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On a Problem Related to the Time of First Reaching a Given Level by a Random Process

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Abstract—This paper considers the problem of estimating the probability of the following event: a continuous random process will first reach a given level at some time from a given variation interval of the independent variable. The general results obtained previously are specified for a smooth Gaussian process. The estimates are calculated for different values of process parameters, and the corresponding numerical results are presented.

Keywords: probability, random process, level crossing

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1. INTRODUCTION. PROBLEM STATEMENT AND PREVIOUS RESULTS

Let $\xi(x)$ be a random process continuous with probability 1, and let y be a given number. As the domain of the process $\xi(t)$ we consider two intervals, namely, a) a half-interval $(x_0, x'']$ or b) a closed interval $[x_0, x'']$. In case a), by assumption,

$$\lim_{x\to x_0} \mathbf{P}\{\xi(x) > y\} = 1, \tag{1}$$

and x_0 can be either a finite number or $-\infty$. In case b), we suppose $\mathbf{P}\{\xi(x_0) > y\} = 1$.

Consider an arbitrary value $x' \in (x_0, x'')$. Let us define the events

$$Z = \{\exists\, \tilde{x} \in (x', x'')\ \ \forall x \in (x_0, \tilde{x})\ \ \xi(x) > y,\, \xi(\tilde{x}) = y\}$$

and

$$L = \{\exists\, \hat{x} \in (x_0, x'']\ \ \forall x \in (x_0, \hat{x})\ \ \xi(x) > y\}.$$

Event L means that at the initial time, the trajectory $\xi(x)$ is above level y; and event Z means that level y will be first reached by the trajectory at some time within the interval (x', x'') (see Fig. 1). It is required to find the conditional probability $\mathbf{P}\{Z|L\}$ of event Z given the occurrence of event L.

This problem is a special case of the one posed in [1]. For non-Markov smooth processes, it was studied, in particular, in the author's publications [2–5]; related problems were considered in many well-known works, e.g., [6–13]. A more detailed bibliography can be found in [5]. For diffusion Markov processes, this problem can be reduced to solving a mixed problem for a partial differential equation, which was shown back in [1]. In this paper, as in [2–5], we consider the case of a non-Markov smooth random process $\xi(x)$. Note that, in contrast to [3–5], a somewhat different procedure for forming lower estimates of the desired probability was proposed in [2]. This paper is a continuation of [2].

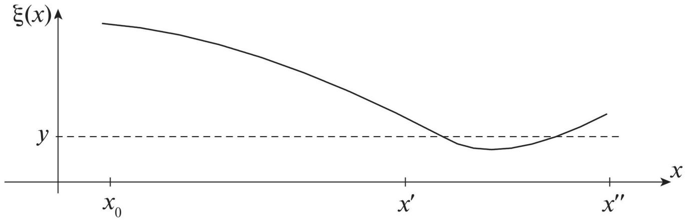

Fig. 1. Some realization of the process $\xi(x)$.

In applications, the above problem arises when investigating various stochastic systems; for example, see [14]. For instance, the probabilistic estimation of the accuracy and safety of aircraft landing reduces to this problem [15–19], which confirms its practical importance.

Following [10], we denote by $G_y(x_0, x'')$ the set of scalar functions continuous on $[x_0, x'']$ or $(x_0, x'']$ (depending on the domain of the process $\xi(x)$ selected) that are not identically equal to y on any subinterval of (x_0, x''). For functions from $G_y(x_0, x'')$, we define the concepts of a crossing of level y, a touching of level y, an upcrossing of level y [10], and a downcrossing of level y [10], as done in [10].

If[1] 1) the sample functions $\xi(x)$ belong to the set $G_y(x_0, x'')$ and have no touchings of level y with probability 1 and 2) the mean[2] number $N(x_0, x'')$ of crossings of level y by the process $\xi(x)$ on the interval (x_0, x'') is finite, then, in view of condition (1), it is not difficult to show that $\mathbf{P}\{Z|L\} = \mathbf{P}\{Z\}$. In other words, one can deal with estimating the unconditional probability $\mathbf{P}\{Z\}$ instead of estimating the conditional one $\mathbf{P}\{Z|L\}$.

For the process $\xi(x)$, we denote by $N^+(x_1, x_2)$ and $N^-(x_1, x_2)$ the mean number of upcrossings and downcrossings of level y, respectively, on an interval (x_1, x_2). Also, we denote by $m_y(x_1, x_2)$ the mean number of local maxima of the process $\xi(x)$ located above level y on an interval (x_1, x_2). The following result was established in [2].

Theorem. *Assume that:*

1) *The sample functions $\xi(x)$ belong to the set $G_y(x_0, x'')$ and have no touchings of level y on the interval (x_0, x'') with probability* 1, *and $N(x_0, x'') < \infty$.*

2) $\mathbf{P}\{\xi(x') = y\} = 0$.

3) *Condition* (1) *holds.*

Let x^* be any point from the interval (x_0, x'') such that $\mathbf{P}\{\xi(x^*) = y\} = 0$, $m_y(x^*, x'') < \infty$. Then

$$N^-(x', x'') - \Big(N^+(x_0, x^*) + m_y(x^*, x'')\Big) \leqslant \mathbf{P}\{Z\} \leqslant N^-(x', x'').$$

If the process $\xi(x)$ is mean-square differentiable, then the numbers $N^-(x_1, x_2)$, $N^+(x_1, x_2)$, and $m_y(x_1, x_2)$ can be calculated by Rice's formulas [6, 8, 10]:

$$N^-(x_1, x_2) = -\int_{x_1}^{x_2} dx \int_{-\infty}^{0} v f_x(y, v) dv, \tag{2}$$

$$N^+(x_1, x_2) = \int_{x_1}^{x_2} dx \int_{0}^{\infty} v f_x(y, v) dv, \tag{3}$$

$$m_y(x_1, x_2) = -\int_{x_1}^{x_2} dx \int_{y}^{\infty} du \int_{-\infty}^{0} z w_x(u, 0, z) dz, \tag{4}$$

[1] Sufficient conditions for assumptions 1) and 2) were formulated in [2].
[2] Here, "mean" refers to the mathematical expectation of a random variable.

where $f_x(u, v)$ is the joint probability density of the random variables $\xi(x)$ and $\xi'(x)$, and $w_x(u, v, z)$ is the joint probability density of the random variables $\xi(x), \xi'(x)$, and $\xi''(x)$. In addition, $\xi'(x)$ and $\xi''(x)$ stand for the first and second mean-square derivatives of the process $\xi(x)$.

2. SPECIFICATION FOR A GAUSSIAN PROCESS

Let the process $\xi(x)$ have the form

$$\xi(x) = a_1 x + a_0 + \eta(x), \quad x \in (-\infty, x''],$$

where $a_1 < 0$ and a_0 are constants, $\eta(x)$ is a stationary centered Gaussian process with continuous realizations and the correlation function

$$r(\tau) \equiv \mathbf{E}\{\eta(x)\eta(x+\tau)\}/\sigma^2,$$

where σ^2 indicates the variance of the processes $\xi(x)$ and $\eta(x)$, and $\mathbf{E}$ is the mathematical expectation operator. If there exists a finite second derivative $r''(0)$, then the process $\xi(x)$ is mean-square differentiable, and Rice's formulas (2) and (3) can be used to calculate the numbers N^- and N^+. If there exists a finite fourth derivative $r^{IV}(0)$, then the process $\xi(x)$ is twice mean-square differentiable, and the number m_y can be calculated by Rice's formula (4).

In the case under consideration, the conditions of the theorem are satisfied, and for any x^* from the interval $(-\infty, x'')$, we have the inequalities

$$N^-(x', x'') - \Big(N^+(-\infty, x^*) + m_y(x^*, x'')\Big) \leqslant \mathbf{P}\{Z\} \leqslant N^-(x', x''), \tag{5}$$

where $N^+(-\infty, x^*) = \lim\limits_{x\to-\infty} N^+(x, x^*)$.

We denote by σ_1^2 and σ_2^2 the variance of the processes $\xi'(x)$ and $\xi''(x)$, respectively, and by ρ the correlation coefficient of the random variables $\xi(x)$ and $\xi''(x)$. Note that

$$\sigma_1^2 = -\sigma^2 r''(0), \quad \sigma_2^2 = \sigma^2 r^{IV}(0), \quad \rho = r''(0)/\sqrt{r^{IV}(0)}.$$

According to these relations, in the nondegenerate case (i.e., $\sigma \neq 0$, $\sigma_1 \neq 0$, and $\sigma_2 \neq 0$), which is studied here, the correlation coefficient ρ takes only negative values.

For the process $\xi(x)$ under consideration, the formulas for $f_x(u, v)$ and $w_x(u, v, z)$ become[3]

$$f_x(u, v) = \frac{1}{2\pi\sigma\sigma_1} \exp\left\{-\frac{(u - a_0 - a_1 x)^2}{2\sigma^2} - \frac{(v - a_1)^2}{2\sigma_1^2}\right\}, \tag{6}$$

$$\begin{gathered} w_x(u, v, z) = \frac{1}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{(v - a_1)^2}{2\sigma_1^2}\right\} \frac{1}{2\pi\sigma\sigma_2\sqrt{1-\rho^2}} \\ \times \exp\left\{-\frac{1}{2(1-\rho^2)}\left[\frac{(u - a_0 - a_1 x)^2}{\sigma^2} - 2\rho\frac{(u - a_0 - a_1 x)z}{\sigma\sigma_2} + \frac{z^2}{\sigma_2^2}\right]\right\}. \end{gathered} \tag{7}$$

[3] Since the process $\xi(x)$ is Gaussian, the probability densities $f_x(u, v)$ and $w_x(u, v, z)$ will be Gaussian (e.g., see [10, 15]); in other words, $f_x(u, v)$ and $w_x(u, v, z)$ are two- and three-dimensional Gaussian probability densities, respectively. In addition, for any x, the correlation coefficient of the random variables $\xi(x)$ and $\xi'(x)$ and that of the random variables $\xi'(x)$ and $\xi''(x)$ are equal to zero due to the stationarity of the process $\eta(x)$ (e.g., see [14, 15]). These circumstances finally bring to formulas (6) and (7).

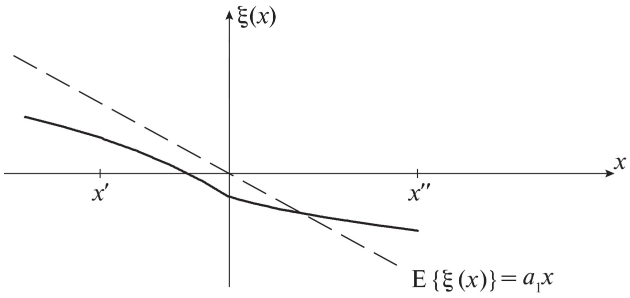

Fig. 2. Crossing of the zero level $y = 0$ by some realization of the process $\xi(x)$ and by its mean $\mathbf{E}\{\xi(x)\}$ for $a_0 = 0$.

For the sake of definiteness, assume that $y = 0$, $a_0 = 0$, and $x'' > 0$ (Fig. 2). Then event Z means that the zero level will be first reached by the process $\xi(x) = a_1 x + \eta(x)$ at some point from the interval (x', x'').

We introduce the parameters[4]

$$\alpha = -a_1/\sigma_1 \quad \text{and} \quad \beta = -a_1 x''/\sigma.$$

Then, see the Appendix, formulas (2)–(4) yield

$$N^-(x', x'') = \left[\frac{1}{\sqrt{2\pi}\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\} + \Phi(\alpha)\right]\left[\Phi(\beta) - \Phi\left(\beta\frac{x'}{x''}\right)\right], \tag{2'}$$

$$N^+(-\infty, x^*) = \left[\frac{1}{\sqrt{2\pi}\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\} - \Phi(-\alpha)\right]\Phi\left(\beta\frac{x^*}{x''}\right), \tag{3'}$$

$$m_y(x^*, x'') = -\frac{\beta}{2\pi\rho\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\}$$
$$\times \int\limits_{x^*/x''}^{1}\left[\Phi\left(-\frac{\beta}{\sqrt{1-\rho^2}}v\right) - \rho\Phi\left(-\frac{\rho\beta v}{\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{\beta^2 v^2}{2}\right\}\right]dv, \tag{4'}$$

where

$$\Phi(z) = \frac{1}{\sqrt{2\pi}}\int\limits_{-\infty}^{z}\exp\{-x^2/2\}dx. \tag{8}$$

3. PROBABILITY ESTIMATES: ACCURACY ANALYSIS

Inequalities (5) can be written as

$$N^-(x', x'') - f(x^*) \leqslant \mathbf{P}\{Z\} \leqslant N^-(x', x''),$$

where

$$f(x^*) = N^+(-\infty, x^*) + m_y(x^*, x''), \quad x^* \in (-\infty, x''].$$

[4] In a physical interpretation, the value of the parameter α shows how strongly, on average, the slope of the realizations of the process $\xi(x)$ differs from that of its mean $\mathbf{E}\{\xi(x)\}$: for small α, this difference is large, and vice versa.

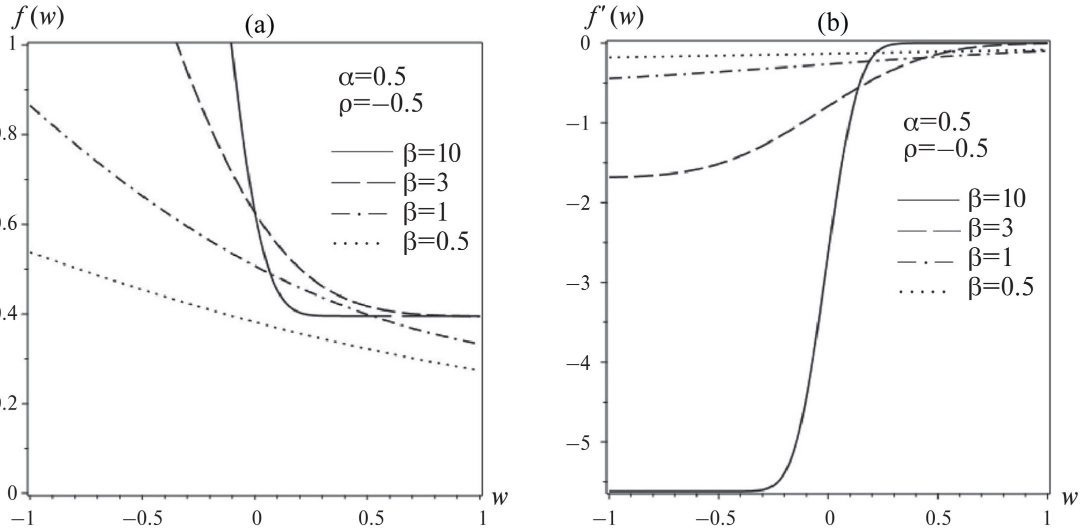

Fig. 3. (a) Curves $f(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.5$ and (b) curves $f'(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.5$.

With the dimensionless variable $w = x^*/x''$, the estimates (5) take the form

$$N^-(x', x'') - f(w) \leqslant \mathbf{P}\{Z\} \leqslant N^-(x', x''), \tag{5'}$$

where $w \in (-\infty, 1]$ and

$$f(w) = N^+(-\infty, w) + m_y(w, 1),$$

$$N^+(-\infty, w) = \left[\frac{1}{\sqrt{2\pi}\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\} - \Phi(-\alpha)\right]\Phi(\beta w),$$

$$m_y(w, 1) = -\frac{\beta}{2\pi\rho\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\}\int\limits_w^1\left[\Phi\left(-\frac{\beta}{\sqrt{1-\rho^2}}v\right) - \rho\Phi\left(-\frac{\rho\beta v}{\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{\beta^2 v^2}{2}\right\}\right]dv.$$

Let us investigate the behavior of the function $f(w)$ on the interval $(-\infty, 1]$. The smaller the values of $f(w)$ are, the higher accuracy the estimates (5′) of the probability $\mathbf{P}\{Z\}$ will have.

Consider the behavior of the function $f(w)$ depending on three parameters: α, β, and ρ. The region of interest is $w \in [-1, 1]$ since, by the physical meaning of the numbers $N^+(-\infty, w)$ and $m_y(w, 1)$, the function $f(w), w < -1$, is monotonically decreasing, so the minimum values of $f(w)$ will be achieved for $w > -1$.

Figure 3a shows the behavior of the function $f(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.5$. As it turned out, for any value combination of the parameters α, β, and ρ, the function $f(w)$ is monotonically decreasing on the interval $(-\infty, 1]$. This conclusion follows from the analysis of the derivative $f'(w)$. Based on the well-known formula for the derivative of a definite integral with variable limits,

$$\frac{d}{dt}\int\limits_{\phi_1(t)}^{\phi_2(t)} h(v)dv = -h(\phi_1(t))\frac{d\phi_1(t)}{dt} + h(\phi_2(t))\frac{d\phi_2(t)}{dt},$$

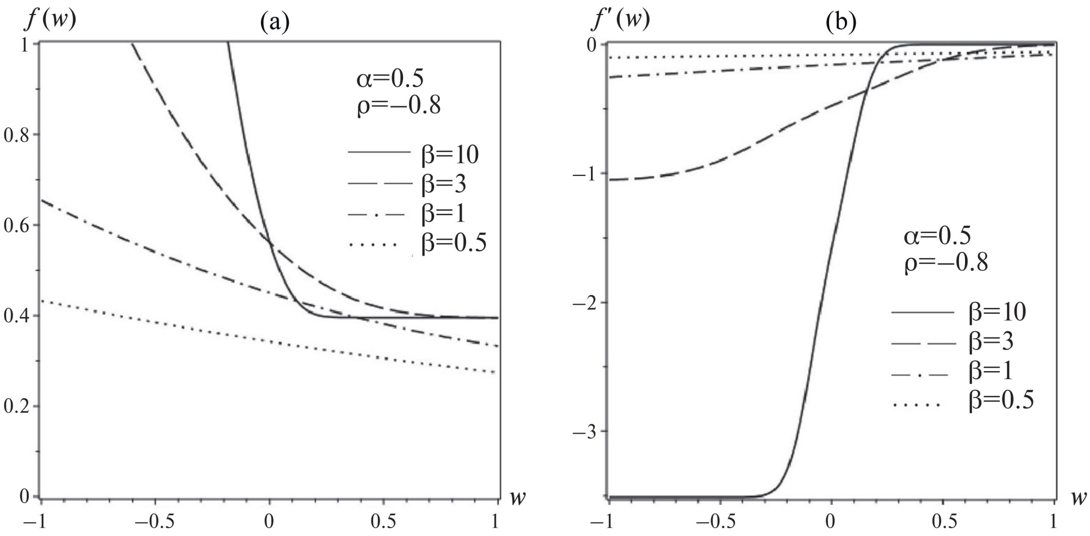

Fig. 4. (a) Curves $f(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.8$ and (b) curves $f'(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.8$.

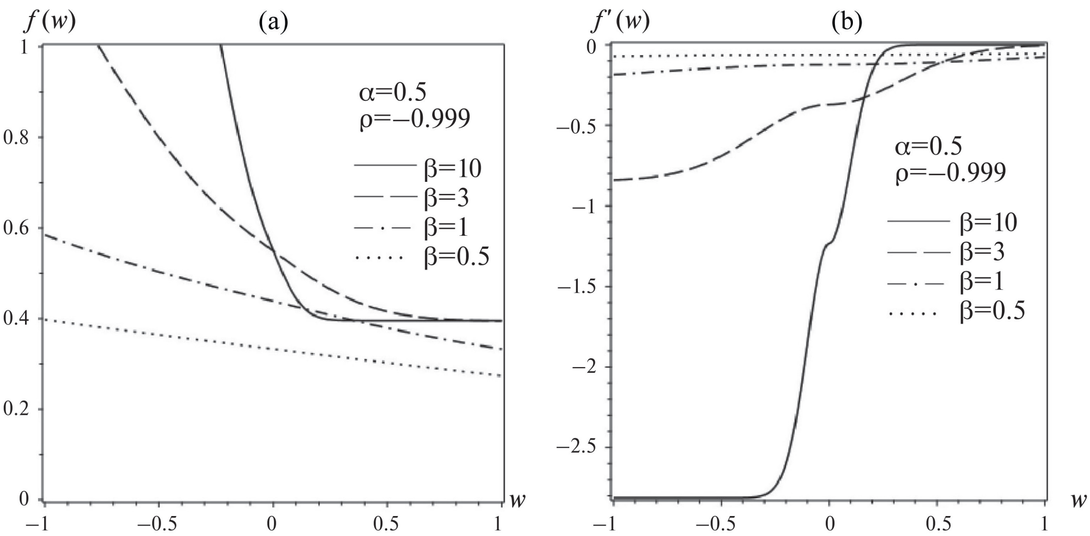

Fig. 5. (a) Curves $f(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.999$ and (b) curves $f'(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.999$.

we obtain the following expression for the derivative of the function $f(w)$:

$$\frac{df(w)}{dw} = \left[\frac{1}{\sqrt{2\pi}\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\} - \Phi(-\alpha)\right]\frac{1}{\sqrt{2\pi}}\exp\left\{-\frac{\beta^2 w^2}{2}\right\}\beta$$
$$+\frac{\beta}{2\pi\rho\alpha}\exp\left\{-\frac{\alpha^2}{2}\right\}\left[\Phi\left(-\frac{\beta w}{\sqrt{1-\rho^2}}\right) - \rho\Phi\left(-\frac{\rho\beta w}{\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{\beta^2 w^2}{2}\right\}\right].$$

Figure 3b shows the behavior of the derivative $f'(w)$ for different β, $\alpha = 0.5$, and $\rho = -0.5$. According to the numerical calculations, $f'(w) < 0$ for all $w \in (-\infty, 1]$. So the minimum values of the function $f(w)$ are achieved at $w = 1$. However, the nature of change of the derivative $f'(w)$ strongly depends on the parameters α, β, and ρ. For example, for $\beta = 10$, the values of $f'(w)$ are negligibly small on the interval $(0.5, 1)$; therefore, $f(w)$ is almost independent of w on this interval.

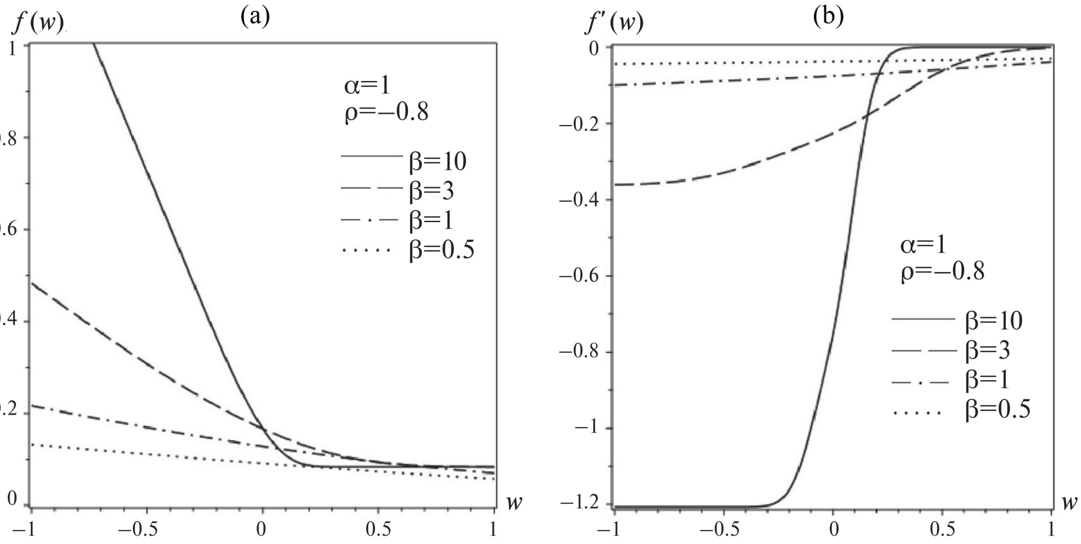

Fig. 6. (a) Curves $f(w)$ for different β, $\alpha = 1$, and $\rho = -0.8$ and (b) curves $f'(w)$ for different β, $\alpha = 1$, and $\rho = -0.8$.

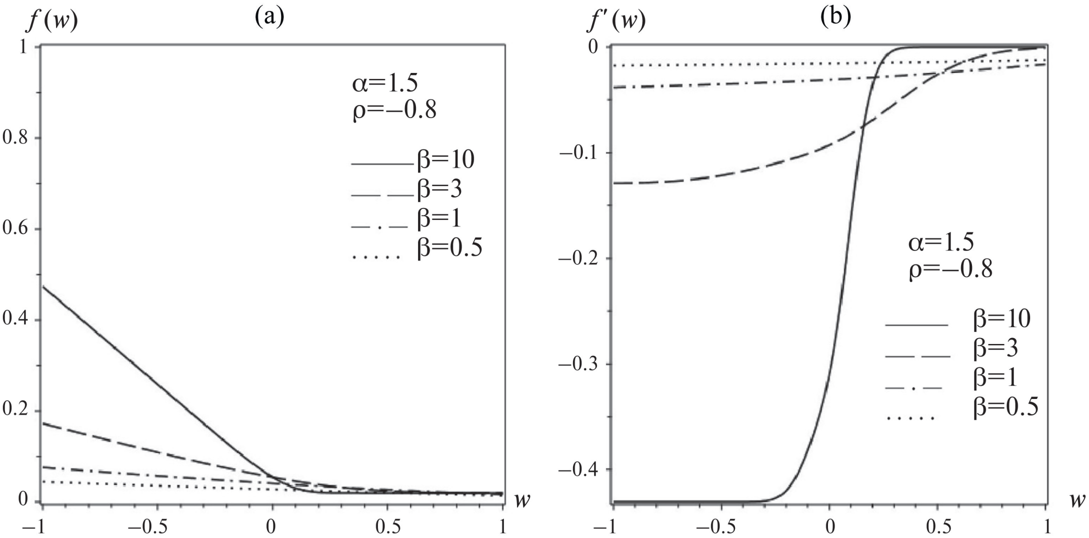

Fig. 7. (a) Curves $f(w)$ for different β, $\alpha = 1.5$, and $\rho = -0.8$ and (b) curves $f'(w)$ for different β, $\alpha = 1.5$, and $\rho = -0.8$.

Direct comparison of Figs. 3a–5a demonstrates changes in the behavior of the function $f(w)$ depending on the correlation coefficient $\rho : \rho = -0.5$ for Fig. 3a, $\rho = -0.8$ for Fig. 4a, and $\rho = -0.999$ for Fig. 5a. The corresponding changes in the behavior of the derivatives $f'(w)$ are shown in Figs. 3b–5b. If w is close to 1, then the values of $f(w)$ change very weakly. If w is below 1, then the values of $f(w)$ decrease under the transition $\{\rho = -0.5\} \to \{\rho = -0.8\} \to \{\rho = -0.999\}$; the smaller w is, the greater this decrease will be. Moreover, this conclusion holds for all β.

Direct comparison of Figs. 4a and 6a–9a demonstrates changes in the behavior of the function $f(w)$ depending on the parameter $\alpha : \alpha = 0.5$ for Fig. 4a, $\alpha = 1$ for Fig. 6a, $\alpha = 1.5$ for Fig. 7a, $\alpha = 2$ for Fig. 8a, and $\alpha = 3$ for Fig. 9a. The corresponding changes in the behavior of the derivatives $f'(w)$ are shown in Figs. 4b and 6b–9b. Clearly, for any fixed β, increasing α leads to a decrease in $f(w)$. If $\alpha = 3$ (Fig. 9a), then the values of $f(w)$ are less than 0.008 for $\beta = 10$ and

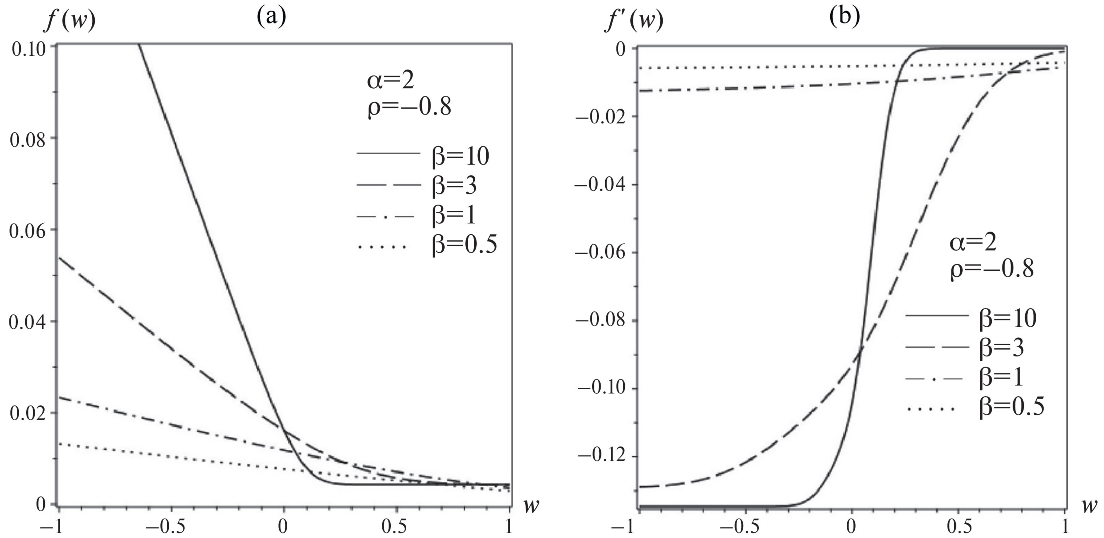

Fig. 8. (a) Curves $f(w)$ for different β, $\alpha = 2$, and $\rho = -0.8$ and (b) curves $f'(w)$ for different β, $\alpha = 2$, and $\rho = -0.8$.

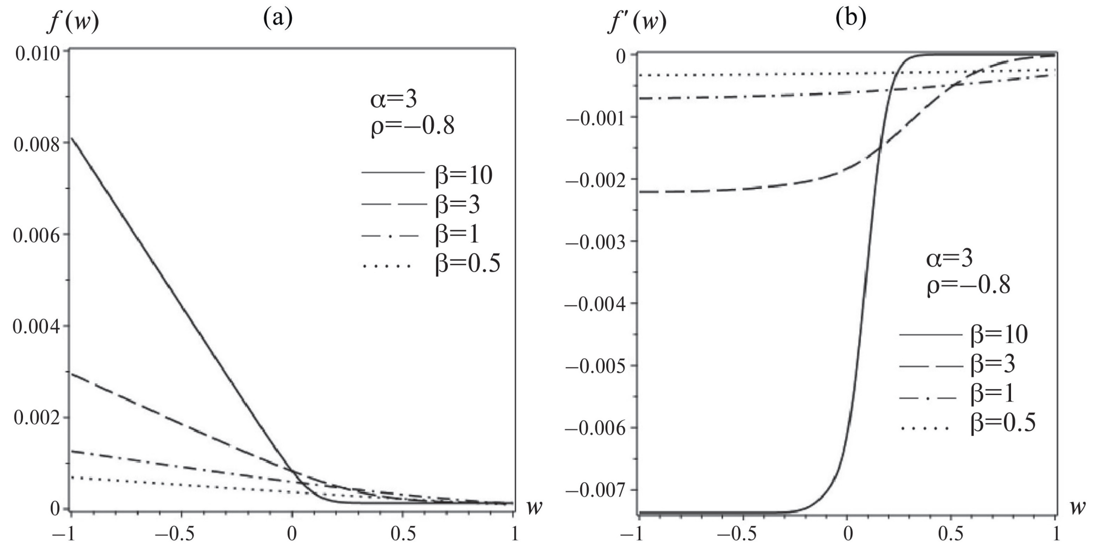

Fig. 9. (a) Curves $f(w)$ for different β, $\alpha = 3$, and $\rho = -0.8$ and (b) curves $f'(w)$ for different β, $\alpha = 3$, and $\rho = -0.8$.

$\beta = 3$ and less than 0.001 for $\beta = 1$ and $\beta = 0.5$. For all value combinations of the parameters α, β, and ρ under consideration, the minimum value of the function $f(w)$ is achieved at $w = 1$. Also note that for $\alpha > 3$, the values of $f(w)$ are negligibly small. In this case, inequalities ($5'$) allow determining the probability $\mathbf{P}\{Z\}$ almost exactly.

The Maple package was used for the numerical calculations.

4. CONCLUSIONS

A special stationary Gaussian process $\xi(x)$ with drift has been considered, and the estimates (5) for the probability $\mathbf{P}\{Z\}$ [2] of the process have been analyzed numerically. That is, the accuracy of these estimates has been studied depending on the choice of the point x^* and the parameters

α, β, and ρ of the process $\xi(x)$. The accuracy is given by $f(w)$, $w \in (-\infty, 1]$, (see (5′)), where the value w is uniquely determined by the choice of the point x^*. The smaller value $f(w)$ takes, the more accurate the resulting probability $\mathbf{P}\{Z\}$ will be. As it turned out, for *Gaussian* processes, the minimum value of $f(w)$ is obtained at $w = 1$ for any value combination of the parameters α, β, and ρ. The values of $f(w)$ strongly depend on the parameter α : the larger α is, the smaller $f(w)$ will be. If $\alpha > 3$, then the values of $f(w)$ are negligibly small, and the probability $\mathbf{P}\{Z\}$ is determined almost exactly.

APPENDIX

We find the number $N^+(x_1, x_2)$ by formula (3), substituting the expression (6) for the density $f_x(u, v)$ with $u = y$. As a result,

$$N^+(x_1, x_2) = \frac{1}{2\pi\sigma\sigma_1} \int\limits_{x_1}^{x_2} \exp\left\{-\frac{(y - a_0 - a_1 x)^2}{2\sigma^2}\right\} dx \int\limits_0^\infty v \exp\left\{-\frac{(v - a_1)^2}{2\sigma_1^2}\right\} dv.$$

Both integrals here are easily calculated:

$$\int\limits_{x_1}^{x_2} \exp\left\{-\frac{(y - a_0 - a_1 x)^2}{2\sigma^2}\right\} dx = -\frac{\sigma}{a_1}\sqrt{2\pi}\left[\Phi\left(\frac{y - a_0 - a_1 x_2}{\sigma}\right) - \Phi\left(\frac{y - a_0 - a_1 x_1}{\sigma}\right)\right],$$

$$\int\limits_0^\infty v \exp\left\{-\frac{(v - a_1)^2}{2\sigma_1^2}\right\} dv = \sigma_1^2 \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\} + \sqrt{2\pi} a_1 \sigma_1 \Phi\left(\frac{a_1}{\sigma_1}\right),$$

where the function $\Phi(\cdot)$ is given by (8). Consequently,

$$N^+(x_1, x_2) = \left[\frac{\sigma_1}{\sqrt{2\pi} a_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\} + \Phi\left(\frac{a_1}{\sigma_1}\right)\right]\left[\Phi\left(\frac{y - a_0 - a_1 x_1}{\sigma}\right) - \Phi\left(\frac{y - a_0 - a_1 x_2}{\sigma}\right)\right].$$

In this relation, letting $y = 0$, $a_0 = 0$, and $x_2 = x^*$ and denoting $\alpha = -a_1/\sigma_1$ and $\beta = -a_1 x''/\sigma$ (by analogy with the main text of the paper), one finally arrives at formula (3′) as $x_1 \to -\infty$ since $a_1 < 0$.

Similarly, substituting the expression (6) for the density $f_x(u, v)$ with $u = y$, by formula (2) we obtain the number

$$N^-(x_1, x_2) = \left[\frac{\sigma_1}{\sqrt{2\pi} a_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\} - \Phi\left(-\frac{a_1}{\sigma_1}\right)\right]\left[\Phi\left(\frac{y - a_0 - a_1 x_1}{\sigma}\right) - \Phi\left(\frac{y - a_0 - a_1 x_2}{\sigma}\right)\right].$$

In this relation, letting $y = 0$, $a_0 = 0$, $x_1 = x'$, and $x_2 = x''$ and denoting $\alpha = -a_1/\sigma_1$ and $\beta = -a_1 x''/\sigma$ (by analogy with the main text of the paper), one finally arrives at formula (2′).

Now, using formula (4), let us determine the number $m_y(x_1, x_2)$ if the density $w_x(u, 0, z)$ is calculated by (7) with $v = 0$. With the new variables $\tilde{u} = u - a_0 - a_1 x$, $\tilde{z} = z$, and $\tilde{x} = x$, we obtain

$$m_y(x_1, x_2) = -\int\limits_{x_1}^{x_2} d\tilde{x} \int\limits_{y - a_0 - a_1\tilde{x}}^{\infty} d\tilde{u} \int\limits_{-\infty}^{0} \frac{\tilde{z}}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\}$$

$$\times \frac{1}{2\pi\sigma\sigma_2\sqrt{1 - \rho^2}} \exp\left\{-\frac{1}{2(1 - \rho^2)}\left[\frac{\tilde{u}^2}{\sigma^2} - 2\rho\frac{\tilde{u}\tilde{z}}{\sigma\sigma_2} + \frac{\tilde{z}^2}{\sigma_2^2}\right]\right\} d\tilde{z}.$$

After additional changes of the variables, $\bar{u} = \tilde{u}/\sigma$, $\bar{z} = \tilde{z}/\sigma_2$, and $\bar{x} = \tilde{x}$, it follows that

$$m_y(x_1, x_2) = -\int_{x_1}^{x_2} d\bar{x} \int_{\frac{y-a_0-a_1\bar{x}}{\sigma}}^{\infty} \sigma d\bar{u} \int_{-\infty}^{0} \frac{\sigma_2 \bar{z}}{\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\}$$

$$\times \frac{1}{2\pi\sigma\sigma_2\sqrt{1-\rho^2}} \exp\left\{-\frac{\bar{u}^2 - 2\rho\bar{u}\bar{z} + \bar{z}^2}{2(1-\rho^2)}\right\} \sigma_2 d\bar{z}$$

$$= -\frac{\sigma_2(1-\rho^2)^{-\frac{1}{2}}}{2\pi\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\} \int_{x_1}^{x_2} d\bar{x} \int_{\frac{y-a_0-a_1\bar{x}}{\sigma}}^{\infty} \exp\left\{-\frac{\bar{u}^2}{2(1-\rho^2)}\right\} d\bar{u} \int_{-\infty}^{0} \bar{z} \exp\left\{-\frac{\bar{z}^2 - 2\rho\bar{u}\bar{z}}{2(1-\rho^2)}\right\} d\bar{z}.$$

The exponent index in the last integral can be transformed as

$$-\frac{\bar{z}^2 - 2\rho\bar{u}\bar{z}}{2(1-\rho^2)} = -\frac{(\bar{z}-\rho\bar{u})^2 - \rho^2\bar{u}^2}{2(1-\rho^2)}.$$

Then we make the change of variables

$$\lambda = \bar{u}, \quad \mu = \bar{z} - \rho\bar{u}, \quad \tau = \bar{x}$$

to get

$$m_y(x_1, x_2) = -\frac{\sigma_2(1-\rho^2)^{-\frac{1}{2}}}{2\pi\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\}$$

$$\times \int_{x_1}^{x_2} d\tau \int_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} \exp\left\{-\frac{\lambda^2}{2(1-\rho^2)}\right\} d\lambda \int_{-\infty}^{-\rho\lambda} (\mu+\rho\lambda) \exp\left\{-\frac{\mu^2 - \rho^2\lambda^2}{2(1-\rho^2)}\right\} d\mu$$

$$= -\frac{\sigma_2(1-\rho^2)^{-\frac{1}{2}}}{2\pi\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\} \int_{x_1}^{x_2} d\tau \int_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} \exp\left\{-\frac{\lambda^2}{2}\right\} d\lambda \int_{-\infty}^{-\rho\lambda} (\mu+\rho\lambda) \exp\left\{-\frac{\mu^2}{2(1-\rho^2)}\right\} d\mu.$$

Let $g(\lambda)$ denote the inner integral with respect to μ. Direct calculations yield

$$g(\lambda) \equiv \int_{-\infty}^{-\rho\lambda} (\mu+\rho\lambda) \exp\left\{-\frac{\mu^2}{2(1-\rho^2)}\right\} d\mu$$

$$= -(1-\rho^2) \exp\left\{-\frac{\rho\lambda^2}{2(1-\rho^2)}\right\} + \rho\lambda \int_{-\infty}^{-\rho\lambda} \exp\left\{-\frac{\mu^2}{2(1-\rho^2)}\right\} d\mu,$$

or, using the function $\Phi(\cdot)$ introduced above,

$$g(\lambda) = -(1-\rho^2) \exp\left\{-\frac{\rho\lambda^2}{2(1-\rho^2)}\right\} + \sqrt{2\pi(1-\rho^2)}\rho\lambda\Phi\left(-\frac{\rho\lambda}{\sqrt{1-\rho^2}}\right).$$

Thus,

$$m_y(x_1, x_2) = -\frac{\sigma_2(1-\rho^2)^{-\frac{1}{2}}}{2\pi\sqrt{2\pi}\sigma_1} \exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\} \int_{x_1}^{x_2} I(\tau) d\tau, \tag{A.1}$$

where

$$I(\tau) = I_1(\tau) + I_2(\tau),$$

$$I_1(\tau) = \int\limits_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} -(1-\rho^2)\exp\left\{-\frac{\lambda^2}{2(1-\rho^2)}\right\} d\lambda,$$

$$I_2(\tau) = \int\limits_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} \sqrt{2\pi(1-\rho^2)}\rho\lambda \exp\left\{-\frac{\lambda^2}{2}\right\} \Phi\left(-\frac{\rho\lambda}{\sqrt{1-\rho^2}}\right) d\lambda.$$

Based on the integration by parts, we bring $I_2(\tau)$ to the form

$$I_2(\tau) = \sqrt{2\pi(1-\rho^2)}\rho\Phi\left(-\frac{\rho(y-a_0-a_1\tau)}{\sigma\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{(y-a_0-a_1\tau)^2}{2\sigma^2}\right\}$$

$$-\rho^2 \int\limits_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} \exp\left\{-\frac{\lambda^2}{2(1-\rho^2)}\right\} d\lambda,$$

which allows writing $I(\tau)$ as

$$I(\tau) = -\int\limits_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} \exp\left\{-\frac{\lambda^2}{2(1-\rho^2)}\right\} d\lambda$$

$$+\sqrt{2\pi(1-\rho^2)}\rho\Phi\left(-\frac{\rho(y-a_0-a_1\tau)}{\sigma\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{(y-a_0-a_1\tau)^2}{2\sigma^2}\right\}.$$

Expressing the first term here through the above function $\Phi(\cdot)$, we find

$$-\int\limits_{\frac{y-a_0-a_1\tau}{\sigma}}^{\infty} \exp\left\{-\frac{\lambda^2}{2(1-\rho^2)}\right\} d\lambda = -\sqrt{2\pi(1-\rho^2)}\left[1-\Phi\left(\frac{y-a_0-a_1\tau}{\sigma\sqrt{1-\rho^2}}\right)\right].$$

Hence, due to (A.1),

$$m_y(x_1,x_2) = -\frac{\sigma_2}{2\pi\sigma_1}\exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\}$$

$$\times\int\limits_{x_1}^{x_2}\left[-1+\Phi\left(\frac{y-a_0-a_1\tau}{\sigma\sqrt{1-\rho^2}}\right)+\rho\Phi\left(-\frac{\rho(y-a_0-a_1\tau)}{\sigma\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{(y-a_0-a_1\tau)^2}{2\sigma^2}\right\}\right]d\tau.$$

In this relation, letting $y=0$, $a_0=0$, $x_1=x^*$, and $x_2=x''$ and utilizing the identity $1-\Phi(z)=\Phi(-z)$, we obtain

$$m_y(x^*,x'') = \frac{\sigma_2}{2\pi\sigma_1}\exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\}\int\limits_{x^*}^{x''}\left[\Phi\left(\frac{a_1\tau}{\sigma\sqrt{1-\rho^2}}\right)-\rho\Phi\left(\frac{\rho a_1\tau}{\sigma\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{(a_1\tau)^2}{2\sigma^2}\right\}\right]d\tau.$$

With the new integration variable $\nu=\tau/x''$, it follows that

$$m_y(x^*,x'') = \frac{\sigma_2 x''}{2\pi\sigma_1}\exp\left\{-\frac{a_1^2}{2\sigma_1^2}\right\}\int\limits_{x^*/x''}^{1}\left[\Phi\left(\frac{a_1x''\nu}{\sigma\sqrt{1-\rho^2}}\right)-\rho\Phi\left(\frac{\rho a_1x''\nu}{\sigma\sqrt{1-\rho^2}}\right)\exp\left\{-\frac{(a_1\nu x'')^2}{2\sigma^2}\right\}\right]d\nu.$$

Finally, letting $\alpha=-a_1/\sigma_1$ and $\beta=-a_1x''/\sigma$ (by analogy with the main text of the paper), one finally arrives at formula (4′).

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