ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2026, Vol. 87, No. 1, pp. 18–28.

On an Algebro-Geometric Approach to the Study of Stability with Respect to a Part of the Variables of Linear Systems with Constant Coefficients

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Received March 11, 2025
Revised July 8, 2025
Accepted July 23, 2025

Abstract—An algebraic-geometric approach to the study of the stability of linear systems with constant coefficients with respect to some of the variables is proposed. The proposed approach establishes a relationship between the conditions of partial stability of a linear system and the existence of invariant subspaces in the original and dual spaces. An example is given to illustrate the results obtained.

Keywords: partial stability, minimal annihilating polynomial, invariant subspace, annihilator

DOI: 10.7868/S1608303226010023

1. INTRODUCTION

As noted in [1], interest in the problem of stability of linear systems with respect to some variables arose at the beginning of the 20th century. By now, the theory of stability of linear systems with constant coefficients is almost complete. The main stages of development of this theory are reflected in [1–8]. In the series of works [9–11] on achieving coordinate consensus in multi-agent systems, it was established that the problem being solved, with a special choice of control action and a linear transformation of the phase space, is reduced to a problem of partial stability of a linear system with constant coefficients. In [12] the problem of partial construction of solutions was considered, which, as noted by the authors, is “consonant” with the stability problems [1, 2]. The results of these studies allow us to obtain necessary and sufficient stability conditions for selected components of the phase vector of an arbitrary system of linear differential equations with constant coefficients. This algebraic approach was further developed in the works [13, 14].

Analyzing the obtained results, it can be noted that the main approaches to the study of partial stability of linear systems with constant coefficients are the Lyapunov function method, the auxiliary μ-systems method and methods based on differential and computer algebra methods. A stability criterion for linear systems was obtained in [2] and formulated in terms of μ-systems. Another stability criterion in [15] relies on the Jordan normal form of the linear operator defined by the system matrix in the original basis space. In [16], the geometric nature of partial stability is noted. A geometric approach to the study of partial stability is proposed in [17, 18].

This article is devoted to the study of stability of linear systems with constant coefficients. Using the algebraic-geometric properties of the original system, necessary and sufficient conditions for the stability of linear systems are obtained, expressed in terms of dual linear spaces on which dual linear operators act. It turns out that if, under certain conditions, some parameters of the system under study lie in the invariant subspace of one of the linear operators, then the problem of partial stability is broken down into two simpler problems. The obtained results establish a connection between the stability of the system and the internal geometry in the space of parameters that determine the dynamics of the system under study. The geometric properties of the system make it possible to obtain conditions under which the property of invariance with respect to the property of partial stability arises.

2. PROBLEM STATEMENT

The stability with respect to a given part of the components of the phase vector of the system is investigated

$$\frac{dx(t)}{dt} = A_* x(t), \tag{1}$$

with the following notations: x is the n-dimensional phase vector, A_* is a constant matrix with elements from the numerical field $\mathbb{F}$, $t \in \mathbb{R}$, $t \geq 0$.

Let's assume that we are investigating the stability of the first m components of the phase vector x of the system (1). We denote the first group of coordinates of the phase vector by the vector y, and the remaining components will form the vector z. In this regard, we represent the system (1) in the form

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + Bz(t), \\ \frac{dz(t)}{dt} &= Cy(t) + Dz(t), \end{aligned} \tag{2}$$

where $y \in \mathbb{F}^m,\ z \in \mathbb{F}^p,\ A \in \mathbb{F}^{m\times m},\ B \in \mathbb{F}^{m\times p},\ C \in \mathbb{F}^{p\times m},\ D \in \mathbb{F}^{p\times p},\ n = m + p,\ p > 0.$

It is required to determine the conditions on the coefficients of the system under study so that the solution $x(t) \equiv 0$ of the system (1), (2) is stable with respect to the components included in the vector y, i.e. y-stable.

3. SOME INFORMATION FROM LINEAR ALGEBRA

3.1. Conjugate Space. Dual Spaces. Annihilators

Let $\mathcal{V}$ be an arbitrary linear space over a field $\mathbb{F}$.

Definition 1 [19, p. 33]. A function $\xi : \mathcal{V} \to \mathbb{F}$ is called a linear functional if it is a homomorphism of linear spaces, i.e., if

$$\xi(\nu_1 + \nu_2) = \xi(\nu_1) + \xi(\nu_2)$$

and

$$\xi(k\nu) = k\xi(\nu)$$

$\forall \nu, \nu_1, \nu_2 \in \mathcal{V}$, $\forall k \in \mathbb{F}$. Linear functionals are also called covectors of the space $\mathcal{V}$.

The set of all linear functionals is a subspace of the space of all functions on $\mathcal{V}$ and is denoted by $\mathcal{V}'$.

Definition 2 [19, p. 33]. The linear space $\mathcal{V}'$ is called the dual space of $\mathcal{V}$.

A basis $e^1, \ldots, e^n$ of $\mathcal{V}'$ that satisfies the relation

$$e^j(e_i) = \delta_i^j, \quad i = 1, \ldots, n, \quad j = 1, \ldots, n,$$

is called the adjoint or dual basis of $e_1, \ldots, e_n$ of $\mathcal{V}$, where δ_i^j is the Kronecker delta.

Let $\mathcal{V}$ and $\mathcal{W}$ be two linear spaces over a field $\mathbb{F}$. Suppose that any two vectors $\nu \in \mathcal{V}$, $w \in \mathcal{W}$ are assigned a number $\nu \mapsto \langle \nu, w\rangle \in \mathbb{F}$ such that the following conditions are satisfied:

(1) For each fixed $w \in \mathcal{W}$, the function $\nu \mapsto \langle \nu, w\rangle$ is a linear functional on $\mathcal{V}$;
(2) For each fixed $\nu \in \mathcal{V}$, the function $w \mapsto \langle \nu, w\rangle$ is a linear functional on $\mathcal{W}$;
(3) For every vector $\nu \in \mathcal{V}$, there exists a vector $w \in \mathcal{W}$ such that $\langle \nu, w\rangle \neq 0$, and conversely, for every vector $w \in \mathcal{W}$, there exists a vector $\nu \in \mathcal{V}$ such that $\langle \nu, w\rangle \neq 0$.

Definition 3 [19, p. 36]. A function $\nu, w \mapsto \langle \nu, w\rangle$ satisfying conditions (a), (b), and (c) is called a pairing between the spaces $\mathcal{V}$ and $\mathcal{W}$. Spaces $\mathcal{V}$ and $\mathcal{W}$ for which at least one pairing exists are called dual. Notation: $\mathcal{V}|\mathcal{W}$.

Proposition 1 [19, p. 36]. *The linear space $\mathcal{V}$ is dual to the dual space $\mathcal{V}'$:*

$$\mathcal{V}|\mathcal{V}'.$$

Setting for any $\nu \in \mathcal{V}$ and $\varphi \in \mathcal{V}'$

$$\langle \nu, \varphi\rangle := \varphi(\nu),$$

we see that the indicated relation defines a pairing of the spaces $\mathcal{V}$ and $\mathcal{V}'$.

Let $S \subset \mathcal{V}$ be an arbitrary subset of the linear space $\mathcal{V}$.

Definition 4 [19, p. 40]. The set of all linear functionals $\xi \in \mathcal{V}'$ that vanish on any vector $\nu \in S$ is called the annihilator of the set S and is denoted by the symbol $\mathrm{Ann}(S)$.

Theorem 1 [20, p. 41]. *If $\mathcal{M}$ is an m-dimensional subspace of an n-dimensional vector space $\mathcal{V}$, then $\mathrm{Ann}(\mathcal{M})$ is an $(n-m)$-dimensional subspace of $\mathcal{V}'$.*

Theorem 2 [20, p. 42]. *If $\mathcal{M}$ is a subspace of a finite-dimensional vector space $\mathcal{V}$, then*

$$\mathrm{Ann}\left(\mathrm{Ann}(\mathcal{M})\right) = \mathcal{M}.$$

Let us list the main properties of annihilators:

1) If $\mathcal{I}$ and $\mathcal{J}$ are subsets of a vector space and $\mathcal{I} \subset \mathcal{J}$, then $\mathrm{Ann}(\mathcal{J}) \subset \mathrm{Ann}(\mathcal{I})$.

2) If $\mathcal{M}$ and $\mathcal{N}-$ are subspaces of a finite-dimensional vector space, then

$$\mathrm{Ann}(\mathcal{M} \cap \mathcal{N}) = \mathrm{Ann}(\mathcal{M}) + \mathrm{Ann}(\mathcal{N}),\ \mathrm{Ann}(\mathcal{M} + \mathcal{N}) = \mathrm{Ann}(\mathcal{M}) \cap \mathrm{Ann}(\mathcal{N}).$$

3) $\dim \mathcal{M} + \dim \mathrm{Ann}(\mathcal{M}) = \dim \mathcal{V}$.

3.2. Cyclic Invariant Subspaces

Consider an n-dimensional vector space $\mathcal{V}$ over some field $\mathbb{F}$ and a linear operator $\mathcal{A}$ on this space.

Definition 5 [21, p. 68]. A polynomial $f(t)$ is said to annihilate a linear operator $\mathcal{A}$ if $f(\mathcal{A}) = \mathcal{O}$. A normalized polynomial of minimal degree that annihilates $\mathcal{A}$ is called a minimal polynomial of $\mathcal{A}$.

Let us denote the minimal polynomial of the operator $\mathcal{A}$ as

$$\mu_{\mathcal{A}}(t) = t^m + \mu_m t^{m-1} + \cdots + \mu_2 t + \mu_1.$$

Let $x \in \mathcal{V}$ be a vector. Let $f(t)$ be a polynomial satisfying the relation

$$f(\mathcal{A})x = 0,$$

is called the annihilating polynomial of vector x, and the polynomial of the least degree that annihilates vector x is called the minimal annihilating polynomial of vector x.

If $f(t) = t^k + \alpha_k t^{k-1} + \cdots + \alpha_2 t + \alpha_1 -$ is the minimal annihilating polynomial of the vector x, then the linear subspace with basis

$$x, \mathcal{A}x, \ldots, \mathcal{A}^{k-1}x$$

is called a cyclic subspace with respect to the linear operator $\mathcal{A}$.

4. MAIN RESULTS

Let us fix some numerical field $\mathbb{F}$. We will investigate the partial stability of the system (1), (2) over the chosen field.

It is obvious that the y-stability property of the system (2) is invariant with respect to z-transformations of the form $z = S\overline{z}$, where $S \in \mathbb{F}^{p\times p}$ is a constant non-singular matrix, i.e. this property does not depend on the choice of basis in the space $\mathbb{F}^p$. Using the algebraic-geometric properties of the system under study, we will try to choose a basis in the phase subspace of variables $z_1, \ldots, z_p$ so as to obtain the desired stability conditions.

By carrying out a non-degenerate replacement of z-variables, we obtain a system of differential equations that is equivalent with respect to the property of partial stability:

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + BS\overline{z}(t), \\ \frac{d\overline{z}(t)}{dt} &= S^{-1}Cy(t) + S^{-1}DS\overline{z}(t). \end{aligned} \tag{3}$$

In connection with the above, we consider the linear operator $\mathcal{D} : \mathbb{F}^p \to \mathbb{F}^p$ and the corresponding adjoint operator $\mathcal{D}^*$, acting in the adjoint space $\mathbb{F}^{p*}$ of linear functionals on the space $\mathbb{F}^p$. In this case, the following relation holds:

$$\langle \mathcal{D}s, \psi\rangle = \langle s, \mathcal{D}^*\psi\rangle = \psi\left(\mathcal{D}s\right) \ \forall s \in \mathbb{F}^p, \forall \psi \in \mathbb{F}^{p*}.$$

Let us introduce into consideration the linear functionals $b_i^* = b_i(s) \in \mathbb{F}^{p*}$, $i = 1, \ldots, m$, and construct invariant cyclic subspaces with respect to the linear operator $\mathcal{D}^*$ with generators b_i^*, $(i = 1, \ldots, m)$:

$$\begin{aligned} U_1^* &= \operatorname{span}\left(b_1^*, \mathcal{D}^* b_1^*, \ldots, \mathcal{D}^{*k_1-1} b_1^*\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ U_i^* &= \operatorname{span}\left(b_i^*, \mathcal{D}^* b_i^*, \ldots, \mathcal{D}^{*k_i-1} b_i^*\right), \\ &\cdots\cdots\cdots\cdots\cdots\cdots \\ U_m^* &= \operatorname{span}\left(b_m^*, \mathcal{D}^* b_m^*, \ldots, \mathcal{D}^{*k_m-1} b_m^*\right). \end{aligned}$$

Thus, the following relationships hold:

$$\mathcal{D}^* U_i^* \subseteq U_i^*, \quad i = 1, \ldots, m.$$

Proposition 2. *The linear subspaces $W_i^* = U_1^* + \cdots + U_i^*$, $i = 1, \ldots, m$, are invariant subspaces of the linear operator $\mathcal{D}^*$.*

Thus,

$$\mathcal{D}^*\left(U_1^* + \cdots + U_i^*\right) \subseteq U_1^* + \cdots + U_i^*, \ \forall i \in \{1, \ldots, m\}.$$

Obviously, there is a sequence of inclusions of subspaces:

$$\{0\} \subseteq U_1^* \subseteq U_1^* + U_2^* \subseteq \cdots \subseteq U_1^* + U_2^* + \cdots + U_m^* \subseteq \mathbb{F}^{p*}.$$

But taking into account invariance, the following chain of inclusions will also be valid:

$$\mathcal{D}^*\{0\} \subseteq \mathcal{D}^* U_1^* \subseteq \mathcal{D}^* \left(U_1^* + U_2^*\right) \subseteq \cdots \subseteq \mathcal{D}^* \left(U_1^* + U_2^* + \cdots + U_m^*\right) \subseteq \mathcal{D}^* \mathbb{F}^{p*}.$$

Using the properties of annihilators, we have:

$$\operatorname{Ann}(\{0\}) \supseteq \operatorname{Ann}(U_1^*) \supseteq \operatorname{Ann}(U_1^* + U_2^*) \supseteq \cdots \supseteq \operatorname{Ann}(U_1^* + U_2^* + \cdots + U_m^*) \supseteq \operatorname{Ann}(\mathbb{F}^{p*}).$$

The last chain of inclusions can be represented as

$$\mathbb{F}^p \supseteq \operatorname{Ann}(U_1^*) \supseteq \cap_{i=1}^2 \operatorname{Ann}(U_i^*) \supseteq \cdots \supseteq \cap_{i=1}^m \operatorname{Ann}(U_i^*) \supseteq \{0\}.$$

Thus, the following inclusions are valid:

$$\mathcal{D}^* \left(U_1^* + \cdots + U_i^*\right) \subseteq U_1^* + \cdots + U_i^*, \ \forall i \in \{1, \ldots, m\}.$$

Theorem 3. *The subspace* $\sum\limits_{i=1}^m U_i^*$ *is an invariant subspace of the operator* $\mathcal{D}^*$, $\dim\left(\sum\limits_{i=1}^m U_i^*\right) = k \Leftrightarrow \cap_{i=1}^m \operatorname{Ann}(U_i^*)$ *is an invariant subspace of the operator* $\mathcal{D}$, $\dim\left(\cap_{i=1}^m \operatorname{Ann}(U_i^*)\right) = p - k$.

Corollary 1. $\sum_{i=1}^m U_i^* = \mathbb{F}^{p*} \Leftrightarrow \cap_{i=1}^m \operatorname{Ann}(U_i^*) = \{0\}$.

Proposition 3. $\dim\left(\sum\limits_{i=1}^m U_i^*\right) = \operatorname{rang}(B, BD, \ldots, BD^{p-1})$.

Theorem 4 (reducibility theorem). *If* $\dim\left(\cap_{i=1}^m \operatorname{Ann}(U_i^*)\right) = k < p$, *then there exists a nondegenerate transformation* $z = S\overline{z}$ *that reduces the system* (2) *to the form*

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + B_{11}\overline{z}_1(t), \\ \frac{d\overline{z}_1(t)}{dt} &= \overline{C}_1 y(t) + \overline{D}_{11}\overline{z}_1(t), \\ \frac{d\overline{z}_2(t)}{dt} &= \overline{C}_2 y(t) + \overline{D}_{21}\overline{z}_1(t) + \overline{D}_{22}\overline{z}_2(t), \end{aligned} \tag{4}$$

where $\overline{z}_1 \in \mathbb{F}^k$, $\overline{z}_2 \in \mathbb{F}^{p-k}$.

Theorem 5 (dual reducibility theorem). *If* $\dim\left(\sum\limits_{i=1}^m U_i^*\right) = p - k > 0$, *then there exists a nondegenerate transformation* $\overline{z} = Wz$ *that reduces the system* (2) *to the form* (4).

Theorem 6 (on partial stability). *Let the conditions of Theorem 4 be satisfied. For system* (1) *to be stable with respect to the first* m *components of the phase vector* x, *it is necessary and sufficient that the system*

$$\begin{aligned} \frac{dy(t)}{dt} &= Ay(t) + \overline{B}_{11}\overline{z}_1(t), \\ \frac{d\overline{z}_1(t)}{dt} &= \overline{C}_1 y(t) + \overline{D}_{11}\overline{z}_1(t) \end{aligned} \tag{5}$$

be stable with respect to all variables.

In some special cases, the study of partial stability of the system (1) can be simplified.

Proposition 4. *If, in addition to the conditions of Theorem 5, the condition*

$$\operatorname{im} C \subseteq \operatorname{Ann}\left(\sum_{i=1}^m U_i^*\right),$$

then the system (2) *is reducible to the form*

$$\begin{aligned}\frac{dy(t)}{dt} &= Ay(t) + \overline{B}_{11}\overline{z}_1(t),\\ \frac{d\overline{z}_1(t)}{dt} &= \overline{D}_{11}\overline{z}_1(t),\\ \frac{d\overline{z}_2(t)}{dt} &= \overline{C}_2 y(t) + \overline{D}_{21}\overline{z}_1(t) + \overline{D}_{22}\overline{z}_2(t),\end{aligned} \tag{6}$$

and the question of y-stability is reduced to the study of the stability of the system

$$\begin{aligned}\frac{dy(t)}{dt} &= Ay(t) + \overline{B}_{11}\overline{z}_1(t),\\ \frac{d\overline{z}_1(t)}{dt} &= \overline{D}_{11}\overline{z}_1(t).\end{aligned} \tag{7}$$

Proofs of the theorems and propositions are given in the Appendix.

Remark 1. If the conditions of proposition 4 are met, the problem of studying partial stability is reduced to studying the stability of two lower-order systems. The subspace $\mathrm{Ann}(U^*)$ is an invariant subspace with respect to y-stability in the sense that the y-stability property is preserved for any choice of the matrix C, whose columns are arbitrary vectors from the subspace $\mathrm{Ann}\left(\sum\limits_{i=1}^{m} U_i^*\right)$. Generally speaking, small variations in the parameters of the system (2) can lead to a violation of the partial stability property.

Remark 2. Note that the proposed approach allows us to investigate the stability of any component of the phase vector of the system under study. It is sufficient to designate the component under study in the system as y, and the remaining components will form the vector z. This implements the algorithm for solving the variable elimination problem proposed in [22].

5. EXAMPLE

Let the system under study have the following initial data:

$$y \in \mathbb{R}^2, \quad z \in \mathbb{R}^4, \quad A \in \mathbb{R}^{2\times 2}, \quad B \in \mathbb{R}^{2\times 4}, \quad C \in \mathbb{R}^{4\times 2}, \quad D \in \mathbb{R}^{4\times 4}, \quad n = 6, \quad p = 4,$$

where

$$A = \begin{pmatrix} a_{11} & a_{12}\\ a_{21} & a_{22}\end{pmatrix}, \quad B = \begin{pmatrix} -1 & 1 & 2 & -2\\ 1 & 1 & 1 & 0\end{pmatrix}, \quad C = \begin{pmatrix} -1 & 1\\ 1 & -1\\ 0 & 0\\ 1 & -1\end{pmatrix}, \quad D = \begin{pmatrix} 9 & -23 & -46 & 8\\ -3 & 5 & 22 & 16\\ -6 & -2 & -10 & -4\\ -15 & 19 & 38 & -10\end{pmatrix}.$$

Based on the data of the problem, we consider the linear space $\mathbb{R}^4$ and the corresponding dual linear space $\mathbb{R}^{4*}$, in which the linear operators $\mathcal{D}$ and $\mathcal{D}^*$ act, respectively.

Covectors $b_1^* = -\xi_1 + \xi_2 + 2\xi_3 - 2\xi_4$ and $b_2^* = \xi_1 + \xi_2 + 2\xi_3$ are generating vectors of invariant cyclic subspaces with respect to the linear operator $\mathcal{D}^*$.

The corresponding subspaces — the linear spans of their bases — have the form:

$$U_1^* = \mathrm{span}(-\xi_1 + \xi_2 + 2\xi_3 - 2\xi_4, 6\xi_1 - 14\xi_2 - 28\xi_3 + 20\xi_4),$$
$$U_2^* = \mathrm{span}(\xi_1 + \xi_2 + \xi_3, -20\xi_2 - 34\xi_3 + 20\xi_4, -36\xi_1 + 348\xi_2 + 660\xi_3 - 384\xi_4).$$

The minimal annihilating polynomials of the covectors b_1^* and b_2^* have the form

$$\mu_1(t) = t^2 + 24t + 108, \quad \mu_2(t) = t^3 + 30t^2 + 252t + 648.$$

At the same time $\dim U_1^* = 2$, $\dim U_2^* = 3$, $\dim (U_1^* + U_2^*) = 3 < 4 = \dim \mathbb{R}^{4*}$.

$$\operatorname{Ann}(U_1^* + U_2^*) : \begin{cases} -\xi_1 + \xi_2 + 2\xi_3 - 2\xi_4 = 0, \\ 6\xi_1 - 14\xi_2 - 28\xi_3 + 20\xi_4 = 0, \\ \xi_1 + \xi_2 + \xi_3 = 0, \\ - 20\xi_2 - 34\xi_3 + 20\xi_4 = 0, \\ -36\xi_1 + \xi_2 + 660\xi_3 - 384\xi_4 = 0. \end{cases}$$

By solving a homogeneous system of linear equations, we find the basis of the solution space:

$$\operatorname{Ann}(U_1^* + U_2^*) = \operatorname{span}(-1, 1, 0, 1).$$

By choosing a basis of the subspace $U_1^* + U_2^*$, partially consistent with the bases of the subspaces U_1^* and U_2^*,

$$e^1 = -\xi_1 + \xi_2 + 2\xi_3 - 2\xi_4, \quad e^2 = 6\xi_1 - 14\xi_2 - 28\xi_3 + 20\xi_4, \quad e^3 = \xi_1 + \xi_2 + \xi_3$$

and complementing it to the basis of the entire space $\mathbb{R}^{4^*}$ with the covector $e^4 = -\xi_1 + \xi_2 + \xi_4$, we obtain the corresponding dual basis of the space $\mathbb{R}^4$:

$$e_1 = (1/4, 9/4, -5/2, -2), \quad e_2 = (1/24, 5/24, -1/4, -1/6),$$
$$e_3 = (2/3, 4/3, -1, -2/3), \quad e_4 = (-1/3, 1/3, 0, 1/3).$$

Then the transition matrix S from the original basis to the new one has the form

$$S = \begin{pmatrix} 1/4 & 1/24 & 2/3 & -1/3 \\ 9/4 & 5/24 & 4/3 & 1/3 \\ -5/2 & -1/4 & -1 & 0 \\ -2 & -1/6 & -2/3 & 1/3 \end{pmatrix}.$$

The matrix of the operator $\mathcal{D}$ in the new basis takes the form

$$D_e = S^{-1}DS = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -108 & -24 & 0 & 0 \\ 0 & 1 & -6 & 0 \\ -162 & -35/2 & -60 & 24 \end{pmatrix}.$$

At the same time

$$BS = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad S^{-1}C = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 3 & -3 \end{pmatrix}.$$

As a result of the transition to a new basis using the non-degenerate transformation $z = S\overline{z}$, we arrive at the system:

$$\frac{dy(t)}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} y(t) + \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix} \overline{z}(t),$$

$$\frac{d\overline{z}(t)}{dt} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 3 & -3 \end{pmatrix} y(t) + \begin{pmatrix} 0 & 1 & 0 & 0 \\ -108 & -24 & 0 & 0 \\ 0 & 1 & -6 & 0 \\ -162 & -\frac{35}{2} & -60 & 24 \end{pmatrix} \overline{z}(t)$$

where $y \in \mathbb{R}^2$, $z \in \mathbb{R}^4$, $A \in \mathbb{R}^{2\times 2}$, $B \in \mathbb{R}^{2\times 4}$, $C \in \mathbb{R}^{4\times 2}$, $D \in \mathbb{R}^{4\times 4}$, $n = 6$, $p = 4$.

Thus, the original problem is divided into two simpler subtasks: studying the stability of two systems

$$\frac{dy(t)}{dt} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} y(t) \quad \text{and} \quad \frac{d\overline{z}_1(t)}{dt} = \begin{pmatrix} 0 & 1 & 0 \\ -108 & -24 & 0 \\ 0 & 1 & -6 \end{pmatrix} \overline{z}_1(t).$$

In this case, for the second system the characteristic polynomial has the form

$$\chi(t) = (-6 - t)\mu_1(t) = -\mu_2(t),$$

which is stable. Thus, for y-stability, the stability of the first system is necessary and sufficient.

6. CONCLUSION

An algebraic-geometric approach to the study of partial stability of linear systems with constant coefficients is proposed, which makes it possible to determine the stability conditions of the y-components of the phase vector by “rotating” the phase subspace of the z-components. By using the geometric properties of this subspace, conditions are obtained under which stability analysis is reduced to solving two problems of lower dimension. It is shown that the system exhibits some invariance properties with respect to partial stability. The proposed approach can be applied to studying the stability of linear discrete and delayed systems, as well as to solving polystability problems.

Further research directions can be carried out in the areas of partial controllability and stabilization of linear systems, as well as in the area of dissemination of this approach to the study of problems of partial stability of linear non-stationary systems and nonlinear systems of differential equations.

APPENDIX

Proof of Proposition 2. Let $\forall w_i^* = u_1^* + \dots + u_i^* \in W_i^* = U_1^* + \dots + U_i^*$. Since $\mathcal{D}^* u_i \in U_i$ $(\forall i = 1, \dots, m)$ due to the invariance of subspaces U_i^*, then $\mathcal{D}^* w_i = \mathcal{D}^*(u_1 + \dots + u_i) = \mathcal{D}^* u_1 + \dots + \mathcal{D}^* u_i \in U_1^* + \dots + U_i^* = W_i^*$ $(\forall i = 1, \dots, m)$. Thus, W_i $(\forall i = 1, \dots, m)$ are invariant subspaces of the operator $\mathcal{D}^*$. Proposition 2 has been proven.

Proof of Proposition 3. The proof follows from the isomorphism of the two dual linear spaces L and L^*, as well as from the properties of annihilators. Indeed, since the relation

$$\mathrm{Ann}\left(\sum_{i=1}^{m} U_i^*\right) = \cap_{i=1}^{m} \mathrm{Ann}\left(U_i^*\right),$$

then in the original dual bases we represent the annihilator in the form

$$\mathrm{Ann}\left(\sum_{i=1}^{m} U_i^*\right) : \begin{cases} B\xi &= 0, \\ BD\xi &= 0, \\ \dots & \\ BD^{p-1}\xi &= 0. \end{cases}$$

But then

$$\dim \mathrm{Ann}\left(\sum_{i=1}^{m} U_i^*\right) = \dim \mathbb{F}^{p*} - \mathrm{rang}(B, BD, \dots, BD^{p-1}).$$

On the other hand, the following relation is valid:

$$\dim \sum_{i=1}^{m} U_i^* + \dim \operatorname{Ann}\left(\sum_{i=1}^{m} U_i^*\right) = \dim \mathbb{F}^{p*},$$

Taking into account the last two relations, we have

$$\dim \sum_{i=1}^{m} U_i^* = \operatorname{rang}(B, BD, \ldots, BD^{p-1}).$$

Proposition 3 has been proven.

Proof of Theorem 3. Let us prove the necessity. From Proposition 2 it follows that $W^* = \sum\limits_{i=1}^{m} U_i^*$ is an invariant subspace with respect to the operator $\mathcal{D}^*$. Let $\forall w^* \in W^*$, $\forall w \in \operatorname{Ann}(W^*) \Leftrightarrow \langle w, w^* \rangle = 0 \Leftrightarrow \langle \mathcal{D}w, w^* \rangle = \langle w, \mathcal{D}^* w^* \rangle = 0$, since $\mathcal{D}^* w^* \in W^*$. But then $\mathcal{D}w \in \operatorname{Ann}(W^*)$. Since $\dim(W^*) + \dim(\operatorname{Ann}(W^*)) = k + \dim(\operatorname{Ann}(W^*)) = p$, then $\dim(\operatorname{Ann}(W^*)) = p - k$. Finally, since $\operatorname{Ann}(W^*) = \operatorname{Ann}\left(\sum_{i=1}^{m} U_i^*\right) = \cap_{i=1}^{m} \operatorname{Ann}\left(U_i^*\right)$, then necessity is proved.

The proof of sufficiency is obtained by verbatim repetition of necessity, provided that the proof proceeds in the opposite direction, i.e., from the end of the proof to its beginning. Theorem 3 is proven.

Proof of Theorem 4. Let $U^* = \sum\limits_{i=1}^{m} U_i^*$, $\dim U^* = p - k > 0$. Thus $U^* = \operatorname{span}\left(e^1, \ldots, e^{p-k}\right)$ and $e^1, \ldots, e^{p-k}$ is a basis of this subspace. We complement this system of vectors to a basis of the space $\mathbb{F}^{p*}$ using the vectors $e^{p-k+1}, \ldots, e^p$. This basis in the space $\mathbb{F}^p$ corresponds to the dual basis $e_1, \ldots, e_p$. Since by Proposition 2 it follows that $\sum\limits_{i=1}^{m} U_i^*$ is an invariant subspace with respect to the operator $\mathcal{D}^*$, we have $\forall u^* \in U^* \Rightarrow \mathcal{D}^* u^* \in U^* \Rightarrow \langle \mathcal{D}v, u^* \rangle = \langle v, \mathcal{D}^* u^* \rangle = u\left(\mathcal{D}v\right)$ $\forall u \in \mathbb{F}^p$, $\forall u^* \in \mathbb{F}^{p*}$. Then

$$\langle \mathcal{D}e_j, e^i \rangle = \langle e_j, \mathcal{D}^* e^i \rangle = e^i\left(\mathcal{D}e_j\right) = 0 \ \ \forall i = 1, \ldots, p-k, \ j = p-k+1, \ldots, p.$$

But this means that the linear operator $\mathcal{D}$ has an invariant subspace $\operatorname{Ann}(U^*) \subseteq \mathbb{F}^p$. Therefore, in the chosen basis, the matrix of the linear operator $\mathcal{D}$ takes the form

$$D_e = \begin{pmatrix} \overline{D}_{11} & 0 \\ \overline{D}_{21} & \overline{D}_{22} \end{pmatrix}.$$

It remains to construct the matrix of the non-degenerate transformation. Since the annihilators of the subspaces U_i^* are representable as a set of homogeneous linear systems

$$\operatorname{Ann}(U_i^*)\ (i = 1, \ldots, m): \begin{cases} \phantom{D^{k_i-1}} b_i \xi &= 0, \\ b_i D \xi &= 0, \\ \quad \cdots \\ b_i D^{k_i-1} \xi &= 0, \end{cases}$$

then the subspace $\cap_{i=1}^{m} \operatorname{Ann}\left(U_i^*\right)$ can be represented as

$$\begin{cases} B\xi &= 0, \\ BD\xi &= 0, \\ \quad \cdots \\ BD^{p-1}\xi &= 0. \end{cases} \tag{A.1}$$

Having found the fundamental system of solutions of the homogeneous system (A.1) $\xi_1, \dots, \xi_{p-k}$ and complementing it to the basis of the space $\mathbb{F}^p$ with the system of vectors $\nu_1, \dots, \nu_k$, we obtain the desired transformation matrix:

$$S = (\nu_1, \cdots, \nu_k, \xi_1, \cdots, \xi_{p-k}).$$

Let's make a change of variables $z = S\overline{z}$ in system (2). System (3) then takes the form (4). Theorem 4 is proven.

Proof of Theorem 5. Let $U^* = \sum\limits_{i=1}^{m} U_i^*$, $\dim U^* = p - k > 0$. Thus, $U^* = \operatorname{span}\left(e^1, \dots, e^{p-k}\right)$ and $e^1, \dots, e^{p-k}-$ are the basis of this subspace. Let us supplement this system of vectors to the basis of the space $\mathbb{F}^{p*}$ using the vectors $e^{p-k+1}, \dots, e^p$. Then, according to Proposition 2, it follows that U^* is an invariant subspace with respect to the operator $\mathcal{D}^*$. Therefore, the matrix of the linear operator $\mathcal{D}^*$ in the chosen basis has the form

$$D_e^* = W^{-1} D^* W = \begin{pmatrix} \overline{D^*}_{11} & \overline{D^*}_{12} \\ 0 & \overline{D^*}_{22} \end{pmatrix}.$$

But then the adjoint operator $\mathcal{D} = (\mathcal{D}^*)^*$ in the dual basis takes the form

$$D = (D_e^*)^T = (W^{-1} D^* W)^T = W^T (D^*)^T W^{-T} = \begin{pmatrix} \overline{D}_{11} & 0 \\ \overline{D}_{21} & \overline{D}_{22} \end{pmatrix}, \quad \overline{D}_{ij} = \overline{D^*}_{ij}^T, \ i, j \in \{1, 2\}.$$

From here $S = W^{-T} \Rightarrow W = S^{-T}$. The required non-degenerate transformation is equal to $\overline{z} = Wz$. By replacing $z = W^{-1}\overline{z}$ in system (2), we arrive at system (4). Theorem 5 is proven.

Proof of Theorem 6. According to Theorem 4, there exists a transformation that reduces system (2) to form (4). Moreover, the transformation found preserves the y-stability property. As a result, a subsystem of $(m + p - k)$th order is identified. According to Corollary 1, the resulting subsystem satisfies the relation $\sum\limits_{i=1}^{m} \overline{U}_i^* = \mathbb{F}^{p-k^*}$. Thus, $\operatorname{rang}(\overline{B}_{11}, \overline{B}_{11}\overline{D}_{11}, \dots, \overline{B}_{11}\overline{D}_{11}^{p-k-1}) = p - k$. Since the order of the resulting subsystem 5 is $m + p - k$, by Corollary 1.1.2 [4, p. 38] we see that this condition is both necessary and sufficient for y-stability of the system (2). Theorem 6 is proved.

Proof of Proposition 4. Let the conditions of theorem 6 be satisfied and, in addition, $\operatorname{im} C \subseteq \operatorname{Ann}\left(\sum\limits_{i=1}^{m} U_i^*\right)$. Then

$$\begin{cases} BC &= 0, \\ BDC &= 0, \\ \dots \\ BD^{p-1}C &= 0. \end{cases} \tag{A.2}$$

As a result, in system (5), the matrix $\overline{C}_1 = 0$. Proposition 4 is proven.

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This paper was recommended for publication by L.B. Rapoport, a member of the Editorial Board