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Admissible and Optimal Control Design for Nonlinear Continuous–Discrete Dynamic Systems

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Abstract—This paper considers nonlinear continuous–discrete (hybrid) dynamic systems with a constant sampling step whose continuous and discrete parts operate within a single loop. Sufficient conditions for the complete controllability and existence of admissible and optimal control laws of such systems are established. Algorithms for designing admissible program control laws, as well as an algorithm for designing an optimal program control law, for such systems are presented. Some examples are provided to confirm the effectiveness of these control design algorithms for nonlinear hybrid systems with a constant sampling step on a finite horizon.

Keywords: continuous–discrete system, hybrid system, discrete system, admissible control, admissible process, optimal control, optimal process

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1. INTRODUCTION. PROBLEM STATEMENT

Solving control problems for nonlinear dynamic systems is topical in many fields since these systems serve as mathematical models of various technical, mechanical, economic, and other processes; for example, see [1–4].

This paper addresses control problems for a dynamic system described by a combination of differential and difference equations; the latter include control, and the continuous and discrete parts of the system operate within a single loop. Such systems are commonly called continuous–discrete or hybrid dynamic control systems [2, 5].

Thus, we consider a nonlinear continuous–discrete control system of the form

$$\begin{cases} x'(t) = f(x(t), y(t_k)), \quad t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = g(x(t_{k+1}), y(t_k), u(t_k)), \quad k = \overline{0, l-1}, \end{cases} \tag{1}$$

with initial conditions

$$x(t_0) = x_0, \quad y(t_0) = y_0, \tag{2}$$

where $x(t) \in R^n$ and $y(t_k) \in R^m$ are the state vectors of system (1) characterizing the behavior of its continuous and discrete parts, respectively; $u(t_k) \in R^q$ is the control vector (input) of the system; the time instants t_k define a uniform grid on the closed interval $[t_0, t_l]$ with a constant step $h = t_{k+1} - t_k > 0$ and nodes $t_k = t_0 + kh$, $0 \leq t_0 < t_l$, $k = 0, 1, \ldots, l$ ($l \in N$ is a fixed number); the functions $f(x, y)$ and $g(x, y, u)$ are continuously differentiable in the aggregate of variables on the sets $R^n \times R^m$ and $R^n \times R^m \times R^q$, respectively, and

$$f(0,0) = 0, \quad g(0,0,0) = 0. \tag{3}$$

In other words, i.e., system (1) with $u = 0$ has the zero (trivial) equilibrium $x = 0,\ y = 0$.

According to (1) and the initial conditions (2), on each closed interval $[t_k, t_{k+1}]$ for a chosen control law u (i.e., for a given finite sequence of $u(t_k)$, $k = \overline{0, l-1}$), the solution $x = \varphi_k(t)$ of the Cauchy problem $x'(t) = f(x(t), y_k)$, $x(t_k) = x_k$ is found, and then the vectors $x_{k+1} = \varphi_k(t_{k+1})$ and $y_{k+1} = g(x_{k+1}, y_k, u_k)$, where $y_k = y(t_k)$ and $u_k = u(t_k)$, with $y(t) = y(t_k)$ for $t \in [t_k, t_{k+1})$, are constructed. The state of system (1) at a time instant t is the pair $z(t) = (x(t), y(t))$, and hence the solution of this system with the initial conditions $z_0 = (x_0, y_0)$ for a chosen control law u is the function

$$z(t) = (x(t), y(t)) = \begin{cases} (\varphi_0(t), y_0), & t_0 \leqslant t < t_1, \\ (\varphi_1(t), y_1), & t_1 \leqslant t < t_2, \\ \vdots & \\ (\varphi_{l-1}(t), y_{l-1}), & t_{l-1} \leqslant t < t_l, \\ (\varphi_{l-1}(t), y_l), & t = t_l. \end{cases} \tag{4}$$

In the solution (4), the component $x(t)$ is continuous for all $t \in [t_0, t_l]$, continuously differentiable on each open interval (t_k, t_{k+1}), $k = \overline{0, l-1}$, but not necessarily differentiable at time instants $t = t_k$, $k = \overline{0, l}$; the component $y(t)$ is piecewise constant and changes its values only at time instants $t = t_k$, $k = \overline{0, l}$.

An example of system (1) is the mathematical model of aircraft flying in a plane [4].

We pose the following problems: 1) the transfer of system (1) from a given initial state $z(t_0) = z^{\{0\}}$ to a given terminal state $z(t_l) = z^{\{1\}}$ $(t_l = t_0 + lh)$ using a suitable control law; 2) optimization, i.e., finding a control law that transfers system (1) from $z(t_0) = z^{\{0\}}$ to $z(t_l) = z^{\{1\}}$ and minimizes the control performance index (cost functional)

$$I(u) = \sum_{k=0}^{l-1} u^\top(t_k) u(t_k). \tag{5}$$

The solution of the posed problems is related to the grid step $h > 0$ since the value of h affects the existence of a control law transferring system (1) from an initial state $z^{\{0\}}$ to a terminal one $z^{\{1\}}$. Therefore, we adopt the following definitions.

Definition 1. The hybrid system (1) is said to be completely controllable for a given $h > 0$ if for any vectors $z^{\{0\}}$, $z^{\{1\}} \in R^{n+m}$, there exists a control law $u(t_k)$, $k = \overline{0, l-1}$, such that $z(t_l) = z^{\{1\}}$ for the solution $z = z(t)$ of system (1) with the initial condition $z(t_0) = z^{\{0\}}$.

Definition 2. A control law $u(t_k)$, $k = \overline{0, l-1}$, that satisfies system (1) for a given $h > 0$ and transfers this system from a given initial state $z(t_0) = z^{\{0\}}$ to a given terminal state $z(t_l) = z^{\{1\}}$ is called an admissible (program) control law of system (1) for $h > 0$.

Definition 3. An admissible control law $u(t_k)$, $k = \overline{0, l-1}$, of system (1) for a given $h > 0$ that minimizes the cost functional (5) is called an optimal (program) control law of system (1) for $h > 0$.

Note that in [6, 7], the author established necessary and sufficient conditions for complete controllability, as well as sufficient conditions and stabilization methods, for a nonlinear hybrid system described by equalities (1) but operating on an infinite horizon (i.e., for $k = 0, 1, 2, \ldots$), in contrast to system (1) that operates on a finite horizon. Therefore, the concepts of complete controllability for system (1) and the one in [6, 7] are defined differently.

The generalizations of system (1) are the hybrid systems considered in [5, 8], where optimization problems with a non-fixed terminal state were posed. Note that in [5], V.F. Krotov's optimality principle [9] was applied jointly to a combination of differential and difference equations to establish sufficient conditions for optimal control of a nonlinear system; and in [8], necessary conditions of optimal control were obtained to solve problems with non-fixed switching instants, which can be chosen during process optimization. The above optimization problem for system (1) assumes a fixed terminal state of the system, with switching instants and their number known in advance for a given $h > 0$; moreover, the continuous subsystem of system (1) contains no control input. As a result, the optimization approaches presented in [5, 8] are not applicable for solving the optimization problem of system (1). In addition, the problem of transferring hybrid systems from one state to another based on designing admissible control laws was not considered in [5, 8].

We also emphasize that scientific literature covers the issues of admissible program control design for nonlinear discrete systems with the cost functional (5) [10, 11] as well as various aspects of admissible and optimal control design for nonlinear continuous systems via the first approximation [12–16], the issues of optimal control for discrete systems [17, 18], and the issues of optimality for continuous and discrete processes [9, 19, 20]. There are also studies on optimal control for certain types of nonlinear hybrid systems (e.g., see [21–24]). As a rule, however, they are devoted to systems with a non-fixed terminal state, and the states of such systems are characterized by only one phase vector, changing continuously between switching instants and discretely at switching instants. In contrast, the state of system (1) is characterized by a set of two different phase vectors, one changing continuously according to differential equations and the other discretely according to the difference equations of the system that contain control. Thus, the issues related to the posed problems are still open.

To solve these problems, we perform a transition from system (1) to an equivalent, in a natural sense, nonlinear discrete system [7]. Transferring the equivalent discrete system from initial to terminal states involves an iterative process of designing an admissible control law, which may be convergent or divergent. Therefore, two admissible control design approaches are considered for the nonlinear hybrid system (1).

The primary objectives of this paper are to establish sufficient conditions for the admissibility and optimality of a control law for the nonlinear hybrid system (1) and design corresponding admissible and optimal control laws for this system.

The novelty of this research is as follows:

1) Sufficient conditions for the complete controllability of system (1) are established. Under these conditions, there exist admissible program control laws transferring the system from any initial state to any terminal one on a finite horizon $[t_0, t_l]$.

2) A sufficient condition for the existence of an optimal control law for system (1) is established.

3) Algorithms for designing admissible and optimal control laws for system (1) are presented.

2. TRANSITION TO A DISCRETE SYSTEM. COMPLETE CONTROLLABILITY CRITERION FOR SYSTEM (1)

Let $x(t_k) = x_k,\ y(t_k) = y_k$, and $u(t_k) = u_k$; using [7] and the relations (3), we pass from system (1) to an equivalent nonlinear discrete system of the form

$$\begin{cases} x_{k+1} = e^{A_1 h}x_k + A_1^{-1}(e^{A_1 h} - I)B_1 y_k + \varepsilon(x_k, y_k; h), \quad k = \overline{0, l-1}, \\ y_{k+1} = A_2 e^{A_1 h}x_k + (A_2 A_1^{-1}(e^{A_1 h} - I)B_1 + B_2)y_k + Cu_k + \delta(x_k, y_k, u_k; h), \end{cases}$$

where $A_1 = f'_x(0,0)$, $B_1 = f'_y(0,0)$, $A_2 = g'_x(0,0,0)$, $B_2 = g'_y(0,0,0)$, and $C = g'_u(0,0,0)$ are matrices of appropriate dimensions, and $det A_1 \neq 0$; in addition,

$$f(x,y) = A_1 x + B_1 y + a(x,y), \quad g(x,y,u) = A_2 x + B_2 y + Cu + b(x,y,u),$$

and the smooth nonlinearities $a(x,y)$ and $b(x,y,u)$ start with quadratic terms in x, y and x, y, u, respectively, satisfying the conditions $a(x,y) = o(\|x\| + \|y\|)$ as $\|x\| + \|y\| \to 0$ and $b(x,y,u) =$

$o(\|x\| + \|y\| + \|u\|)$ as $\|x\| + \|y\| + \|u\| \to 0$ (from this point onwards, $\|\cdot\|$ means the Euclidean norm of a vector in the corresponding space of real numbers or the norm of a matrix);

$$\varepsilon(x_k, y_k; h) = e^{(t_k+h)A_1} \int\limits_{t_k}^{t_k+h} e^{-sA_1} a(p(s, x_k, y_k), y_k) ds, \tag{6}$$

where $x = p(t, x_k, y_k)$ is the solution of the Cauchy problem $x' = A_1x + B_1y_k + a(x, y_k)$, $x(t_k) = x_k$; finally,

$$\delta(x_k, y_k, u_k; h) = A_2\varepsilon(x_k, y_k; h) + b(x_{k+1}, y_k, u_k). \tag{7}$$

The equivalent nonlinear discrete system can be compactly written as

$$z_{k+1} = A(h)z_k + Bu_k + \xi(z_k, u_k, h), \quad k = \overline{0, l-1}, \tag{8}$$

where

$$z_k = \begin{bmatrix} x_k \\ y_k \end{bmatrix}, \quad B = \begin{bmatrix} O \\ C \end{bmatrix}, \quad \xi(z_k, u_k, h) = \begin{bmatrix} \varepsilon(x_k, y_k; h) \\ \delta(x_k, y_k, u_k; h) \end{bmatrix},$$

and $A(h)$ is a block matrix (of order $n + m$) given by

$$A(h) = \begin{bmatrix} e^{A_1h} & A_1^{-1}(e^{A_1h} - I)B_1 \\ A_2e^{A_1h} & A_2A_1^{-1}(e^{A_1h} - I)B_1 + B_2 \end{bmatrix}. \tag{9}$$

System (8) with $u = 0$ has the zero (trivial) equilibrium, and the function $\xi(z, u, h)$ satisfies the condition $\xi(z, u, h) = o(\|z\| + \|u\|)$ as $\|z\| + \|u\| \to 0$ [7, 25].

Definition 4. The discrete system (8) is said to be completely controllable for a given $h > 0$ if for any states $z^{\{0\}}$, $z^{\{1\}} \in R^{n+m}$ there exists a control law u_k, $k = \overline{0, l-1}$, such that $z_l = z^{\{1\}}$, $z_l = z(t_l)$, for the solution z_k, $k = \overline{0, l}$, of system (8) with the initial condition $z_0 = z^{\{0\}}$.

We have the following result.

Theorem 1. *The hybrid system* (1) *is completely controllable for a given* $h > 0$ *if and only if the discrete system* (8) *is completely controllable for* $h > 0$.

The proof of Theorem 1 and other main statements is provided in the Appendix.

3. OPTIMAL CONTROL OF THE LINEAR HYBRID SYSTEM

Let the first approximation system of the nonlinear discrete system (8) be completely controllable for $h = h_0 > 0$. In other words, the discrete system

$$z_{k+1} = A_0z_k + Bu_k, \quad k = \overline{0, l-1}, \tag{10}$$

where $A_0 = A(h_0)$ due to (9), is completely controllable. Then, according to the complete controllability criterion of linear discrete systems [10, 11],

$$\operatorname{rank}[B, A_0B, A_0^2B, \ldots, A_0^{n+m-1}B] = n + m. \tag{11}$$

System (10) is equivalent, in a natural sense [7], to the linear hybrid system

$$\begin{cases} x'(t) = A_1x(t) + B_1y(t_k), & t_k \leqslant t < t_{k+1}, \quad t_{k+1} - t_k = h_0, \\ y(t_{k+1}) = A_2x(t_{k+1}) + B_2y(t_k) + Cu(t_k), & k = \overline{0, l-1}. \end{cases} \tag{12}$$

Based on [11], we arrive at the following result.

Theorem 2. *If system* (10) *is completely controllable, then there exists an optimal control law* $u^{(0)}(t_k) = u_k^{(0)}$ *transferring the hybrid system* (12) *from any initial state* $z(t_0) = z^{\{0\}}$ *to any terminal state* $z(t_l) = z^{\{1\}}$ *and minimizing the cost functional* (5), *with*

$$u_k^{(0)} = S^\top(k)F^+(0)d(0, z^{\{0\}}), \quad S(k) = A_0^{l-k-1}B, \quad k = \overline{0, l-1},$$
$$d(0, z^{\{0\}}) = z^{\{1\}} - A_0^l z^{\{0\}}, \quad F(0) = \sum_{k=0}^{l-1} S(k)S^\top(k), \tag{13}$$

and $F^+(0)$ *as the pseudoinverse of the matrix* $F(0)$.

This optimal control law of system (12) leads to the optimal motion $z^{(0)}(t) = (x^{(0)}(t), y^{(0)}(t))$, $t \in [t_0, t_l]$, of this system and the optimal motion $z_k^{(0)} = [x_k^{(0)} \;\; y_k^{(0)}]^\top$ $(k = \overline{0, l})$ of system (10), with $x^{(0)}(t_k) = x_k^{(0)}$ and $y^{(0)}(t_k) = y_k^{(0)}$.

Example 1. For the linear hybrid system

$$\begin{cases} x'(t) = x(t) + 2y(t_k), \quad t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = -x(t_{k+1}) + y(t_k) + 3u(t_k), \quad k = 0, 1, 2, \end{cases} \tag{14}$$

it is required to design an optimal control law $u^{(0)}(t_k)$ transferring this system in three steps ($l = 3$) from the initial state $z^{\{0\}} = [2 \;\; 1]^\top$ to the terminal one $z^{\{1\}} = [1 \;\; 2]^\top$.

The hybrid system (14) is equivalent to the linear discrete system

$$\begin{cases} x_{k+1} = e^h x_k + 2(e^h - 1)y_k \\ y_{k+1} = -e^h x_k + (3 - 2e^h)y_k + 3u_k, \quad k = 0, 1, 2. \end{cases}$$

For $h_0 = ln2$, it takes the form (10) with the matrices $A_0 = \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}$, and $B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}$. Condition (11) holds, i.e., this system is completely controllable for $h_0 = ln2$. Using formulas (13), we find

$$S(0) = A_0^2 B = [6 \;\; -9]^\top, \quad S(1) = A_0 B = [6 \;\; -3]^\top, \quad S(2) = B = [0 \;\; 3]^\top,$$
$$d(0, z^{\{0\}}) = z^{\{1\}} - A_0^3 z^{\{0\}} = [11 \;\; -1]^\top,$$
$$F(0) = \sum_{k=0}^{2} S(k)S^\top(k) = \begin{bmatrix} 72 & -72 \\ -72 & 99 \end{bmatrix}, \quad F^+(0) = F^{-1}(0) = \begin{bmatrix} 11/216 & 1/27 \\ 1/27 & 1/27 \end{bmatrix}.$$

For $t_{k+1} - t_k = h_0$ and $t_0 = 0$, system (14) has the optimal control law $u^{(0)}(t_0) = u^{(0)}(0) = -7/36$, $u^{(0)}(t_1) = u^{(0)}(ln2) = 73/36$, $u^{(0)}(t_2) = u^{(0)}(ln4) = 10/9$, and the corresponding optimal motion

$$z(t) = (x(t), y(t)) = \begin{cases} (4e^t - 2, \;\; 1), \quad 0 \leqslant t < ln2, \\ \left(-\dfrac{31}{12}e^t + \dfrac{67}{6}, -\dfrac{67}{12}\right), \quad ln2 \leqslant t < ln4, \\ \left(\dfrac{1}{24}e^t + \dfrac{2}{3}, -\dfrac{1}{3}\right), \quad ln4 \leqslant t < ln8, \\ \left(\dfrac{1}{24}e^t + \dfrac{2}{3}, 2\right), \quad t = ln8. \end{cases}$$

At $t_0 = 0$, $t_1 = ln2$, $t_2 = ln4$, and $t_3 = ln8$, this motion coincides with the solution of the equivalent discrete system $z_0 = [2 \;\; 1]^\top$, $z_1 = [6 \;\; -67/12]^\top$, $z_2 = [5/6 \;\; -1/3]^\top$, $z_3 = [1 \;\; 2]^\top$. That is, the control law designed transfers system (14) in three steps from the initial state $z^{\{0\}}$ to the terminal one $z^{\{1\}}$ and, by Theorem 2, minimizes the cost functional (5).

4. THE COMPLETE CONTROLLABILITY OF SYSTEMS (1) AND (8) ON A CLOSED SUBSET OF THE EUCLIDEAN SPACE. FEATURES OF THE NONLINEARITY $\xi(Z, U, H)$

Consider the interconnection between the complete controllability of systems (1) and (8) for some $h = h_0 > 0$ on closed subsets of the Euclidean space R^{n+m}.

Let $S \subset R^{n+m}$ be the set of states of system (1). We introduce several notions as follows.

Definition 5. System (1) is said to be completely controllable on the set S for $h = h_0 > 0$ if all states $z^* \in S$ are reachable from any state $z^{\{0\}} \in S$.

Definition 6. A state $z^* \in S$ of system (1) for $h = h_0 > 0$ is said to be reachable from a state $z^{\{0\}} \in S$ if there exists an admissible control law $u(t_k)$, $k = \overline{0, l-1}$, such that the solution $z = z(t)$ of system (1) with the initial condition $z(t_0) = z^{\{0\}}$ satisfies the following conditions:
1) $z(t) \in S$ for $t \in [t_0, t_l]$; 2) $z(t_l) = z^*$, where $t_l = t_0 + lh_0$.

Similar definitions can be formulated for the nonlinear discrete system (8), taking its specifics into account.

Assume that for $h = h_0 > 0$, system (8) is completely controllable on a set $K \subset R^{n+m}$, and system (1) is completely controllable on a set $S \subset R^{n+m}$.

Theorem 3. *If for $h = h_0 > 0$ systems (1) and (8) are completely controllable on sets S and K, respectively, then $K \subset S$.*

Let $U_0 = S_0 \times S^0$, where S_0 and S^0 are closed subsets of the neighborhoods of the points $z = 0$ and $u = 0$, respectively. In view of the features of the functions $f(x, y)$ and $g(x, y, u)$ in system (1), the next result can be proved for any $h > 0$.

Theorem 4. *The nonlinearity $\xi(z, u, h)$ in system (8) satisfies the Lipschitz condition in the variables z, u on any closed subset U_0 of the neighborhood of the point $z = 0$, $u = 0$, i.e.,*

$$\left\|\xi\left(z_k^{(1)}, u_k^{(1)}, h\right) - \xi\left(z_k^{(2)}, u_k^{(2)}, h\right)\right\| \leqslant L\left(\left\|z_k^{(1)} - z_k^{(2)}\right\| + \left\|u_k^{(1)} - u_k^{(2)}\right\|\right), \tag{15}$$

where $(z_k^{(i)}, u_k^{(i)}) \in U_0$, $k = \overline{0, l-1}$, $i = 1, 2$; and the Lipschitz constant L is sufficiently small if the diameter of the subset U_0 is small.

5. THE EXISTENCE AND DESIGN OF AN ADMISSIBLE CONTROL LAW FOR SYSTEM (1)

Let system (10) be completely controllable. We pose the problem of designing an admissible control law $u(t_k)$, $k = \overline{0, l-1}$, transferring system (1) from a given initial state $z(t_0) = z^{\{0\}} \in R^{n+m}$ to a given terminal state $z(t_l) = z^{\{1\}} \in R^{n+m}$ and investigate the issue of the complete controllability of system (1) for $h = h_0 > 0$.

Let

$$m = \max\left\{\max_{0 \le k \le l-1} \|A_0^{l-k-1}B\| \max_{0 \le k \le l-1} \|S^\top(k)F^+(0)\| \max_{0 \le k \le l-1} \|A_0^{l-k-1}\|,\right.$$
$$\left.\max_{0 \le k \le l-1} \|A_0^{l-k-1}\|, \max_{0 \le k \le l-1} \|S^\top(k)F^+(0)\| \max_{0 \le k \le l-1} \|A_0^{l-k-1}\|\right\},$$

and let the function $\xi(z, u, h)$, $h = h_0 > 0$, in system (8) satisfy inequality (15) on the set $R^{n+m} \times R^q$, with a Lipschitz constant L such that

$$0 < L < \frac{1}{ml(l+2)}. \tag{16}$$

The following result is true.

Theorem 5. *Assume that system* (10) *is completely controllable and the function* $\xi(z, u, h_0)$ *in system* (8) *satisfies the Lipschitz condition in the variables* z, u *on the set* $R^{n+m} \times R^q$, *with a Lipschitz constant obeying* (16). *Then the hybrid system* (1) *is completely controllable for* $h = h_0$.

Consider again the closed subset $U_0 = S_0 \times S^0$ of the neighborhood of the point $z = 0$, $u = 0$. According to Section 4, we arrive at the following statement.

Corollary 1. *Assume that system* (10) *is completely controllable on the set* S_0 *and the Lipschitz constant for the function* $\xi(z, u, h_0)$ *of system* (8) *satisfies inequality* (16) *on the set* U_0. *Then for* $h = h_0$ *there exists an admissible program control law transferring the hybrid system* (1) *from any initial state* $z(t_0) = z^{\{0\}} \in S_0$ *to any terminal state* $z(t_l) = z^{\{1\}} \in S_0$.

Based on the proof of Theorem 5, we describe an algorithm for designing an admissible program control law for system (1) with $h = h_0 > 0$.

Algorithm 1.

1. Verify the complete controllability of system (10), i.e., check condition (11).

2. Using formulas (13), construct the optimal control law $u_k^{(0)}$, $k = \overline{0, l-1}$, for system (10) and the corresponding optimal motion $z_k^{(0)}$, $k = \overline{0, l}$.

3. Apply the simple iteration method to construct the sequences of optimal control actions $\{u_k^{(q)}\}$ and optimal motions $\{z_k^{(q)}\}$ of the systems:

$$z_{k+1}^{(q)} = A_0 z_k^{(q)} + B u_k^{(q)} + \xi\left(z_k^{(q-1)}, u_k^{(q-1)}, h_0\right), \quad k = \overline{0, l-1}, \quad q = 1, 2, \ldots,$$

where $z_k^{(0)}$ and $u_k^{(0)}$ are obtained at Step 2 above, using formulas (A.2)–(A.4) (see the Appendix).

4. Find the admissible program control law of system (1) with $h = h_0 > 0$:

$$u^*(t_k) = u_k^* = \lim_{q \to \infty} u_k^{(q)}, \quad k = \overline{0, l-1}.$$

Note that in a small neighborhood of the point $x = 0$, $y = 0$, $u = 0$, condition (16) surely holds for the function $\xi(z, u, h_0)$ (see Theorem 4). Therefore, for system (1) with $h = h_0 > 0$, it is always possible to design an admissible control law if the states $z^{\{0\}}$ and $z^{\{1\}}$ lie in a small neighborhood of the point $x = 0$, $y = 0$ and the condition of Step 1 of Algorithm 1 is valid.

Example 2. For the nonlinear hybrid system

$$\begin{cases} x'(t) = x(t) + 2y(t_k), \quad t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = -x(t_{k+1}) + y(t_k) + 3u(t_k) + x(t_{k+1})y(t_k), \quad k = 0, 1, 2, \end{cases} \tag{17}$$

it is required to design an admissible program control law transferring this system in three steps ($l = 3$) from the initial state $z^{\{0\}} = [2 \;\; 1]^\top$ to the terminal one $z^{\{1\}} = [1 \;\; 2]^\top$.

The first approximation system of the equivalent nonlinear discrete system for $h_0 = ln2$ is completely controllable, and its optimal control law is given by $u_0^{(0)} = -7/36$, $u_1^{(0)} = 73/36$, $u_2^{(0)} = 10/9$. The corresponding motion of the system is $z_0^{(0)} = [2 \;\; 1]^\top$, $z_1^{(0)} = [6 \;\; -67/12]^\top$, $z_2^{(0)} = [5/6 \;\; -1/3]^\top$, $z_3^{(0)} = [1 \;\; 2]^\top$ (see Example 1).

The discrete system equivalent to (17) for $h = h_0$ has the form

$$z_{k+1} = A_0 z_k + B u_k + \xi(z_k, u_k, h_0), \quad k = 0, 1, 2,$$

where

$$A_0 = \begin{bmatrix} 2 & 2 \\ -2 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 3 \end{bmatrix}, \quad \xi(z_k, u_k, h_0) = \begin{bmatrix} 0 \\ 2x_k y_k + 2y_k^2 \end{bmatrix}.$$

We obtain the following result by applying Algorithm 1: on the 49th iteration with accuracy up to 10^{-15}, the iterative process converges to the control law

$$u^*(t_0) = -1.421310674166737, \quad u^*(t_1) = 7.208203932499369, \quad u^*(t_2) = 4.314757303333053.$$

For the sake of comparison, note that when transferring system (17) with $h = h_0$ from the state $z^{\{0\}} = [0.1 \quad -0.02]^\top$ to the state $z^{\{1\}} = [0.03 \quad 0.2]^\top$ in three steps, the iterative process with accuracy up to 10^{-15} converges on the 11th iteration. That is, if the states $z^{\{0\}}$ and $z^{\{1\}}$ of system (17) lie in a neighborhood of the point $x = 0$, $y = 0$, the iterative process will converge faster.

However, when transferring system (17) with $h = h_0$ from the state $z^{\{0\}} = [-2 \quad 5]^\top$ to the state $z^{\{1\}} = [3 \quad 4]^\top$ in three steps, the iterative process diverges.

Consider the existence and design of an admissible program control law for system (1) when the conditions of Theorem 5 fail. For this purpose, we introduce an auxiliary system of the form

$$z_{k+1} = A_0 z_k + B u_k, \quad k = 0, 1, 2, \ldots, \tag{18}$$

differing from (10) in the range of k, assuming that the former system is completely controllable [7]. This is true if system (10) is completely controllable [26].

6. THE EXISTENCE AND DESIGN OF AN ADMISSIBLE CONTROL LAW FOR SYSTEM (1) CONSIDERING STABILIZATION OF SYSTEM (18)

According to [10], the iterative process of designing an admissible control law for system (8) with $h = h_0 > 0$ is convergent if the function $\xi(z, u, h_0)$ satisfies the global Cauchy–Lipschitz condition in the variables z, u with a sufficiently small Lipschitz constant L. When this requirement fails, there may arise rapidly growing motions tending to infinity at a high rate; in this case, the iterative process will diverge, which is confirmed by the analysis of system (17) being transferred from the state $z^{\{0\}} = [-2 \quad 5]^\top$ to the state $z^{\{1\}} = [3 \quad 4]^\top$ in three steps.

One way to improve the convergence of the iterative process is the preliminary stabilization [27] of the system defined by (8) with $h = h_0 > 0$, or, equivalently, of the system defined by (1) with $h = h_0 > 0$, assuming $k = 0, 1, 2, \ldots$ in these systems [7]. Any of these systems is stabilized by stabilizing the completely controllable system (18), as the three systems mentioned have the same stabilizing control law [7].

Let us design a stabilizing control law $w(z_k) = D z_k$, $k = 0, 1, 2, \ldots$, where D is a constant matrix of dimensions $q \times (n + m)$, for system (18), under which the initial $z(t_0) = z^{\{0\}}$ and terminal $z(t_l) = z^{\{1\}}$ states will fall into the attraction domain of the equilibrium $z = 0$ of the system

$$z_{k+1} = A z_k + B v_k + \xi(z_k, D z_k + v_k, h_0), \quad k = \overline{0, l-1}, \tag{19}$$

with $v_k \equiv 0$, where $A = A_0 + BD$ and $v_k \in R^q$ is the control input. This ensures the existence of an admissible control law for system (8) with $h = h_0$ [10].

Assume that the function $\xi(z, Dz + v, h_0)$ in system (19) satisfies the Lipschitz condition in the variables z, $Dz + v$ on the set $R^{n+m} \times R^q$. If an estimate of the Lipschitz constant L for the function $\xi(z, Dz + v, h_0)$ on $R^{n+m} \times R^q$ is found (e.g., see [28]), we compare it with the condition

$$0 < L < \frac{1}{ml(1 + (\|D\| + 1)(l + 1))}, \tag{20}$$

where

$$m = \max\left\{\max_{0\le k\le l-1}\|S^{\top}(k)F^{+}(0)\|\max_{0\le k\le l-1}\|A^{l-k-1}\|, \max_{0\le k\le l-1}\|A^{l-k-1}\|,\right.$$

$$\left.\max_{0\le k\le l-1}\|S^{\top}(k)F^{+}(0)\|\max_{0\le k\le l-1}\|A^{l-k-1}\|\max_{0\le k\le l-1}\|A^{l-k-1}B\|\right\},$$

$$S(k) = A^{l-k-1}B.$$

These considerations are based on the following result.

Theorem 6. *Assume that system* (18) *is completely controllable and the stabilizing control law* $w(z_k) = Dz_k$ *of system* (18) *is chosen so that the Lipschitz constant for the function* $\xi(z, Dz+v, h_0)$ *in system* (19) *satisfies condition* (20) *for* $(z, Dz+v) \in R^{n+m}\times R^{q}$*. Then the hybrid system* (1) *is completely controllable for* $h = h_0$*, and an admissible control* $u^*(t_k)$ *transferring it from any initial state* $z(t_0) = z^{\{0\}}$ *to any terminal state* $z(t_l) = z^{\{1\}}$ *is given by*

$$u^*(t_k) = w(z^*(t_k)) + v^*(t_k), \quad k = \overline{0, l-1},$$

where $v^*(t_k) = v_k^*$ *and* $z^*(t_k) = z_k^*$ *are an admissible control law and the corresponding motion of system* (19)*, respectively.*

In this case, according to the proof of Theorem 6, we present the following algorithm for designing an admissible control law for system (1) with $h = h_0 > 0$.

Algorithm 2.

1. Verify the complete controllability of system (18) and design the stabilizing control law $w(z_k) = Dz_k$ for this system.

2. Using formulas (13), construct the optimal control law $v_k^{(0)}$, $k = \overline{0, l-1}$, for the system $z_{k+1} = Az_k + Bv_k$, where $A = A_0 + BD$, $k = \overline{0, l-1}$, and the corresponding optimal motion $z_k^{(0)}$, $k = \overline{0, l}$.

3. Apply the simple iteration method to construct the sequences of optimal control actions $\{v_k^{(q)}\}$ and optimal motions $\{z_k^{(q)}\}$ of the systems $z_{k+1}^{(q)} = Az_k^{(q)} + Bv_k^{(q)} + \xi\left(z_k^{(q-1)}, Dz_k^{(q-1)} + v_k^{(q-1)}, h_0\right)$, $k = \overline{0, l-1}$, $q = 1, 2, \ldots$, where $z_k^{(0)}$ and $v_k^{(0)}$ are obtained at Step 2 above, using formulas (A.12)–(A.14) (see the Appendix).

4. Find $v^*(t_k) = \lim_{q\to\infty} v_k^{(q)}$, $k = \overline{0, l-1}$; $z_k^* = \lim_{q\to\infty} z_k^{(q)}$, $k = \overline{0, l}$.

5. Design the admissible program control law of system (1) with $h = h_0 > 0$: by formula $u^*(t_k) = w(z_k^*) + v^*(t_k)$, $k = \overline{0, l-1}$.

Example 3. a) It is required to transfer system (17) with $h_0 = ln2$ from the state $z(t_0) = z^{\{0\}} = [-2\ \ 5]^{\top}$ to the state $z(t_3) = z^{\{1\}} = [3\ \ 4]^{\top}$ (see Example 2).

Recall that Algorithm 1 has failed to solve this problem. Now we apply Algorithm 2 and design, for the corresponding system (18), a stabilizing control law such that the matrix A has eigenvalues $\lambda_{1,2} = 0.001$. Then, on the 27th iteration with accuracy up to 10^{-15}, the admissible control law of system (17) becomes

$$u_0^* = -2.459959701325314, \quad u_1^* = -404.806942255234989, \quad u_2^* = 73.986989056064154.$$

Note that for $\lambda_{1,2} = 0.0001$, the iterative process with accuracy up to 10^{-15} will converge on the 12th iteration.

b) Consider the nonlinear hybrid system

$$\begin{cases} x'(t) = x(t) - y(t_k) + y^2(t_k), & t_k \leqslant t < t_{k+1}, \\ y(t_{k+1}) = -x(t_{k+1}) - y(t_k) + 2u(t_k) + u(t_k)y(t_k), & k = 0, 1, 2, 3, 4. \end{cases} \tag{21}$$

This system for $h_0 = ln\frac{3}{2}$ is equivalent to the nonlinear discrete system

$$\begin{cases} x_{k+1} = \dfrac{3}{2}x_k - \dfrac{1}{2}y_k + \dfrac{1}{2}y_k^2 \\ y_{k+1} = -\dfrac{3}{2}x_k - \dfrac{1}{2}y_k + 2u_k - \dfrac{1}{2}y_k^2 + u_k y_k, \quad k = 0, 1, 2, 3, 4. \end{cases}$$

The first approximation system for this discrete system is completely controllable. Algorithm 1 will not help to transfer system (21) from the state $z(t_0) = z^{\{0\}} = [0.1 \ \ -0.2]^\top$ to the state $z(t_5) = z^{\{1\}} = [0.4 \ \ 0.6]^\top$. Application of Algorithm 2 leads to the following results:

1) For $\lambda_1 = 0.1$ and $\lambda_2 = 0.2$, the iterative process converges on the 80th iteration, and with accuracy up to 10^{-18} we obtain the admissible control law

$u^*(t_0) = 0.324385159473078927$, $u^*(t_1) = 0.501724268550236387$,
$u^*(t_2) = 0.502341831544914188$, $u^*(t_3) = 0.399479636515260206$,
$u^*(t_4) = 0.544220425724104726$.

2) For $\lambda_1 = 0.2$ and $\lambda_2 = 0.3$, the iterative process diverges.

3) For $\lambda_1 = 0.1$ and $\lambda_2 = 0.3$, the iterative process with accuracy up to 10^{-18} converges on the 88th iteration.

4) For $\lambda_1 = 0.01$ and $\lambda_2 = 0.1$, as well as for $\lambda_1 = 0.01$ and $\lambda_2 = -0.001$, the iterative process diverges, meaning that the matrices D corresponding to the above $\lambda_{1,2}$ violate condition (20).

Thus, in particular cases (see Example 3a)), stabilization of the corresponding system (18) with $\lambda_{1,2}$ being near zero accelerates the convergence of the iterative process; however, in the general case, this is false.

7. SUFFICIENT CONDITIONS FOR OPTIMAL CONTROL OF THE HYBRID SYSTEM (1)

The algorithms presented in Sections 5 and 6 can be used to design an admissible control law for system (1). The following questions arise accordingly. Is an admissible control law optimal? What are the sufficient conditions for optimal control of system (1)? How to design an optimal control law for such a system?

To answer these questions, we will assume the complete controllability of system (1) for some $h = h_0 > 0$ and, due to the equivalence of systems (1) and (8), utilize sufficient optimality conditions for controlled discrete processes [19], with an appropriate modification to the problem of finding an optimal process with a fixed right endpoint of the trajectory.

Let $M = \{(z_k, u_k) : k = \overline{0, l-1}\}$ be the set of *admissible processes* of system (8) with $h = h_0 > 0$, i.e., pairs (z_k, u_k) satisfying, for $h = h_0 > 0$, this system and the boundary conditions $z_0 = z^{\{0\}}$, $z_l = z^{\{1\}}$.

Assume that the performance of a control process of system (8) with $h = h_0 > 0$ is generally evaluated by the cost functional

$$I(z_k, u_k) = \sum_{k=0}^{l-1} f^0(z_k, u_k), \tag{22}$$

where the function $f^0(z, u)$ is differentiable in z and u on the set M.

A process $(z_k, u_k) \in M$ is said to be *optimal* if $I(z_k, u_k) \to \min$.

By denoting $\tilde{f}(z_k, u_k) = A_0 z_k + B u_k + \xi(z_k, u_k, h_0)$, we write system (8) with $h = h_0 > 0$ as $z_{k+1} = \tilde{f}(z_k, u_k)$, $k = \overline{0, l-1}$, where the function $\tilde{f}(z, u)$ is continuously differentiable in z and u on the set $R^{n+m} \times R^q$.

Let D be the set of *possible processes* [19] of system $z_{k+1} = \tilde{f}(z_k, u_k)$, $k = \overline{0, l-1}$, i.e., pairs (z_k, u_k) satisfying this system; then $M \subset D$. Following [19], we introduce auxiliary mathematical constructs:

1) the function

$$R(z_k, u_k) = \varphi(\tilde{f}(z_k, u_k)) - \varphi(z_k) - f^0(z_k, u_k), \tag{23}$$

where $\varphi(z)$ is some function;

2) the functional

$$L(z_k, u_k, \varphi) = -\sum_{k=0}^{l-1} R(z_k, u_k) + \varphi(z^{\{1\}}) - \varphi(z^{\{0\}}). \tag{24}$$

Consider several auxiliary statements.

Lemma 1. *For any function $\varphi(z)$, the values of the cost functional* (22) *and functional* (24) *coincide on the set M, i.e., $L(z_k, u_k, \varphi) = I(z_k, u_k)$ for $(z_k, u_k) \in M$.*

Lemma 2. *If an admissible process $(z_k^*, u_k^*) \in M$ and some function $\varphi(z)$ satisfy the condition*

$$R(z_k^*, u_k^*) = \max_{(z_k, u_k) \in D} R(z_k, u_k), \quad k = \overline{0, l-1}, \tag{25}$$

then

$$L(z_k^*, u_k^*, \varphi) = \min_{(z_k, u_k) \in D} L(z_k, u_k, \varphi), \quad k = \overline{0, l-1}.$$

Let the cost functional (22) take the form (5), i.e., $I(z_k, u_k) = I(u_k) = \sum_{k=0}^{l-1} u_k^\top u_k$. We arrive at the following result.

Theorem 7. *Assume that an admissible process $(z_k^*, u_k^*) \in M$ and some function $\varphi(z)$ satisfy condition* (25), *with $f^0(z_k, u_k) = u_k^\top u_k$. Then $u^*(t_k) = u_k^*$ is the optimal control law of system* (1) *with $h = h_0$.*

The optimization problem of the cost functional can be formulated in two settings [19]:

1) It is required to determine the optimal process (z_k^*, u_k^*) minimizing the cost functional (22); in this setting, if $f^0(z_k, u_k) = u_k^\top u_k$, then $u^*(t_k) = u_k^*$ is the optimal control law of system (1) with $h = h_0 > 0$.

2) It is required to determine a minimizing sequence of admissible processes $\{z_k^{(s)}, u_k^{(s)}\} \subset M$, $s = 1, 2, \ldots$, on which the cost functional (22) tends to its greatest lower bound, i.e.,

$$I\left(z_k^{(s)}, u_k^{(s)}\right) \to \inf_{(z_k, u_k) \in M} I(z_k, u_k) \quad \text{as} \quad s \to \infty;$$

in this setting, if $f^0(z_k, u_k) = u_k^\top u_k$, then $\{u^{(s)}(t_k)\}$ is a minimizing sequence of admissible control actions of system (1) with $h = h_0 > 0$ on which the cost functional (5) tends to its greatest lower bound.

As a rule, the optimal control problem is formulated in the first setting, but does not always have a solution since the cost functional (22) may generally possess no minimum on the set M. Meanwhile, the problem in the second setting always has a solution if the cost functional (22) is bounded below [19].

The cost functional (5) is nonnegative, i.e., bounded below. Hence, there exists a minimizing sequence of admissible control actions of system (1) for the cost functional (5). And the following result is valid.

Theorem 8. *Let* $\{\underline{z_k^{(s)}, u_k^{(s)}}\} \subset M$, $s = 1, 2, \ldots$, *be a sequence of admissible processes of the system* $z_{k+1} = \tilde{f}(z_k, u_k)$, $k = \overline{0, l-1}$. *Assume the existence of a function* $\varphi(z)$ *such that, as* $s \to \infty$,

$$R\left(z_k^{(s)}, u_k^{(s)}\right) \to \sup_{(z_k, u_k) \in D} R(z_k, u_k), \quad k = \overline{0, l-1},$$

with $f^0(z_k, u_k) = u_k^\top u_k$. *Then* $\{u^{(s)}(t_k)\}$ *is a minimizing sequence of admissible control actions of system* (1) *with* $h = h_0$ *on which the cost functional* (5) *tends to its greatest lower bound.*

Suppose that system (1) with $h = h_0 > 0$ is known to have an optimal control. Let us study the issue of designing such a control.

8. OPTIMAL CONTROL DESIGN FOR THE HYBRID SYSTEM (1)

Due to the equivalence of systems (1) and (8), let us pose the following problem: for the system $z_{k+1} = \tilde{f}(z_k, u_k)$, $k = \overline{0, l-1}$, it is required to find an admissible process (z_k^*, u_k^*) on which the cost functional (22) (or the corresponding functional (5)) is minimized without imposing additional constraints on the control input $u_k \in R^q$.

To solve this problem, we use Lagrange's method of multipliers for discrete control processes with a one-dimensional (scalar) argument and an unbounded control input [19] as well as the sufficient optimality conditions provided by Theorem 7.

Let $\varphi(z)$ be a function differentiable in z on the set R^{n+m}. Then the necessary conditions for the maximum of the function $R(z, u)$ in z, u are given by

$$\frac{\partial R(z_k^*, u_k^*)}{\partial z_{k,i}} = 0, \quad i = \overline{1, n+m}, \quad k = \overline{0, l-1}, \tag{26}$$

$$\frac{\partial R(z_k^*, u_k^*)}{\partial u_{k,j}} = 0, \quad j = \overline{1, q}, \quad k = \overline{0, l-1}. \tag{27}$$

Let a vector function $\psi(k) = (\psi_1(k), \psi_2(k), \ldots, \psi_{n+m}(k))$, $k = \overline{0, l-1}$, be such that

$$\psi_i(k) = \frac{\partial \varphi(z_k^*)}{\partial z_{k,i}}, \quad i = \overline{1, n+m}.$$

We compile the Hamiltonian of the problem:

$$H(\psi(k), z_k, u_k) = \sum_{i=1}^{n+m} \psi_i(k) \tilde{f}_i(z_k, u_k) - f^0(z_k, u_k).$$

In view of (23), the equality $z_{k+1} = \tilde{f}(z_k, u_k)$, and the last two equalities above, it can be shown that conditions (26) and (27) take the form

$$\psi_i(k) = \frac{\partial H(\psi(k+1), z_k^*, u_k^*)}{\partial z_{k,i}}, \quad i = \overline{1, n+m}, \tag{28}$$

and

$$\frac{\partial H(\psi(k+1), z_k^*, u_k^*)}{\partial u_{k,j}} = 0, \quad j = \overline{1, q}, \tag{29}$$

respectively. Equalities (28) and (29) are necessary optimality conditions for discrete control processes with a scalar argument [19], with equality (28) called the adjoint system of finite-difference equations [29].

These conditions allow selecting, from the set of all admissible processes of the system $z_{k+1} = \tilde{f}(z_k, u_k)$, $k = \overline{0, l-1}$, a subset of potentially optimal processes [19]. If an optimal process is known to exist, and the resulting subset consists of a single element, it will be the desired optimal process. In this case, we obtain the following algorithm for designing an optimal control law for system (1) with $h = h_0 > 0$.

Algorithm 3.

1. Perform the transition from system (1) with $h = h_0 > 0$ to the system $z_{k+1} = \tilde{f}(z_k, u_k)$, $k = \overline{0, l-1}$.
2. Compile the Hamiltonian of the problem and conditions (29).
3. Find from conditions (29) the control law u_k^* with the parameters $\psi(k+1)$ and z_k^*, $k = \overline{0, l-1}$.
4. Compile the adjoint system (28).
5. Find the solutions $z_1^*, z_2^*, \ldots, z_{l-1}^*$ of the system $z_{k+1} = \tilde{f}(z_k, u_k)$ considering the condition $z_0^* = z^{\{0\}}$ and the control law $u_k^* = \{u_{k,j}^*(\psi(k+1), z_k^*)\}$, $k = \overline{0, l-1}$, $j = \overline{1, q}$.
6. Solve the adjoint system together with the system $z_l^* = z^{\{1\}}$, considering the solutions z_1^*, $z_2^*, \ldots, z_{l-1}^*$, and find $\psi_i(k+1)$, z_k^*, and u_k^*, $i = \overline{1, n+m}$, $k = \overline{0, l-1}$.
7. If the resulting solution is unique, then $u^*(t_k) = u_k^*$ is the optimal control law of system (1) with $h = h_0$; otherwise, find the value of the cost functional (5) on the found solutions and choose the control law corresponding to the solution that minimizes this functional.

Example 4. It is required to design an optimal control law that transfers system (17) with a sampling step of $h_0 = ln2$ from the initial state $z(t_0) = [2 \;\; 1]^\top$ to the terminal one $z(t_3) = [1 \;\; 2]^\top$ and minimizes the cost functional (5).

Recall that (Example 2) system (17) with $h_0 = ln2$ is equivalent to the nonlinear discrete system

$$\begin{cases} x_{k+1} = 2x_k + 2y_k \\ y_{k+1} = -2x_k - y_k + 3u_k + 2x_k y_k + 2y_k^2, \quad k = 0, 1, 2, \end{cases} \tag{30}$$

and the problem is to design an optimal control law transferring system (30) from the state $z_0 = [x_0 \;\; y_0]^\top = [2 \;\; 1]^\top$ to the state $z_3 = [x_3 \;\; y_3]^\top = [1 \;\; 2]^\top$.

According to (5), $f^0(z_k, y_k) = u_k^2$ since $u_k \in R^1$, and the set of admissible processes M for system (30) is compact. Therefore, the posed problem has a solution as the continuous cost functional (5) will reach its minimum value on M.

Note that $z_{k,1} = x_k$ and $z_{k,2} = y_k$; the Hamiltonian of the problem takes the form

$$H(\psi(k), z_k, u_k) = \psi_1(k)(2x_k + 2y_k) + \psi_2(k)(-2x_k - y_k + 3u_k + 2x_k y_k + 2y_k^2) - u_k^2,$$

and condition (29) yields

$$u_k^* = \frac{3}{2}\psi_2(k+1), \quad k = 0, 1, 2. \tag{31}$$

Due to (28), the adjoint system is given by

$$\begin{cases} \psi_1(k) = 2\psi_1(k+1) + 2\psi_2(k+1)(y_k^* - 1) \\ \psi_2(k) = 2\psi_1(k+1) + \psi_2(k+1)(4y_k^* + 2x_k^* - 1), \quad k = 0, 1, 2. \end{cases} \tag{32}$$

Considering the first boundary-value condition as well as equalities (30) and (31), we find the solutions

$$\begin{cases} x_1^* = 6 \\ y_1^* = 1 + 4.5\psi_2(1), \end{cases} \quad \begin{cases} x_2^* = 14 + 9\psi_2(1) \\ y_2^* = 40.5\psi_2^2(1) + 67.5\psi_2(1) + 4.5\psi_2(2) + 1. \end{cases}$$

Then the system $z_3^* = [1 \;\; 2]^\top$ turns into

$$\begin{cases} 29 + 153\psi_2(1) + 81\psi_2^2(1) + 9\psi_2(2) = 0 \\ 729\psi_2^3(1) + 2308.5\psi_2^2(1) + 1822.5\psi_2(1) + 121.5\psi_2(2) + 4.5\psi_2(3) + 81\psi_2(1)\psi_2(2) + 2y_2^{*2} = 3. \end{cases}$$

Based on the first equality in (32), (31), and the obtained solutions, we find

$$\psi_1(2) = 2\psi_1(3) + 2\psi_2(3)(40.5\psi_2^2(1) + 67.5\psi_2(1) + 4.5\psi_2(2)); \tag{33}$$

and based on the second equality in (32), $\psi_2(1) = 2\psi_1(2) + \psi_2(2)(15 + 18\psi_2(1))$. Then

$$\psi_1(2) = 0.5\psi_2(1) - \psi_2(2)(7.5 + 9\psi_2(1)) \tag{34}$$

and $\psi_2(2) = 2\psi_1(3) + \psi_2(3)(162\psi_2^2(1) + 288\psi_2(1) + 18\psi_2(2) + 31)$, consequently,

$$\psi_1(3) = 0.5\psi_2(2) - \psi_2(3)(81\psi_2^2(1) + 153\psi_2(1) + 15.5). \tag{35}$$

Let us substitute conditions (34) and (35) into (33). In view of the first equation in the system $z_3^* = [1 \;\; 2]^\top$, we get

$$\psi_2(2)(8.5 + 9\psi_2(1)) - 0.5\psi_2(1) = 2\psi_2(3). \tag{36}$$

Solving (36) together with the system $z_3^* = [1 \;\; 2]^\top$ yields three solutions and, accordingly, three control laws satisfying equality (31), system (30), and the boundary value conditions. Only one of them is optimal, particularly for system (17) with $h_0 = ln2$, as it minimizes the cost functional (5). With accuracy up to 10^{-15}, the optimal control law is given by $u^*(t_0) = -2.498184215120807$, $u^*(t_1) = 0.190252087647811$, $u^*(t_2) = 0.007263139516772$.

9. CONCLUSIONS

In this paper, we have established sufficient conditions for the complete controllability and existence of admissible and optimal control laws for nonlinear continuous–discrete dynamic systems with a constant sampling step. Algorithms for designing admissible and optimal program control laws for the above systems on a finite horizon have been presented. The effectiveness of these algorithms has been confirmed by numerical examples.

The conditions and algorithms will serve for control and stable operation of real systems, most topical in aviation, engineering, and economics.

APPENDIX

Proof of Theorem 1. Systems (1) and (8) are equivalent, and there exists a connection between their solutions [7, p. 867]. Based on this connection, the desired result follows from Definitions 1 and 4.

The proof of Theorem 1 is complete.

Proof of Theorem 2. According to [11], by the complete controllability of system (10), there exists the optimal control law (13) transferring system (10) from the state $z_0 = z^{\{0\}}$ to the state $z_l = z^{\{1\}}$ and minimizing the cost functional (5). And since systems (10) and (12) are equivalent and have the same control performance index, the optimal control of system (10) will also be optimal for system (12).

The proof of Theorem 2 is complete.

Proof of Theorem 3. In view of Definitions 5 and 6 and the equivalence of systems (1) and (8), the sets S and K can be described as $S = \{(x(t), y_k) : t \in [t_k, t_{k+1}),\ k = \overline{0, l-1}\} \cup \{(x(t_l), y_l)\}$, where $x(t_k) = x_k$, and $K = \{(x_k, y_k) : k = \overline{0, l}\}$. Consequently, $K \subset S$, and the proof of Theorem 3 is complete.

Proof of Theorem 4. The smooth nonlinearities $a(x, y)$ and $b(x, y, u)$ in the expansions of the functions $f(x, y)$ and $g(x, y, u)$ start with the quadratic terms in x, y and in x, y, u, respectively. Therefore, on the sets S_0 and U_0 of small diameter, they satisfy the Lipschitz condition with sufficiently small Lipschitz constants L_1 and L_2, respectively. The function $\xi(z, u, h)$ is continuously differentiable on the set U_0, and condition (15) holds for it. Since the components $\varepsilon(x, y; h)$ and $\delta(x, y, u; h)$ of the function $\xi(z, u, h)$ are linearly expressed through the functions $a(x, y)$ and $b(x, y, u)$ (see (6) and (7)), the Lipschitz constant L in condition (15) will be expressed through the constants L_1 and L_2 and will be sufficiently small.

The proof of Theorem 4 is complete.

Proof of Theorem 5. According to [11], by the complete controllability of system (10), there exists the optimal control law $u_k^{(0)}$ (13) transferring system (10) from $z_0 = z^{\{0\}} \in R^{n+m}$ to any state $z_l = z^{\{1\}} \in R^{n+m}$ and minimizing the cost functional (5). Then $z_k^{(0)}$ $(k = \overline{0, l})$ is the optimal motion of system (10) generated by the control law $u_k^{(0)}$, $k = \overline{0, l-1}$.

We design an admissible control law for system (8) with $h = h_0$ by the simple iteration method applied to solving the optimal control problems

$$z_{k+1}^{(q)} = A_0 z_k^{(q)} + B u_k^{(q)} + \xi\left(z_k^{(q-1)}, u_k^{(q-1)}, h_0\right), \quad k = \overline{0, l-1}, \quad q = 1, 2, \ldots, \tag{A.1}$$

$$I(u^{(q)}) = \sum_{k=0}^{l-1} u_k^{(q)\top} u_k^{(q)} \to \min,$$

intended to transfer systems (A.1) from the state $z_0 = z^{\{0\}}$ to the state $z_l = z^{\{1\}}$, where $z_k^{(0)}$ and $u_k^{(0)}$ are the optimal motion and control law of system (10), respectively. Then, using [11], we obtain a sequence of optimal controls $\{u_k^{(q)}\}$, $q = 1, 2, \ldots$, given by the formulas

$$u_k^{(q)} = S^\top(k) F^+(0) d^{(q)}\left(0, z^{\{0\}}\right), \quad k = \overline{0, l-1}, \tag{A.2}$$

$$d^{(q)}\left(0, z^{\{0\}}\right) = z^{\{1\}} - A_0^l z^{\{0\}} - \sum_{i=0}^{l-1} A_0^{l-i-1} \xi\left(z_i^{(q-1)}, u_i^{(q-1)}, h_0\right), \tag{A.3}$$

and the corresponding sequence of optimal motions $\{z_k^{(q)}\}$, $q = 1, 2, \ldots$,

$$z_k^{(q)} = A_0^k z^{\{0\}} + \sum_{i=0}^{k-1} A_0^{k-i-1} B u_i^{(q)} + \sum_{i=0}^{k-1} A_0^{k-i-1} \xi\left(z_i^{(q-1)}, u_i^{(q-1)}, h_0\right), \quad k = \overline{0, l}, \tag{A.4}$$

with $z_0^{(q)} = z^{\{0\}}$ and $z_l^{(q)} = z^{\{1\}}$.

Indeed, from (A.4) it follows that $z_0^{(q)} = z^{\{0\}}$, and based on [11, pp. 13–14], the control law $u_k^{(q)}$ transfers systems (A.1) from the state $z_0 = z^{\{0\}}$ to the state $z_l = z^{\{1\}}$ if and only if $\sum_{k=0}^{l-1} S(k) u_k^{(q)} = d^{(q)}(0, z^{\{0\}})$, where $S(k) = A_0^{l-k-1} B$. Then, taking (A.4) and (A.3) into account, we get $z_l^{(q)} = z^{\{1\}}$.

To prove the convergence of the iterative process, let us treat the pair of functions $\{z_k^{(q)}, u_k^{(q)}\}$ as an element of the Euclidean space R^{n+m+q} with the norm

$$\rho\left(\left\{z_k^{(q)}, u_k^{(q)}\right\}\right) = \max_{0 \le k \le l} \left\| z_k^{(q)} \right\| + \max_{0 \le k \le l-1} \left\| u_k^{(q)} \right\|. \tag{A.5}$$

By utilizing (A.2), (A.3), the properties of the norm, inequality (15), and (A.5), we derive the upper bound

$$\max_{0\le k\le l-1}\left\|u_k^{(q)}-u_k^{(q-1)}\right\|$$
$$\le lL\max_{0\le k\le l-1}\left\|S^\top(k)F^+(0)\right\|\max_{0\le k\le l-1}\left\|A_0^{l-k-1}\right\|\rho\left(\left\{z_k^{(q-1)},u_k^{(q-1)}\right\}-\left\{z_k^{(q-2)},u_k^{(q-2)}\right\}\right).$$

Similarly, by utilizing (A.4), the properties of the norm, inequality (15), the above estimate, and (A.5), we obtain the upper bound

$$\max_{0\le k\le l}\left\|z_k^{(q)}-z_k^{(q-1)}\right\|$$
$$\le lL\left(l\max_{0\le k\le l-1}\left\|A_0^{l-k-1}B\right\|\max_{0\le k\le l-1}\left\|S^\top(k)F^+(0)\right\|\max_{0\le k\le l-1}\left\|A_0^{l-k-1}\right\|+\max_{0\le k\le l-1}\left\|A_0^{l-k-1}\right\|\right)$$
$$\times\rho\left(\left\{z_k^{(q-1)},u_k^{(q-1)}\right\}-\left\{z_k^{(q-2)},u_k^{(q-2)}\right\}\right).$$

Let

$$m=\max\Big\{\max_{0\le k\le l-1}\|A_0^{l-k-1}B\|\max_{0\le k\le l-1}\|S^\top(k)F^+(0)\|\max_{0\le k\le l-1}\|A_0^{l-k-1}\|,$$
$$\max_{0\le k\le l-1}\|A_0^{l-k-1}\|,\max_{0\le k\le l-1}\|S^\top(k)F^+(0)\|\max_{0\le k\le l-1}\|A_0^{l-k-1}\|\Big\}.$$

Summing these bounds, according to (A.5), yields

$$\rho\left(\left\{z_k^{(q)},u_k^{(q)}\right\}-\left\{z_k^{(q-1)},u_k^{(q-1)}\right\}\right)\le lLm(l+2)\rho\left(\left\{z_k^{(q-1)},u_k^{(q-1)}\right\}-\left\{z_k^{(q-2)},u_k^{(q-2)}\right\}\right). \quad (A.6)$$

Due to condition (16), we have $0<lLm(l+2)<1$; then the sequence of vector functions $\{z_k^{(q)},u_k^{(q)}\}$ converges, uniformly in k, to some vector function $\{z_k^*,u_k^*\}$, and the operator $F(z,u):R^{n+m+q}\to R^{n+m+q}$ (A.2)–(A.4) is a contraction. According to the principle of contractive mappings, it has a unique fixed point, i.e., the limit vector function $\{z_k^*,u_k^*\}$ with

$$z_k^*=\lim_{q\to\infty}z_k^{(q)},\quad u_k^*=\lim_{q\to\infty}u_k^{(q)}, \quad (A.7)$$

$$u_k^*=S^T(k)F^+(0)\left(z^{\{1\}}-A_0^lz^{\{0\}}-\sum_{i=0}^{l-1}A_0^{l-i-1}\xi(z_i^*,u_i^*,h_0)\right),\quad k=\overline{0,l-1}, \quad (A.8)$$

$$z_k^*=A_0^kz^{\{0\}}+\sum_{i=0}^{k-1}A_0^{k-i-1}Bu_i^*+\sum_{i=0}^{k-1}A_0^{k-i-1}\xi(z_i^*,u_i^*,h_0),\quad k=\overline{0,l}. \quad (A.9)$$

In addition, (A.7) implies $z_0^*=z_0$ and $z_l^*=z_l$.

Direct substitution of (A.8) with $k=0,1,\ldots,l-1$ into (A.9) with $k=l$ shows that $z_l^*=z_l$ if the product $F(0)F^+(0)$ acts as an identity matrix, which is true because $\sum_{k=0}^{l-1}S(k)u_k^{(q)}=d^{(q)}(0,z^{\{0\}})$.

Due to the equivalence of systems (1) and (8), the admissible control law $u_k^*=u^*(t_k)$ for system (8) with $h=h_0$ is also an admissible control law for system (1) with $h=h_0$. For any states $z^{\{0\}}$, $z^{\{1\}}\in R^{n+m}$, one can design an admissible control law for system (8) with $h=h_0$; therefore, system (8) with $h=h_0$ is completely controllable. By Theorem 1, system (1) is completely controllable for $h=h_0$, and the proof of Theorem 5 is finished.

Proof of Theorem 6. The complete controllability of system (18) implies its stabilizability [7]. Let us choose a stabilizing control law $w(z_k) = Dz_k$ $(k = 0, 1, 2, \ldots)$ for system (18) so that all eigenvalues of the matrix $A = A_0 + BD$ lie inside the unit circle [7]. Then the linear system

$$z_{k+1} = Az_k + Bv_k, \quad k = \overline{0, l-1}, \tag{A.10}$$

is completely controllable [10, 13].

By employing formulas (13) of Theorem 2, we design an optimal control law $v_k^{(0)}$ transferring the linear system (A.10) from any state $z_0 = z^{\{0\}}$ to any state $z_l = z^{\{1\}}$ and ensuring $I(v^{(0)}) \to \min$, where $I(v^{(0)})$ is the cost functional (5) of the control variable $v_k^{(0)}$; then $z_k^{(0)}$ is the corresponding optimal motion. Next, we design an admissible control law transferring system (19) from the state $z_0 = z^{\{0\}}$ to the state $z_l = z^{\{1\}}$ by the simple iteration method applied to solving the optimal control problems

$$\begin{gathered} z_{k+1}^{(q)} = Az_k^{(q)} + Bv_k^{(q)} + \xi\left(z_k^{(q-1)}, Dz_k^{(q-1)} + v_k^{(q-1)}, h_0\right), \quad k = \overline{0, l-1}, \quad q = 1, 2, \ldots, \\ I(v^{(q)}) = \sum_{k=0}^{l-1} v_k^{(q)\top} v_k^{(q)} \to \min, \end{gathered} \tag{A.11}$$

where $z_k^{(0)}$ and $v_k^{(0)}$ are the optimal motion and control law of system (A.10), respectively.

As a result, we obtain the following sequences of optimal control actions $\{v_k^{(q)}\}$ and optimal motions $\{z_k^{(q)}\}$, $q = 1, 2, \ldots$:

$$v_k^{(q)} = S^\top(k) F^+(0) d^{(q)}\left(0, z^{\{0\}}\right), \text{ where } S(k) = A^{l-k-1}B, \; k = \overline{0, l-1}, \tag{A.12}$$

$$d^{(q)}\left(0, z^{\{0\}}\right) = z^{\{1\}} - A^l z^{\{0\}} - \sum_{i=0}^{l-1} A^{l-i-1} \xi\left(z_i^{(q-1)}, Dz_i^{(q-1)} + v_i^{(q-1)}, h_0\right), \tag{A.13}$$

$$z_k^{(q)} = A^k z^{\{0\}} + \sum_{i=0}^{k-1} A^{k-i-1} B v_i^{(q)} + \sum_{i=0}^{k-1} A^{k-i-1} \xi\left(z_i^{(q-1)}, Dz_i^{(q-1)} + v_i^{(q-1)}, h_0\right), \quad k = \overline{0, l}, \tag{A.14}$$

with $z_0^{(q)} = z^{\{0\}} = z_0$ and $z_l^{(q)} = z^{\{1\}} = z_l$ for $q = 1, 2, \ldots$.

To justify the convergence of the iterative process, let us treat the pair of functions $\{z_k^{(q)}, v_k^{(q)}\}$ as an element of the Euclidean space R^{n+m+q} with the norm

$$\rho\left(\left\{z_k^{(q)}, v_k^{(q)}\right\}\right) = (\|D\| + 1) \max_{0 \le k \le l} \left\|z_k^{(q)}\right\| + \max_{0 \le k \le l-1} \left\|v_k^{(q)}\right\|. \tag{A.15}$$

Similar to the proof of Theorem 5, we derive the following upper bounds:

$$\begin{gathered} \max_{0 \le k \le l-1} \left\|v_k^{(q)} - v_k^{(q-1)}\right\| \\ \le lL \max_{0 \le k \le l-1} \left\|S^\top(k) F^+(0)\right\| \max_{0 \le k \le l-1} \left\|A^{l-k-1}\right\| \rho\left(\left\{z_k^{(q-1)}, v_k^{(q-1)}\right\} - \left\{z_k^{(q-2)}, v_k^{(q-2)}\right\}\right), \end{gathered}$$

$$\begin{gathered} \max_{0 \le k \le l} \left\|z_k^{(q)} - z_k^{(q-1)}\right\| \\ \le lL \left(l \max_{0 \le k \le l-1} \left\|A^{l-k-1}B\right\| \max_{0 \le k \le l-1} \left\|S^\top(k) F^+(0)\right\| \max_{0 \le k \le l-1} \left\|A^{l-k-1}\right\| \right. \\ \left. + \max_{0 \le k \le l-1} \left\|A^{l-k-1}\right\|\right) \rho\left(\left\{z_k^{(q-1)}, v_k^{(q-1)}\right\} - \left\{z_k^{(q-2)}, v_k^{(q-2)}\right\}\right). \end{gathered}$$

Let

$$m = \max\Big\{\max_{0\le k\le l-1} \|S^\top(k)F^+(0)\| \max_{0\le k\le l-1} \|A^{l-k-1}\|, \max_{0\le k\le l-1} \|A^{l-k-1}\|,$$
$$\max_{0\le k\le l-1} \|S^\top(k)F^+(0)\| \max_{0\le k\le l-1} \|A^{l-k-1}\| \max_{0\le k\le l-1} \|A^{l-k-1}B\|\Big\}.$$

Adding the first bound to the second one multiplied by $(\|D\|+1)$, in view of (A.15), gives

$$\rho\left(\left\{z_k^{(q)}, v_k^{(q)}\right\} - \left\{z_k^{(q-1)}, v_k^{(q-1)}\right\}\right)$$
$$\le Llm(1+(l+1)(\|D\|+1))\rho\left(\left\{z_k^{(q-1)}, v_k^{(q-1)}\right\} - \left\{z_k^{(q-2)}, v_k^{(q-2)}\right\}\right). \tag{A.16}$$

Under condition (20) we establish $0 < Llm(1+(l+1)(\|D\|+1)) < 1$; then the sequence of vector functions $\{z_k^{(q)}, v_k^{(q)}\}$ converges, uniformly in k, to some vector function $\{z_k^*, v_k^*\}$, and the operator $F(z,v) : R^{n+m+q} \to R^{n+m+q}$ (A.12)–(A.14) is a contraction. According to the principle of contractive mappings, it has a unique fixed point, i.e., the limit vector function $\{z_k^*, v_k^*\}$ with $z_k^* = \lim_{q\to\infty} z_k^{(q)}$, $v_k^* = \lim_{q\to\infty} v_k^{(q)}$. In addition,

$$v_k^* = S^\top(k)F^+(0)\left(z^{\{l\}} - A^l z^{\{0\}} - \sum_{i=0}^{l-1} A^{l-i-1}\xi(z_i^*, Dz_i^* + v_i^*, h_0)\right), \quad k = \overline{0, l-1},$$

$$z_k^* = A^k z^{\{0\}} + \sum_{i=0}^{k-1} A^{k-i-1}Bv_i^* + \sum_{i=0}^{k-1} A^{k-i-1}\xi(z_i^*, Dz_i^* + v_i^*, h_0), \ k = \overline{0, l},$$

are the admissible control law and the corresponding motion of system (19), respectively.

For $h = h_0$, system (8) has the form $z_{k+1} = A_0 z_k + Bu_k + \xi(z_k, u_k, h_0)$, $k = \overline{0, l-1}$. We substitute the control law $u_k^* = Dz_k^* + v_k^*$ and the motion z_k^*, $k = \overline{0, l-1}$, into this system to obtain

$$z_{k+1}^* = Az_k^* + Bv_k^* + \xi(z_k^*, Dz_k^* + v_k^*, h_0), \quad k = \overline{0, l-1}.$$

Actually, this is system (19) with $z_k = z_k^*$ and $v_k = v_k^*$; in addition, $z_0^* = z^{\{0\}} = z_0$ and $z_l^* = z^{\{1\}} = z_l$. Hence, $u_k^* = w(z_k^*) + v_k^*$ $(k = \overline{0, l-1})$ is an admissible control law for system (8) with $h = h_0$; and due to the equivalence of systems (1) and (8), it is the desired admissible control law for system (1) with $h = h_0$. Moreover, system (1) is completely controllable for $h = h_0$, and the proof of Theorem 6 is finished.

Proof of Lemma 1. Let $(z_k, u_k) \in M$. Due to equalities (24), (23), $z_{k+1} = \tilde f(z_k, u_k)$, and the boundary value conditions $z_0 = z^{\{0\}}$, $z_l = z^{\{1\}}$, we obtain

$$L(z_k, u_k, \varphi) = -\sum_{k=0}^{l-1}\left(\varphi\left(\tilde f(z_k, u_k)\right) - \varphi(z_k) - f^0(z_k, u_k)\right) + \varphi(z^{\{1\}}) - \varphi(z^{\{0\}})$$
$$= -\sum_{k=0}^{l-1}\left(\varphi(z_{k+1}) - \varphi(z_k) - f^0(z_k, u_k)\right) + \varphi(z_l) - \varphi(z_0)$$
$$= -\varphi(z_1) + \varphi(z_0) + f^0(z_0, u_0) - \varphi(z_2) + \varphi(z_1) + f^0(z_1, u_1) - \ldots - \varphi(z_{l-1})$$
$$+ \varphi(z_{l-2}) + f^0(z_{l-2}, u_{l-2}) - \varphi(z_l) + \varphi(z_{l-1}) + f^0(z_{l-1}, u_{l-1}) + \varphi(z_l) - \varphi(z_0)$$
$$= \sum_{k=0}^{l-1} f^0(z_k, u_k) = I(z_k, u_k).$$

The proof of Lemma 1 is complete.

Proof of Lemma 2. It rests on Theorem 5.1 from [19], stating (as applied to minimization of the cost functional (22)) that a process (z_k^*, u_k^*) is optimal if and only if

$$f^0(z_k^*, u_k^*) = \min_{(z_k,u_k)\in D} f^0(z_k, u_k), \quad k = \overline{0, l-1}.$$

Since $\varphi(z^{\{1\}}) - \varphi(z^{\{0\}}) \equiv \text{const}$, it follows from (24) that only the expression $\sum_{k=0}^{l-1} R(z_k, u_k)$ affects the minimization of the functional $L(z_k, u_k, \varphi)$. Then, by taking (25) into account and applying Theorem 5.1 from [19] to the functional (24), we arrive at

$$-\sum_{k=0}^{l-1} R(z_k^*, u_k^*) = \min_{(z_k,u_k)\in D}\left(-\sum_{k=0}^{l-1} R(z_k, u_k)\right) \text{ and}$$

$$\min_{(z_k,u_k)\in D} L(z_k, u_k, \varphi) = \min_{(z_k,u_k)\in D}\left(-\sum_{k=0}^{l-1} R(z_k, u_k)\right) + \varphi(z^{\{1\}}) - \varphi(z^{\{0\}})$$
$$= -\sum_{k=0}^{l-1} R(z_k^*, u_k^*) + \varphi(z^{\{1\}}) - \varphi(z^{\{0\}}) = L(z_k^*, u_k^*, \varphi), \quad k = \overline{0, l-1}.$$

The proof of Lemma 2 is complete.

Proof of Theorem 7. Letting

$$l_\varphi = \min_{(z_k,u_k)\in D} L(z_k, u_k, \varphi)$$

gives

$$L(z_k, u_k, \varphi) \geq l_\varphi \text{ for } (z_k, u_k) \in D, \tag{A.17}$$

and, by Lemma 2,

$$L(z_k^*, u_k^*, \varphi) = l_\varphi. \tag{A.18}$$

According to (22) and Lemma 1, we have $L(z_k, u_k, \varphi) = I(z_k, u_k)$ for $(z_k, u_k) \in M$. Then from (A.17) and (A.18) it follows that $I(z_k, u_k) \geq l_\varphi$ for $(z_k, u_k) \in M$, and $I(z_k^*, u_k^*) = l_\varphi$. Hence, $I(z_k^*, u_k^*) \leq I(z_k, u_k)$ for $(z_k, u_k) \in M$, and (z_k^*, u_k^*) is the optimal process for system (8) with $h = h_0$; moreover, u_k^* is the optimal control law for this system. Since $I(z_k, u_k) = I(u_k)$ and the equivalent systems (1) and (8) have a common control performance index, $u^*(t_k) = u_k^*$ is the optimal control law for system (1) with $h = h_0$. The proof of Theorem 7 is complete.

Proof of Theorem 8. Consider the cost functional (24) on the set D and let

$$l_\varphi = \inf_{(z_k,u_k)\in D} L(z_k, u_k, \varphi).$$

According to Lemma 1, we have $L(z_k, u_k, \varphi) = I(z_k, u_k)$ for $(z_k, u_k) \in M$, $M \subset D$. Then

$$\inf_{(z_k,u_k)\in D} L(z_k, u_k, \varphi) \leq \inf_{(z_k,u_k)\in M} L(z_k, u_k, \varphi)$$

and

$$\inf_{(z_k,u_k)\in M} I(z_k, u_k) \geq l_\varphi. \tag{A.19}$$

On the other hand, the condition of this theorem implies

$$-\sum_{k=0}^{l-1} R\left(z_k^{(s)}, u_k^{(s)}\right) \to \inf_{(z_k,u_k)\,\in\, D} \left(-\sum_{k=0}^{l-1} R(z_k, u_k)\right) \quad \text{as} \quad s \to \infty,$$

and because $\varphi(z^{\{1\}}) - \varphi(z^{\{0\}}) \equiv \text{const}$, $L(z_k^{(s)}, u_k^{(s)}, \varphi) \to l_\varphi$ as $s \to \infty$. Due to (A.19) and the definition of the infimum of a functional, we establish the convergence

$$I\left(z_k^{(s)}, u_k^{(s)}\right) \to \inf_{(z_k,u_k)\,\in\, M} I(z_k, u_k),$$

i.e., $\{z_k^{(s)}, u_k^{(s)}\}$ is a minimizing sequence of admissible processes for the functional $I(z_k, u_k) = \sum_{k=0}^{l-1} u_k^\top u_k$. Then $\{u^{(s)}(t_k)\}$ is a minimizing sequence of admissible control actions of system (1) with $h = h_0$ on which the functional (5) tends to its greatest lower bound. The proof of Theorem 8 is complete.

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