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Generalized $\mathcal{H}_2$ Control of a Continuous-Time Markov Jump Linear System on a Finite Horizon

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Abstract—For continuous-time Markov jump linear systems, the concept of the generalized $\mathcal{H}_2$ norm is introduced as the worst-case value of the maximum of the expected squared Euclidean norm of the target output on a finite horizon, provided that the sum of the squared energy of an exogenous disturbance and a quadratic form of the initial state is equal to one. This norm is characterized in terms of coupled Riccati differential matrix equation solutions and in terms of linear matrix inequalities. Linear dynamic state-feedback controllers ensuring an upper bound for the $\mathcal{H}_2$ norm of the closed-loop system are designed by solving a semidefinite programming problem. The effectiveness of the approach is demonstrated by the results of numerical simulations.

Keywords: generalized $\mathcal{H}_2$ norm, linear matrix inequalities (LMIs), homogeneous Markov chains, multi-objective control

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1. INTRODUCTION

Random structure systems, particularly Markov jump systems [1–6], are widespread in modern control problems. Such systems have a finite number of distinct operation modes, and the dynamics in each mode are described by a specific system of differential equations. Jumps between modes occur at random time instants, determined by the evolution of a homogeneous Markov chain (Markov jumps, also called Markovian switching in the literature). The simplest problems leading to random structure systems are control with failures and disruptions [2], synchronization in variable topology networks [3, 4], multi-agent control [5, 6], and others.

The stability problem for random structure systems was pioneered by I.Ia. Kats and N.N. Krasovskii [7]. Later, various formulations of control problems for such systems were considered in [8–10]; in particular, a linear-quadratic controller was designed. In [1, 11–14], $\mathcal{H}_\infty$ and $\mathcal{H}_2$ control problems were solved for Markov jump systems. The $\mathcal{H}_\infty$ and $\mathcal{H}_2$ norms allow assessing the quality of transients on average. However, it is often necessary to guarantee that the maximum value of a transient (target output) will not exceed a given threshold, i.e., to estimate the maximum value of the system's target output. One possible approach to solving this problem is based on the generalized $\mathcal{H}_2$ norm.

For continuous-time systems, the concept of the generalized $\mathcal{H}_2$ norm, corresponding to the maximum deviation of a system under an exogenous disturbance of bounded energy and zero initial conditions, was introduced in [15]. The generalized $\mathcal{H}_2$ norm characterizes the system gain when the input signal has a bounded L_2 norm and the output signal is measured by the L_∞ norm (the maximum value of the Euclidean norm of the target output over time). For a linear continuous time-varying system, the concept of the maximum deviation was introduced in [16, 17] as a natural extension of the generalized $\mathcal{H}_2$ norm to systems with nonzero initial conditions, and an algorithm for its calculation was presented therein as well. As shown in [16, 17], multi-objective control problems with generalized $\mathcal{H}_2$ norms as performance criteria can be effectively solved using the apparatus of linear matrix inequalities (LMIs).

In [18], an estimate of the generalized $\mathcal{H}_2$ norm was calculated for continuous-time Markov jump linear systems on an infinite horizon. For such systems, suboptimal generalized $\mathcal{H}_2$ control was designed in [19]. Another definition of the generalized $\mathcal{H}_2$ norm, differing from the classical one, was used in [20]: estimates for the first absolute moment of the system output components under bounded-energy disturbances were constructed. In [21], the problem of generalized $\mathcal{H}_2$ filtering for semi-Markovian systems was considered. Problems of generalized $\mathcal{H}_2$ filtering and control for discrete-time Markov jump systems were studied in [22–24]. In all the works mentioned, the dynamics of systems in each operation mode were described by time-invariant systems.

In this paper, the concept of the generalized $\mathcal{H}_2$ norm for linear continuous-time Markov jump systems is considered on a finite horizon, and the systems are generally supposed to be time-varying in each operation mode. Several algorithms are proposed for calculating this characteristic. Also, we demonstrate how to find its upper bound rather simply; hence, suboptimal generalized $\mathcal{H}_2$ control can be designed in the case where the state of the Markov chain is available to the controller.

The remainder of this paper is organized as follows. In Section 2, we introduce the concept of the generalized $\mathcal{H}_2$ norm for continuous-time Markov jump linear systems on a finite horizon and present algorithms for its calculation. Section 3 provides the solution of the suboptimal generalized $\mathcal{H}_2$ control design problem in the cases where the state of the Markov chain is available and unavailable to the controller. In Section 4, we solve the multi-objective control problem. Numerical simulations demonstrating the above results are given in Section 5.

2. GENERALIZED $\mathcal{H}_2$ NORM

Let $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\in[t_s,t_f]}, \mathsf{P})$ be a probability space with filtration $(\mathcal{F}_t)_{t\in[t_s,t_f]}$. We denote by $L_2^\Omega([t_s,t_f],\mathbb{R}_2^{n_v})$ the space of $\mathcal{F}_t$-adapted processes $v = \{v(t) \in \mathbb{R}_2^{n_v}, t \in [t_s,t_f]\}$ such that

$$\mathbb{E}\|v\|_2^2 = \mathbb{E}\int_{t_s}^{t_f} v^\top(t)v(t)dt < \infty,$$

where $\mathbb{E}(\cdot)$ indicates the mathematical expectation operator.

On a fixed time interval (finite horizon) $[t_s,t_f]$, we consider a continuous-time linear system with a random structure changing according to the evolution of a stationary Markov chain:

$$\begin{aligned} \dot{x} &= A_{\theta(t)}(t)x + B_{\theta(t)}(t)v, \qquad x(t_s) = x_0, \\ z &= C_{\theta(t)}(t)x, \end{aligned} \tag{1}$$

where $x \in \mathbb{R}_2^{n_x}$ is the system state; $v \in L_2^\Omega([t_s,t_f],\mathbb{R}_2^{n_v})$ is an exogenous disturbance; $z \in \mathbb{R}_2^{n_z}$ is the target output; $\theta(t)$ is a homogeneous continuous-time Markov chain defined by the initial distribution $\pi_j = \mathsf{P}\{\theta(t_s) = j\}$ and the matrix of transition rates $P(\tau) = (p_{ij}(\tau))$, $i,j \in \mathfrak{S} = \{1,\dots,S\}$, where $p_{ij}(\tau)$ is the probability that the system, being in state i at some time instant t, will pass to state j in time τ, i.e. $p_{ij}(\tau) = \mathsf{P}\{\theta(t+\tau) = j|\theta(t) = i\}$. Assume that

$$\mathsf{P}\{\theta(t+\tau) = j|\theta(t) = i\} = \begin{cases} \lambda_{ij}\tau + o(\tau), & i \neq j, \\ 1 + \lambda_{ij}\tau + o(\tau), & i = j, \end{cases}$$

where λ_{ij} are the elements of the stationary intensity matrix Λ with the following properties:

$$\sum_{j=1}^{S}\lambda_{ij}=0,\quad \lambda_{ij}\geqslant 0,\quad \lambda_{ii}<0.$$

Note that a continuous-time homogeneous Markov chain can be defined using the intensity matrix, which is related to the transition rate matrix by $P(\tau)=e^{\Lambda\tau}$ [1].

Let the target output of the system be represented as

$$z=\text{column}(z_1,z_2,\ldots,z_M),\quad z_m=C_{m,\theta(t)}(t)x,\quad m=1,\ldots,M.$$

We define the generalized ∞ norm of the target output by the relation

$$\|z\|_{g\infty}^2=\sup_{t\in[t_s,t_f]}\max_{m=1,\ldots,M}\mathbb{E}\{|z_m(t)|_2^2\},\quad |z_m(t)|_2^2=z_m^\top(t)z_m(t). \tag{2}$$

System (1) generates a linear operator mapping the initial conditions and exogenous disturbance into the target output, i.e., $\mathcal{S}:(x_0,v)\mapsto z$. We define the generalized $\mathcal{H}_2$ norm of system (1) as the norm of the operator $\mathcal{S}$ as follows:

$$\|\mathcal{S}\|_{g2}^2=\sup_{(x_0,v)\neq 0}\frac{\|z\|_{g\infty}^2}{\mathbb{E}\|v\|_2^2+x_0^\top Rx_0}, \tag{3}$$

where $R=R^\top\succ 0$ is a given weight matrix reflecting the relative importance of considering uncertainties in the initial conditions and exogenous disturbances.

Theorem 1. *The generalized $\mathcal{H}_2$ norm of the Markov jump linear system* (1) *on the finite horizon* $[t_s,t_f]$ *can be calculated as*

$$\|\mathcal{S}\|_{g2}=\sup_{T\in[t_s,t_f]}\gamma(T),$$

where $\gamma(T)$ is the solution of the following semidefinite programming problem with respect to the unknown matrices $Q_l(t)=Q_l^\top(t)\succcurlyeq 0$:

$$\begin{gathered}\inf\gamma^2\\ \begin{bmatrix}\dot Q_l(t)+A_l^\top(t)Q_l(t)+Q_l(t)A_l(t)+\sum_{j=1}^{S}\lambda_{lj}Q_j(t) & Q_l(t)B_l(t)\\ B_l^\top(t)Q_l(t) & -I\end{bmatrix}\preccurlyeq 0,\quad t\in[t_s,T],\\ \sum_{l=1}^{S}\pi_lQ_l(t_s)-R\preccurlyeq 0,\quad \begin{bmatrix}Q_l(T) & C_{m,l}^\top(T)\\ C_{m,l}(T) & \gamma^2I\end{bmatrix}\succcurlyeq 0,\quad m=1,\ldots,M,\quad l\in\mathfrak{S}.\end{gathered} \tag{4}$$

Let

$$t^*=\arg\sup_{T\in[t_s,t_f]}\gamma(T),$$

and let the matrix functions $Q_{\theta(t)}(t)$ be obtained by solving inequalities (4) *on the horizon* $[t_s,t^*]$. *Then the worst-case exogenous disturbance $v^*(t)$ and the vector of initial conditions x_0^* are given by*

$$x_0^*=e_{\max}\left(R^{-1}\sum_{l=1}^{S}\pi_lQ_l(t_s)\right),\quad v^*(t)=\begin{cases}B_{\theta(t)}^\top(t)Q_{\theta(t)}(t)x(t), & t\in[t_s,t^*],\\ 0, & t\in(t^*,t_f],\end{cases} \tag{5}$$

where $x(t)$ is the solution of the Cauchy problem for the system

$$\dot x=(A_{\theta(t)}+B_{\theta(t)}B_{\theta(t)}^\top(t)Q_{\theta(t)}(t))x(t),\quad x(t_s)=x_0^*. \tag{6}$$

Remark 1. Note that in the case of deterministic exogenous disturbances $v \in L_2([t_s, t_f], \mathbb{R}_2^{n_v})$, the generalized $\mathcal{H}_2$ norm satisfies the estimate

$$\sup_{(x_0,v)\neq 0} \frac{\|z\|_{g\infty}^2}{\|v\|_{L_2}^2 + x_0^\top R x_0} \leqslant \|\mathcal{S}\|_{g2}, \tag{7}$$

since the domain for calculating the supremum is reduced. Thus, by calculating the generalized $\mathcal{H}_2$ norm for stochastic disturbances (3), one obtains an upper bound for the case of deterministic disturbances (7).

To find the generalized $\mathcal{H}_2$ norm using the matrix inequalities (4), we perform discretization. Let us introduce, e.g., a uniform grid with step h :

$$t_0 = t_s, \qquad t_k = t_{k-1} + h, \qquad k = 1, \dots, K; \qquad h = \frac{T - t_s}{K}. \tag{8}$$

Then the discrete counterparts of inequalities (4) have the form

$$\begin{gathered} \begin{bmatrix} Q_{l,k+1} - Q_{l,k} + h(A_{l,k}^\top Q_{l,k} + Q_{l,k} A_{l,k} + \sum_{j=1}^{S} \lambda_{lj} Q_{j,k}) & h Q_{l,k} B_{l,k} \\ h B_{l,k}^\top Q_{l,k} & -hI \end{bmatrix} \preccurlyeq 0, \\ \sum_{l=1}^{S} \pi_l Q_{l,0} - R \preccurlyeq 0, \quad \begin{bmatrix} Q_{l,K} & C_{m,l,K}^\top \\ C_{m,l,K} & \gamma^2 I \end{bmatrix} \succcurlyeq 0, \quad m = 1, \dots, M, \quad l \in \mathfrak{S}, \end{gathered} \tag{9}$$

where $A_{l,k} = A_l(t_k)$, $B_{l,k} = B_l(t_k)$, $C_{m,l,k} = C_{m,l}(t_k)$, and $Q_{l,k} = Q_l(t_k)$, $k = 0, \dots, K-1$.

Corollary 1. *The generalized $\mathcal{H}_2$ norm of the Markov jump linear system* (1) *on the finite horizon $[t_s, t_f]$ can be calculated as*

$$\|\mathcal{S}\|_{g2} = \sup_{T \in [t_s, t_f]} \gamma(T),$$

where $\gamma(T)$ is the solution of the semidefinite programming problem

$$\begin{gathered} \inf \gamma^2 \\ \begin{bmatrix} -\dot{Y}_l(t) + A_l(t) Y_l(t) + Y_l(t) A_l^\top(t) + B_l(t) B_l^\top(t) + \lambda_{ll} Y_l(t) & V_l(t) \\ V_l^\top(t) & -W_l(t) \end{bmatrix} \preccurlyeq 0, \\ \begin{bmatrix} Y_l(T) & Y_l(T) C_{m,l}^\top(T) \\ C_{m,l}(T) Y_l(T) & \gamma^2 I \end{bmatrix} \succcurlyeq 0, \quad m = 1, \dots, M, \quad l \in \mathfrak{S}, \quad t \in [t_s, T], \\ \mathcal{L}\big(Y_1(t_s), \dots, Y_S(t_s)\big) \succcurlyeq 0, \end{gathered} \tag{10}$$

where

$$\begin{gathered} V_l(t) = \left[\sqrt{\lambda_{l1}} Y_l(t) \ \dots \ \sqrt{\lambda_{l,l-1}} Y_l(t) \ \ \sqrt{\lambda_{l,l+1}} Y_l(t) \ \dots \ \sqrt{\lambda_{l,S}} Y_l(t)\right], \\ W_l(t) = \operatorname{diag}(Y_1(t), \dots, Y_{l-1}(t), Y_{l+1}(t), \dots, Y_S(t)), \\ \mathcal{L}(Y_1(t_s), \dots, Y_S(t_s)) = \begin{bmatrix} R & \sqrt{\pi_1} I & \dots & \sqrt{\pi_S} I \\ \sqrt{\pi_1} I & Y_1(t_s) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \sqrt{\pi_S} I & 0 & \dots & Y_S(t_s) \end{bmatrix}. \end{gathered} \tag{11}$$

To find the generalized $\mathcal{H}_2$ norm of system (1) using Corollary 1, it is necessary to solve problem (10) for each time instant $T \in [t_s, t_f]$, which is a computationally intensive process. We formulate a theorem yielding an upper bound for the generalized $\mathcal{H}_2$ norm of system (1) rather easily. It will also be used below to design suboptimal generalized $\mathcal{H}_2$ control in the case where the state of the Markov chain is available to the controller.

Theorem 2. *The generalized $\mathcal{H}_2$ norm of the Markov jump linear system* (1) *on the finite horizon $[t_s, t_f]$ satisfies the inequality*

$$\|\mathcal{S}\|_{g2} \leqslant \gamma,$$

where γ is the solution of the following semidefinite programming problem with respect to the unknown matrices $Y_l = Y_l^\top \succcurlyeq 0$, $l \in \mathfrak{S}$:

$$\begin{gathered}
\inf \gamma^2 \\
\begin{bmatrix} -\dot{Y}_l(t) + A_l(t)Y_l(t) + Y_l(t)A_l^\top(t) + B_l(t)B_l^\top(t) + \lambda_{ll}Y_l(t) & V_l(t) \\ V_l^\top(t) & -W_l(t) \end{bmatrix} \preccurlyeq 0, \\
\begin{bmatrix} Y_l(t) & Y_l(t)C_{m,l}^\top(t) \\ C_{m,l}(t)Y_l(t) & \gamma^2 I \end{bmatrix} \succcurlyeq 0, \quad m = 1, \ldots, M, \quad l \in \mathfrak{S}, \quad \in [t_s, t_f], \\
\mathcal{L}(Y_1(t_s), \ldots, Y_S(t_s)) \succcurlyeq 0,
\end{gathered} \tag{12}$$

where $V_l(t)$ and $W_l(t)$ are given by formula (11).

3. CONTROL LAW DESIGN

Consider a linear controlled plant with a random structure and dynamics described by the equations

$$\begin{aligned} \dot{x} &= A_{\theta(t)}(t)x + B_{\theta(t)}(t)v + B^u_{\theta(t)}(t)u, \qquad x(t_s) = x_0, \\ z &= C_{\theta(t)}(t)x + D_{\theta(t)}(t)u, \end{aligned} \tag{13}$$

where $x \in \mathbb{R}_2^{n_x}$ is the plant's state; $v(t) \in \mathbb{R}_2^{n_v}$ is a stochastic exogenous disturbance; $z \in \mathbb{R}_2^{n_z}$ is the target output; $u \in \mathbb{R}_2^{n_u}$ is the control vector (input); $\theta(t)$ is a continuous-time homogeneous Markov chain defined by the initial distribution $\pi_j = \mathsf{P}\{\theta(t_s) = j\}$ and the transition rate matrix $P(\tau) = (p_{ij}(\tau))$, $i, j \in \mathfrak{S}$.

3.1. Feedback Control Considering the Markov Chain State

Let us pose the following problem: it is required to design a linear state-feedback controller considering the Markov chain state,

$$u(t) = \Theta_{\theta(t)}(t)x(t), \quad \theta(t) \in \mathfrak{S}, \tag{14}$$

that minimizes the generalized $\mathcal{H}_2$ norm of the closed-loop system

$$\begin{aligned} \dot{x} &= (A_{\theta(t)}(t) + B^u_{\theta(t)}(t)\Theta_{\theta(t)}(t))x + B_{\theta(t)}(t)v, \quad x(t_s) = x_0, \\ z &= (C_{\theta(t)}(t) + D_{\theta(t)}(t)\Theta_{\theta(t)}(t))x \end{aligned} \tag{15}$$

(system (13) with the controller (14)).

Substituting the matrices of the closed-loop system (15) into inequalities (10), we arrive at the following result.

Theorem 3. *The gain matrices* $\mathbf{\Theta}(t) = (\Theta_1(t), \ldots, \Theta_S(t))$ *of the controllers* (14) *minimizing the generalized* $\mathcal{H}_2$ *norm of system* (13) *are obtained by solving the problem*

$$\|\mathcal{S}\|_{g2} = \inf_{\mathbf{\Theta}(t), t\in[t_s,t_f]} \sup_{T\in[t_s,t_f]} \gamma_{\mathbf{\Theta}}(T), \tag{16}$$

where $\gamma_{\mathbf{\Theta}}(T)$ *is the solution of the semidefinite programming problem*

$$\begin{gathered}\inf \gamma^2 \\ \begin{bmatrix} -\dot{Y}_l + A_l Y_l + Y_l A_l^\top + B_l^u \Theta_l Y_l + Y_l \Theta_l^\top B_l^{u\top} + B_l B_l^\top + \lambda_{ll} Y_l & * \\ V_l^\top & -W_l \end{bmatrix} \preccurlyeq 0, \\ \begin{bmatrix} Y_l(T) & * \\ C_{m,l}(T) Y_l(T) + D_{m,l}(T)\Theta_l Y_l(T) & \gamma^2 I \end{bmatrix} \succcurlyeq 0, \\ \mathcal{L}(Y_1(t_s), \ldots, Y_S(t_s)) \succcurlyeq 0, \quad t \in [t_s, T], \quad m = 1, \ldots, M, \quad l \in \mathfrak{S}. \end{gathered} \tag{17}$$

For brevity, the argument t of matrix functions in the first inequality is omitted, and $*$ *indicates a symmetric element.*

Problem (16) cannot be solved using the existing apparatus since it is necessary to search for the infimum over all possible controller parameters $\mathbf{\Theta}(t)$. This problem vanishes under the conditions of Theorem 2: the feedback parameters can be calculated within the semidefinite programming problem (12). In this case, it seems reasonable to find a controller minimizing the norm bound, the so-called suboptimal generalized $\mathcal{H}_2$ controller.

Theorem 4. *The gain matrices* $\Theta_l(t)$ *of the controllers* (14) *minimizing the bound of the generalized* $\mathcal{H}_2$ *norm of system* (13) *have the form* $\Theta_l(t) = Z_l(t) Y_l^{-1}(t)$, *where* $Y_l = Y_l^\top \succcurlyeq 0$ *and* Z_l *are obtained by solving the semidefinite programming problem*

$$\begin{gathered}\inf \gamma^2 \\ \begin{bmatrix} -\dot{Y}_l + A_l Y_l + Y_l A_l^\top + B_l^u Z_l + Z_l^\top B_l^{u\top} + B_l B_l^\top + \lambda_{ll} Y_l & * \\ V_l^\top & -W_l \end{bmatrix} \preccurlyeq 0, \\ \begin{bmatrix} Y_l(t) & * \\ C_{m,l}(t) Y_l(t) + D_{m,l}(t) Z_l(t) & \gamma^2 I \end{bmatrix} \succcurlyeq 0, \\ \mathcal{L}(Y_1(t_s), \ldots, Y_S(t_s)) \succcurlyeq 0, \quad t \in [t_s, t_f], \quad m = 1, \ldots, M, \quad l \in \mathfrak{S}. \end{gathered} \tag{18}$$

3.2. Feedback Control Independent of the Markov Chain State

Next, we design a controller whose parameters are independent of the Markov chain state:

$$u(t) = \Theta(t) x(t). \tag{19}$$

In this case, the corresponding closed-loop system (system (13) with the controller (19)) takes the form

$$\begin{aligned} \dot{x} &= (A_{\theta(t)}(t) + B^u_{\theta(t)}(t)\Theta(t))x + B_{\theta(t)}(t) v, \qquad x(t_s) = x_0, \\ z &= (C_{\theta(t)}(t) + D_{\theta(t)}(t)\Theta(t))x. \end{aligned} \tag{20}$$

Theorem 5. *The generalized $\mathcal{H}_2$ norm of system (20) satisfies the inequality $\|\mathcal{S}\|_{g2} \leqslant \gamma$ with some positive γ if there exists $\rho > 0$ such that the following LMIs are valid for the matrices $P(t) = P^\top(t)$, $Z(t)$, and $Y_l(t) = Y_l^\top \succcurlyeq 0$, $l \in \mathfrak{S}$:*

$$
\begin{gathered}
\begin{bmatrix}
-\dot{Y}_l + A_l Y_l + Y_l A_l^\top + B_l^u Z + Z^\top B_l^{u\top} + B_l B_l^\top + \lambda_{ll} Y_l & * & * \\
V_l^\top & -W_l & * \\
\rho Z^\top B_l^{u\top} + Y_l - P & 0 & -2\rho P
\end{bmatrix} \preccurlyeq 0, \\
\begin{bmatrix}
Y_l(t) & * & * \\
C_{m,l}(t) Y_l(t) + D_{m,l}(t) Z(t) & \gamma^2 I & * \\
P - Y_l & \rho Z^\top(t) D_{m,l}^\top(t) & 2\rho P
\end{bmatrix} \succcurlyeq 0, \\
\mathcal{L}(Y_1(t_s), \ldots, Y_S(t_s)) \succcurlyeq 0, \qquad t \in [t_s, t_f], \quad m = 1, \ldots, M, \quad l \in \mathfrak{S},
\end{gathered}
\tag{21}
$$

where $\Theta(t) = Z(t) P^{-1}(t)$.

4. MULTI-OBJECTIVE CONTROL

Consider a linear controlled plant with a random structure and dynamics described by the equations

$$
\begin{aligned}
\dot{x} &= A_{\theta(t)}(t) x + B_{\theta(t)}(t) v + B^u_{\theta(t)}(t) u, \qquad x(t_s) = x_0, \\
z^{(k)} &= C^{(k)}_{\theta(t)}(t) x + D^{(k)}_{\theta(t)}(t) u, \qquad k = 1, \ldots, N,
\end{aligned}
\tag{22}
$$

where $x \in \mathbb{R}_2^{n_x}$ is the plant's state; $v(t) \in \mathbb{R}_2^{n_v}$ is a stochastic exogenous disturbance; $z^{(k)} \in \mathbb{R}_2^{n_{z_k}}$ are the target outputs; $u \in \mathbb{R}_2^{n_u}$ is the control vector (input); $\theta(t)$ is a continuous-time homogeneous Markov chain defined by the initial distribution $\pi_j = \mathsf{P}\{\theta(t_s) = j\}$ and the transition rate matrix $P(\tau) = (p_{ij}(\tau))$, $i, j \in \mathfrak{S}$.

Each target output (vector) $z^{(k)}$, $k = 1, \ldots, N$, is represented as the set of vectors

$$
\begin{aligned}
z^{(k)} &= \operatorname{column}(z_1^{(k)}, z_2^{(k)}, \ldots, z_{M_k}^{(k)}), \\
z_m^{(k)} &= C^{(k)}_{m,\theta(t)}(t) x + D^{(k)}_{m,\theta(t)}(t) u, \qquad m = 1, \ldots, M_k.
\end{aligned}
$$

Assume that the impact of the exogenous disturbance on the kth target output is characterized by the performance criterion

$$
J_k^2 = \sup_{(x_0, v) \neq 0} \frac{\|z^{(k)}\|^2_{g\infty}}{\mathbb{E}\|v\|_2^2 + x_0^\top R x_0},
\tag{23}
$$

i.e., it represents the generalized $\mathcal{H}_2$ norm. Therefore, it is possible to formulate and solve a multi-objective control problem: $\min\{J_1, \ldots, J_N\}$. Following [16], we define the auxiliary performance criterion J_α; then, according to Theorem 4.1 [16], Pareto optimal solutions of the above multi-objective problem can be obtained by solving the problem

$$
\begin{gathered}
\min_u J_\alpha, \quad J_\alpha = \max_{k=1,\ldots,N} \frac{J_k}{\alpha_k}, \\
\alpha \in \mathcal{A} := \left\{ \alpha = (\alpha_1, \ldots, \alpha_N) : \alpha_k > 0, \sum_{k=1}^N \alpha_k = 1 \right\},
\end{gathered}
\tag{24}
$$

where the performance criterion J_α represents the Germeier convolution and, moreover, is the generalized $\mathcal{H}_2$ norm of the system

$$\begin{aligned}\dot x &= (A_{\theta(t)}(t) + B^u_{\theta(t)}(t)\Theta_{\alpha,\theta(t)}(t))x + B_{\theta(t)}(t)v, \qquad x(t_s) = x_0,\\ \zeta &= (\mathcal{C}_{\theta(t)}(t) + \mathcal{D}_{\theta(t)}(t)\Theta_{\alpha,\theta(t)}(t))x,\end{aligned} \tag{25}$$

where

$$\zeta = \text{column}\Big(\alpha_1^{-1}z_1^{(1)},\dots,\alpha_1^{-1}z_{M_1}^{(1)},\alpha_2^{-1}z_1^{(2)},\dots,\alpha_2^{-1}z_{M_2}^{(2)},\dots,\alpha_N^{-1}z_1^{(N)},\dots,\alpha_N^{-1}z_{M_N}^{(N)}\Big),$$

$$\mathcal{C}_{\theta(t)} = \text{column}\Big(\alpha_1^{-1}C^{(1)}_{\theta(t)},\dots,\alpha_N^{-1}C^{(N)}_{\theta(t)}\Big), \quad \mathcal{D}_{\theta(t)} = \text{column}\Big(\alpha_1^{-1}D^{(1)}_{\theta(t)},\dots,\alpha_N^{-1}D^{(N)}_{\theta(t)}\Big).$$

A controller $u(t) = \Theta_{\alpha,\theta(t)}(t)x(t)$, $\alpha \in \mathcal{A}$, will be called a multi-objective optimal generalized $\mathcal{H}_2$ controller if it ensures the minimum possible values of the generalized $\mathcal{H}_2$ norm of system (25). Based on Theorem 4, we arrive at the following statement.

Theorem 6. *The gain matrices $\Theta_{\alpha,l}(t)$ of the Pareto suboptimal controllers in terms of the performance criteria J_k, $k = 1,\dots,N$, have the form $\Theta_{\alpha,l}(t) = Z_l(t)Y_l^{-1}(t)$, where $Y_l = Y_l^\top \succcurlyeq 0$ and Z_l are obtained by solving the semidefinite programming problem*

$$\begin{aligned}&\inf \gamma^2\\ &\begin{bmatrix} -\dot Y_l + A_lY_l + Y_lA_l^\top + B_l^uZ_l + Z_l^\top B_l^{u\top} + B_lB_l^\top + \lambda_{ll}Y_l & * \\ V_l^\top & -W_l\end{bmatrix} \preccurlyeq 0,\\ &\begin{bmatrix} Y_l(t) & * \\ C^{(k)}_{m,l}(t)Y_l(t) + D^{(k)}_{m,l}(t)Z_l(t) & \alpha_k^2\gamma^2 I\end{bmatrix} \succcurlyeq 0, \qquad \begin{matrix} m = 1,\dots,M,\\ k = 1,\dots,N,\end{matrix}\\ &\mathcal{L}(Y_1(t_s),\dots,Y_S(t_s)) \succcurlyeq 0, \qquad t \in [t_s,t_f], \qquad l \in \mathfrak{S}.\end{aligned} \tag{26}$$

5. NUMERICAL SIMULATIONS

As an illustration of the above results, we consider a continuous-time linear system with Markov jumps between two states:

$$\begin{aligned}\dot x &= A_{\theta(t)}(t)x + B_{\theta(t)}(t)v + B^u_{\theta(t)}(t)u, \qquad x(t_s) = x_0,\\ z^{(k)} &= C^{(k)}_{\theta(t)}(t)x + D^{(k)}_{\theta(t)}(t)u, \qquad k = 1,2,\end{aligned} \tag{27}$$

with the matrices

$$A_1 = \begin{bmatrix} 1.2 & -2.0\\ 0.1 & 1.1\end{bmatrix}, \quad A_2 = \begin{bmatrix} 0.2 & 0.0\\ 2.0 & 0.3\end{bmatrix}, \quad B_1 = B_1^u = \begin{bmatrix} 0\\ 1\end{bmatrix}, \quad B_2 = B_2^u = \begin{bmatrix} 1\\ 0\end{bmatrix},$$

$$C_1^{(1)} = C_2^{(1)} = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}, \quad D_1^{(1)} = D_2^{(1)} = 0, \quad C_1^{(2)} = C_2^{(2)} = 0, \quad D_1^{(2)} = D_2^{(2)} = \begin{bmatrix} 1\\ 1\end{bmatrix}.$$

This system is considered on the horizon $[0,2]$, the parameters of the Markov chain and the weight matrix are given by

$$\pi = \begin{bmatrix} 0.3\\ 0.7\end{bmatrix}, \quad \Lambda = \begin{bmatrix} -10 & 10\\ 5 & -5\end{bmatrix}, \quad R = \begin{bmatrix} 1 & 0\\ 0 & 1\end{bmatrix}.$$

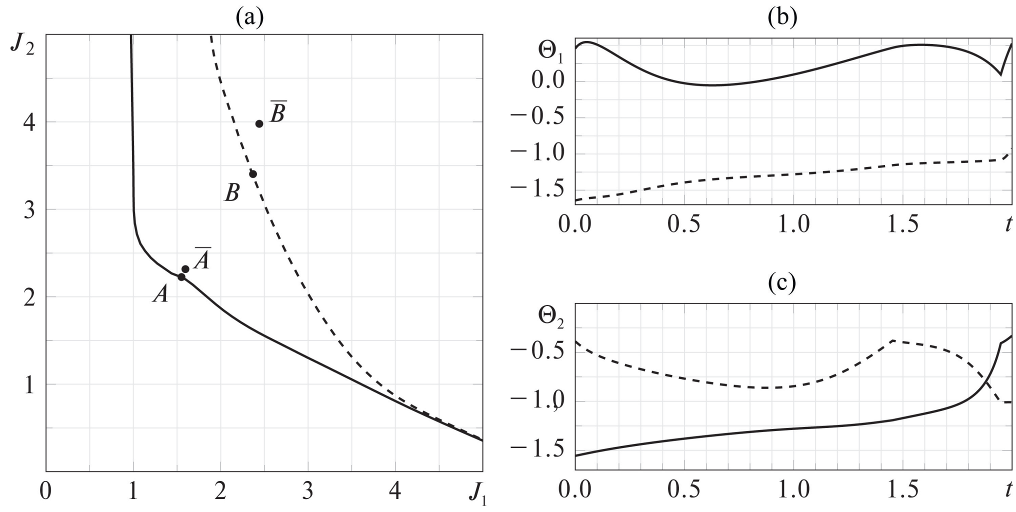

Fig. 1. The Pareto set on the criteria plane (J_1, J_2).

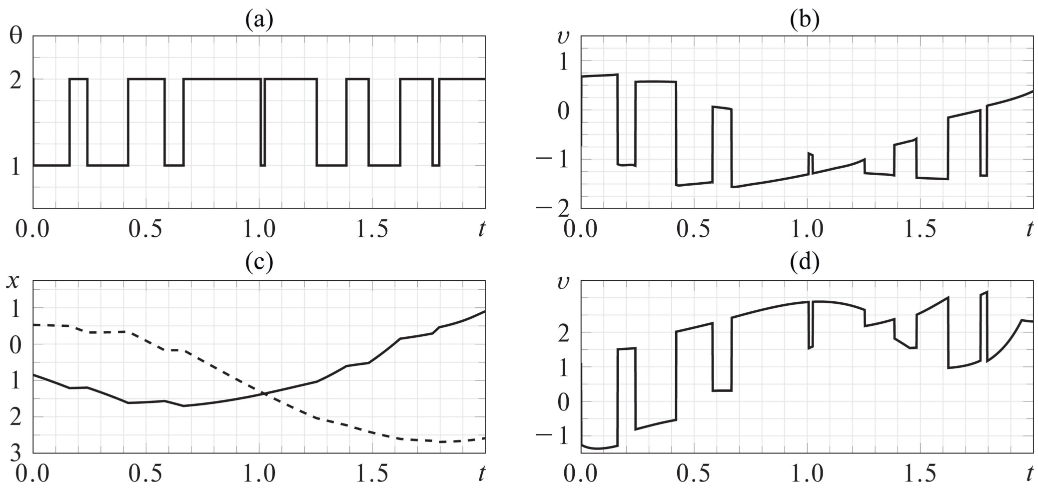

Fig. 2. An example of a Markov process realization.

We introduce two performance criteria J_1 and J_2, which are the generalized $\mathcal{H}_2$ norms of the system with respect to the target outputs $z^{(1)}$ and $z^{(2)}$, respectively:

$$J_1^2 = \sup_{(x_0,v)\neq 0} \frac{\|z^{(1)}\|_{g\infty}^2}{\mathbb{E}\|v\|_2^2 + x_0^\top R x_0}, \qquad J_2^2 = \sup_{(x_0,v)\neq 0} \frac{\|z^{(2)}\|_{g\infty}^2}{\mathbb{E}\|v\|_2^2 + x_0^\top R x_0}. \tag{28}$$

Let us perform discretization on the horizon $[0, 2]$ with a step of $h = 0.001$ and calculate the generalized $\mathcal{H}_2$ norm of system (27) without control. Since the target output is $z^{(2)} = 0$, the generalized $\mathcal{H}_2$ norm of the system coincides with the criterion J_1. The value $\|\mathcal{S}\|_{g2} = 5.7882$ was obtained by solving inequalities (10).

The controllers $\Theta_{\alpha,l}(t) = [\Theta^1_{\alpha,l}, \Theta^2_{\alpha,l}]$, $l = 1, 2$, were designed using Theorem 4, and the corresponding values of the criteria J_1 and J_2 were calculated using Corollary 1. In Fig. 1a, the solid line depicts the Pareto optimal curve on the criteria plane (J_1, J_2). Point $A(1.5509; 2.2492)$ corresponds to the convolution parameter $\alpha = 0.4$. Figures 1b and 1c show the graphs of the Pareto optimal gains $\Theta_{\alpha,l}(t)$ depending on time: the solid curve corresponds to the gain $\Theta^1_{\alpha,l}(t)$ whereas the dashed one to the gain $\Theta^2_{\alpha,l}(t)$.

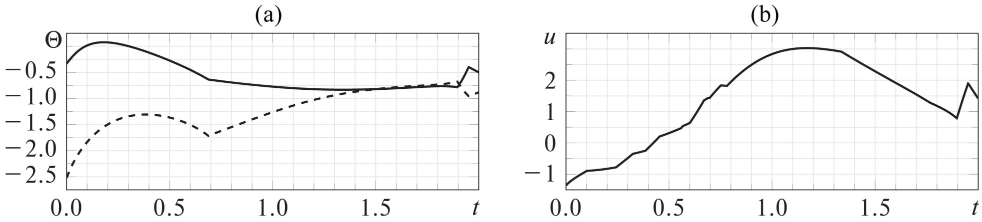

Fig. 3. The coefficients of the gain matrix $\Theta(t)$ and control input $u(t)$.

Next, Fig. 2 presents an example of a Markov process realization: the graphs of the Markov chain state (Fig. 2a), the worst-case disturbances (Fig. 2b), the components x_1 and x_2 (curve) of the system state vector (the solid and dashed curves in Fig. 2c, respectively), and the control input (Fig. 2d).

Note that the gains can be considered slowly varying on the horizon selected. Therefore, it is interesting to compare the values of the criteria under the suboptimal dynamic controller and the static controller corresponding to the average values $\overline{\Theta}_1(t) \equiv [0.2402;\ -1.2928]$ and $\overline{\Theta}_2(t) \equiv [-1.2366;\ -0.6797]$ (see point $\overline{A}(1.5959; 2.3158)$ in Fig. 1a). According to the data presented, the losses in control performance can be considered acceptable, and they are compensated for by the relatively simple-to-implement static controller.

5.1. Control Independent of the Markov Chain State

Now we design a controller whose parameters are independent of the state of the Markov process (19). To solve inequalities (21), let us choose the parameter $\rho = 0.2$, leaving the other simulation parameters unchanged. In Fig. 1a, the upper bound of the Pareto optimal front is plotted by the dashed curve; point $B(2.3697; 3.4030)$ corresponds to the criteria values for $\alpha = 0.4$. For the chosen α, Fig. 3a shows the graphs of the Pareto optimal gains $\Theta_\alpha(t)$ depending on time (the solid curve corresponds to the gain $\Theta^1_\alpha(t)$ whereas the dashed one to the gain $\Theta^2_\alpha(t)$). In Fig. 3b, the control graph for a Markov process realization is presented.

Similar to the previous case, we analyze the behavior of the performance criteria if the dynamic controller is replaced by the static one corresponding to the average values $\overline{\Theta}(t) \equiv [-0.5538;\ -1.2106]$ (see point $\overline{B}(2.4413; 3.9778)$ in Fig. 1a). Clearly, the resulting values of the criteria are worse, but the static controller is simpler to implement, so this approach can be considered justified.

6. CONCLUSIONS

For linear systems with a random structure on a finite horizon, the concept of the generalized $\mathcal{H}_2$ norm has been introduced, and algorithms for its calculation have been presented based on solving both coupled matrix Riccati differential equations and systems of LMIs. Suboptimal generalized $\mathcal{H}_2$ dynamic linear state-feedback control has been designed in the cases where the state of the Markov chain is available and unavailable to the controller. Also, it has been demonstrated how to solve multi-objective control problems if the criteria are generalized $\mathcal{H}_2$ norms.

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APPENDIX

Before proceeding to the proofs of the above theorems, for convenience, we introduce the notation

$$\mathbf{X} = (X_1, X_2, \dots, X_S), \qquad X_l(t) = X_l^\top(t) \succcurlyeq 0, \qquad l \in \mathcal{S}, \tag{A.1}$$

$$\mathcal{R}_l(\mathbf{X}) = \dot{X}_l(t) + A_l^\top(t)X_l(t) + X_l(t)A_l(t) + X_l(t)B_l(t)B_l^\top(t)X_l(t) + \sum_{j=1}^{S}\lambda_{lj}X_j(t) \tag{A.2}$$

and establish an auxiliary result.

Lemma 1. *Let $\mathbf{X} = (X_1, X_2, \dots, X_S)$ be the solution of the equations*

$$\mathcal{R}_l(\mathbf{X}) + M_l = 0, \quad X_l(t_f) = X_l^0, \quad l \in \mathcal{S}, \qquad t \in [t_s, t_f], \tag{A.3}$$

and let $\mathbf{Y} = (Y_1, Y_2, \dots, Y_S)$ be the solution of the equations

$$\mathcal{R}_l(\mathbf{Y}) + N_l = 0, \quad X_l(t_f) = Y_l^0, \quad l \in \mathcal{S}, \quad t \in [t_s, t_f], \tag{A.4}$$

where $0 \preccurlyeq M_l(t) \preccurlyeq N_l(t)$ and $0 \preccurlyeq X_l^0 \preccurlyeq Y_l^0$, $l \in \mathcal{S}$, $t \in [t_s, t_f]$. Then

$$X_l(t) \preccurlyeq Y_l(t), \quad l \in \mathcal{S}, \quad t \in [t_s, t_f]. \tag{A.5}$$

Proof of Lemma 1. Consider the system of Lyapunov equations

$$\begin{gathered}\dot{P}_l + (A_l + B_lB_l^\top X_l)^\top P_l + P_l(A_l + B_lB_l^\top X_l) + \sum_{j=1}^{S}\lambda_{lj}P_j \\ + (X_l - Y_l)B_lB_l^\top(X_l - Y_l) + N_l - M_l = 0\end{gathered} \tag{A.6}$$

on the time interval $[t_s, t_f]$ with the boundary conditions $P_l(t_f) = Y_l^0 - X_l^0$, $l \in \mathcal{S}$. This equation has a unique solution $P_l(t)$, $l \in \mathcal{S}$, with $P_l(t) \succcurlyeq 0$ since $(X_l - Y_l)B_lB_l^\top(X_l - Y_l) + N_l - M_l \succcurlyeq 0$ and $P_l(t_f) \succcurlyeq 0$ [1, 25]. Note that $P_l(t) = Y_l(t) - X_l(t)$ is the solution of equation (A.6), which finally gives (A.5).

Proof of Theorem 1. We write the functional (3) in the following form:

$$\|\mathcal{S}\|_{g2} = \sup_{T\in[t_s,t_f]}\max_{m=1,\dots,M}\gamma_m(T), \qquad \gamma_m^2(T) = \sup_{(x_0,v)\neq 0}\frac{\mathbb{E}|z_m(T)|_2^2}{\mathbb{E}\|v\|_2^2 + x_0^\top Rx_0}.$$

Due to the linearity of the operator $\mathcal{S}$, the last equality can be written as

$$\sup_{(x_0,v)\neq 0}\mathbb{E}\left\{|z_m(T)|_2^2 - \gamma^2\left(\|v\|_2^2 + x_0^\top Rx_0\right)\right\} = 0. \tag{A.7}$$

(Hereinafter, for brevity, the argument and index $\gamma_m(T)$ are omitted.) We introduce the Bellman function at a time instant t :

$$V(t, x_t, l) = \sup_v \mathbb{E}\left\{z_m^\top(T)z_m(T) - \gamma^2\int_t^T v^\top(\tau)v(\tau)d\tau\,\Big|\,x(t) = x_t,\ \theta(t) = l\right\}. \tag{A.8}$$

Then the relation (A.7) becomes

$$\sup_{x_0\neq 0}\mathbb{E}\left\{V(t_s, x_0, \theta_0) - \gamma^2 x_0^\top Rx_0\right\} = 0, \qquad \theta_0 = \theta(t_s). \tag{A.9}$$

Let us calculate $V(t_s, x_0, \theta_0)$ using the stochastic Bellman equation [26]

$$\max_v \{\mathcal{L}^v V(t, x(t), l) - \gamma^2 v^\top(t) v(t)\} = 0,$$
$$V(T, x(T), l) = x^\top(T) C_{m,l}^\top(T) C_{m,l}(T) x(T), \tag{A.10}$$

where the infinitesimal generator $\mathcal{L}^v$ has the form

$$\mathcal{L}^v g(t, x, l) = \frac{\partial g(t, x, l)}{\partial t} + (A_l(t)x + B_l(t)v)^\top \nabla_x g(t, x, l) + \sum_{j=1}^{S} \lambda_{lj} g(t, x, j). \tag{A.11}$$

We seek a solution in the class of quadratic forms $V(t, x(t), l) = x^\top(t) X_l(t) x(t)$, $X_l(t) = X_l^\top(t) \succcurlyeq 0$. From this point onwards, for brevity again, the arguments of the functions $x(t)$ and $v(t)$ will be omitted. Substituting the infinitesimal generator (A.11) into equation (A.10) yields

$$\max_v \left\{ x^\top \dot{X}_l(t) x + 2(A_l(t)x + B_l(t)v)^\top X_l(t) x + \sum_{j=1}^{S} \lambda_{lj} x^\top X_j(t) x - \gamma^2 v^\top v \right\} = 0. \tag{A.12}$$

Since the expression in curly braces is a concave functional in the variable v, a solution of this problem does exist. To obtain it, we find the stationary point v^* :

$$v^* = \gamma^{-2} B_l^\top(t) X_l(t) x. \tag{A.13}$$

Omitting the arguments of matrix functions and substituting (A.13) into (A.12), after straightforward simplifications, we get

$$x^\top \left(\dot{X}_l + A_l^\top X_l + X_l A_l + \gamma^{-2} X_l B_l B_l^\top X_l + \sum_{j=1}^{S} \lambda_{lj} X_j \right) x = 0. \tag{A.14}$$

Equality (A.14) must hold for any value of x; therefore, one arrives at the differential matrix equation

$$\dot{X}_l + A_l^\top X_l + X_l A_l + \gamma^{-2} X_l B_l B_l^\top X_l + \sum_{j=1}^{S} \lambda_{lj} X_j = 0 \tag{A.15}$$

with the boundary conditions $X_l(T) = C_{m,l}^\top(T) C_{m,l}(T)$. Then, at the initial time, we have $V(t_s, x_0, \theta_0) = x_0^\top X_{\theta_0}(t_s) x_0$, and the calculation of (A.9) reduces to

$$\sup_{x_0 \neq 0} x_0^\top \left(\sum_{l=1}^{S} \pi_l X_l(t_s) - \gamma^2 R \right) x_0. \tag{A.16}$$

This expression is a quadratic form in the variable x_0 and reaches its maximum at the point

$$x_0^* = e_{\max} \left(R^{-1} \sum_{l=1}^{S} \pi_l X_l(t_s) \right) \tag{A.17}$$

under the condition

$$\sum_{l=1}^{S} \pi_l X_l(t_s) - \gamma^2 R \preccurlyeq 0; \tag{A.18}$$

in this case, the value of γ is given by

$$\gamma = \lambda_{\max}^{1/2}\left(R^{-1}\sum_{l=1}^{S}\pi_l X_l(t_s)\right). \tag{A.19}$$

Writing condition (A.19) using an LMI with the minimum value of γ, after the change of variables $X_l(t) = \gamma^2 Q_l(t)$, we obtain the following optimization problem on the interval $[t_s, T]$ for calculating $\gamma_m(T)$:

$$\begin{aligned}&\inf \gamma^2\\ &\mathcal{R}_l(\mathbf{Q}) = 0,\ \ \sum_{l=1}^{S}\pi_l Q_l(t_s) - R \preccurlyeq 0,\ \ Q_l(T) = \gamma^{-2}C_{m,l}^{\top}(T)C_{m,l}(T),\ \ l \in \mathfrak{S}.\end{aligned} \tag{A.20}$$

Next, we show that $\gamma_m(T)$ can be found by solving the following semidefinite programming problem on the interval $[t_s, T]$:

$$\begin{aligned}&\inf \gamma^2\\ &\mathcal{R}_l(\mathbf{Q}) \preccurlyeq 0,\ \ \sum_{l=1}^{S}\pi_l Q_l(t_s) - R \preccurlyeq 0,\ \ Q_l(T) \succcurlyeq \gamma^{-2}C_{m,l}^{\top}(T)C_{m,l}(T),\ \ l \in \mathfrak{S}.\end{aligned} \tag{A.21}$$

Let γ_1 be the solution of problem (A.20), and let γ_2 and $\mathbf{Q} = (Q_1, Q_2, \ldots, Q_S)$ be obtained by solving (A.21). The solution of (A.20) is a solution of (A.21) if the corresponding inequalities hold as equalities, therefore $\gamma_2 \leqslant \gamma_1$.

Assume that $\gamma_2 < \gamma_1$. Let $\mathbf{X} = (X_1, X_2, \ldots, X_S)$ be the solution of the equations $\mathcal{R}_l(\mathbf{X}) = 0$, $l \in \mathfrak{S}$, on the interval $[t_s, T]$ with the boundary conditions $X_l(T) = \gamma_2^{-2}C_{m,l}^{\top}(T)C_{m,l}(T)$. Then, due to Lemma 1, we have

$$X_l(t) \preccurlyeq Q_l(t), \quad l \in \mathcal{S}, \quad t \in [t_s, T]. \tag{A.22}$$

Since $\pi_l \geqslant 0$, $l \in \mathfrak{S}$, the relations

$$\sum_{l=1}^{S}\pi_l X_l(t_s) \preccurlyeq \sum_{l=1}^{S}\pi_l Q_l(t_s) \preccurlyeq R \tag{A.23}$$

hold at the initial time instant. Hence, $\mathbf{X}$ is a solution of the problem (A.20) for $\gamma_2 < \gamma_1$, which contradicts the condition $\gamma_1^2 = \inf \gamma^2$; consequently, the assumption $\gamma_2 < \gamma_1$ is false, and $\gamma_1 = \gamma_2$.

Note that for a fixed m, solving problem (A.21) yields $\gamma_m(T)$. To find $\max\limits_{m=1,\ldots,M}\gamma_m(T)$, it is necessary to supplement inequalities (A.21) with enumeration over all possible values of m, i.e., solve (A.21) for $m = 1, \ldots, M$. Applying the Schur complement lemma to inequalities (A.21), we finally arrive at conditions (4). The proof of Theorem 1 is complete.

Proof of Corollary 1. In inequalities (A.21), we make the change of variables $Y_l(t) = Q_l^{-1}(t)$, $l \in \mathfrak{S}$, and multiply the first and third inequalities by $Y_l(t)$ on the left and right. After that, using the Schur complement lemma, we get the expressions (10). The proof of Corollary 1 is complete.

Proof of Theorem 2. Let the generalized $\mathcal{H}_2$ norm of system (1), $\|\mathcal{S}\|_{g2}$, be achieved at a time instant t^*, i.e.,

$$\|\mathcal{S}\|_{g2} = \sup_{T\in[t_s,t_f]}\gamma(T), \quad t^* = \arg\sup_{T\in[t_s,t_f]}\gamma(T),$$

where $\gamma(T)$ is the solution of problem (10). Thus, for $T = t^*$, we obtain $\inf\gamma^2 = \gamma^2(t^*) = \|\mathcal{S}\|_{g2}^2$ in problem (10).

Let solving problem (12) yield a value $\widehat{\gamma}$ and matrices $\widehat{Y}_l(t)$, $t \in [t_s, t_f]$. Assume that $\widehat{\gamma} < \|\mathcal{S}\|_{g2}$; then for $T = t^*$ and $Y_l(t) = \widehat{Y}_l(t)$, $t \in [t_s, t^*]$, problem (10) has a solution $\widehat{\gamma}^2 < \|\mathcal{S}\|_{g2}^2$, which contradicts the condition $\inf \gamma^2 = \|\mathcal{S}\|_{g2}^2$. Hence, the assumption $\widehat{\gamma} < \|\mathcal{S}\|_{g2}$ is false, and $\|\mathcal{S}\|_{g2} \leqslant \widehat{\gamma}$. The proof of Theorem 2 is complete.

Proof of Theorem 4. We substitute the matrices of the closed-loop system (15) into inequalities (12) and apply the changes $\Theta_l(t)Y_l(t) = Z_l(t)$, making the inequalities linear. These transformations lead to inequalities (18). The proof of Theorem 4 is complete.

Proof of Theorem 5. Let us utilize the approach outlined in [27]. Multiplying the first and second inequalities of (21) on the left and right by

$$\begin{bmatrix} I & 0 & \rho^{-1}(Y_l - P)P^{-1} \\ 0 & I & 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} I & 0 & \rho^{-1}(Y_l - P)P^{-1} \\ 0 & I & 0 \end{bmatrix}^\top,$$

respectively, we obtain

$$\begin{aligned} &\begin{bmatrix} -\dot{Y}_l + (A_l + B_l^u\Theta)Y_l + Y_l(A_l + B_l^u\Theta)^\top + B_lB_l^\top + \lambda_{ll}Y_l & V_l \\ V_l^\top & -W_l \end{bmatrix} \preccurlyeq 0, \\ &\begin{bmatrix} Y_l & Y_l(t)(C_{m,l} + D_{m,l}\Theta)^\top \\ (C_{m,l} + D_{m,l}\Theta)Y_l & \gamma^2 I \end{bmatrix} \succcurlyeq 0. \end{aligned} \tag{A.24}$$

Together with the third inequality of (21), they are the conditions for calculating the bound of the generalized $\mathcal{H}_2$ norm of system (20) using Theorem 2. The proof of Theorem 5 is complete.

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