ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 1, pp. 65–76.




# On Optimality of Singular Controls in One Optimal Control Problem of the First Order Stochastic Hyperbolic Equations

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Received August 16, 2024
Revised September 20, 2025
Accepted September 23, 2025

**Abstract**—The optimal control problem described by a system of the first order stochastic hyperbolic equations and arising in the modeling of a number of chemical-technological processes under the influence of random noise is considered. The optimality of singular controls in the sense of the Pontryagin maximum principle is investigated.

*Keywords*: stochastic first order equations hyperbolic system, two parameter Wiener process, optimality, Pontryagin maximum principle, singular control, second order optimality conditions

**DOI:** 10.7868/S1608303226010058

## 1. INTRODUCTION

Various quality aspects, including derivation of necessary optimality conditions of first order and study of singular cases of optimal control problems for determinate dynamical systems described by the first order hyperbolic equations, have been studied in [1–6] and others.

Such an optimal control problem in the stochastic case has been considered in [7] and first-order necessary optimality conditions (the analogue of Pontryagin's maximum principle, linearized maximum principle, analogue of the Euler's equation [8]) have been established there.

The relevance of research in this direction is determined by the need for the most accurate description, for example, of automatic control systems, a number of chemical technological processes [9–14], whose realistic option is a stochastic description that takes into account the impact of random factors on the controlled object.

The present work is devoted to the study of the singular case in the sense of Pontryagin's maximum principle and derivation of the second order necessary optimality conditions for singular controls in the stochastic control problem described by the system of nonlinear stochastic first order hyperbolic equations in canonical form.

The applied scheme of the study is a modification and development of the schemes from [5, 6], allowing for stochastic properties of the problem under consideration.

## 2. PROBLEM STATEMENT

Assume that the control process in the given domain $D = [t_0, t_1][x_0, x_1]$ is described by the following system of stochastic nonlinear partial differential equations of first order

$$\begin{aligned} \frac{\partial z(t,x)}{\partial t} &= f(t,x,z,y,u) + p(t,x,z)\frac{\partial W_1(t,x)}{\partial t}, \\ \frac{\partial y(t,x)}{\partial x} &= g(t,x,z,y,u) + q(t,x,y)\frac{\partial W_2(t,x)}{\partial x}, \quad (t,x) \in D, \end{aligned} \tag{1}$$

with Goursat type boundary conditions

$$z(t_0,x) = a(x), \quad x \in [x_0,x_1], \quad y(t,x_0) = b(t), \quad t \in [t_0,t_1]. \tag{2}$$

Here $(z(t,x), y(t,x))$ is $(n+m)$-dimensional sought vector-function: $f(t,x,z,y,u)$ $(g(t,x,z,y,u))$ is a given $n(m)$-dimensional vector-function continuous in totality of variables together with partial derivative with respect to $(z,y)$ up to the second order inclusively; $p(t,x,z)$ $(q(t,x,y))$ is $(n \times n)$ $((m \times m))$-dimensional measurable and bounded matrix-function; white noises $\frac{\partial W_1(t,x)}{\partial t}$, $\frac{\partial W_2(t,x)}{\partial x}$ are the derivatives with respect to $t$ and $x$, respectively, of the two-parameter Wiener's process $W_1(t,x)$, $W_2(t,x)$ [10, 13]; $a(x)$, $b(t)$ are the given measurable and bounded on $[x_0,x_1]$, $[t_0,t_1]$ vector-function of appropriate dimensions.

As admissible controls we take a class of measurable with respect to non-decreasing Borel $\sigma$ algebra $\mathcal{F} = \overline{s}(W(t,s), t_0 \leqslant t \leqslant t_1, x_0 \leqslant s \leqslant x_1)$ and bounded in $D$ $r$-dimensional vector-functions $u(t,x)$ with the values from the given non-empty and bounded set $U \subset R^r$ $(u(t,x) \in L_\infty(D,U))$.

The solution $(z(t,x), y(t,x))$ of system (1)–(2) corresponding to the definite admissible control $u(t,x)$ is understood in the sense [15].

Everywhere it is assumed that for each admissible control the corresponding solution to the boundary value problem (1)–(2) exists and is unique in the domain $D$.

Let us consider the problem of the minimization of the functional

$$S(u) = E\left\{ \int_{t_0}^{t_1}\int_{x_0}^{x_1} F_3(t,x,z(t,x),y(t,x),u(t,x))\,dxdt + \int_{x_0}^{x_1} F_1(x,z(t_1,x))\,dx + \int_{t_0}^{t_1} F_2(t,y(t,x_1))\,dt \right\}, \tag{3}$$

determined on the solutions of the boundary value problem (1)–(2), generated by all admissible controls.

Here $F_1(x,z)$, $F_2(t,y)$, $F_3(t,x,z,y,u)$ are the given scalar functions continuous in totality of variables together with partial derivatives with respect to the state vector (i.e. with respect to $(z,y)$) up to the second order inclusively, $E$ stands for mathematical expectation.

An admissible control $u(t,x)$ that minimizes the functional (3) subject to (1) and (2) is called an optimal control. The corresponding process $(u(t,x), z(t,x), y(t,x))$ is then called an optimal process.

It is assumed that in the considered stochastic problem there exists an optimal control.

The main goal of the paper is to derive second order necessary conditions for optimality (the case of singular controls) in the studied stochastic control problem with distributed parameters.

## 3. SECOND ORDER INCREMENT FORMULA OF THE QUALITY FUNCTIONAL

Let $(u(t,x), z(t,x), y(t,x))$ be fixed, $(\overline{u}(t,x) = u(t,x) + \Delta u(t,x), \overline{z}(t,x) = z(t,x) + \Delta z(t,x),$ $\overline{y}\,(t,x) = y\,(t,x) + \Delta y\,(t,x))$ be arbitrary admissible processes.

We introduce the analogue of the Hamilton–Pontryagin function

$$H\,(t,x,z,y,u,\psi,\xi) = -F_3\,(t,x,z,y,u) + \psi' f\,(t,x,z,y,u) + \xi' g\,(t,x,z,y,u)\,,$$

and the denotation of type:

$$\Delta_v f\,[t,x] = f\,(t,x,z\,(t,x)\,,y\,(t,x)\,,v) - f\,(t,x,z\,(t,x)\,,y\,(t,x)\,,u\,(t,x))\,,$$
$$H_z\,[t,x] = H_z(t,x,z(t,x),y(t,x),u(t,x),\psi(t,x),\xi(t,x)).$$

Here $(\psi(t,x), \xi(t,x), \alpha(t,x), \beta(t,x)) \in L_\infty(D,R^n) \times L_\infty(D,R^m) \times L_\infty(D,R^{n\times n}) \times L_\infty(D,R^{m\times m})$ are the solutions of the following stochastic conjugate problem

$$\psi_t\,(t,x) = -\frac{\partial H\,(t,x,z,y,u,\psi,\xi)}{\partial z} + \alpha\,(t,x)\,\frac{\partial W_1\,(t,x)}{\partial t}, \quad \psi\,(t_1,x) = \frac{\partial F_1\,(x,z\,(t_1,x))}{\partial z},$$
$$\xi_x\,(t,x) = -\frac{\partial H\,(t,x,z,y,u,\psi,\xi)}{\partial y} + \beta\,(t,x)\,\frac{\partial W_2\,(t,x)}{\partial x}, \quad \xi\,(t,x_1) = \frac{\partial F_2\,(t,y\,(t,x_1))}{\partial y}.$$

Now, applying the Taylor formula and taking into account the introduced denotations, we can write increment of quality criterion (3) corresponding to the admissible controls $u\,(t,x)$ and $\overline{u}\,(t,x)$ in the form

$$\begin{aligned}\Delta S\,(u) = -E\Bigg\{ &\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\overline{u}} H\,[t,x]\,dxdt \\ &+ \frac{1}{2}\int_{x_0}^{x_1} \Delta z'\,(t_1,x)\,\frac{\partial^2 F_1\,(x,z\,(t_1,x))}{\partial z^2}\Delta z\,(t_1,x)dx \\ &+ \frac{1}{2}\int_{t_0}^{t_1} \Delta y'\,(t,x_1)\,\frac{\partial^2 F_2\,(t,y\,(t,x_1))}{\partial y^2}\Delta y\,(t,x_1)dt \\ &+ \int_{t_0}^{t_1}\int_{x_0}^{x_1} \Big[\Delta_{\overline{u}} H_z'\,[t,x]\,\Delta z\,(t,x) + \Delta_{\overline{u}} H_y'\,[t,x]\,\Delta y\,(t,x)\Big]\,dxdt \\ &+ \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Big[\Delta z'(t,x) H_{zz}[t,x]\Delta z(t,x) + \Delta z'(t,x) H_{zy}[t,x]\Delta y(t,x) \\ &+ \Delta y'\,(t,x)\,H_{yy}\,[t,x]\,y\,(t,x)\Big] dxdt + \eta_1\,(t,x;\Delta u)\,, \end{aligned} \tag{4}$$

where $\eta_1(t,x;\Delta u)$ is determined by the formula

$$\eta_1(t,x;\Delta u) = E\left\{ \int\limits_{x_0}^{x_1} o_1\left(\|\Delta z(t_1,x)\|^2\right)dx + \int\limits_{t_0}^{t_1} o_2\left(\|\Delta y(t,x_1)\|^2\right)dt \right.$$
$$- \int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1} o_3\left([\|\Delta z(t,x)\| + \|\Delta y(t,x)\|]^2\right)dxdt$$
$$-\frac{1}{2}\int\limits_{t_0}^{t_1}\int\limits_{x_0}^{x_1}\Big[\Delta z'(t,x)\,\Delta_{\overline{u}}H_{zz}[t,x]\,\Delta z(t,x) + \Delta z'(t,x)\,\Delta_{\overline{u}}H_{zy}[t,x]\,\Delta y(t,x)$$
$$\left. + \Delta y'(t,x)\Delta_{\overline{u}}H_{yz}[t,x]\Delta z(t,x) + \Delta y'(t,x)\Delta_{\overline{u}}H_{yy}[t,x]\Delta y(t,x)\Big]dxdt \right\},$$

$\|\Delta z\|$ is a norm of the vector $\Delta z = (\Delta z_1, \Delta z_2, \ldots, \Delta z_n)$ in $R^n$ defined by the formula $\|\Delta z\| = \sum\limits_{i=1}^{n} |\Delta z_i|$, and everywhere the condition $\frac{o_i(\propto^2)}{a^2} \to 0$, holds as $\propto \to 0$.

On the other hand, by linearization from boundary value problem (1)–(2), we obtain that $(\Delta z(t,x), \Delta y(t,x))$ is the solution of the following stochastic linearized systems of equations

$$\begin{aligned} \Delta z_t &= f_z[t,x]\,\Delta z(t,x) + \Delta_{\overline{u}}f_y[t,x]\,\Delta y(t,x) + \Delta_{\overline{u}}f[t,x] \\ &\quad + p_z[t,x]\Delta z(t,x)\frac{\partial W_1(t,x)}{\partial t} + \eta_2(t,x;\Delta u), \\ \Delta y_x &= g_z[t,x]\,\Delta z(t,x) + \Delta_{\overline{u}}g_y[t,x]\,\Delta y(t,x) + \Delta_{\overline{u}}g[t,x] \\ &\quad + q_y[t,x]\,\Delta y(t,x)\frac{\partial W_2(t,x)}{\partial x} + \eta_3(t,x;\Delta u) \end{aligned} \tag{5}$$

with boundary conditions

$$\Delta z(t_0,x) = 0, \quad x \in [x_0,x_1]; \quad \Delta y(t,x_0) = 0, \quad t \in [t_0,t_1], \tag{6}$$

where by definition

$$\eta_2(t,x;\Delta u) = \Delta_{\overline{u}}f_z[t,x]\,\Delta z + \Delta_{\overline{u}}f_y[t,x]\,\Delta y + o_4(\|\Delta z + \Delta y\|) + o_5(\|\Delta z(t,x)\|)\frac{\partial W_1(t,x)}{\partial t},$$

$$\eta_3(t,x;\Delta u) = \Delta_{\overline{u}}g_z[t,x]\,\Delta z + \Delta_{\overline{u}}g_y[t,x]\,\Delta y + o_6(\|\Delta z + \Delta y\|) + o_7(\|\Delta y(t,x)\|)\frac{\partial W_1(t,x)}{\partial t}.$$

Here the quantities $o_i(.)$, $i = \overline{4,7}$ are determined from the expansion of functions $f(.)$, $p(.)$, $g(.)$, $q(.)$, respectively by the first order Taylor formula corresponding to the controls $\overline{u}(t,x)$ and $u(t,x)$.

Interpreting equations (5) as linear inhomogeneous stochastic equations with respect to $\Delta z(t,x)$, $\Delta y(t,x)$, and taking into account boundary conditions (6) based on the analogue of the Cauchy formula from [15], we obtain

$$\Delta z(t,x) = \int\limits_{t_0}^{t} V_{11}(t,x;\tau,x)\,\Delta_{\overline{u}}f[\tau,x]dt + \eta_4(t,x;\Delta u), \tag{7}$$

$$\Delta y(t,x) = \int\limits_{x_0}^{x} V_{22}(t,x;t,s)\,\Delta_{\overline{u}}g[t,s]ds + \eta_5(t,x;\Delta u). \tag{8}$$

Here, by definition

$$\eta_4(t,x;\Delta u)=\int_{t_0}^{t}\int_{x_0}^{x}\left\{\frac{\partial V_{11}(t,x;\tau,s)}{\partial x}\left[\Delta_{\overline{u}}f[\tau,s]+\eta_2(\tau,s;\Delta u)\right]\right.$$

$$\left.+\frac{\partial V_{12}(t,x;\tau,s)}{\partial x}\left[\Delta_{\overline{u}}g[\tau,s]+\eta_3(\tau,s;\Delta u)\right]\right\}dsd\tau$$

$$+\int_{t_0}^{t}V_{11}(t,x;\tau,x)\,\eta_2(\tau,x;\Delta u)\,d\tau,$$

$$\eta_5(t,x;\Delta u)=\int_{t_0}^{t}\int_{x_0}^{x}\left\{\frac{\partial V_{21}(t,x;\tau,s)}{\partial t}\left[\Delta_{\overline{u}}f[\tau,s]+\eta_2(\tau,s;\Delta u)\right]\right.$$

$$\left.+\frac{\partial V_{22}(t,x;\tau,s)}{\partial t}\left[\Delta_{\overline{u}}g[\tau,s]+\eta_3(\tau,s;\Delta u)\right]\right\}dsd\tau$$

$$+\int_{t_0}^{t}V_{22}(t,x;\tau,s)\,\eta_3(t,s;\Delta u)ds,$$

while $V_{ij}(t,x;\tau,s)$, $(t_0\leqslant t\leqslant t\leqslant t_1, x_0\leqslant s\leqslant x\leqslant x_1)$, $i,j=1,2$ are matrix functions that are the solutions of the following stochastic problems [15]:

$$\frac{\partial V_{11}(t,x;\tau,s)}{\partial \tau}=-V_{11}(t,x;\tau,s)\,f_z[\tau,s]-V_{12}(t,x;\tau,s)\,g_z[\tau,s]-V_{11}(t,x;\tau,s)\,p[\tau,s]\frac{\partial W_1(\tau,s)}{\partial \tau},$$

$$\frac{\partial V_{12}(t,x;\tau,s)}{\partial s}=-V_{11}(t,x;\tau,s)\,f_y[t,s]-V_{12}(t,x;\tau,s)\,g_y[\tau,s]-V_{12}(t,x;\tau,s)\,q[\tau,s]\frac{\partial W_2(\tau,s)}{\partial s},$$

$$\frac{\partial V_{21}(t,x;\tau,s)}{\partial t}=-V_{21}(t,x;\tau,s)\,f_z[\tau,s]-V_{22}(t,x;\tau,s)\,g_z[\tau,s]-V_{21}(t,x;\tau,s)\,p[\tau,s]\frac{\partial W_1(\tau,s)}{\partial \tau},$$

$$\frac{\partial V_{22}(t,x;\tau,s)}{\partial s}=-V_{21}(t,x;\tau,s)\,f_z[\tau,s]-V_{21}(t,x;\tau,s)\,g_z[\tau,s]-V_{22}(t,x;\tau,s)\,q[\tau,s]\frac{\partial W_2(\tau,s)}{\partial s},$$

$$V_{11}(t,x;t,s)=E_1,\quad V_{12}(t,x;\tau,x)=0,\quad t_0\leqslant\tau\leqslant t,$$

$$V_{21}(t,x;\tau,s)=0,\quad V_{22}(t,x;t,s)=E_2,$$

$x_0\leqslant s\leqslant x$, where $E_1,\ E_2$ are unit matrices of corresponding dimensions.

For further presentations, we introduce into consideration the $(n\times n)$-dimensional matrix function $R(x,\tau,s)$ and $(m\times m)$-dimensional function $Q(t,\tau,s)$ defined as:

$$R(x,\tau,s)=\int_{\max(\tau,s)}^{t_1}V_{11}^{'}(t,x;\tau,x)H_{zz}[t,x]\,V_{11}(t,x;s,x)\,dt$$

$$-V_{11}^{'}(t_1,x;\tau,x)\,\frac{\partial^2 F_1(x,z(t_1,x))}{\partial z^2}V_{11}(t_1,x;s,x)\,,$$

$$Q(t,\tau,s)=\int_{\max(\tau,s)}^{x_1}V_{22}^{'}(t,x;t,x)H_{yy}[t,x]\,V_{22}(t,x;t,s)\,dx$$

$$-V_{22}^{'}(t,x_1;\tau,t)\,\frac{\partial^2 F_2(t,y(t,x_1))}{\partial y^2}V_{22}(t,x_1;t,s)\,.$$

Taking into account the introduced denotations and following [5, 6], using representations (7), (8) of the solutions $\Delta z$ and $\Delta y$, we can represent the second order increment formula (4) of the quality criterion (3) in the following form:

$$\Delta S(u) = E\left\{ -\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\overline{u}} H[t,x]\,dxdt - \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1}\int_{t_0}^{t_1} \Delta_{\overline{u}} f'[\tau,x] R(x,\tau,s)\,\Delta_{\overline{u}} f[s,x]\,dsdxd\tau - \right.$$
$$- \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1}\int_{x_0}^{x_1} \Delta_{\overline{u}} g'[\tau,x] Q(t,\tau,s)\,\Delta_{\overline{u}} g[t,s]\,dtdsd\tau -$$
$$- \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{t}^{t_1} \Delta_{\overline{u}} H_z'[\tau,x]\,V_{11}(\tau,x;t,x)\,dt\right]\Delta_{\overline{u}} f[t,x]\,dxdt - \tag{9}$$
$$\left. - \int_{t_0}^{t_1}\int_{x_0}^{x_1}\left[\int_{x}^{x_1} \Delta_{\overline{u}} H_y'[t,s]\,V_{22}(t,s;t,x)\,ds\right]\Delta_{\overline{u}} g[t,x]\,dxdt \right\} + \eta(t,x;\Delta u),$$

where

$$\eta(t,x;\Delta u) = E\left\{ \eta_1(t,x;\Delta u) - \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \Big[\Delta z'(t,x) H_{zy}[t,x]\,\Delta y(t,x) \right.$$
$$+ \Delta y'(t,x)\,H_{yz}[t,x]\,\Delta z(t,x) + \eta_4(t,x)\,H_{zz}[t,x]\,\Delta z(t,x)\Big]dxdt$$
$$- \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left(\int_{t_0}^{t} V_{11}(t,x;\tau,x)\,\Delta_{\overline{u}} f[t,x]\,dt\right)' H_{zz}[t,x]\,\eta_4(t,x)\,dxdt$$
$$+ \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} V_{11}(t_1,x;\tau,x)\,\Delta_{\overline{u}} f[t,x]\,\frac{\partial^2 F_1(x,z(t_1,x))}{\partial z^2}\eta_4(t_1,x;\Delta u)\,dxdt$$
$$+ \frac{1}{2}\int_{x_0}^{x_1} \eta_4'(t_1,x;\Delta u)\,\frac{\partial^2 F_1(x,z(t_1,x))}{\partial z^2}\Delta z(t_1,x)\,dx - \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} \eta_5'(t,x;\Delta u)\,H_{yy}[t,x]\,\Delta y(t,x)\,dxdt$$
$$- \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1}\left(\int_{x_0}^{x} V_{22}(t,x;\tau,s)\,\Delta_{\overline{u}} g[t,s]\,ds\right)' H_{yy}[t,x]\,\eta_5(t,x;\Delta u)\,dxdt \tag{10}$$
$$+ \frac{1}{2}\int_{t_0}^{t_1}\int_{x_0}^{x_1} V_{22}(t,x_1;\tau,s)\,\Delta_{\overline{u}} g[t,s]\,\frac{\partial^2 F_2(t,y(t,x_1))}{\partial y^2}\eta_5(t,x_1;\Delta u)dsdt$$
$$+ \frac{1}{2}\int_{t_0}^{t_1} \eta_5'(t,x_1)\,\frac{\partial^2 F_2(t,y(t,x_1))}{\partial y^2}\Delta y(t,x_1)\,dt$$
$$\left. - \int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\overline{u}} H_z'[t,x]\eta_4(t,x;\Delta u)\,dxdt + \int_{t_0}^{t_1}\int_{x_0}^{x_1} \Delta_{\overline{u}} H_y'[t,x]\eta_5(t,x;\Delta u)\,dxdt \right\}.$$

Note that the constructed increment formula (9) allows one to get Pontryagin's maximum principle type first order necessary optimality conditions and to investigate the case of degeneration of the maximum principle and its consequences from a unified standpoint.

## 4. STOCHASTIC ANALOGUE OF PONTRYAGIN'S MAXIMUM PRINCIPLE

First we give an auxiliary statement.

**Lemma 1.** *For almost all* $(t,x)\in D$ *the following estimations are valid*

$$\|\Delta z(t,x)\| \leqslant K_1 E\left(\int\limits_{t_0}^{t}\|\Delta_{\overline{u}} f[\tau,x]\|\,d\tau+\int\limits_{t_0}^{t}\int\limits_{x_0}^{x}\|\Delta_{\overline{u}} g[\tau,s]\|\,dsd\tau\right), \tag{11}$$

$$\|\Delta y(t,x)\| \leqslant K_2 E\left(\int\limits_{x_0}^{x}\|\Delta_{\overline{u}} g[t,s]\|\,ds+\int\limits_{t_0}^{t}\int\limits_{x_0}^{x}\|\Delta_{\overline{u}} f[\tau,s]\|\,dsd\tau\right), \tag{12}$$

*where* $K_i=\mathrm{const}>0,\ i=1,2.$

The proof of the lemma is in the Appendix.

The validity of the statement [7] follows from the increment formula (9).

**Theorem 1.** *For the admissible control* $u(t,x)$ *be optimal in the studied distributed parameter stochastic control problem* (1)–(3) *it is necessary that the inequality*

$$E\Delta_v H[\theta,\gamma]\leqslant 0 \tag{13}$$

*be fulfilled for all* $(\theta,\gamma)\in[t_0,t_1)\times[x_0,x_1)$ *and for* $v\in U$.

*Here*

$$\begin{aligned}\Delta_v H[\theta,\gamma]&=H(\theta,\gamma,z(\theta,\gamma),y(\theta,\gamma),v,\psi(\theta,\gamma),\xi(\theta,\gamma))\\&-H(\theta,\gamma,z(\theta,\gamma),y(\theta,\gamma),u(\theta,\gamma),\psi(\theta,\gamma),\xi(\theta,\gamma)),\end{aligned}$$

*while* $(\theta,\gamma)\in[t_0,t_1)\times[x_0,x_1)$ *is an arbitrary Lebesgue point* (*a regular control point* [16] $u(t,x)$).

The proof of Theorem 1 is given in the Appendix.

Inequality (13) is a first-order necessary condition for optimality and is a stochastic analogue of Pontryagin's maximum principle for control problem (1)–(3) under consideration.

## 5. NECESSARY CONDITIONS FOR OPTIMALITY OF CONTROLS SINGULAR IN THE SENSE OF PONTRYAGIN'S MAXIMUM PRINCIPLE

Although Pontryagin's maximum principle (13) is the strongest first-order necessary condition for optimality in the control problem (1)–(3) under consideration, in some cases it holds trivially (the singular case [3–6, 17]).

Following [3–6, 17], we introduce the definition of singular control for the problem under consideration.

**Definition 1.** The admissible control $u(t,x)$ is said to be singular in the sense of Pontryagin's maximum principle if for all $(\theta,\gamma)\in[t_0,t_1)\times[x_0,x_1)$ and $v\in U$

$$E\Delta_v H[\theta,\gamma]=0. \tag{14}$$

In the singular case, i.e., if relation (14) is fulfilled, studying the increment formula (9), we prove the validity of the following statement.

**Theorem 2.** *For optimality of the control $u(t,x)$, singular in the sense of Pontryagin's maximum principle, in the considered distributed parameters stochastic control problem* (1)–(3), *it is necessary that for any $v \in U$, $(\theta,\gamma) \in [t_0,t_1) \times [x_0,x_1)$ the following relations be fulfilled:*

$$A(\theta,\gamma,v) = E\left[\Delta_v f'[\theta,\gamma] R(\gamma,\theta,\theta) + \Delta_v H_z'[\theta,\gamma]\right] \Delta_v f[\theta,\gamma] \leqslant 0, \tag{15}$$

$$B(\theta,\gamma,v) = E\left[\Delta_v g'[\theta,\gamma] Q(\theta,\gamma,\gamma) + \Delta_v H_y'[\theta,\gamma]\right] \Delta_v g[\theta,\gamma] \leqslant 0. \tag{16}$$

The proof of Theorem 2 is given in the Appendix.

Now let us consider one special case that requires separate study.

Let in equation (1)

$$\begin{aligned} f(t,x,z,y,u) &= A_1(t,x)z + B_1(t,x)y + f_1(t,x,u), \\ g(t,x,z,y,u) &= A_2(t,x)z + B_2(t,x)y + f_2(t,x,u). \end{aligned} \tag{17}$$

And the minimizing functional is in the form:

$$S(u) = E\left\{\int_{x_0}^{x_1} C'(x) z(t_1,x)\,dx + \int_{t_0}^{t_1} F_2(t,y(t,x_1))\,dt\right\}. \tag{18}$$

We introduce the denotations

$$K(t,\gamma,v) = V_{22}(t,x_1;t,\gamma)\Delta_{\overline{u}} g[t,\gamma]$$

$$+\int_{t_0}^{t}\left[\frac{\partial V_{21}(t,x_1;\tau,\gamma)}{\partial t}\Delta_v f[\tau,\gamma] + \frac{\partial V_{22}(t,x_1;\tau,\gamma)}{\partial t}\Delta_{\overline{u}} g[\tau,\gamma]\right] d\tau. \tag{19}$$

The following theorem is valid

**Theorem 3.** *For optimality of the control $u(t,x)$ singular in the sense of Pontryagin's maximum principle, in the considered distributed parameters stochastic control problem* (1), (2), (17), (18) *it is necessary that the inequality*

$$E\int_{t_0}^{t_1} K(t,\gamma,v)\,\frac{\partial^2 F_2(t,y(t,x_1))}{\partial y^2} K(t,\gamma,v)\,dt \geqslant 0$$

*be fulfilled for all $\gamma \in [x_0,x_1)$, $v(t) \in U, t \in [t_0,t_1)$.*

The proof of Theorem 3 is given in the Appendix.

Note that the similar result holds in the case of the functional of the form

$$S(u) = E\left\{\int_{x_0}^{x_1} F_1(x,z(t_1,x))\,dx + \int_{t_0}^{t_1} D'(t) y(t,x_1)\,dt\right\}.$$

## 6. CONCLUSION

In order to develop a stochastic analogue of the increment method, in the considered problem optimal control of the systems of the first order hyperbolic equations, we set a second-order increment formula for the quality functional. Based on the obtained incremental formula, we proved Pontryagin's maximum principle type necessary condition for optimality.

Further, using special variations of admissible controls, we proved necessary conditions of optimality for the optimality of the singular (in the sense of Pontryagin's maximum principle) controls for the considered distributed parameters stochastic control problem.

*APPENDIX*

**Proof of Lemma.** From system of equations (5) considering (6) we obtain

$$\Delta z\,(t,x)=\int\limits_{t_0}^{t}[f\,(\tau,x,\overline{z},\overline{y},\overline{u})-f\,(\tau,x,z,y,u)]d\tau+\int\limits_{t_0}^{t}[p\,(\tau,x,\overline{z})\;-p(\tau,x,z\;)]\,\frac{\partial W_1\,(\tau,x)}{\partial\tau}d\tau,$$

$$\Delta y\,(t,x)=\int\limits_{x_0}^{x}[g\,(t,s,\overline{z},\overline{y},\overline{u})-f\,(t,s,z,y,u)]ds+\int\limits_{x_0}^{x}[q\,(t,s,\overline{z})\;-q(t,s,z\;)]\,\frac{\partial W_2\,(t,s)}{\partial s}ds.$$

Hence, passing to the norm and using the Lipschitz condition, and also accepting the mathematical expectations from both parts of the obtained inequalities, we obtain

$$E\left\|\Delta z(t,x)\right\|\leqslant E\left\{\int\limits_{t_0}^{t}\|\Delta_{\overline{u}}f[t,x]\|dt+L_1\int\limits_{t_0}^{t}\left(\|\Delta z(t,x)\|+\|\Delta y(t,x)\|\right)dt\right\},$$

$$E\left\|\Delta y(t,x)\right\|\leqslant E\left\{\int\limits_{x_0}^{x}\|\Delta_{\overline{u}}g\,[t,s]\|ds+L_2\int\limits_{x_0}^{x}\left(\|\Delta z\,(t,s)\|+\|\Delta y\,(t,s)\|\right)ds\right\}.$$

Applying the Gronwall–Vendroff's lemma to the last two inequalities (see, e.i. [18]), we have

$$E\left\|\Delta z(t,x)\right\|\leqslant E\left\{\int\limits_{t_0}^{t}\|\Delta_{\overline{u}}f[t,x]\|dt+L_3\int\limits_{t_0}^{t}\|\Delta y\,(t,x)\|\,dt\right\},\tag{A.1}$$

$$E\left\|\Delta y(t,x)\right\|\leqslant E\left\{\int\limits_{x_0}^{x}\|\Delta_{\overline{u}}g\,[t,s]\|ds+L_4\int\limits_{x_0}^{x}\|\Delta z\,(t,s)\|\,ds\right\},\tag{A.2}$$

where $L_i=\text{const}>0$, $i=\overline{1,4}$. Enhancing inequalities (A.1), (A.2) with each other and again using the Gronwall–Vendroff lemma, we obtain the validity of the estimation (11) for $\|\Delta z(t,x)\|$, and estimate (12) for $\|\triangle y(t,x)\|$.

The lemma is proved.

**Proof of Theorem 1.** Assuming that $u\,(t,x)$ is a fixed admissible control, we define its special increment in the form

$$\Delta u_\varepsilon\,(t,x)=\begin{cases} v-u\,(t,x)\,,\;(t,x)\in D_\varepsilon=[\theta,\theta+\varepsilon)\times\left[\gamma,\gamma+\varepsilon^2\right),\\ \qquad\qquad 0,(t,x)\in D/D_\varepsilon.\end{cases}\tag{A.3}$$

Here and in the sequel, $(\theta,\gamma)\in[t_0,t_1)\times[x_0,x_1)$ is an arbitrary regular point of the control $u\,(t,x)$, $v\in U$ is an arbitrary vector, while $\varepsilon>0$ is an arbitrary, rather small number.

By $(\triangle z_\varepsilon\,(t,x)\,,\Delta y_\varepsilon\,(t,x))$ we denote a special increment of the system (1)–(2), corresponding to the special increment (A.3) of the control $u\,(t,x)$.

Using estimations (11), (12) for $(\Delta z_\varepsilon(t,x), \Delta\, y_\varepsilon(t,x))$, we obtain

$$E\left\|\Delta z_\varepsilon(t,x)\right\| \leqslant \begin{cases}0,(t,x)\in D\setminus D_\varepsilon,\\ K\varepsilon,\ (t,x)\in D_\varepsilon,\end{cases}\qquad E\left\|\Delta y_\varepsilon(t,x)\right\| \leqslant \begin{cases}0,(t,x)\in D\setminus D_\varepsilon,\\ K\varepsilon,\ (t,x)\in D_\varepsilon.\end{cases}$$

Taking into account these estimation in formula (10), we see that the remainder term $\eta(\triangle u_\varepsilon(t,x))$ is a quantity of order $o\left(\varepsilon^4\right)$.

Consequently,

$$\begin{aligned}\Delta S_\varepsilon(u) = E\Bigg\{ & -\int_\theta^{\theta+\varepsilon}\int_\gamma^{\gamma+\varepsilon^2}\Delta_v H[t,x]\,dxdt\\ & -\frac{1}{2}\int_\theta^{\theta+\varepsilon}\int_\theta^{\gamma+\varepsilon^2}\int_\theta^{\theta+\varepsilon}\Delta_v f'[\tau,x]R(x,\tau,s)\,\Delta_v f[s,x]\,dsdxd\tau\\ & -\frac{1}{2}\int_\theta^{\theta+\varepsilon}\int_\gamma^{\gamma+\varepsilon^2}\int_\gamma^{\varepsilon^2}\Delta_v g'[\tau,x]Q(t,\tau,s)\,\Delta_v g[t,s]\,d\tau dsdt\\ & -\int_\theta^{\theta+\varepsilon}\int_\gamma^{\gamma+\varepsilon^2}\left[\int_t^{t_1}\Delta_v H_z'[\tau,x]\,V_{11}(\tau,x;t,x)\,dt\right]\Delta_v f[t,x]\,dxdt\\ & -\int_\theta^{\theta+\varepsilon}\int_\gamma^{\gamma+\varepsilon^2}\left[\int_x^{x_1}\Delta_v H_y'[t,s]\,V_{22}(t,s;t,x)\,ds\right]\Delta_v g[t,x]\,dxdt\Bigg\}+o\left(\varepsilon^4\right).\end{aligned}\tag{A.4}$$

Hence, after applying the mean value theorem and allowing for $\triangle S_\varepsilon(u) = S(u+\Delta u_\varepsilon) - S(u) \geqslant 0$, the validity of the theorem statement follows. Theorem 1 is proved.

**Proof of Theorem 2.** Considering (14), from expansion (A.4) we obtain

$$\Delta S_\varepsilon(u) = S(u+\Delta u_\varepsilon) - S(u)$$

$$= E\varepsilon^4\left[\Delta_v f'[\theta\,,\gamma]R(\gamma,\theta,\theta) + \Delta_v H_z'[\theta,\gamma]\right]\Delta_v f[\theta,\gamma] + o(\varepsilon^4) \geqslant 0.$$

Hence the statement (15) follows.

Now we determine the special increment of the singular control by the formula

$$u_\varepsilon(t,x) = \begin{cases} v - u(t,x),\ (t,x)\in D_\varepsilon = [\theta,\theta+\varepsilon^2)\times[\gamma,\gamma+\varepsilon),\\ \qquad\qquad 0,(t,x)\in D/D_\varepsilon.\end{cases}$$

Then expansion (9) yields that condition (16) is fulfilled along the process $(u(t,x), z(t,x), y(t,x))$, singular in the sense of Pontryagin's maximum principle. Thus, Theorem 2 is proved.

**Proof of Theorem 3.** In the case of problem (1), (2), (17), (18) formulas (7)–(9) yield that

$$\begin{aligned}
\Delta S(u) = &- E\int_{t_0}^{t_1}\int_{x_0}^{x_1}\Delta_{\overline{u}}H[t,x]\,dxdt \\
&+\frac{1}{2}\int_{t_0}^{t_1}\Delta y'(t,x_1)\frac{\partial^2 F_2(t,y(t,x_1))}{\partial y^2}\Delta y(t,x_1)dt+\int_{t_0}^{t_1}o_2\left(\|\Delta y(t,x_1)\|^2\right)dt, \\
\Delta z(t_1,x) = &\int_{t_0}^{t_1}V_{11}(t_1,x;\tau,x)\,\Delta_{\overline{u}}f[t,x]dt \\
&+\int_{t_0}^{t}\int_{x_0}^{x}\left[\frac{\partial V_{11}(t_1,x;\tau,s)}{\partial x}\Delta_{\overline{u}}f[t,s]+\frac{\partial V_{12}(t_1,x;\tau,s)}{\partial x}\Delta_{\overline{u}}g[t,s]\right]dsdt, \\
\Delta y(t,x_1) = &\int_{x_0}^{x_1}V_{22}(t,x_1;\tau,s)\,\Delta_{\overline{u}}g[t,s]ds \\
&+\int_{t_0}^{t}\int_{x_0}^{x}\left[\frac{\partial V_{21}(t,x_1;\tau,s)}{\partial t}\Delta_{\overline{u}}f[t,s]+\frac{\partial V_{22}(t,x_1;\tau,s)}{\partial t}\Delta_{\overline{u}}g[t,s]\right]dsdt.
\end{aligned} \tag{A.5}$$

We determine the special increment of the control $u(t,x)$ by the formula

$$\Delta u_\varepsilon(t,x)=\begin{cases} v(t)-u(t,x), & (t,x)\in D_\varepsilon=[t_0,t_1)\times[\gamma,\gamma+\varepsilon), \\ & 0,(t,x)\in D\backslash D_\varepsilon,\end{cases}$$

where $\varepsilon>0$ is an arbitrary rather number $v(t)\in U$, $\gamma\in[x_0,x_1)$. Then (A.5) yields that for $(t,x)\in[t_0,t_1]\times[\gamma,x_1]$

$$\Delta y_\varepsilon(t,x_1)=\varepsilon\left[V_{22}(t,x_1;t,\gamma)\,\Delta_v g[t,\gamma]\right.$$

$$\left.+\int_{t_0}^{t}\left[\frac{\partial V_{21}(t,x_1;\tau,s)}{\partial t}\Delta_v f[\tau,s]+\frac{\partial V_{22}(t,x_1;\tau,\gamma)}{\partial t}\Delta_{\overline{u}}g[\tau,\gamma]\right]d\tau+o(\varepsilon)\right. .$$

Therefore, taking into account the denotation $K(t,\gamma,v)$ determined by formula (19), we obtain

$$\Delta S_\varepsilon(u)=-E\int_{t_0}^{t_1}\int_{\gamma}^{\gamma+\varepsilon}\Delta_{v(t)}H[t,x]\,dxdt$$

$$+\frac{1}{2}\varepsilon^2\int_{t_0}^{t_1}K(t,\gamma,v)\frac{\partial^2 F_2(t,y(t,x_1))}{\partial y^2}K(t,\gamma,v)dt+o\left(\varepsilon^2\right).$$

The statement of Theorem 3 follows from this expansion. Theorem 3 is proved.

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*This paper was recommended for publication by A.G. Kushner, a member of the Editorial Board*