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NONLINEAR SYSTEMS

Boundary Control of a Rod Heating Process Using Current and Preceding Time Feedback

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Abstract—This paper considers the problem of optimal boundary control with feedback for a rod heating process. Devices to measure the current temperature values are placed on the rod. To design the boundary control, a linear dependence of its value on the measured temperature values, both at the current and previous time instants, is proposed. The constant coefficients in this dependence are the feedback parameters under optimization. As a result, the original feedback control design problem is formulated as a parametric optimal control problem for a distributed parameter plant. The optimal values of the above parameters are determined via formulas derived for the components of the gradient of the objective functional. With these formulas, it is possible to use efficient first-order finite-dimensional optimization methods for parameter determination. Numerical results of computer experiments are presented and analyzed.

Keywords: boundary control, rod heating, feedback, measurement point, feedback parameters, the gradient of a functional

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1. INTRODUCTION

In this paper, we investigate the problem of optimal control of a boundary value condition for a rod heating process. Feedback is implemented through temperature measurements at internal points of the rod, and the measured values are used to form the current value of the boundary control function. The peculiarity of the feedback considered here is that the dependence of the boundary control value includes the temperature values at the measurement points, both at the current and previous time instants. In this case, the boundary control design problem reduces to determining the optimal values of the coefficients in the dependence of the boundary control value on the temperature values at the measurement points.

Note that the number of publications devoted to the analysis and, especially, to the numerical solution of optimal control problems for plants described by partial differential equations (PDEs) [1–4] is substantially inferior to that of the publications related to control design problems for lumped parameter systems [5–9]. This is due to, first, the complexity of analyzing and numerically solving boundary value problems described by systems of PDEs and, second, the complexity of implementing control systems for distributed parameter plants [10–12]. With the development of measurement tools, remote control means, computer technologies, and methods of computational mathematics in recent decades, it has become possible to design and implement real-time feedback control and regulation systems for complex facilities and industrial processes, both with lumped and distributed parameters [13–18].

Note that the feedback control approach proposed below for the rod heating process using the history of previous measurements is, of course, more demanding in terms of implementation than the one using only the current measurement values. However, considering the state-of-the-art of technology and engineering, the implementation of such systems has become quite feasible.

In what follows, we derive explicit formulas for the components of the gradient of the objective functional with respect to the feedback parameters. With these formulas, efficient numerical methods and software packages for first-order finite-dimensional optimization [5, 19] can be used to determine the optimal parameter values. The results of computer experiments for test boundary control problems with feedback are provided and analyzed.

2. PROBLEM STATEMENT

Consider the problem of optimal boundary control (regulation) of a heating process for a rod heated from one end. The temperature at the points of the rod is described by a parabolic differential equation of the form

$$\frac{\partial u(x,t)}{\partial t} = a^2 \frac{\partial^2 u(x,t)}{\partial x^2} + \mu_1 \left[\theta - u(x,t)\right], \quad (x,t) \in \Omega = (0,l) \times (0,t_f], \tag{2.1}$$

with an initial condition

$$u(x,t) = \gamma(x;\varphi), \quad \varphi = \text{const} \in \mathbb{R}^m, \quad x \in (0,l), \quad t \le 0, \tag{2.2}$$

and boundary value conditions

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = -\mu_2(\vartheta(t) - u(0,t)), \quad 0 \le t \le t_f, \tag{2.3}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=l} = \mu_2(\theta - u(l,t)), \quad 0 \le t \le t_f. \tag{2.4}$$

Here, $u(x,t)$ is a function continuously differentiable with respect to $t \in (0,t_f]$ and twice continuously differentiable with respect to $x \in (0,l)$, which defines the temperature of the rod at a time instant $t \in [0,t_f]$ at a point $x \in [0,l]$; l is the length of the rod; t_f is the terminal time of the heating process; θ is an imprecisely known constant temperature of the environment defined by the set Θ of possible values of θ and a (probability) density function $\rho_\Theta(\theta)$ such that

$$\rho_\Theta(\theta) \ge 0, \quad \int\limits_\Theta \rho_\Theta(\theta)d\theta = 1;$$

finally, μ_1 and μ_2 are given heat transfer coefficients between the rod and the environment.

According to condition (2.2), the temperature at the points of the rod before the heating process is defined by a known parametrically given function $\gamma(x;\varphi)$. It depends on an imprecisely known constant m-dimensional vector of parameters φ defined by the set Φ of possible values of φ and a (probability) density function $\rho_\Phi(\varphi)$ such that

$$\rho_\Phi(\varphi) \ge 0, \quad \int\limits_\Phi \rho_\Phi(\varphi)d\varphi = 1. \tag{2.5}$$

In particular, the initial temperature may be constant along the length of the rod, i.e., $\gamma(x;\varphi) = \varphi \in \Phi \in R$.

The sets Θ and/or Φ may be finite sets

$$\Theta = \{\theta_i : i = 1, \ldots, N_\theta\}, \quad \Phi = \{\varphi_j : j = 1, \ldots, N_\varphi\}$$

with given probability values for their elements, i.e.,

$$p_i^\theta = p(\theta = \theta_i), \quad p_j^\varphi = p(\varphi = \varphi_j), \quad i = 1, \ldots, N_\theta, \quad j = 1, \ldots, N_\varphi,$$
$$\sum_{i=1}^{N_\theta} p_i^\theta = 1, \quad \sum_{j=1}^{N_\varphi} p_j^\varphi = 1. \tag{2.6}$$

The piecewise continuous function $\vartheta(t)$, $t \in [0, t_f]$, is the control action optimized for the rod heating process under study. It defines the boundary value condition (2.4) at the left end of the rod and is called the boundary control function [5]. Assume that, based on technical or technological considerations, it satisfies the constraint

$$\underline{\vartheta} \le \vartheta(t) \le \bar{\vartheta}, \quad t \in [0, t_f], \tag{2.7}$$

where $\underline{\vartheta}$ and $\bar{\vartheta}$ are given values.

The control problem for this heating process is to find a control function $\vartheta(t)$ that affects the temperature of the rod at the left end according to (2.3), satisfies condition (2.7), and minimizes the objective functional

$$J(\vartheta) = \int\limits_\Phi \int\limits_\Theta I(\vartheta; \varphi, \theta) \rho_\Phi(\varphi) d\theta d\varphi,$$
$$I(\vartheta; \varphi, \theta) = \int\limits_0^l \mu(x) \left(u\left(x, t_f; \vartheta, \varphi, \theta\right) - U(x)\right)^2 dx + \sigma \int\limits_0^{t_f} \vartheta^2(t) dt. \tag{2.8}$$

Here, $u(x, t) = u(x, t; \vartheta, \varphi, \theta)$ is a function continuously differentiable with respect to $t \in [0, t_f]$ and twice continuously differentiable with respect to $x \in (0, l)$ that represents the solution of the initial-boundary value problem (2.1)–(2.4) under arbitrarily given admissible control function $\vartheta(t)$, parameters φ of the initial function $\gamma(x; \varphi)$, and environment temperature θ; $U(x)$, $x \in (0, l)$, is a piecewise continuous function defining the desired temperature distribution on the rod at the end of the heating process; $\mu(x) \ge 0$, $x \in (0, l)$, is a given continuous weight function; finally, $\sigma > 0$ is the regularization parameter of the objective functional.

A specific feature of this setting is that the objective functional (2.8) assesses the control function $\vartheta(t)$ not by the behavior of a single phase trajectory $u(x, t)$ under any given admissible parameters φ of the initial function $\gamma(x; \varphi)$ and environment temperature θ, but in the whole for a pencil of phase trajectories. The pencil contains the phase trajectories obtained by solving the boundary value problem (2.1)–(2.5) under all admissible parameters φ and θ. Thus, in the optimal control problem under consideration, we seek for a boundary control function $\vartheta(t)$ minimizing the functional (2.8) on average over the entire set Φ of the parameters φ of the initial function $\gamma(x; \varphi)$ and the value set Θ of the environment temperature θ.

Let the heating process be controlled via a state feedback law. More precisely put, at some L points $\xi_i \in (0, l)$, $i = 1, \ldots, L$, the temperature values are measured, either in continuous time:

$$u_i(t) = u(\xi_i, t), \quad i = 1, \ldots, L, \quad t \in (0, t_f],$$

or at given discrete time instants $\bar{t}_j \in (0, t_f]$, $j = 1, \ldots, M$:

$$u_{ij} = u(\xi_i, \bar{t}_j), \quad i = 1, \ldots, L, \quad j = 1, \ldots, M. \tag{2.9}$$

The measured temperature values are employed to form the current value of the control function. In the case of continuous measurements, the following dependence is used:

$$\vartheta(t) = \sum_{i=1}^{L} [k_{1i} (u(\xi_i, t) - U_i) + k_{2i} (u(\xi_i, t - \tau) - U_i)], \quad t \in [0, t_f], \tag{2.10}$$
$$U_i = U(\xi_i), \quad i = 1, \ldots, L.$$

Here, $U_i = U(\xi_i)$, $i = 1, \ldots, L$, and $\tau > 0$ is a given value determined experimentally depending on the heating process dynamics. That is, under strong dynamics, the value τ should be small; under weak dynamics, the value τ should be chosen large enough so that the values $u(\xi_i, t)$ and $u(\xi_i, t - \tau)$ differ sufficiently.

The optimized constant coefficients k_{1i} and k_{2i}, $i = 1, \ldots, L$, will be called the feedback parameters.

In the case of the discrete feedback law (2.10), the control values $\vartheta(t)$ will be formed using the dependence

$$\vartheta(t) = \vartheta_{ij} = \sum_{i=1}^{L} [k_{1i}(u(\xi_i, \bar{t}_j) - U_i) + k_{2i}(u(\xi_i, \bar{t}_{j-1}) - U_i)], \quad t \in [\bar{t}_j, \bar{t}_{j+1}). \tag{2.11}$$

Clearly, the boundary control is, in general, a piecewise constant function with discontinuities at the discrete time instants of measurements.

Substituting the dependence (2.10) into boundary value condition (2.3) yields

$$\frac{\partial u(0,t)}{\partial x} = -\mu_2 \left(\sum_{i=1}^{L} [k_{1i}(u(\xi_i, t) - U_i) + k_{2i}(u(\xi_i, t - \tau) - U_i)] - u(0,t) \right), \tag{2.12}$$

where $t \in [0, t_f]$.

The new boundary value condition (2.12) has two features. First, it contains a time delay; second, it represents a nonlocal boundary value condition connecting the behavior of the unknown function on the boundary of the rod with its values inside the rod at the current and preceding time instants [20–23].

In the discrete feedback case, the boundary value condition (2.3) takes the form

$$\frac{\partial u(0,t)}{\partial x} = -\mu_2 \left(\sum_{i=1}^{L} [k_{1i} (u_{ij} - U_i) + k_{2i} (u_{ij-1} - U_i)] - u(0,t) \right), \quad t \in [t_{j-1}, t_j], \quad j = 1, \ldots, M.$$

Now we analyze the constraint (2.7) on the control function $\vartheta(t)$ taking the dependence (2.10) into account. Based on technological considerations, assume that the temperature at the points of the rod under all possible admissible heating modes, parameters of the initial conditions, and environment temperature values satisfies the condition

$$\underline{u} \leq u(x, t; \vartheta, \varphi, \theta) \leq \bar{u}, \quad \varphi \in \Phi, \quad \theta \in \Theta, \quad x \in (0, l), \quad t \in [0, t_f]. \tag{2.13}$$

Here, $\underline{u}$ and $\bar{u}$ are given values defining the range of possible temperature values at the points of the rod. In view of (2.7), from (2.10) we obtain the following conditions on the parameters of the continuous feedback law:

$$\underline{\vartheta} \leq \sum_{i=1}^{L} [k_{1i} (u(\xi_i, t) - U_i) + k_{2i} (u(\xi_i, t - \tau) - U_i)] \leq \bar{\vartheta}. \tag{2.14}$$

Let us choose τ sufficiently small so that for each $t \in [0, t_f]$, the values of $u_i(t) = u(\xi_i, t)$ and $u_i(t-\tau) = u(\xi_i, t-\tau)$ are close to each other and cannot, from practical considerations, lie at different end points of the interval $[\underline{u}, \bar{u}]$. Then from (2.14) we have

$$\underline{\vartheta} \le \sum_{i=1}^{L} [(k_{1i} + k_{2i})\, u_i(t) - (k_{1i} + k_{2i})\, U_i] \le \overline{\vartheta}, \quad t \in (0, t_f]. \tag{2.15}$$

In addition, temperature values at different measurement points can take values at different endpoints of the interval $[\underline{u}, \bar{u}]$. We introduce in the L-dimensional space a cube with $N = 2^L$ vertices at the points

$$T_1 = (\underline{u},\, \underline{u}, \ldots, \underline{u})^{\mathrm{T}} \in \mathbb{R}^L, \quad T_2 = (\underline{u},\, \underline{u}, \ldots, \bar{u})^{\mathrm{T}} \in \mathbb{R}^L, \ \ldots,$$
$$T_{N-1} = (\underline{u},\, \bar{u}, \ldots, \bar{u})^{\mathrm{T}} \in \mathbb{R}^L, \quad T_N = (\bar{u},\, \bar{u}, \ldots, \bar{u})^{\mathrm{T}} \in \mathbb{R}^L.$$

Note that inequalities (2.15) are linear with respect to the feedback coefficients. By denoting

$$U = (U_1, U_2, \ldots, U_L)^{\mathrm{T}}, \quad K_1 = (k_{11}, k_{12}, \ldots, k_{1L})^{\mathrm{T}}, \quad K_2 = (k_{21}, k_{22}, \ldots, k_{2L})^{\mathrm{T}},$$

from (2.15) we derive N linear constraints on the parameters K_1 and K_2 :

$$\underline{\vartheta} \le \langle K_1 + K_2, T_s \rangle - \langle K_1 + K_2, U \rangle \le \bar{\vartheta}, \quad s = 1, \ldots, N. \tag{2.16}$$

Here, $\langle a, b \rangle$ stands for the inner product of vectors a and b.

As is easily shown, the constraints (2.16) on the continuous feedback parameters will also be valid in the case of the discrete feedback law (2.11).

Let the optimized feedback coefficients and locations of measurement points be compiled into the vector

$$\mathbf{K} = (K_1,\ K_2) \in \mathbb{R}^{2L}.$$

For the feedback control problem, the objective functional (2.8) can be written as

$$J(\mathbf{K}) = \int_{\Phi} \int_{\Theta} I(\mathbf{K}; \varphi, \theta) \rho_{\Phi}(\varphi) d\theta d\varphi, \tag{2.17}$$

$$I(\mathbf{K}; \varphi, \theta) = \int_0^l \mu(x) \left(u\left(x, t_f; \mathbf{K}, \varphi, \theta\right) - U(x) \right)^2 dx + \sigma \left\| \mathbf{K} \right\|^2_{\mathbb{R}^{2L}} . \tag{2.18}$$

Thus, the original optimal control problem for the boundary value condition, with both continuous and discrete feedback, has been reduced to a parametric optimal control problem.

According to (2.10) and (2.11), the feedback control approach proposed here involves not only current but also previous information about the process state at measurement points. Owing to this peculiarity of the approach, the process dynamics can be considered during control. Obviously, the above dependencies (2.10) and (2.11) can be further complicated, e.g., by adding more previous measurements.

An important feature of the resulting parametric optimal control problem is the possible nonconvexity (multi-extremality) of the objective functional (2.17), (2.18). Indeed, the dependencies (2.10) and (2.11) incorporate the product of the feedback coefficients and the temperature values of the rod at measurement points, which implicitly depend on the former coefficients. Hence, the objective functional (2.17), (2.18) should be minimized using nonlocal (global) optimization methods. We applied the multistart technique with a local optimization method launched from different starting points.

The peculiarity of the boundary value condition (2.12), i.e., the presence of a time delay, is considered without particular difficulties by the well-known method of steps [23]. When numerically solving the boundary value problem by the grid method, the nonlocality of this condition can be taken into account using sweep schemes [24, 25].

The gradient projection method will be used to solve the parametric optimal control problem with the linear constraints (2.16) on the optimized parameters numerically. For this purpose, in the next section, we prove the differentiability of the objective functional and derive formulas for the components of its gradient. The formulas will serve to formulate necessary conditions for the local optimality of the optimized parameters [5, 19].

3. THE NUMERICAL SOLUTION APPROACH TO THE CONTROL DESIGN PROBLEM

For the numerical solution of the parametric optimal control problem (2.1), (2.2), (2.4), (2.12), (2.17) with the linear constraints (2.13)–(2.16) on the optimized parameters (consequently, a convex admissible domain $\mathbf{K}$ of the parameters), we propose the gradient projection method [5]:

$$\mathbf{K}^{s+1} = \Pr_{(2.14)} \Big(\mathbf{K}^s - \alpha_s \, grad_K \; J(\mathbf{K}^s)\Big), \tag{3.1}$$

$$\alpha_s = \arg\min_{\alpha \geq 0} J\left(\Pr_{(2.14)} \Big(\mathbf{K}^s - \alpha \, grad_K \; J(\mathbf{K}^s)\Big)\right), \quad s = 0, 1, \ldots . \tag{3.2}$$

Here, $grad_K J(\mathbf{K})$ is the $2L$-dimensional vector formed by the components of the gradient of the objective functional (2.17), (2.18); α_s is the step size of one-dimensional optimization along the antigradient of the functional at the sth iteration, determined, e.g., by the golden section method; finally, $\mathbf{K}^0$ is some initial value of the sought parameter vector. (If this value violates the constraints (2.12), (2.14), it must first be projected onto the admissible domain.) There are known constructive formulas [5, 26] to obtain the projection operator $\Pr_{(2.14)}(\mathbf{K})$ of an arbitrary vector $\mathbf{K} \in \mathbb{R}^{2L}$ onto the admissible domain defined by the linear inequalities (2.12), (2.16).

As noted above, the parametric optimal control problem may be multi-extremal with respect to the optimized parameters $\mathbf{K}$, and the method (3.1), (3.2) yields only the local minimum point of the objective functional nearest to the point $\mathbf{K}^0 \in \mathbb{R}^{2L}$. Therefore, the multistart technique was selected for computations: the iterative procedure (3.1), (3.2) was applied several times for different admissible initial points $\mathbf{K}^0$. The solution is the local minimum vector $\mathbf{K}^*$ corresponding to the smaller value of the objective functional.

Clearly, the gradient $grad_{\mathbf{K}} \; J(\mathbf{K})$ of the objective functional plays an important role in implementing the procedure (3.1), (3.2). The next result establishes the differentiability of the objective functional $J(\mathbf{K})$ and provides formulas for the components of its gradient.

Theorem 1. *Under the above assumptions on the functions and parameters of the optimal control problem* (2.1), (2.2), (2.12), (2.4), (2.17), (2.18) *with the feedback law* (2.10), *the objective functional* (2.17), (2.18) *is differentiable with respect to the admissible values of the parameters* $\mathbf{K}$, *and the components of its gradient are given by*

$$\frac{\partial J(\mathbf{K})}{\partial k_{1i}} = \int\limits_{\Phi} \int\limits_{\Theta} \left[\int\limits_0^{t_f} a^2 \mu_2 \psi(0,t) \left(u(\xi_i, t) - U_i\right) \, dt + 2\sigma \, k_{1i} \right] \rho_\Phi(\varphi) d\theta d\varphi, \tag{3.3}$$

$$\frac{\partial J(\mathbf{K})}{\partial k_{2i}} = \int\limits_{\Phi} \int\limits_{\Theta} \left[\int\limits_0^{t_f - \tau} a^2 \mu_2 \psi(0, t+\tau) \; \left(u(\xi_i, t+\tau) - U_i\right) dt + 2\sigma \, k_{2i} \right] \rho_\Phi(\varphi) d\theta d\varphi, \tag{3.4}$$

where $i = 1, \dots, L$ *and* $u(x,t) = u(x,t;\mathbf{K},\varphi,\theta)$ *is the solution of the initial-boundary value problem* (2.1), (2.2), (2.4), (2.14); *an almost everywhere twice differentiable, with respect to* $x \in (0,l)$ *and* $t \in [0,t_f]$, *function* $\psi(x,t;\mathbf{K},\varphi,\theta)$ *is the solution of the following adjoint initial-boundary value problem under given admissible values of the parameters* $\mathbf{K},\varphi,\theta$:

$$\frac{\partial\psi(x,t)}{\partial t} = -a^2\frac{\partial^2\psi(x,t)}{\partial x^2} - \mu_1\psi(x,t), \quad (x,t) \in \Omega = (0,l) \times (0,t_f], \tag{3.5}$$

$$\psi(x,t_f) = -2\mu(x)\left(u\left(x,t_f;\mathbf{K},\varphi,\theta\right) - U(x)\right), \quad 0 \le x \le l, \tag{3.6}$$

$$\psi_x(0,t) = \mu_2\psi(0,t), \quad 0 \le t \le t_f, \tag{3.7}$$

$$\psi_x(l,t) = -\mu_2\psi(l,t), \quad 0 \le t \le t_f, \tag{3.8}$$

and it satisfies the conditions

$$\begin{gathered}\frac{\partial\psi(\xi_i^-,t)}{\partial x} = \frac{\partial\psi(\xi_i^+,t)}{\partial x} + \frac{\partial\psi(\xi_i^+,t+\tau)}{\partial x} - \frac{\partial\psi(\xi_i^-,t+\tau)}{\partial x} \\ - k_{1i}\mu_2\psi(0,t) - k_{2i}\,\mu_2\psi(0,t+\tau), \quad 0 \le t \le t_f - \tau,\end{gathered} \tag{3.9}$$

$$\psi_x(\xi_i^-,t) = \psi_x(\xi_i^+,t) - k_{1i}\mu_2\psi(0,t), \quad t_f - \tau \le t \le t_f, \tag{3.10}$$

$$\psi(\xi_i^-,t) = \psi(\xi_i^+,t), \quad 0 \le t \le t_f, \tag{3.11}$$

at the points ξ_i, $i = 1,\dots,L$, *for* $t \in [0,t_f)$.

In the formulas presented, $\chi_{[0,t_f-\tau]}(t)$ *denotes the characteristic function:*

$$\chi_{[0,t_f-\tau]}(t) = \begin{cases} 0, & t \notin [0,t_f-\tau], \\ 1, & t \in [0,t_f-\tau]. \end{cases}$$

The proof of Theorem 1 is postponed to the Appendix.

In the case of the discrete-time feedback law (2.11), we have the following.

Theorem 2. *Under the above assumptions on the functions and parameters of problem* (2.1), (2.2), (2.14), (2.4), (2.17), (2.18), *the objective functional* (2.17), (2.18) *for the discrete feedback law* (2.11) *is differentiable with respect to the parameters* $\mathbf{K}$ *under their admissible values, and the components of the gradient of the objective functional are given by*

$$\frac{\partial J(\mathbf{K})}{\partial k_{1i}} = \int\limits_\Phi\int\limits_\Theta\left[\sum_{j=0}^{M-1}\int\limits_{\bar t_j}^{\bar t_j+\Delta t} a^2\mu_2\psi(0,t)\left(u(\xi_i,t) - U_i\right)dt + 2\sigma\,k_{1i}\right]\rho_\Phi(\varphi)d\theta d\varphi, \;\; \bar t_j \le t \le \bar t_j + \Delta t,$$

$$\frac{\partial J(\mathbf{K})}{\partial k_{2i}} = \int\limits_\Phi\int\limits_\Theta\left[\sum_{j=0}^{M-1}\int\limits_{\bar t_j}^{\bar t_j+\Delta t-\tau}\left(a^2\mu_2\psi(0,t+\tau)\left(u(\xi_i,t+\tau) - U_i\right)dt\right) + 2\sigma\,k_{2i}\right]\rho_\Phi(\varphi)d\theta d\varphi,$$

$$\bar t_j - \tau \le t \le \bar t_j + \Delta t - \tau,$$

where $i = 1,\dots,L$ *and, for given admissible values of the parameters* $\mathbf{K},\varphi,\theta$ *and the corresponding function* $u(x,t;\mathbf{K},\varphi,\theta)$ (*the solution of the initial-boundary value problem* (2.1), (2.2), (2.12), (2.4)), *the function* $\psi(x,t) = \psi(x,t;\mathbf{K},\varphi,\theta)$ *is the solution of the adjoint initial-boundary value problem*

$$\frac{\partial\psi(x,t)}{\partial t} = -a^2\frac{\partial^2\psi(x,t)}{\partial x^2} - \mu_1\psi(x,t), \quad \bar t_j \le t \le \bar t_j + \Delta t, \quad j = 0,\dots,M-1.$$

$$\psi(x,t_f) = -2\mu(x)\left(u\left(x,t_f;\mathbf{K},\varphi,\theta\right) - U(x)\right), \quad 0 \le x \le l,$$

$$\psi_x(0,t) = \mu_2\psi(0,t), \quad 0 \le t \le t_f, \quad \bar t_j \le t \le \bar t_j + \Delta t, \quad j = 0,\dots,M-1,$$

$$\psi_x(l,t) = -\mu_2\psi(l,t), \quad 0 \le t \le t_f, \quad \bar t_j \le t \le \bar t_j + \Delta t, \quad j = 0,\dots,M-1,$$

and it satisfies the conditions

$$\frac{\partial\psi(\xi_i^-,t)}{\partial x} = \frac{\partial\psi(\xi_i^+,t)}{\partial x} + \frac{\partial\psi(\xi_i^+,t+\tau)}{\partial x} - \frac{\partial\psi(\xi_i^-,t+\tau)}{\partial x}$$

$$-\sum_{j=1}^{M-1}\delta(t-t_j)\int\limits_{t_{j-1}}^{t_j}[k_{1i}\mu_2\psi(0,t)+k_{2i}\,\mu_2\psi(0,t+\tau)]dt,$$

$$\bar t_j \le t \le \bar t_j + \Delta t - \tau, \quad i=1,\ldots,L,$$

$$\frac{\partial\psi(\xi_i^-,t)}{\partial x} = \frac{\partial\psi(\xi_i^+,t)}{\partial x} - \sum_{j=1}^{M-1}\delta(t-t_j)\int\limits_{t_{j-1}}^{t_j}k_{1i}\mu_2\psi(0,t)dt,$$

$$\bar t_j + \Delta t - \tau \le t \le \bar t_j + \Delta t, \quad i=1,\ldots,L,$$

$$\psi(\xi_i^+,t) = \psi(\xi_i^-,t), \quad \bar t_j \le t \le \bar t_j + \Delta t, \quad i=1,\ldots,L,$$

at the points ξ_i, $i=1,\ldots,L$, *for* $\bar t_j \le t \le \bar t_j+\Delta t$ *and the conditions*

$$\psi(x,t_j^-) = \psi(x,t_j^+), \quad j=0,\ldots,M-1,$$

at the points $t_j, j=0,\ldots,M-1$ *for* $x\in[0,l]$.

Let us state necessary optimality conditions for the parameters $\mathbf{K}$ in variational form.

Theorem 3. *For the local optimality of the parameters* $\mathbf{K}^*\in\mathbb{R}^{2L}$ *in the control design problem* (2.1), (2.2), (2.4), (2.12), (2.17), (2.18) *with the continuous feedback law* (2.10), *it is necessary that*

$$(grad_{\mathbf{K}}J(\mathbf{K}), \mathbf{K}-\mathbf{K}^*) \ge 0$$

for an arbitrary admissible vector $\mathbf{K}\in\mathbb{R}^{2L}$ *satisfying conditions* (2.14), *where the gradient of the objective functional* (2.17), (2.18) *is given by formulas* (3.3), (3.4).

In the discrete feedback case, necessary optimality conditions for the parameters are formulated by analogy. The proofs of Theorems 2 and 3 can be found in the Appendix.

4. THE RESULTS OF COMPUTER EXPERIMENTS

In this section, we present the results of numerical experiments for the following functions and parameter values of the problem statement:

$$t_0=0, \quad t_f=2, \quad a=1, \quad \mu_1=0.1, \quad \mu(x)=0.1, \quad l=1,$$

$$\Phi=\{40,\,45,\,50\}, \quad \Theta=\{5,\,6,\,7\}, \quad \rho_\Phi(\varphi_i)=\rho_\Theta(\theta_i)=\frac{1}{3}, \quad i=1,2,3,$$

$$\underline{\vartheta}=10, \quad \bar\vartheta=750, \quad U(x)=70, \quad x\in[0,1].$$

The direct and adjoint initial-boundary value problems were numerically solved using an implicit scheme of the grid method [19]. The experiments were carried out for different values of the grid step. The experimental results below correspond to the steps $h_x=0.01$ and $h_t=0.05$.

The nonlocality in the left boundary value condition (2.12) was taken into account using an approach based on the boundary condition transfer scheme [20–22]. To consider the time delay in the boundary value condition, the method of steps with a step size of τ was applied [23].

Table 1. The components of the normalized gradients of the functional computed by formulas (3.5)–(3.7) and (4.1) for the parameter values given in Table 2

N	Formulas	$\nabla^{norma}_{k_{11}} J$	$\nabla^{norma}_{k_{12}} J$	$\nabla^{norma}_{k_{21}} J$	$\nabla^{norma}_{k_{22}} J$
1	(3.3), (3.4)	0.286391	0.437807	0.354151	0.775162
	(4.1)	0.286693	0.437915	0.354402	0.774879
2	(3.3), (3.4)	0.008363	−0.55936	−0.698830	−0.445737
	(4.1)	0.012542	−0.55928	−0.699544	−0.444614
3	(3.3), (3.4)	0.322336	−0.585921	0.276813	0.823887
	(4.1)	0.322786	−0.586143	0.278189	0.824126

Table 2. The initial and resulting values of the optimized parameters and the corresponding values of the functional for $L = 2$

N	$(k^0_{11};\, k^0_{12});\quad (k^0_{21}; k^0_{22})$	$J(\mathbf{K})$	$(k^*_{11};\, k^*_{12});\quad (k^*_{21};\, k^*_{22})$	$J(\mathbf{K}^*)$
1	(2.00; 5.00); (4.00; 7.00)	2132.76	(1.497; 2.499); (3.496; 3.001)	0.00043142
2	(0.20; 0.70); (0.40; 0.90)	2714.32	(1.399; 2.600); (3.194; 2.800)	0.00056845
3	(2.50; 1.50); (0.40; 3.00)	2478.19	(1.502; 2.4563); (3.3212; 2.925)	0.00048367

The feedback control problem was solved for two different initial values of the optimized parameters.

For each set of the initial values of the optimized parameters, Table 1 provides the components of the normalized gradients computed by formulas (3.3), (3.4) and using the central finite-difference approximation of the derivative of the functional:

$$\partial J(\mathbf{K})/\partial \mathbf{K}_j \approx (J(\mathbf{K} + \varepsilon e_j) - J(\mathbf{K} - \varepsilon e_j))/2\varepsilon, \qquad (4.1)$$

where $\mathbf{K}_j$ is the jth component of the n-dimensional optimized vector $\mathbf{K}$ (the aggregate of the optimized parameters k_{11}, k_{12}, k_{21}, and k_{22}); and e_j is the n-dimensional vector consisting of zeros, except for the jth component equal to one. The value of ε in the experiments was varied, and the table contains the most acceptable results.

In Table 2, the reader can find the initial and optimal values of the parameters $\mathbf{K}$ for the feedback control law (2.10) with $L = 2$ measurement points.

Numerical experiments were carried out in which the exact values $u(\xi_1, t)$ and $u(\xi_2, t)$ of the process states observed at the $L = 2$ measurement points were corrupted by random noise:

$$u(t; \mathbf{K}^*) = u(\xi_i, t; \mathbf{K}^*) = u(\xi_i, t; \mathbf{K}^*)(1 + \chi(2\sigma_i - 1)), \quad i = 1, 2,$$

where $u(\xi_i, t; \mathbf{K}^*)$ is the solution of the direct initial-boundary value problem computed at the point $x = \xi_i$ for $t \in [0, t_f]$, σ_i is a random variable with the uniform distribution on the interval $[0, 1]$, and χ is the noise level.

Figure 1 shows the graphs of the functions representing, along the rod, the deviations $\Delta u(x, t_f; \mathbf{K}^*, \chi) = u(x, t_f; \mathbf{K}^*, \chi) - U(x)$ of the resulting temperatures from the desired one under different noise levels: 0% (no noise), 1%, 3%, and 5%, corresponding to χ equal to 0, 0.01, 0.03, and 0.05.

Also, the control design approach proposed above was experimentally compared with the solution of the feedback control design problem without using previous measurements in the form [16]:

$$\vartheta(t; \mathbf{K}) = \sum_{i=1}^{L} k_{1j}(u(\xi_i, t) - U_i), \quad t \in [t_0, t_f]. \qquad (4.2)$$

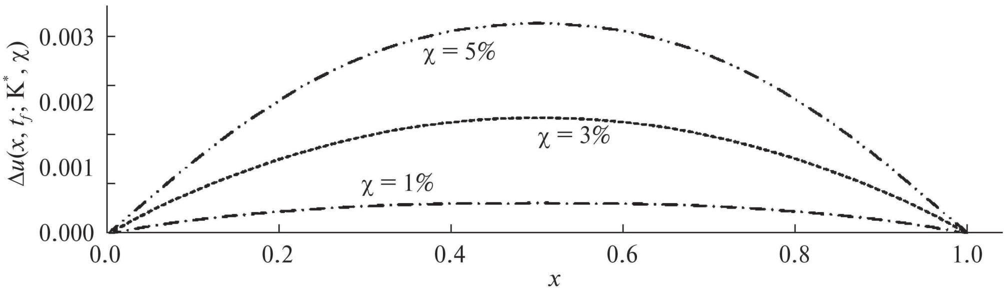

Fig. 1. The deviations of the temperature distribution functions along the rod from the desired one $U(x)$ under different noise levels χ.

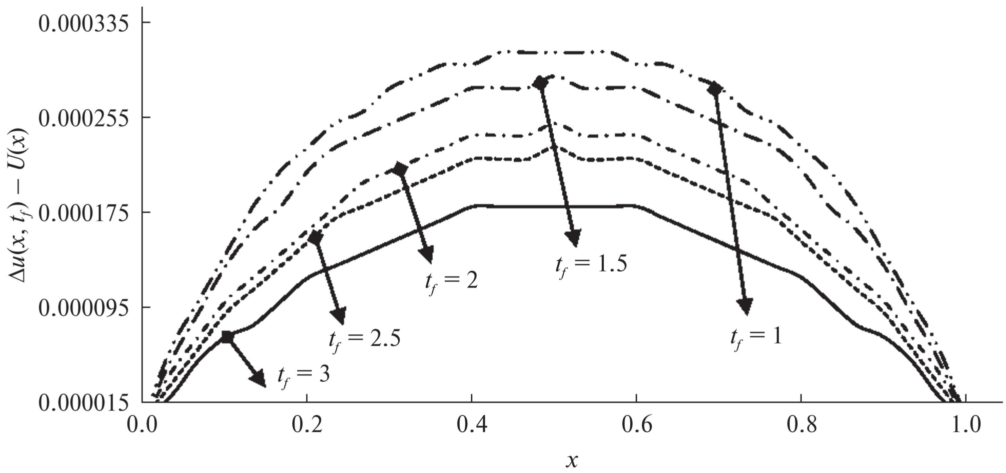

Fig. 2. Temperature distribution along the rod for different $t \in [1, 3]$. The ordinate axis has enlarged scale.

Clearly, the phase state at previous time instants is not used in (4.2) to form the current value of the control function $\vartheta(t)$.

Table 3 presents the numerical solution of the boundary control design problem using the feedback law (4.2). The experiments were carried out for three different initial values of the optimized parameters.

Table 3. The initial and resulting values of the feedback parameters (4.2) and the objective functional

N	$(k_{11}^0;\ k_{12}^0)$	$J(\mathbf{K^0})$	$(k_{11}^*;\ k_{12}^*)$	$J(\mathbf{K}^*)$
1	(2.00; 5.00)	2012.37	(1.2412; 4.4215)	0.0866534
2	(0.20; 0.70)	1987.13	(1.6512; 3.1265)	0.0872112
3	(2.50; 1.50)	2179.33	(2.2451; 1.1532)	0.0858423

Figure 2 demonstrates the graphs of the temperature deviations from the desired one for the obtained parameters and $t > t_f = 1$. According to the graphs, the heating process continues to settle, despite that the design problem was solved on the interval $[0, 1]$, i.e., $t_f = 1$.

5. CONCLUSIONS

In this paper, the optimal boundary control design problem for a rod heating process has been investigated. The problem statement includes a linear dependence of the boundary control value on the rod temperature values at measurement points, both at the current and previous time instants. The parameters optimized in this problem are the constant feedback parameters representing the coefficients in the dependence of the control value on the temperature values at the measurement points.

Determining the values of the feedback parameters has reduced the original problem to a parametric optimal control problem with distributed parameters. First-order numerical optimization methods have been applied to solve it. Computer experiments have been carried out.

The boundary control design approach and analysis method proposed in this paper can be extended to other types of PDEs with other initial-boundary value conditions.

APPENDIX

Proof of Theorem 1. Due to the mutual independence of all possible values of the parameters of the initial conditions $\gamma(x;\varphi)$ under different parameters φ and external influences θ, we have

$$grad_K\, J(\mathbf{K}) = grad_{\mathbf{K}} \int\limits_{\Phi}\int\limits_{\Theta} I(\mathbf{K};\varphi,\theta)\rho_{\Phi}(\varphi)d\theta d\varphi = \int\limits_{\Phi}\int\limits_{\Theta} grad_{\mathbf{K}} I(\mathbf{K};\varphi,\theta)\rho_{\Phi}(\varphi)d\theta d\varphi. \quad \text{(A.1)}$$

Therefore, let us study the differentiability of the functional $I(\mathbf{K};\varphi,\theta)$ and derive formulas for the components of its gradient with respect to the optimized parameters $\mathbf{K}$ under any given admissible parameters $\varphi\in\Phi$ and $\theta\in\Theta$.

To prove this theorem, we utilize the well-known method of incrementing the optimized parameters and estimate the corresponding increment of the functional [5, 27].

Let the parameter vector $\mathbf{K}=(k_1,\, k_2)$, associated with the solution $u(x,t)=u(x,t,\mathbf{K})$ of the boundary value problem (2.1), (2.2), (2.4), be incremented to $\tilde{\mathbf{K}}=\mathbf{K}+\Delta\mathbf{K}=(k_1+\Delta k_1,\ k_2+\Delta k_2)$, and the latter vector is associated with the solution $\tilde{u}(x,t)=\tilde{u}(x,t,\tilde{\mathbf{K}})=u(x,t)+\Delta u(x,t)$ of the boundary value problem.

From (2.1), (2.2), (2.4), and (2.12) it follows that $\Delta u(x,t)$ is the solution of the boundary value problem

$$\frac{\partial \Delta u(x,t)}{\partial t} = a^2\frac{\partial^2 \Delta u(x,t)}{\partial x^2} - \mu_1 \Delta u(x,t), \quad (x,t)\in\Omega=(0,l)\times(0,t_f], \quad \text{(A.2)}$$

$$\Delta u(x,0)=0, \quad x\in[0,l], \quad \text{(A.3)}$$

$$\begin{aligned}\left.\frac{\partial \Delta u(x,t)}{\partial x}\right|_{x=0} &= \mu_2\Delta u(0,t) - \sum_{i=1}^{L}\left[k_{1i}\Delta u(\xi_i,t)+k_{2i}\Delta u(\xi_i,t-\tau)\right] \\ &+\sum_{i=1}^{L}\left[\left(u(\xi_i,t)-U_i\right)\Delta k_{1i} + \left(u(\xi_i,t-\tau)-U_i\right)\Delta k_{2i}\right], \quad t\in[0,t_f]\,,\end{aligned} \quad \text{(A.4)}$$

$$\left.\frac{\partial \Delta u(x,t)}{\partial x}\right|_{x=l} = -\mu_2\Delta u(l,t), \quad t\in[0,t_f]\,. \quad \text{(A.5)}$$

The increment of the functional (2.17) can be, in a straightforward way, represented as

$$\begin{aligned}\Delta I(\mathbf{K};\varphi,\theta) &= I(\mathbf{K}+\Delta\mathbf{K};\varphi,\theta) - I(\mathbf{K};\varphi,\theta) \quad \text{(A.6)} \\ &= 2\int\limits_0^l \mu(x)\left[u\left(x,t_f;\mathbf{K},\varphi,\theta\right)-U(x)\right]\,\Delta u\left(x,t_f\right)dx \\ &+ 2\sigma\sum_{i=1}^{L}\left[k_{1i}\Delta k_{1i}+k_{2i}\Delta k_{2i}\right] + R\left(\|\Delta u\|_{L^2(\Omega)}\,,\ \|\Delta\mathbf{K}\|_{\mathbb{R}^{2L}}\right),\end{aligned}$$

where the residual $R\left(\|\Delta u\|_{L^2(\Omega)}\,,\ \|\Delta\mathbf{K}\|_{\mathbb{R}^{2L}}\right)$ includes all terms of the second order of smallness with respect to $\|\Delta\mathbf{K}\|_{\mathbb{R}^{2L}}$ and $\|\Delta u\|_{L^2(\Omega)}$. Some estimates of the increment of the solution of an initial-boundary value problem for a parabolic equation depending on the variations of the problem parameters were obtained in [5, 27]. According to these estimates,

$$\|\Delta u\|_{L^2(\Omega)} \leq \eta\ \|\Delta\mathbf{K}\|_{\mathbb{R}^{2L}}\,,$$

where the value of $\eta > 0$ is independent of the parameters $\mathbf{K}$. We obtain formulas for the components of the gradient of the objective functional with respect to the parameters $\mathbf{K}$. Let $\psi(x,t)$ be some, for now arbitrary, function continuous everywhere in Ω, twice differentiable with respect to $x \in (\xi_i, \xi_{i+1})$, $i = 0,\ldots,L$, $\xi_0 = 0$, $\xi_{L+1} = l$, and differentiable with respect to $t \in (0, t_f)$. We multiply (A.2) by $\psi(x,t)$ and integrate the result over the rectangle Ω. Under the above assumptions and conditions (A.4)–(A.5), the result is

$$\int_0^{t_f}\int_0^{l} \psi(x,t)\,\frac{\partial \Delta u(x,t)}{\partial t}\,dxdt - a^2\sum_{i=0}^{L}\int_{\xi_i}^{\xi_{i+1}}\int_0^{t_f}\psi(x,t)\,\frac{\partial^2\Delta u(x,t)}{\partial x^2}\,dtdx \tag{A.7}$$

$$-\,\mu_1\int_0^{t_f}\int_0^{l}\psi(x,t)\Delta u(x,t)\,dxdt = 0.$$

Let us split the interval $[0,l]$ into the subintervals $[\xi_i,\xi_{i+1}]$, $i = 0,\ldots,L$, and perform integration by parts separately for the first and second terms of (3.5). In view of (A.3)–(A.5), we have

$$\int_0^{t_f}\int_0^{l}\psi(x,t)\,\frac{\partial\Delta u(x,t)}{\partial t}\,dxdt = \int_0^{l}\psi(x,t_f)\Delta u(x,t_f)\,dx - \int_0^{t_f}\int_0^{l}\frac{\partial\psi(x,t)}{\partial t}\,\Delta u(x,t)\,dxdt, \tag{A.8}$$

$$a^2\sum_{i=0}^{L}\int_{\xi_i}^{\xi_{i+1}}\int_0^{t_f}\psi(x,t)\,\frac{\partial^2\Delta u(x,t)}{\partial x^2}\,dtdx \tag{A.9}$$

$$= a^2\int_0^{t_f}\left[\left(\frac{\partial\psi(0,t)}{\partial x} - \mu_2\psi(0,t)\right)\Delta u(0,t) - \left(\frac{\partial\psi(l,t)}{\partial x} + \mu_2\psi(l,t)\right)\Delta u(l,t)\right]dt$$

$$+\,a^2\sum_{i=1}^{L}\int_0^{t_f}\left[\frac{\partial\psi(\xi_i^-,t)}{\partial x} - \frac{\partial\psi(\xi_i^+,t)}{\partial x} + k_{1i}\mu_2\psi(0,t)\right]\Delta u(\xi_i,t)\,dt$$

$$+\,a^2\sum_{i=1}^{L}\int_0^{t_f-\tau}\left[\frac{\partial\psi(\xi_i^-,t+\tau)}{\partial x} - \frac{\partial\psi(\xi_i^+,t+\tau)}{\partial x} + k_{2i}\,\mu_2\psi(0,t+\tau)\right]\Delta u(\xi_i,t)\,dt$$

$$+\,a^2\sum_{i=1}^{L}\int_0^{t_f}\left[\left(\psi(\xi_i^+,t) - \psi(\xi_i^-,t)\right)\frac{\partial\Delta u(\xi_i,t)}{\partial x}\right]dt - a^2\sum_{i=1}^{L}\Delta k_{1i}\int_0^{t_f}\mu_2\psi(0,t)\left(u(\xi_i,t) - U_i\right)dt$$

$$-\,a^2\sum_{i=1}^{L}\Delta k_{2i}\int_0^{t_f-\tau}\mu_2\psi(0,t+\tau)\left(u(\xi_i,t+\tau) - U_i\right)dt + a^2\int_0^{t_f}\int_0^{l}\frac{\partial^2\psi(x,t)}{\partial x^2}\,\Delta u(x,t)\,dxdt.$$

Based on (A.6)–(A.9), the increment of the functional takes the form

$$\Delta I(\mathbf{K};\varphi,\theta) = \int_0^l \Big[\psi(x,t_f) + 2\mu(x)(u(x,t_f;\mathbf{K},\varphi,\theta) - U(x))\Big]\Delta u(x,t_f)\,dx \tag{A.10}$$

$$- a^2 \int_0^{t_f} \Big[\Big(\frac{\partial\psi(0,t)}{\partial x} - \mu_2\psi(0,t)\Big)\Delta u(0,t) - \Big(\frac{\partial\psi(l,t)}{\partial x} + \mu_2\psi(l,t)\Big)\Delta u(l,t)\Big]\,dt$$

$$+ \int_0^{t_f}\int_0^l \Big[-\frac{\partial\psi(x,t)}{\partial t} - a^2\frac{\partial^2\psi(x,t)}{\partial x^2} - \mu_1\psi(x,t)\Big]\Delta u(x,t)\,dxdt$$

$$+ a^2\sum_{i=1}^{L}\int_0^{t_f-\tau}\Big[\frac{\partial\psi(\xi_i^-,t)}{\partial x} - \frac{\partial\psi(\xi_i^+,t)}{\partial x} + \frac{\partial\psi(\xi_i^-,t+\tau)}{\partial x}$$

$$-\frac{\partial\psi(\xi_i^+,t+\tau)}{\partial x} + k_{1i}\mu_2\psi(0,t) + k_{2i}\,\mu_2\psi(0,t+\tau)\Big]\Delta u(\xi_i,t)\,dt$$

$$+ a^2\sum_{i=1}^{L}\int_{t_f-\tau}^{t_f}\Big[\frac{\partial\psi(\xi_i^-,t)}{\partial x} - \frac{\partial\psi(\xi_i^+,t)}{\partial x} + k_{1i}\mu_2\psi(0,t)\Big]\Delta u(\xi_i,t)\,dt$$

$$-\sum_{i=1}^{L}\Bigg\{a^2\int_0^{t_f}\mu_2\psi(0,t)\ (u(\xi_i,t) - U_i)\,dt + 2\sigma k_{1i}\Bigg\}\Delta k_{1i}$$

$$-\sum_{i=1}^{L}\Bigg\{a^2\int_0^{t_f-\tau}\psi(0,t+\tau)\ (u(\xi_i,t+\tau) - U_i)\ dt + 2\sigma\, k_{2i}\Bigg\}\Delta k_{2i}$$

$$+ R\left(\|\Delta u\|,\ \|\Delta\mathbf{K}\|_{\mathbb{R}^{2L}}\right).$$

Due to the arbitrariness of the function $\psi(x,t)$, we require it to be almost everywhere the solution of the initial-boundary value problem (3.5)–(3.11).

Note that the components of the gradient of the functional are determined by the linear part of the increment of the functional at the increments of the corresponding arguments. Therefore,

$$\frac{\partial I(\mathbf{K};\varphi,\theta)}{\partial k_{1i}} = \int_0^{t_f}\Big(a^2\mu_2\psi(0,t)(u(\xi_i,t) - U_i)\Big)\ dt + 2\sigma k_{1i},\ \ i = 1,\dots,L,\quad 0 \le t \le t_f,$$

$$\frac{\partial I(\mathbf{K};\varphi,\theta)}{\partial k_{2i}} = \int_0^{t_f-\tau}\Big(a^2\mu_2\psi(0,t+\tau)\,(u(\xi_i,t+\tau) - U_i)\Big)\,dt + 2\sigma\, k_{2i},\ \ i = 1,\dots,L,\quad 0 \le t \le t_f - \tau.$$

By the expression (A.1), we finally arrive at the desired formulas of Theorem 2. The proof of Theorem 1 is complete.

Theorem 2 is proved similarly to Theorem 1. The main difference is related to formulas (2.11) and (2.12), which necessitate replacing integration over t on the entire interval $[0,t_f]$ with a sum of integrals over the intervals $[t_{j-1},t_j]$, $j = 1,\dots,M$, $t_0 = 0$, $\bar t_M = t_f$.

Theorem 3 follows from the well-known theorem on integral necessary conditions for optimality in optimization problems [5, 27].

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