| = \sqrt{(a_1 - b_1)^2 + \ldots + (a_n - b_n)^2},$$
$$r(a, B) = \inf_{b \in B} r(a, b), \quad r(A, B) = \inf_{a \in A, b \in B} r(a, b).$$

A δ-neighborhood M^δ of a set M is the set of points x such that $r(x, M) < \delta$.

Definition 7 [7]. A set $M \subset R^n$ (not necessarily stationary) is said to be stable if for any $\varepsilon > 0$ it is possible to find $\delta > 0$ such that each solution $x(t)$ with an initial condition $x(t_0)$, $t_0 \leqslant t < \infty$, from the δ-neighborhood of the set M exists and satisfies the inequality $r(x(t), M) < \varepsilon$, $t_0 \leqslant t < \infty$, where $r(x, M)$ is the distance from the point x to the set M.

Definition 8 [7]. A stationary set is said to be stable in the large if it is stable and each solution infinitely approaches this set as $t \to \infty$. A stationary set is said to be pointwise stable in the large if it is stable and each solution tends to the stationary point as $t \to \infty$.

This paper addresses only planar systems, i.e., $n = 2$ in the above definitions. A vector on the plane will be written as $x = [x_1; x_2] \in R^2$, where x_1 and x_2 are the projections of the vector onto the coordinate axes, respectively.

The inner product of two vectors x and y will be designated by $x \cdot y = [x_1; x_2] \cdot [y_1; y_2] = x_1y_1 + x_2y_2$.

2. SMOOTH SWITCHING CURVE

Consider the problem of stabilizing the second-order integrator

$$\dot{x}_1 = x_2,\ \dot{x}_2 = u(x) \tag{5}$$

using a discontinuous feedback of the form

$$u(x) = -\mathrm{sgn}(S(x_1, x_2)), \tag{6}$$

where $\mathrm{sgn}(w) = 1$ for $w > 0$, $\mathrm{sgn}(w) = -1$ for $w < 0$, and $\mathrm{sgn}(w) = [-1, 1]$ for $w = 0$. The function $S(x_1, x_2)$ defines a switching (discontinuity) curve dividing the phase plane (x_1, x_2) into domains G^- $(S < 0)$ and G^+ $(S > 0)$. According to definitions (3) and (4), in this case, $U(t, x) = [-1, 1]$, and a solution of system (5), (6) is a solution of the differential inclusion (3), where $F(t, x) = [x_2; -\mathrm{sgn}(S(x_1, x_2))]$.

Let us partition the phase space R^2 into sets (see Figs. 1–3) so that $R^2 = D_1 \cup D_2^+ \cup D_2^- \cup D_3^+ \cup D_3^-$, where

- $D_1 = \{(x_1, x_2) : S(x_1, x_2) = 0\}$ is the set of points lying on the discontinuity curve;
- $D_2^+ = \{(x_1, x_2) : S(x_1, x_2) > 0 \,\text{and}\, x_1 < 0\}$;
- $D_2^- = \{(x_1, x_2) : S(x_1, x_2) < 0,\ x_1 > 0\}$;
- $D_3^+ = \{(x_1, x_2) : S(x_1, x_2) > 0 \,\text{and}\, x_1 \geqslant 0\}$;
- $D_3^- = \{(x_1, x_2) : S(x_1, x_2) < 0,\ x_1 \leqslant 0\}$.

Then, in the new designations, $G^- = D_2^- \cup D_3^-$ and $G^+ = D_2^+ \cup D_3^+$.

Theorem 1. *Let*

$$S(x_1, x_2) = x_2 + \phi(x_1), \tag{7}$$

where a continuously differentiable function $\phi(\xi)$, $\xi \in R$, satisfies the following properties:

1) $0 \leqslant \dfrac{d\phi(\xi)}{d\xi} < +\infty$;
2) $\exists\, 0 < \delta \leqslant 1$ *such that* $\dfrac{d\phi(\xi)}{d\xi} > 0,\ |\xi| < \delta$;
3) $\phi(0) = 0$;
4) $\phi(\xi) = -\phi(-\xi)$.

Then system (5)–(7) *with the function $\phi(x_1)$ is globally asymptotically stable.*

Remark 1. The discontinuity surface for the control law is chosen in the form (7) to avoid solvability and uniqueness problems in the system with a discontinuous right-hand side [13]. That is, the switching curve for system (5), (6) has the form $x_2 = -\phi(x_1)$.

Proof. Let us prove that function (1) is a Lyapunov function for system (5)–(7).

1. First, it is necessary to show that $\dot{V}(x) \leqslant 0$ for $S \neq 0$ and $x_1 \neq 0$. Differentiating $V(x)$ along the trajectories of system (5), (6) with $x_1 \neq 0$ yields

$$\dot{V} = -x_2\mathrm{sgn}(x_2 + \phi(x_1)) + \mathrm{sgn}(x_1)x_2 = -[\mathrm{sgn}(x_2 + \phi(x_1)) - \mathrm{sgn}(\phi(x_1))]x_2. \tag{8}$$

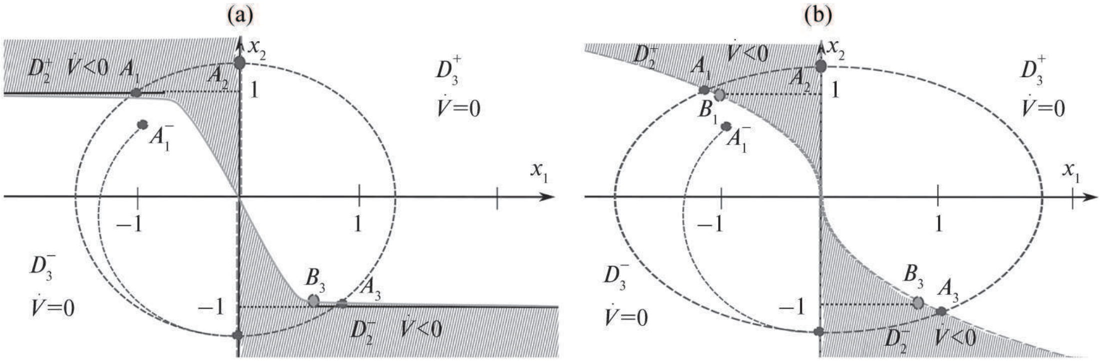

Fig. 1. The partition of the phase plane R^2 into the sets D_1, $D_2^{\pm}$, and $D_3^{\pm}$ for a smooth discontinuity surface: (a) sigmoid and (b) the polynomial function $y = -|\xi|^{1/n-1}\xi$, $n > 1$.

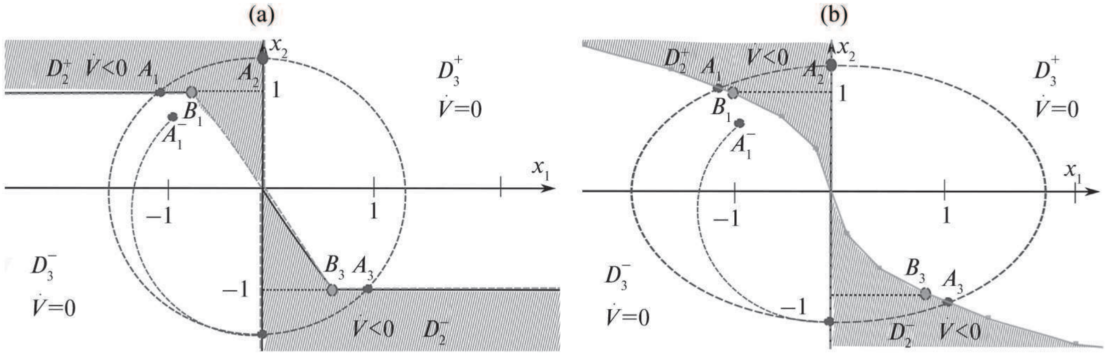

Fig. 2. The partition of the phase plane R^2 into the sets D_1, $D_2^{\pm}$, and $D_3^{\pm}$ for a continuous discontinuity surface: (a) saturator and (b) the piecewise-linear approximation of the polynomial function $y = -|\xi|^{1/n-1}\xi$, $n > 1$.

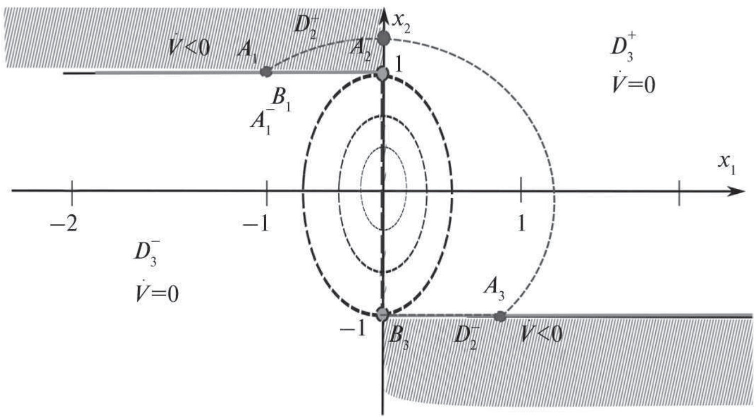

Fig. 3. The partition of the phase plane R^2 into the sets $D_1 = D_1^0 \cup D_1^+ \cup D_1^-$, and $D_2^{\pm}$, $D_3^{\pm}$ for a discontinuous switching function.

Due to the monotonicity of sgn, the inequality $[\operatorname{sgn}(s+s_0)-\operatorname{sgn}(s_0)]s \geqslant 0$ is valid for all $(s+s_0) \neq 0$ and all $s_0 \neq 0$, implying the non-positivity of the derivative: $\dot V(x) \leqslant 0$ outside the switching curve for $x_1 \neq 0$. The domains where the strict inequality holds are shaded in Figs. 1 and 2, and they correspond to subsets of the domains $D_2^{\pm}$.

2. Next, we establish that the set $\dot{V}(x) = 0$ does not contain entire trajectories for $S \neq 0$ and $x_1 \neq 0$. The derivative of the function $V(x)$ vanishes for $x_2 = 0$ on the sets D_3^+ and D_3^-, where both bracketed terms are simultaneously equal to $+1$ and -1, respectively. Obviously, these sets cannot contain entire trajectories. Indeed, the trajectories of the system in D_3^- and D_3^+ are the parabolas

$$x_1 = \mp\frac{1}{2}x_2^2 + C. \tag{9}$$

Since no parabola can lie entirely in D_3^- or D_3^+ (see Fig. 1), and the motion occurs with a constant acceleration, the system inevitably reaches the set D_1 (the switching curve) in finite time. Now assume that $x_2 = 0$, then $x_1 = c_1$. Let $c_1 > 0$, then integrating the system gives $x_2 = c_1 t + c_2 = 0$. The latter is true only if $c_1 = 0$ for $t > 0$. This contradiction means that there are no entire trajectories on the set $x_2 = 0$ as well.

3. To proceed, we demonstrate that $\dot{V}(x) < 0$ for $S \neq 0$ and $x_1 = 0$. On this set, $V(x) = x_2^2/2$, and the derivative along the trajectories of the system is $\dot{V}(x) = -x_2\mathrm{sgn}(x_2 + \phi(x_1))_{x_1=0} = -|x_2| < 0$, $x_2 \neq 0$.

4. Now, we prove that $\dot{V}(x) \leqslant 0$ for $S = 0$ and $x_1 \neq 0$. First, it is necessary to extend the vector field to the discontinuity curve. For this purpose, we construct the convex hull for each pair of vectors on the discontinuity curve. To utilize Filippov's result, the system should be written in new designations to find the projections of the vector field onto the normal at each point of the switching curve. The right-hand side of system (5)–(7) has the vector form $f^+ = [x_2; -1]$ for $S > 0$ and $f^- = [x_2; 1]$ for $S < 0$, with the null discontinuity set in the definition being $M = D_1$. Since S is a smooth surface, let us find the projections of the vector field f_N^+ $(S > 0)$ and f_N^- $(S < 0)$ onto the normal to the discontinuity curve at each point. Using Filippov's formulas [7], we write the equations

$$\begin{aligned} f_N^{\pm}(x_1) &= \frac{\nabla S(x_1) \times f^{\pm}(x_1)}{\|\nabla S(x_1)\|} = -\left(\rho(x_1) \pm 1\right)/\|\nabla S(x_1)\|, \\ \rho(x_1) &= \phi(x_1)\frac{d\phi(x_1)}{dx_1}, \end{aligned} \tag{10}$$

$$\nabla S(x_1) = \left[\frac{d\phi(x_1)}{dx_1}; 1\right], \quad \|\nabla S(x_1)\| = \sqrt{1 + \left(\frac{d\phi(x_1)}{dx_1}\right)^2} \neq 0. \tag{11}$$

By construction of system (5), (6) (only one physical equilibrium and one switching curve), the following point and linear singularities are possible here [7]:

1) $f_N^- f_N^+ > 0$: traversal of the surface (crossing the line), and it is not a linear singularity (case AA_0), e.g., in [14];
2) $f_N^- > 0$, $f_N^+ < 0$, and $f^0 \neq [0; 0]$: a stable sliding mode in which the system moves according to the right-hand side $f^0 = \alpha f^+ + (1 - \alpha) f^-$, $\alpha = f_N^-/(f_N^- - f_N^+) \in (0, 1)$ (case AA_1), e.g., in [15];
3) $f_N^- f_N^+ < 0$ and $f^0 = [0; 0]$: the curve consists of stationary points (case AA_2), e.g., a pendulum with dry friction [16];
4) $f_N^- < 0$, $f_N^+ > 0$, and $f^0 \neq [0; 0]$: an unstable sliding mode, the system constantly leaves the discontinuity surface and cannot pierce it (case AA_3).

First, we show that case AA_2 is impossible for system (5), (6) with the vector field projections (10), (11). In the case under consideration, the vector f_0 has the form

$$f^0 = [-\phi(x_1); 1 - 2\alpha]. \tag{12}$$

For any $0 \leqslant \alpha \leqslant 1$ and $x_1 \neq 0$, $f^0 \neq [0; 0]$, i.e., case AA_2 is impossible since the closed-loop system has no additional stationary points besides the origin.

Let us prove the impossibility of case AA_3 as well. Assume that $f_N^- < 0$; it follows from (10) that $\rho(x_1) > 1$ and, by the monotonicity and oddness of $\phi(x_1)$, we have $f_N^+ < 0$, i.e., case AA_0. Similarly, the assumption $f_N^+ > 0$ leads to $f_N^- > 0$ and, hence, case AA_0. Case AA_3 is impossible.

Now we obtain the conditions of a sliding mode. The above formula for α yields

$$\alpha = \frac{1}{2}(1 - \rho(x_1)). \tag{13}$$

By (13) and the definition of a sliding mode, the latter is possible only for $|\rho(x_1)| < 1$. Clearly, in this case, we have $f_N^- > 0$ and $f_N^+ < 0$.

For $\alpha = 0$ and $\alpha = 1$, $\rho(x_1) = \pm 1$, and for a point $x_1 > 0$ ($x_1 < 0$) we have $f_N^- > 0$ and $f_N^+ = 0$ ($f_N^- = 0$ and $f_N^+ < 0$, respectively). In both cases, this is a point singularity of type *ab* [7], meaning the start or end of a sliding mode. In the case under consideration, this point singularity is not an equilibrium because the magnitude of the vector field at this point is nonzero, and the right-hand side of the system is uniquely defined: either f^+ or f^-.

Thus, in the closed-loop system, only two modes are possible on the switching curve: 1) and 2). A traversal of the discontinuity curve means the continuity of trajectories on it; consequently, at these points, the derivative of the Lyapunov function can be extended by continuity, i.e., in view of items 1–3, it can only be $\dot V(x) \leqslant 0$.

It remains to find $\dot V$ for $S = 0$. We compute f_0 using (12), (13) and form a new system of differential equations governing the motion of the original system on the discontinuity curve. According to the definition [7], it has the form $\dot x = f_0$ and can be written as

$$\dot x_1 = -\phi(x_1), \quad \dot x_2 = \rho(x_1). \tag{14}$$

After eliminating x_2, the derivative of function (1) along the trajectories of system (14) is given by

$$\dot V = -\rho(x_1)\phi(x_1) - \operatorname{sgn}(x_1)\phi(x_1) = -\phi(|x_1|)\left(\phi(|x_1|)\frac{d\phi(x_1)}{dx_1} + 1\right) < 0, \quad x_1 \neq 0. \tag{15}$$

If $\frac{d\phi(x_1)}{dx_1} = 0$, it follows that $x_2 = \text{const}$ and x_1 constantly decreases. This motion will continue until the derivative $\frac{d\phi(x_1)}{dx_1}$ becomes positive, after which the variable x_2 will also start decreasing.

5. The positivity of the function $V(x)$ in the entire space R^2, except for the origin, is obvious, and $V(x)$ tends to infinity as $\|x\| \to \infty$. Thus, the function $V(x)$ satisfies all conditions of the Barbashin–Krasovskii theorem [8, 9] (LaSalle's theorem [17]): the origin is an asymptotically stable equilibrium of system (5)–(7) in the large. The proof of Theorem 1 is complete.

Remark 2. The Lyapunov function (1) for system (5), (6) is not a Lur'e–Postnikov function [18] because system (5), (6) cannot be reduced to the Lur'e form. (A linear part cannot be separated in it.)

Remark 3. In the problem under consideration, the classical approaches to constructing Lyapunov functions for discontinuous systems [19, 20] allow proving only either local stability [21–23] or stability in the large [10, 24].

3. CONTINUOUS SWITCHING CURVE

Theorem 2. *Let us define the set* $Q = R\backslash I_L$, *where* $I_L = \{-\xi^k, -\xi^{k-1}, \ldots, -\xi^1, \xi^1, \ldots, \xi^{k-1}, \xi^k,$ $\xi^k \in R,\ k = 1, \ldots, L,\ L \in Z^+\}$. *Assume that a continuously differentiable function* $\phi_1(\xi)$, $\xi \in Q$, *satisfies the following properties:*

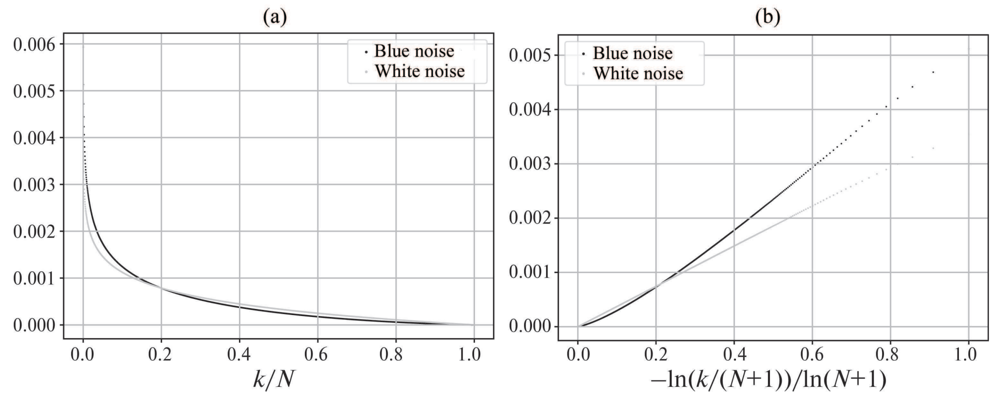

Fig. 4. Examples of functions satisfying the conditions of Theorem 1 or Theorem 2.

1) $0 \leqslant \dfrac{d\phi_1(\xi)}{d\xi} < +\infty$;
2) $\exists\, 0 < \delta \leqslant \xi^1$ *such that* $0 < \dfrac{d\phi_1(\xi)}{d\xi}$, $|\xi| < \delta$;
3) $\phi_1(0) = 0$;
4) $\phi_1(\xi) = -\phi_1(-\xi)$.

In addition, assume the validity of the equality $\phi_1(\xi) = -\phi_1(-\xi)$, $\xi \in I_L$. *Then system* (5)–(7) *with the function* $\phi(x_1) = \phi_1(x_1)$ *is globally asymptotically stable.*

Proof. Since the function $\phi_1(x_1)$ is not smooth, the switching curve $x_2 = -\phi_1(x_1)$ will be nonsmooth. Thus, the difference between the functions $\phi(x_1)$ in Theorems 1 and 2 is the absence of derivatives at particular points ξ^k, $k = 1, \ldots, L$, on a null set. Since system (5)–(7) is planar, each such point on the discontinuity curve can be interpreted as the intersection set of only two smooth discontinuity curves, and the vector field can be extended using Fillipov's method in the case of several smooth discontinuity surfaces [7]. We choose any point from the set I_L, e.g., with subscript l; the projection of the vector field $f^{\pm}_{N_l}$ at this point of the phase space, $(x^l_1, -\phi_1(x^l_1))$, is composed of the projections computed for the two adjacent smooth segments of the discontinuity curve. For $\epsilon > 0$, let $f^{\pm}_{N_{l-1}} = f^{\pm}_N(x^l_1 - \epsilon)$ and $f^{\pm}_{N_{l+1}} = f^{\pm}_N(x^l_1 - \epsilon)$ denote the projections of the vector field computed by formulas (10) to the left and right, respectively, of the discontinuity point. Then the projections of the vector field are given by

$$f^{\pm}_{N_l} = \frac{1}{2}(f^{\pm}_{N_{l-1}} + f^{\pm}_{N_{l+1}}). \tag{16}$$

Due to the continuity of the solution, the global stability of the origin fails only if this point is stationary, i.e., under the conditions of case 3) in Theorem 1. More precisely put, we need to check that $f^0_l = \alpha_l f^+ + (1 - \alpha_l) f^- \neq [0; 0]$, $\alpha_l = f^-_{N_l}/(f^-_{N_l} - f^+_{N_l})$. Substituting the corresponding function into (12) yields

$$f^0_l = [-\phi_1(x^l_1); 1 - 2\alpha_l]. \tag{17}$$

Since $x^l_1 \neq 0$ by the conditions of the theorem, equality (17) implies $f^0_l \neq [0; 0]$ for any $0 \leqslant \alpha_l \leqslant 1$. Hence, the point $(x^l_1, -\phi_1(x^l_1))$ is not stationary. Theorem 1 applies to the remaining points. The proof of Theorem 2 is complete.

The characteristic trajectories of systems (5)–(7) with the function $\phi(x_1)$ and $\phi_1(x_1)$ are schematically shown by dashed lines in Figs. 1 and 2, respectively. In both cases, starting the dynamics from the point $A_1 \in D^+_2$ to the point $A_2 \in D^+_3$, the inequality $V(A_1) > V(A_2)$ holds by the negativity of the derivative of the Lyapunov function. Then the system evolves along a parabola in the domain D_3^+, where $\dot V = 0$, i.e., the inequality $V(A_1) > V(A_3) = V(A_2)$ is true with $A_3 \in D_1$. Next, two options are possible: the trajectory traverses the discontinuity curve and reaches D_2^- or a stable sliding mode and moves along it to the point B_3. At the final point, the strict inequality $V(A_1) > V(B_3)$ holds.

Consider some examples of functions satisfying the conditions of Theorem 1. In Fig. 4, the smooth saturation functions $\Phi_1(\xi) = \sin(\psi(k\xi))$ with $\psi(k\xi) = \mathrm{sat}_{(\frac{\pi}{2k})^{1/3}}(k\xi^3)$ and $\Phi_2(\xi) = \sin(\psi(k\xi))$ with $\psi(k\xi) = \mathrm{sat}_{\xi^*}(\xi^3 + \xi/4)$ are indicated by nos. 1 and 2, respectively. In the latter formula, ξ^* stands for the solution of the polynomial equation $\xi^3 + \xi/4 = \pi/2$.

As an example of a function $\phi_1(x_1)$ satisfying the conditions of Theorem 2, one can take a piecewise-linear function with L bends. For instance, in Fig. 4, no. 3 corresponds to the function $\Phi_3(\xi) = \frac{1}{3}(\mathrm{sat}(k_1\xi) + \mathrm{sat}(k_2\xi) + \mathrm{sat}(k_3\xi))$, $k_1 = 0.3$, $k_2 = 2.65$, $k_3 = 5$, with three bends, and no. 4 to the function $\Phi_4(\xi) = \frac{1}{4}\left(\sum_{i=1}^4 \mathrm{sat}(k_i\xi)\right)$, $k_i = \frac{i}{3}$, $i = 1, \ldots, 4$, with four bends. The simplest function of this class is sat, the function indicated by no. 5 in Fig. 4.

4. DISCONTINUOUS SWITCHING CURVE

Theorem 3. *Let*

$$\phi_2(\xi) = K\mathrm{sgn}(\xi), \quad \xi \in R, \quad 0 < K < +\infty. \tag{18}$$

System (5)–(7) *with the function* $\phi(x_1) = \phi_2(x_1)$ *has a globally semi-stable cycle passing through the points* $(0, \pm K)$, *and the origin is a sewn center according to Filippov's classification.*

Proof. By analogy with Theorem 2, we first divide the curve $S = 0$ into three smooth segments: $D_1^+ = \{(x_1, x_2) : S(x_1, x_2) = 0, x_1 > 0\}$, $D_1^- = \{(x_1, x_2) : S(x_1, x_2) = 0, x_1 < 0\}$, and $D_1^0 = \{(x_1, x_2) : x_2 \in (-K, K), x_1 = 0\}$.

Since these are straight lines, their normals can be written explicitly: $n^+ = n^- = [0; 1]$ for $D_1^\pm$ and $n^0 = [1; 0]$ for D_1^0. (By definition, the normal direction is chosen towards the set $S > 0$.)

By substituting $\nabla S = n_0$ into (10), we find the projections of the vector field on the set D_1^0 :

$$f_{n_0}^\pm = [1; 0][x_2; \pm 1]_{x_2 \in D_1^0} = x_2, \quad x_2 \in D_1^0. \tag{19}$$

From (19) it follows that $f_{n_0}^+ f_{n_0}^- > 0$, $x_2 \in D_1^0 \backslash 0$. In other words, case AA_0 (see Theorem 1) occurs on the set D_1^0, except for $x_2 = 0$. Thus, the right-hand side of the system is $f = f^+$ for $x_2 > 0$ and $f = f^-$ for $x_2 < 0$ on the set D_1^0, except for $x_2 = 0$. We choose any point on the set D_1^0, e.g., $(0, \delta)$, $0 < \delta < K$. Since system (5)–(7), (18) has an extended right-hand side, it can be integrated from this point. Straightforward transformations bring us to the parabola (9) with $C = \pm\delta^2/2$. The same phase curve is satisfied by the point $(0, -\delta)$ with the opposite sign but the same-magnitude projection of the vector field onto the discontinuity curve. Therefore, in Filippov's terminology [7], the origin is a center. The first part of Theorem 1 is proved.

Now let $\nabla S = [0; 1]$. Using (10), we find the projections of the vector field on the set D_1^+ :

$$f_n^\pm = [0; 1][x_2; \pm 1] = \pm 1 \in D_1^+. \tag{20}$$

On the set D_1^-, we obtain the same result, meaning that a sliding mode occurs on these sets and, moreover, $\alpha_n = \frac{1}{2}$. Let us compute the new vector field f^0 :

$$f_n^0 = \frac{1}{2}[x_2; -1] + \frac{1}{2}[x_2; 1] = [x_2; 0], \quad x_2 \in D_1^\pm. \tag{21}$$

Thus, $f_n^0 = [1; 0]$ on the set D_1^-, and $f_n^0 = [-1; 0]$ on the set D_1^+, and its magnitude does not change. Next, we compute the projection of the vector field at the points $(0, -K)$ and $(0, K)$. For the point $x^l = (0, K)$ in the notation of Theorem 2, we apply formula (16) with the corresponding values of the vector field projections on the normals from the sets D_1^+ and D_1^0 to get

$$f_{N_K}^{\pm} = \frac{1}{2}(x_2 \pm 1)|_{x_2=K} = K \pm 1, \quad \alpha = \frac{1}{2}(1 - K). \tag{22}$$

For the point $(0, K)$,

$$f_K^0 = \frac{1}{2}(1 - K)[x_2; -1] + \frac{1}{2}(1 + K)[x_2; 1] = [K; K]. \tag{23}$$

Obviously, $f_K^0 \neq [0; 0]$, so the point $(0, K)$ is not stationary.ng The above considerations repeated for the point $(0, -K)$ give $f_{-K}^0 = [-K; -K] \neq [0; 0]$, so point $(0, -K)$ is not stationary as well. We construct the solution passing through these points, which has the form (9) with the parameter $C = \pm K^2/2$. Sewing the two parabolas along the x_2 axis yields a closed trajectory representing a limiting cycle [7]. Let us show its stability for all points on the plane outside it. Consider function (1) for the domain outside the cycle. Assume that an arbitrary trajectory starts at the point $A_1 \in D_2^+$ outside the cycle (Fig. 3). By Theorem 1, the derivative of the Lyapunov function at the point $A_1 \in D_2^+$ is negative, vanishing at the point $A_2 \in D_3^+$; hence, the inequality $V(A_1) > V(A_2)$ holds. Since $\dot{V} = 0$ in D_3^+, the system further moves along a parabola to the point $A_3 \in D_1$. In view of the first inequality, we obtain $V(A_3) = V(A_2) < V(A_1)$, i.e., the trajectory has approached the cycle in the sense of the norm of $V(x)$. On the discontinuity curve A_3B_3, a stable sliding mode with the vector field (21) occurs, and hence the trajectory arrives at the point $(0, -K)$, reaching the cycle in finite time without crossing the switching curve. The points from D_3^- are considered similarly. Thus, the cycle of two sewn parabolas is stable for all points on the plane outside it. The proof of Theorem 3 is complete.

5. NUMERICAL EXAMPLES

As an illustration, the level lines of the Lyapunov function (1) were constructed for system (5)–(7) with the functions $\phi = x_1^{1/3}$, $\phi_1 = \mathrm{sat}(x_1)$, and ϕ_2 (18) with $K = 1$. Figure 5 shows one of the level lines (solid line) and several phase trajectories (dashed lines) starting on it. The initial points of the trajectories are marked with circles. According to the figure, none of the trajectories leaves the invariant set bounded by the level line. The trajectory segments evolving along the boundary of the set lie in the subsets D_3^- and D_3^+, where the derivative of the Lyapunov function along the trajectories of the system is zero.

In Fig. 5a, after the trajectory reaches the discontinuity curve, a stable sliding mode occurs until a certain time; and the trajectory subsequently reaches a domain where the discontinuity curve is crossed and a domain where the Lyapunov function's derivative becomes negative, and then the situation repeats. In this case, the origin is a sewn stable focus [7].

In Fig. 5b, after the trajectory reaches the discontinuity curve, a stable sliding mode occurs up to zero. In this case, the origin is a sewn stable node or a linear singularity of type AA_1 [7].

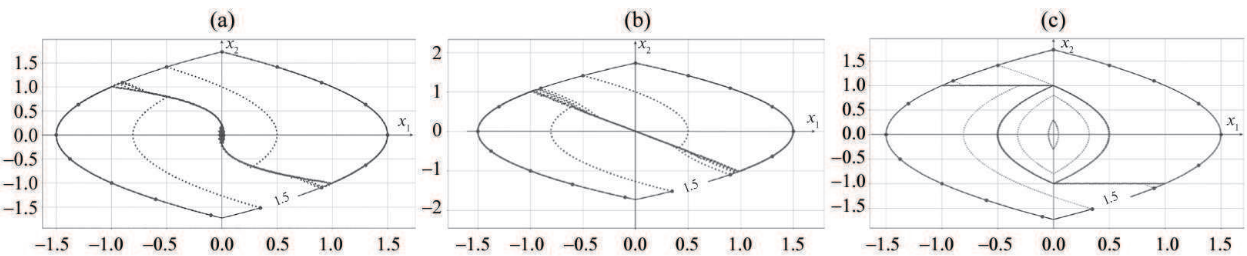

Fig. 5. The level line of the Lyapunov function and phase trajectories on the plane (x_1, x_2). The points indicate the initial conditions corresponding to the phase trajectories plotted.

In Fig. 5c, after the trajectory reaches the discontinuity curve, a stable sliding mode occurs up to the point $(0, 1)$ or $(0, -1)$ (depending on the initial conditions); then the trajectories wind onto the cycle. Unfortunately, due to rounding errors, the trajectory may evolve along a cycle of smaller size. If a point inside the cycle is chosen, another cycle of smaller size is obtained, and so on down to zero. In this case, the origin is a sewn center [7].

6. CONCLUSIONS

This paper has considered the problem of stabilizing a chain of two integrators by a discontinuous feedback. This is a limiting case of nested saturation functions, the feedback studied previously. A Lyapunov function for the closed-loop system has been proposed and then utilized, together with the results of Filippov's theory, to prove the global asymptotic stability of the origin for different types of continuous switching curves. In the case where the switching curve is discontinuous, it has been established that the origin is a sewn center according to Filippov's classification; in addition, the closed-loop system has a semi-stable cycle enclosing this equilibrium. The main result of the work has been formulated as three theorems. As numerical examples, the level line of the Lyapunov function and phase trajectories on the plane have been plotted for different types of switching curves.

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