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LINEAR SYSTEMS

An Algorithm for Constructing Gain Matrices in the Spectrum Assignment Problem of a Linear Control System

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Abstract—This paper proposes an algorithm for constructing gain matrices in the spectrum assignment problem of a continuous-time linear dynamic control system without any constraints on the matrix coefficients of the system. The algorithm is based on constructing eigenvectors and adjoined vectors corresponding to the given eigenvalues of the corresponding matrix. An algorithm with a minimum number of simple algebraic operations is developed for their construction. As a result, to solve the above problem, a complete set of gain matrices is constructed, depending on a certain number of arbitrary scalar parameters. Cases of the uniqueness of such a matrix are determined. Illustrative examples are provided in the cases of a simple spectrum and a multiple one. Gain matrices are constructed for a dynamic system describing the operation of a multi-chamber heating furnace.

Keywords: continuous-time linear systems, state feedback, spectrum assignment problem, cascade decomposition method, algorithm for constructing a set of gain matrices

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1. INTRODUCTION

The problem of stabilizing the programmed motion of a dynamic system is one of the most important challenges in control theory, dictated by pressing practical demands.

The solution of the stabilization problem has a long history of over 150 years and an extensive bibliography, including both purely theoretical and practical works. The history up to 2019 was described in detail in [1], with a comprehensive list of references containing 107 items.

A more general problem is also of great interest: if a linear dynamic feedback control system is described by a set of differential equations with respect to the unknown components of the state vector of the system, it is required to determine an appropriate feedback (gain matrix) under which the spectrum of the matrix coefficient at the state vector will possess definite properties. For example, it should be located in a desired region of the complex plane. In particular, if this region lies in the left half-plane of the complex plane, then the corresponding gain matrix is constructed by stabilizing the programmed motion of the control system under consideration.

In a special case, the region for locating the spectrum points can be a finite set of arbitrarily given numbers. In this case, the problem of finding the corresponding gain matrix is called pole assignment, spectrum control, or spectrum assignment in the literature.

In the monograph [2], V.I. Zubov studied a nonlinear time-varying control system and established sufficient conditions for the existence of additional components that can be introduced into the system to stabilize it; also, he provided a method for finding such additional components.

For a linear time-invariant control system of the form

$$\dot{x} = Ax + Bu, \tag{1.1}$$

where $x = x(t)$, $x \in \mathbb{R}^n$, $u = u(t)$, $u \in \mathbb{R}^m$, $\dot{x} = \frac{dx}{dt}$, $A : \mathbb{R}^n \to \mathbb{R}^n$, $B : \mathbb{R}^m \to \mathbb{R}^n$, and $t \in [t_0, t_k]$, the spectrum assignment problem consists in the following:

For arbitrarily given numbers $\{\lambda_j\}_{j=1}^n$, it is required to construct a state-feedback control law

$$u = Kx \tag{1.2}$$

under which the spectrum of the matrix $A + BK$ will coincide with $\{\lambda_j\}_{j=1}^n$, i.e.,

$$\det(A + BK - \lambda_j I) = 0. \tag{1.3}$$

In the case of complex values λ_j, they must be pairwise ordered and complex conjugate.

For a linear time-invariant system with a one-dimensional control vector (scalar control), V.I. Zubov essentially derived a sufficient condition for the solvability of the spectrum assignment problem: this is the controllability condition of the system. The spectrum was considered arbitrary, and the author passed to the spectrum located in the left half-plane of the complex plane only to stabilize the system [2, pp. 153 and 154].

The complete result on the existence of a solution of problem (1.1)–(1.3) was obtained by W.M. Wonham [3, 4, p. 79]:

Theorem 1. *Problem* (1.1)–(1.3) *is solvable if and only if the pair* (A, B) *is controllable.*

Consequently, the condition

$$\operatorname{rank}(B\,AB\,\ldots\,A^{n-1}B) = n$$

must hold.

The history of the proof of this theorem by some authors, with the aim of simplification, was presented in [1].

A practical solution of problem (1.1)–(1.3) was obtained for the case $m = 1$ as a solution of the equation

$$\det(A + BK - \lambda I) = f(\lambda),$$

where $f(\lambda)$ is a polynomial of degree n with corresponding real coefficients that contains the components of the row matrix $K = (k_1 \quad k_2 \quad \ldots \quad k_n)$. This equation is uniquely solvable with respect to k_i, $i = 1, \ldots, n$, and the solution is given by the exact Bass–Gura and Ackermann formulas [1, p. 579].

A numerical implementation of problem (1.1)–(1.3) in MATLAB, based on the full pole placement method [5], was provided in [6].

For the controllable system (1.1) with given A and B, $\operatorname{Ker} B = \{0\}$, a method for constructing a set of gain matrices was described in [7]. For this purpose, Jordan chains of vectors of the matrix $(A - \lambda I \quad B)$ for each $\lambda = \lambda_j$ were used, and additional parameterizing matrices were introduced.

The so-called cascade decomposition method for solving problem (1.1)–(1.3) without any constraints on A and B was proposed in [8]. (For brevity, it will be referred to as the cascade method.) This method allows one to establish that either system (1.1) is uncontrollable (and the matrix K cannot be constructed) or system controllability holds (in this case, the method yields a complete manifold of all matrices K corresponding to the given A, B, and $\{\lambda_j\}_{j=1}^n$). Thus, another proof of Theorem 1 was obtained and, moreover, a method for constructing a complete set of gain matrices was developed.

Problem (1.1)–(1.3) is solved as follows [8].

The equation

$$(A + BK)v_j = \lambda_j v_j \tag{1.4}$$

is solved with respect to v_j and K; first, a set of n linearly independent vectors v_j corresponding to the eigenvalues λ_j is constructed; then, the components of the matrix K are found. The cascade method is used only to construct v_j.

However, application of the cascade method is rather computationally intensive: it involves decompositions of spaces into subspaces, projectors onto subspaces, and semi-inverse matrices.

The goal of this paper is to maximally simplify the solution of problem (1.1)–(1.3); the idea is to create an algorithm for constructing n linearly independent eigenvectors and adjoined vectors of the matrix $A + BK$ that requires only solving systems of linear algebraic equations and checking the linear independence of the resulting vectors at the last step of the algorithm, e.g., by computing determinants.

Such a design of gain matrices appreciably simplifies the computational process and creates the necessary prerequisites for developing simpler computational programs.

For the sake of comparison, note the following: Moore–Penrose pseudoinverse matrices were used in [6] to solve the applied problem (1.1)–(1.3) in MATLAB; they were computed as the limit of a sequence of certain matrices. In this case, only one matrix K is constructed, although there exists a set of such matrices [8, p. 2024] for the problem considered in [6].

In this paper, we reveal a complete set of gain matrices for a particular problem by establishing the dependence of the matrix on some numerical parameters, some being arbitrary while the others satisfying certain conditions.

The manifold of gain matrices is very useful in applications: one can select an appropriate matrix, e.g., with a smaller norm or with fewer nonzero components.

The number of steps in the algorithm for finding the eigenvectors and adjoined vectors of the matrix $A + BK$ can be determined in advance: there are exactly p of them, where $p = \min q$ and the number q is given by the condition

$$\operatorname{rank}(B\,AB\,\ldots\,A^qB) = n. \tag{1.5}$$

The algorithm proposed below is based on the cascade method [8] and represents its significant simplification. The algorithm for constructing the eigenvectors and adjoined vectors of the matrix $A + BK$ is justified in the Appendix.

As illustrative examples, we find gain matrices in the cases of a simple spectrum and a multiple one and construct gain matrices for a dynamic system describing the operation of a multi-chamber heating furnace.

2. SOLVING A LINEAR ALGEBRAIC EQUATION WITH A SINGULAR MATRIX AT THE UNKNOWN

A linear system of the form

$$Cx = y \tag{2.1}$$

with a singular matrix C can be solved with respect to x only under some constraint (condition) imposed on y :

$$Qy = 0. \tag{2.2}$$

The solution x may be nonunique, i.e.,

$$x = C^{-}y + z, \tag{2.3}$$

where C^{-} is some matrix, and z is an arbitrary vector such that $Cz = 0$.

The constraint (2.2) will be called the well-posedness condition for system (2.1), and the vector (2.3) will be called the solution of system (2.1).

For example, let $C = \begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 2 & 0 & 2 \\ 0 & 0 & 0 \end{pmatrix}$, i.e.,

$$\begin{gathered} x_1 + x_3 = y_1, \\ x_2 = y_2, \\ 2x_1 + 2x_3 = y_3, \\ 0 = y_4; \end{gathered} \tag{2.4}$$

then $x = \begin{pmatrix} y_1 \\ y_2 \\ 0 \end{pmatrix} + \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$ is the solution of system (2.4) with an arbitrary value x_3 under the conditions

$$\begin{gathered} -2y_1 + y_3 = 0, \\ y_4 = 0. \end{gathered}$$

Consequently, $Qy = 0$, where $Q = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ -2 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$. In this case, $z = \begin{pmatrix} -x_3 \\ 0 \\ x_3 \end{pmatrix}$.

System (2.4) can also be solved as follows:

$x = \begin{pmatrix} 0 \\ y_2 \\ y_1 \end{pmatrix} + \begin{pmatrix} x_1 \\ 0 \\ -x_1 \end{pmatrix}$ for any x_1 under the well-posedness condition

$$\begin{gathered} y_1 - \frac{1}{2}y_3 = 0, \\ y_4 = 0. \end{gathered}$$

But the solution set and the well-posedness condition are equivalent in this case; hence, the particular form of the solution with the corresponding well-posedness condition does not matter in practice.

3. THE ALGORITHM FOR CONSTRUCTING A GAIN MATRIX IN THE CASE OF A SIMPLE SPECTRUM

3.1. Preliminary Transformations

A_1. In the case of simple eigenvalues λ_j, $j = 1, \ldots, n$, in equation (1.4)

$$(A + BK)v(\lambda_j) = \lambda_j v(\lambda_j), \tag{3.1}$$

introduce the designation

$$Kv(\lambda_j) = f(\lambda_j). \tag{3.2}$$

A_2. Write equation (3.1) as

$$Bf(\lambda_j) = (\lambda I - A)v(\lambda_j). \tag{3.3}$$

A_3. Solve equation (3.3) with respect to $f(\lambda_j)$:

$$f(\lambda_j) = B^-(\lambda_j I - A)v(\lambda_j) + z(\lambda_j) \tag{3.4}$$

with an arbitrary vector $z = z(\lambda_j)$ under the well-posedness condition

$$Q(\lambda_j I - A)v(\lambda_j) = 0. \tag{3.5}$$

As a result of preliminary transformations, the relation (3.1) is replaced by an equivalent system of the three relations (3.2), (3.4), and (3.5).

3.2. The Algorithm for Solving Equation (3.5) *with Respect to the Vector* $v = v(\lambda_j)$

Forward pass

The first step. A_{11}. In equation (3.5) with the subscript j omitted for now, denote by $w_1 = w_1(\lambda)$ the coefficient at λ (i.e., Qv); then (3.5) is equivalent to the system

$$\begin{cases} Qv = w_1 \\ QAv = \lambda w_1. \end{cases} \tag{3.6}$$

A_{12}. Obtain the solution v of system (3.6) in the form

$$v = G_1(\lambda, w_1) + z_1 \tag{3.7}$$

with some matrix G_1 and an arbitrary vector $z_1 = z_1(\lambda)$ under the well-posedness condition

$$Q_1(\lambda, w_1) = 0. \tag{3.8}$$

The second step. A_{21}. In the relation (3.8), denote by $w_2 = w_2(\lambda)$ the coefficient at λ (i.e., $Q_1 w_1$); then (3.8) is equivalent to the system

$$\begin{cases} Q_1 w_1 = w_2 \\ Q_{11} w_1 = \lambda w_2 \end{cases} \tag{3.9}$$

with some matrix Q_{11}.

A_{22}. Obtain the solution of this system in the form

$$w_1 = G_2(\lambda, w_2) + z_2$$

with some matrix G_2 and an arbitrary vector $z_2 = z_2(\lambda)$ under the well-posedness condition

$$Q_2(\lambda, w_2) = 0.$$

And so on $\ldots$.

The ith step. A_{i1}. In the well-posedness condition identified at the $(i-1)$th step,

$$Q_{i-1}(\lambda, w_{i-1}) = 0, \qquad i = 2, 3, \ldots, \tag{3.10}$$

denote by $w_i = w_i(\lambda)$ the coefficient at λ; as a result, (3.10) is equivalent to the system

$$\left\{ \begin{array}{l} Q_{i-1}w_{i-1} = w_i \\ Q_{i-11}w_{i-1} = \lambda w_i. \end{array} \right. \tag{3.11}$$

A_{i2}. Obtain the solution of this system in the form

$$w_{i-1} = G_i(\lambda, w_i) + z_i \tag{3.12}$$

with some matrix G_i and an arbitrary vector $z_i = z_i(\lambda)$ under the well-posedness condition

$$Q_i(\lambda, w_i) = 0.$$

And so on ....

The pth step. A_{p2}. As proved in [8], if system (1.1) is completely controllable, then for $i = p$ (see (1.5)) system (3.11) is solvable with respect to w_{p-1} without any well-posedness condition:

$$w_{p-1} = G_p(\lambda, w_p) + z_p \tag{3.13}$$

with some matrix G_p and an arbitrary vector $z_p = z_p(\lambda)$. The element $w_p = w_p(\lambda)$ is also arbitrary.

Construct w_{p-1}.

3.3. The Algorithm for Constructing the Vector $v(\lambda)$

Backward pass

B_1. Substitute the vector w_{p-1} constructed at step A_p into (3.12) with $i = p - 1$, thereby determining the vector w_{p-2}.

B_2. Substitute the vector w_{p-2} constructed at the previous step into (3.12) with $i = p - 2$, thereby determining the vector w_{p-3}.

And so on ....

B_{p-2}. Substitute the vector w_2 constructed at step B_{p-3} into (3.12) with $i = 2$, thereby determining the vector w_1.

B_{p-1}. Substitute the vector w_1 constructed at step B_{p-2} into (3.7), thereby determining the vector v :

$$v = G(w_p(\lambda), z_p(\lambda), z_{p-1}(\lambda), \ldots, z_1(\lambda)) \tag{3.14}$$

with some matrix G and arbitrary vector functions $w_p(\lambda)$ and $z_s(\lambda)$, $s = 1, \ldots, p$.

3.4. Constructing the System of n Linearly Independent Vectors $v(\lambda_j)$

If system (1.1) is completely controllable, then for any set $\{\lambda_j\}_{j=1}^n$ there exist elements $w_p(\lambda), z_p(\lambda), z_{p-1}(\lambda), \ldots, z_1(\lambda)$ such that the vectors $v(\lambda_j)$ (3.14) with the above $w_{p-1}(\lambda), z_p(\lambda), z_{p-1}(\lambda), \ldots, z_1(\lambda)$ are linearly independent [8].

The eigenvalues λ_j can also be complex, pairwise ordered.

If system (1.1) is not completely controllable, then n linearly independent vectors $v(\lambda_j)$ cannot be constructed, and the same applies to the matrix K [8].

Here, one should construct a vector $v(\lambda_j)$ such that the vectors $v(\lambda_1)$, $v(\lambda_2)$, ..., $v(\lambda_n)$ are linearly independent for any set $\{\lambda_j\}_{j=1}^n$.

3.5. The Algorithm for Constructing a Gain Matrix in the Case of a Simple Spectrum

C_1. Substitute the vectors $v_j = v(\lambda_j)$ constructed in Section 3.4 into formula (3.4), thereby determining $f_j = f_j(\lambda_j)$, $j = 1, \ldots, n$.

C_2. Substitute v_j and f_j into (3.2) for each value of j. As a result, n linear equations with unknowns $k_{ij}, i = 1, \ldots, m$ (the components of matrix K) are obtained.

C_3. Extract from the above equations those containing the components k_{1j} as the unknowns, thereby forming a system of linear algebraic equations with the principal determinant $\triangle_1$ made up of the components of the linearly independent vectors $v(\lambda_j)$; hence, $\triangle_1 \neq 0$.

C_4. Solve the system obtained at step C_3 with respect to k_{1j} (the components of the first row of the matrix K).

C_5. Repeat steps C_3 and C_4 for $i = 2, 3, \ldots, m$, thereby determining the components of the ith rows of the matrix K.

Example 1. Let $A = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 1 \\ 1 & 1 \\ 1 & 1 \end{pmatrix}$, and $K = (k_{ij})$, $i = 1, 2$, $j = \overline{1, 3}$.

We solve the equation $Bf = (\lambda I - A)v$, where $f = (f_1, f_2)$ and $v = (v_1, v_2, v_3)$, with respect to f (step A_3):

$$f_1 + f_2 = \lambda v_1,$$
$$f_1 + f_2 = -v_1 + \lambda v_2,$$
$$f_1 + f_2 = -v_2 + \lambda v_3.$$

Consequently,

$$f_1 = \lambda v_1 - z(\lambda), \quad f_2 = z(\lambda) \tag{3.15}$$

with an arbitrary vector $z(\lambda)$ ((3.15) is (3.4)) under the well-posedness condition (3.5):

$$\lambda v_1 = -v_1 + \lambda v_2,$$
$$\lambda v_1 = -v_2 + \lambda v_3,$$

or

$$\begin{cases} \lambda(v_1 - v_2) = -v_1 \\ \lambda(v_1 - v_3) = -v_2. \end{cases} \tag{3.16}$$

Following step A_{11}, we denote

$$v_1 - v_2 = w_{11}, \quad v_1 - v_3 = w_{12}. \tag{3.17}$$

Then (3.16) becomes

$$\begin{cases} -v_1 = \lambda w_{11} \\ -v_2 = \lambda w_{12}. \end{cases} \tag{3.18}$$

The system composed of (3.17) and (3.18) is system (3.6).

We solve system (3.17), (3.18) with respect to v_1, v_2, and v_3 (see step A_{12}, formula (3.7)):

$$v_1 = -\lambda w_{11}, \quad v_2 = -\lambda w_{12}, \quad v_3 = -\lambda w_{11} - w_{12} \tag{3.19}$$

(here $z_1 = 0$) under the well-posedness condition of system (3.17), (3.18): $-\lambda w_{11} + \lambda w_{12} = w_{11}$. As a result, we arrive at the relation (3.8):

$$\lambda(-w_{11} + w_{12}) = w_{11}.$$

Following step A_{21}, we denote

$$-w_{11} + w_{12} = w_2;$$

then

$$w_{11} = \lambda w_2.$$

The last two equalities are system (3.9).

We solve this system with respect to w_{11} and w_{12} (see step A_{22}) :

$$\begin{aligned} w_{11} &= \lambda w_2, \\ w_{12} &= (1+\lambda) w_2. \end{aligned} \tag{3.20}$$

A solution exists for any w_2 without any well-posedness condition; hence, step A_{22} is step A_{p2} with $p = 2$.

Substituting (3.20) into (3.19) yields formula (3.14):

$$v(\lambda) = (\lambda^2,\ \lambda + \lambda^2,\ 1 + \lambda + \lambda^2). \tag{3.21}$$

For different values of λ_1, λ_2, and λ_3, the vectors $v(\lambda_j) = (\lambda_j^2,\ \lambda_j + \lambda_j^2,\ 1 + \lambda_j + \lambda_j^2)$, $j = 1, 2, 3$, are linearly independent since the determinant $\triangle_1$ made up of the components of the above vectors for each j reduces to a Vandermonde determinant [9, p. 33], which is nonzero.

Then $v(\lambda_j)$ are substituted into (3.15) (step C_1) and, together with f_1 and f_2, are substituted into (3.2) (step C_2).

Two systems are formed from the resulting equalities.

For $i = 1$ (see step C_3),

$$\begin{cases} \lambda_1^2 k_{11} + (\lambda_1 + \lambda_1^2) k_{12} + (1 + \lambda_1 + \lambda_1^2) k_{13} = \lambda_1^3 - z(\lambda_1) \\ \lambda_2^2 k_{11} + (\lambda_2 + \lambda_2^2) k_{12} + (1 + \lambda_2 + \lambda_2^2) k_{13} = \lambda_2^3 - z(\lambda_2) \\ \lambda_3^2 k_{11} + (\lambda_3 + \lambda_3^2) k_{12} + (1 + \lambda_3 + \lambda_3^2) k_{13} = \lambda_3^3 - z(\lambda_3); \end{cases}$$

for $i = 2$ (step C_4),

$$\begin{cases} \lambda_1^2 k_{21} + (\lambda_1 + \lambda_1^2) k_{22} + (1 + \lambda_1 + \lambda_1^2) k_{23} = z(\lambda_1) \\ \lambda_2^2 k_{21} + (\lambda_2 + \lambda_2^2) k_{22} + (1 + \lambda_2 + \lambda_2^2) k_{23} = z(\lambda_2) \\ \lambda_3^2 k_{21} + (\lambda_3 + \lambda_3^2) k_{22} + (1 + \lambda_3 + \lambda_3^2) k_{23} = z(\lambda_3). \end{cases}$$

And the components of the matrix K are determined accordingly.

The nonunique form of the matrix K is ensured by the arbitrary function $z(\lambda)$.

4. THE ALGORITHM FOR CONSTRUCTING A GAIN MATRIX IN THE CASE OF A MULTIPLE SPECTRUM

For example, let λ_1 have multiplicity k, $k \leqslant n$, and let $\lambda_{k+1}, \lambda_{k+2}, \ldots, \lambda_n$ be distinct arbitrary numbers not equal to λ_1.

The vectors $v_1 = v(\lambda_1)$, $v_{k+1} = v(\lambda_{k+1}), \ldots, v_n = v(\lambda_n)$ are constructed as described in Section 3; as $k - 1$ vectors $\tau_j = \tau_j(\lambda)$ we take the vectors $\tau_1, \tau_2, \ldots, \tau_{k-1}$ of the matrix $A + BK$ that are adjoined to v_1 and correspond to the eigenvalue λ_1 :

$$(A + BK)\tau_j = \lambda_1 \tau_j + \tau_{j-1}, \qquad j = 1, \ldots, k-1, \qquad \tau_0 = v_1.$$

4.1. Preliminary Transformations

D_1. Introduce the designation

$$K\tau_j = g_j. \tag{4.1}$$

D_2. For each $j = 1, \dots, k-1$, solve the equations $Bg_j = (\lambda_1 I - A)\tau_j + \tau_{j-1}$ with respect to g_j :

$$g_j = B^-((\lambda_1 I - A)\tau_j + \tau_{j-1}) + z_j, \tag{4.2}$$

with an arbitrary vector z_j such that $Bz_j = 0$, under the well-posedness conditions

$$Q((\lambda_1 I - A)\tau_j + \tau_{j-1}) = 0. \tag{4.3}$$

4.2. The Algorithm for Solving Equations (4.3) *with Respect to* τ_j

<u>Forward pass</u>

The first step. D_{11}. In (4.3), denote by w_1^j, $j = 1, \dots, k-1$, the coefficients at λ_1 (i.e., $Q\tau_j$) :

$$Q\tau_j = w_1^j; \tag{4.4}$$

as a result, (4.3) takes the form

$$QA\tau_j = \lambda_1 w_1^j + Q\tau_{j-1}, \tag{4.5}$$

where $\tau_0 = v_1$ is a known function (see Section 3).

D_{12}. Find the solutions of systems (4.4) and (4.5) in the form

$$\tau_j = G_1^j(\lambda_1, w_1^j) + z_1^j \tag{4.6}$$

with some matrices G_1^j and arbitrary vectors z_1^j under the well-posedness conditions

$$Q_1^j(\lambda_1, w_p, w_1^j) = 0.$$

The ith step. D_{i1}. In the relations obtained at the $(i-1)$th step of the algorithm,

$$Q_{i-1}^j(\lambda_1, w_p, w_{i-1}^j) = 0, \tag{4.7}$$

denote by w_i^j the coefficients at λ_1 :

$$Q_{i-1}^j w_{i-1}^j = w_i^j. \tag{4.8}$$

The expressions (4.7) become

$$Q_{i-11}^j w_{i-1}^j = \lambda_1 w_i^j. \tag{4.9}$$

D_{i2}. Solve systems (4.8) and (4.9) with respect to w_{i-1}^j :

$$w_{i-1}^j = G_i^j(\lambda_1, w_i^j) + z_i^j \tag{4.10}$$

with some matrices G_i^j and arbitrary vectors z_i^j under the well-posedness conditions

$$Q_i^j(\lambda_1, w_p, w_i^j) = 0, \tag{4.11}$$

and so on . . . .

The pth step. D_{p2}. As proved in [8], if system (1.1) is completely controllable, then for each $j = 1, \dots, k-1$ and $i = p$, equations (4.11) are solvable with respect to w_{p-1}^j :

$$w_{p-1}^j = G_p^j(\lambda_1, w_p, w_p^j) + z_p^j, \tag{4.12}$$

where G_p^j are some matrices and z_p^j, w_p, and w_p^j are arbitrary vectors. In this case, well-posedness conditions are absent.

Form w_{p-1}^j.

Backward pass

E_1. For each $j = 1, \dots, k-1$, substitute the vectors w_{p-1}^j (4.12), with as yet arbitrary w_p^j and r_p^j, into (4.10) with $i = p-1$, thereby determining the vectors w_{p-2}^j.

E_2. Substitute w_{p-2}^j into (4.10) with $i = p-2$. Thereby, the vectors w_{p-3}^j are obtained.

And so on....

E_{p-2}. Substitute w_2^j (found at step E_{p-3}) into (4.10) with $i = 2$. Thereby, the vectors w_1^j are obtained.

E_{p-1}. Determine τ_j from (4.6):

$$\tau_j = G(w_p, w_p^j, z_p^j, z_{p-1}^j, \dots, z_1^j), \quad j = 1, \dots, k-1. \tag{4.13}$$

4.3. Constructing the System of n Linearly Independent Vectors in the Case of a Multiple Spectrum

According to [8], in the case of the complete controllability of system (1.1), there exist vectors $w_p, w_p^j, z_p^j, z_{p-1}^j, \dots, z_1^j$ such that the vectors $v_1, \tau_1, \dots, \tau_{k-1}, v_{k+1}, \dots, v_n$ are linearly independent. Here, the vectors $\tau_1, \dots, \tau_{k-1}$ are constructed using formulas (4.13), and $v_j = v(\lambda_j)$, $j = 1, k+1, \dots, n$, using formulas (3.14).

If system (1.1) is not completely controllable, then n linearly independent vectors $v_1, \tau_1, \dots, \tau_{k-1}, v_{k+1}, \dots, v_n$ cannot be constructed for arbitrary values $\lambda_1, \lambda_{k+1}, \dots, \lambda_n$, where λ_1 is an eigenvalue of multiplicity k [8].

Select elements $w_p, w_p^j, z_p^j, z_{p-1}^j, \dots, z_1^j$ such that the vectors $v_1, \tau_1, \dots, \tau_{k-1}, v_{k+1}, \dots, v_n$ are linearly independent.

4.4. The Algorithm for Constructing a Gain Matrix in the Case of a Multiple Spectrum

Let λ_1 be an eigenvalue of multiplicity k of the matrix $A + BK$, and let the other $\lambda_{k+1}, \dots, \lambda_n$ be simple eigenvalues.

G_1. For each $j = 1, k+1, \dots, n$, substitute the vectors $v_j = v(\lambda_j)$ constructed in Section 3 into (3.4), thereby determining f_j.

G_2. Substitute f_j and v_j into (3.2):

$$Kv(\lambda_j) = f(\lambda_j), \quad j = 1, k+1, \dots, n. \tag{4.14}$$

G_3. Substitute the vectors $\tau_1, \tau_2, \dots, \tau_{k-1}$ obtained at step 4.3 into formulas (4.2), thereby determining the vectors g_j.

G_4. Substitute g_j and τ_j into (4.1):

$$K\tau_j = g_j, \quad j = 1, \dots, k. \tag{4.15}$$

G_5. From equalities (4.14) and (4.15) extract those containing the components k_{1j} of the first row of the matrix K, $j = 1, \ldots, n$. Thereby, a linear algebraic system with the unknowns k_{1i} is formed, with the principal determinant $\triangle_2$ made up of the components of the linearly independent vectors $v_1, \tau_1, \ldots, \tau_{k-1}, v_{k+1}, \ldots, v_n$; hence, $\triangle_2 \neq 0$.

G_6. Solve the system constructed at the previous step, thereby obtaining the components of the first row of the matrix K.

G_7. Repeat steps G_5 and G_6 for $i = 2, 3, \ldots, m$, thereby determining the components of the ith rows of matrix K.

Example 2. Let A and B be the matrices from Example 1, and let $\lambda_1 = \lambda_2 = \lambda_3$. It is required to construct three linearly independent vectors v_1, τ_1, τ_2 such that

$$(A + BK)v_1 = \lambda_1 v_1, \tag{4.16}$$

$$(A + BK)\tau_1 = \lambda_1 \tau_1 + v_1, \tag{4.17}$$

$$(A + BK)\tau_2 = \lambda_1 \tau_2 + \tau_1. \tag{4.18}$$

The vector v_1 has been constructed in Section 3: $v_1 = (\lambda_1^2, \lambda_1 + \lambda_1^2, 1 + \lambda_1 + \lambda_1^2)$ (formula (3.21).

To construct $\tau_1 = (\tau_{11}, \tau_{12})$, in equation (4.17) we introduce the designation (equality (4.1) with $j = 1$)

$$K\tau_1 = g_1, \quad g_1 = (g_{11}, g_{12}). \tag{4.19}$$

This equation becomes

$$Bg_1 = (\lambda_1 I - A)\tau_1 + v_1;$$

therefore,

$$\begin{aligned} g_{11} &= \lambda_1 \tau_{11} + \lambda_1^2 - z_1^1, \\ g_{12} &= z_1^1 \end{aligned} \tag{4.20}$$

(this is (4.2) with $j = 1$) with an arbitrary value z_1^1 under the well-posedness condition ((4.3) with $j = 1$)

$$\begin{cases} \lambda_1(\tau_{11} - \tau_{12} - 1) = -\tau_{11} \\ \lambda_1(\tau_{11} - \tau_{13} - 1) = -\tau_{12} + 1. \end{cases}$$

Next, the coefficients at λ_1 in the last two relations are denoted as the new unknowns (see step D_{11}), e.g.,

$$\begin{aligned} \tau_{11} - \tau_{12} - 1 &= a, \\ \tau_{11} - \tau_{13} - 1 &= b. \end{aligned}$$

(Here, a and b are used instead of w_{11}^1 and w_{12}^1 for the sake of simple notation, see (4.4).) The further actions are performed as described in subsection 4.2. As a result, the vector $\tau_1 = (2\lambda_1, 1 + 2\lambda_1, 1 + 2\lambda_1)$ is determined.

Next (see subsection 4.1, $j = 2$), with the designations

$$K\tau_2 = g_2, \quad g_2 = (g_{21}, g_{22}), \tag{4.21}$$

equation (4.18) takes the form

$$Bg_2 = (\lambda \cdot I - A)\tau_2 + \tau_1. \tag{4.22}$$

From this equation we find

$$\begin{aligned} g_{21} &= \lambda_1 \tau_{21} + 2\lambda_1 - z_2, \\ g_{22} &= z_2 \end{aligned} \tag{4.23}$$

with an arbitrary value z_2 under the condition

$$\begin{cases} \lambda_1(\tau_{21} - \tau_{22}) = -\tau_{21} + 1 \\ \lambda_1(\tau_{21} - \tau_{23}) = -\tau_{22} + 1. \end{cases}$$

Continuing to solve the last system by the method described in subsection 4.4, we get

$$\tau_2 = (1, 1, 1).$$

As is easily verified, the determinant $\triangle_2$ made up of the components of the vectors v_1, τ_1, τ_2 is nonzero.

Next, using formulas (4.20) and (4.23), we determine $g_1 = (3\lambda_1^2 - z_1^1, z_1^1)$ and $g_2 = (3\lambda^2 - z_2, z_2)$ with arbitrary values z_1^1 and z_2. The vector f is given by (3.15): $f = (\lambda_1^3 - z, z)$ with an arbitrary value z.

Finally, a system consisting of the equations $Kv_1 = f_1$, (4.19), and (4.21) is formed.

As a result, two systems are obtained:

$$\begin{cases} \lambda_1^2 k_{11} + (\lambda_1 + \lambda_1^2)k_{12} + (1 + \lambda_1 + \lambda_1^2)k_{13} = \lambda_1^3 - z \\ 2\lambda_1^2 k_{11} + (1 + 2\lambda_1)k_{12} + (1 + 2\lambda_1)k_{13} = 3\lambda_1^2 - z_1^1 \\ k_{11} + k_{12} + k_{13} = 3\lambda_1 - z_2 \end{cases}$$

and

$$\begin{cases} \lambda_1^2 k_{21} + (\lambda_1 + \lambda_1^2)k_{22} + (1 + \lambda_1 + \lambda_1^2)k_{23} = z \\ 2\lambda_1 k_{21} + (1 + 2\lambda_1)k_{22} + (1 + 2\lambda_1)k_{23} = z_1^1 \\ k_{21} + k_{22} + k_{23} = z_2. \end{cases}$$

They yield the components k_{ij}, $i = 1, 2, 3$, $j = 1, 2, 3$, of the matrix K in a nonunique way, depending on arbitrary values z, z_1^1, and z_2.

In a special case where $\lambda = -1$ is a triple eigenvalue of the matrix $A + BK$, we have $K = \begin{pmatrix} c_1 & -2 + c_2 & -1 - c_3 \\ -c_1 & -c_2 & c_3 \end{pmatrix}$, where c_1, c_2, and c_3 are linear combinations of arbitrary values z, z_1^1, and z_2, therefore being arbitrary as well.

Direct verification shows that, for such K,

$$\det(A + BK - \lambda I) = -(\lambda + 1)^3.$$

5. ON THE NONUNIQUENESS OF THE GAIN MATRIX

The nonuniqueness of K for given A, B, and $\{\lambda_j\}_{j=1}^n$ arises if Ker $B \neq \{0\}$, i.e., when the equation $Bf = h$ with a fixed vector $h \in \mathbb{R}^n$ has a nonunique solution $f = B^- h + z$ with an arbitrary vector $z \in \mathbb{R}^m$.

Also, the nonuniqueness of K arises if at least one of systems (3.11) or systems (4.8), (4.9) with $i = 1, \ldots, p$ has a nonunique solution w_{i-1} or w_{i-1}^j, $j = 1, \ldots, k$, where k is the multiplicity of λ_s.

However, if Ker $B = \{0\}$ and all systems (3.11) or (4.8), (4.9) are uniquely solvable, then K has a unique form; see Example 2 in [8].

Note that the higher the multiplicity of an eigenvalue is, the more arbitrary parameters K may contain.

For example, a unique matrix K was constructed in [6] for system (1.1) with the matrices $A = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 5 & 0 & 0 \\ 7 & 0 & 0 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$ and the eigenvalue $\lambda = -1$ of multiplicity 4.

However, for this case, the algorithm proposed here yields a matrix K depending on eight parameters, $K = K(c_1, c_2, \ldots, c_8)$, connected only by the condition

$$\begin{vmatrix} c_1 & c_2 & 0 & 0 \\ c_3 & c_4 & c_1 & c_2 \\ c_5 & c_6 & c_3 & c_4 \\ c_7 & c_8 & c_5 & c_6 \end{vmatrix} \neq 0.$$

This condition arises due to the linear independence of the vectors v_1, τ_1, τ_2, and τ_3 [8].

For a particular dynamic system of the form (1.1) and a given arbitrary set $\{\lambda_j\}_{j=1}^n$, the method proposed in this paper constructs precisely the **complete** set of gain matrices. This conclusion follows from the equivalence of the transformations applied.

6. AN ILLUSTRATIVE EXAMPLE: GAIN MATRICES FOR STABILIZING THE OPERATION OF A MULTI-CHAMBER HEATING FURNACE

The dynamic model and operation scheme of three furnaces for heating three chambers were presented in [10].

Consider the case of three furnaces and five chambers:

$$\begin{aligned} \dot{x}_1 &= 2ax_1 + bu_1 + cu_2, \\ \dot{x}_2 &= ax_2 + cu_2 + du_3, \\ \dot{x}_3 &= ax_3 + bu_1 + 2cu_2 + du_3, \\ \dot{x}_4 &= 3ax_4 + 2cu_2 + 2du_3, \\ \dot{x}_5 &= 3ax_5 + 2bu_1 + 3cu_2 + du_3, \end{aligned} \tag{6.1}$$

where $a \leqslant -\frac{1}{10}$, $b = \frac{1}{4}$, $c = \frac{1}{9}$, and $d = \frac{1}{5}$. In the case of the unreachability of a given initial temperature $x_i(t_0) = x_{0i}$ in the ith chambers, $i = 1, 2, \ldots, 5$, it is required to correct the program control (fuel supply) $u_r = u_r(t)$ of the rth furnaces, $r = 1, 2, 3$, so that at a specified time instant $t = t_k$ the temperature $x_i(t_k)$ in the chambers is close to a desired one x_{ki}. For this purpose, a stabilizing control law with a gain matrix K can be used to make the state of system (6.1) exponentially tending to the program state.

Here, $A = \begin{pmatrix} 2a & 0 & 0 & 0 & 0 \\ 0 & a & 0 & 0 & 0 \\ 0 & 0 & a & 0 & 0 \\ 0 & 0 & 0 & 3a & 0 \\ 0 & 0 & 0 & 0 & 3a \end{pmatrix}$ and $B = \begin{pmatrix} 1/4 & 1/9 & 0 \\ 0 & 1/9 & 1/5 \\ 1/4 & 2/9 & 1/5 \\ 0 & 2/9 & 2/5 \\ 2/4 & 3/9 & 1/5 \end{pmatrix}$.

The components of the matrix $K = (k_{ij})$ are found from the equation $Kv = f$, where $f = f(\lambda)$ and $v = (v_1, \ldots, v_5)$, and $v_i = v_i(\lambda)$ are obtained from the equation $Bf = (\lambda I - A)v$, or

$$\begin{aligned} f_1 + f_2 &= (\lambda - 2a)v_1, \\ f_2 + f_3 &= (\lambda - a)v_2, \\ f_1 + 2f_2 + f_3 &= (\lambda - a)v_3, \\ 2f_2 + 2f_3 &= (\lambda - 3a)v_4, \\ 2f_1 + 3f_2 + f_3 &= (\lambda - 3a)v_5, \end{aligned} \tag{6.2}$$

where $f = (4f_1, 9f_2, 5f_3)$ to get rid of fractions in the matrix B.

The solution of system (6.2) is

$$\begin{aligned} f_1 &= (\lambda - 2a)v_1 - z, \\ f_2 &= z, \\ f_3 &= (\lambda - a)v_2 - z \end{aligned} \tag{6.3}$$

(formula (3.4)) with an arbitrary value z under conditions (3.5):

$$\begin{gathered}2(\lambda-a)v_2=(\lambda-3a)v_4,\\(\lambda-2a)v_1+(\lambda-a)v_2=(\lambda-a)v_3,\\2(\lambda-2a)v_1+(\lambda-a)v_2=(\lambda-3a)v_5,\end{gathered}$$

or

$$\begin{gathered}\lambda(2v_2-v_4)=2av_2-3av_4,\\\lambda(v_1+v_2-v_3)=2av_1+av_2-av_3,\\\lambda(2v_1+v_2-v_5)=4av_1+av_2-3av_5.\end{gathered}\tag{6.4}$$

Constructing the vector $v(\lambda)$

The first step. We introduce the designations (see step A_{11})

$$\begin{gathered}2v_2-v_4=aw_{11},\\v_1+v_2-v_3=aw_{12},\\2v_1+v_2-v_5=aw_{13}.\end{gathered}\tag{6.5}$$

Then from (6.4) it follows that

$$\begin{gathered}2v_2-3v_4=\lambda w_{11},\\2v_1+v_2-v_3=\lambda w_{12},\\4v_1+v_2-3v_5=\lambda w_{13}.\end{gathered}\tag{6.6}$$

$\mathbf{A_{12}}$. System (6.5), (6.6), which contains six equations and five unknowns, is used to find v_j (step A_{12}) :

$$\begin{aligned}v_1&=(\lambda-a)w_{12},\\v_2&=-\frac{1}{4}(\lambda-3a)w_{11},\\v_3&=-\frac{1}{4}(\lambda-3a)w_{11}+(\lambda-2a)w_{12},\\v_4&=-\frac{1}{2}(\lambda-a)w_{11},\\v_5&=-\frac{1}{4}(\lambda-3a)w_{11}+2(\lambda-a)w_{12}-aw_{13}\end{aligned}\tag{6.7}$$

(this is formula (3.7)) under the well-posedness condition of this system:

$$(\lambda-3a)w_{11}-4(\lambda-a)w_{12}-2(\lambda-3a)w_{13}=0;$$

hence,

$$\lambda(w_{11}-4w_{12}-2w_{13})=a(3w_{11}-4w_{12}-6w_{13}).$$

The second step. **$\mathbf{A_{21}}$**. Denoting (see step A_{21})

$$w_{11}-4w_{12}-2w_{13}=aw_{21},\tag{6.8}$$

we have

$$3w_{11}-4w_{12}-6w_{13}=\lambda w_{21}.\tag{6.9}$$

$\mathbf{A_{22}}$. System (6.8), (6.9) is solved with respect to w_{1i} (step A_{22}) :

$$\begin{aligned} w_{11} &= \frac{1}{2}(\lambda - a)w_{21} + 2w_{13}, \\ w_{12} &= \frac{1}{8}(\lambda - 3a)w_{21} \end{aligned} \tag{6.10}$$

for any w_{13} and w_{21}. There is no well-posedness condition; consequently, $p = 2$.

Next, we substitute the values w_{11} and w_{12} (6.10) into formulas (6.7):

$$\begin{aligned} v_1 &= \frac{1}{8}(\lambda - a)(\lambda - 3a)w_{21}, \\ v_2 &= -\frac{1}{2}(\lambda - 3a)w_{13} - \frac{1}{8}(\lambda - a)(\lambda - 3a)w_{21}, \\ v_3 &= -\frac{1}{2}(\lambda - 3a)w_{13} - \frac{1}{8}a(\lambda - 3a)w_{21}, \\ v_4 &= -(\lambda - a)w_{13} - \frac{1}{4}(\lambda - a)^2 w_{21}, \\ v_5 &= -\frac{1}{2}(\lambda - a)w_{13} + \frac{1}{8}(\lambda - a)(\lambda - 3a)w_{21} \end{aligned} \tag{6.11}$$

(this is (3.14) for $p = 2$)) with any $w_{13}(\lambda)$ and $w_{21}(\lambda)$ such that $v(\lambda_j)$ are linearly independent:

$$\det(v(\lambda_1)\ v(\lambda_2)\ v(\lambda_3)\ v(\lambda_4)\ v(\lambda_5)) \neq 0.$$

In particular, for $w_{13} = -2$ and $w_{21} = 8\lambda^2$,

$$v(\lambda) = \begin{pmatrix} (\lambda - a)(\lambda - 3a)\lambda^2 \\ (\lambda - 3a)(1 - (\lambda - a)\lambda^2) \\ (\lambda - 3a)(1 - a\lambda^2) \\ 2(\lambda - a)(1 - (\lambda - a)\lambda^2) \\ (\lambda - a)(1 + (\lambda - 3a)\lambda^2) \end{pmatrix}. \tag{6.12}$$

As is easily verified, the determinant $\triangle$ made up of the components of this vector for different values of $\lambda = \lambda_j$, $j = 1, 2, \ldots, 5$, by transformations reduces to the Vandermonde determinant $W(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_5)$ and is therefore nonzero.

Thus, the vectors $v(\lambda_j)$, $j = 1, 2, \ldots, 5$, are linearly independent.

Constructing the gain matrix in the case of a simple spectrum

For each $\lambda = \lambda_j$, $j = 1, 2, \ldots, 5$, we substitute the values $v(\lambda_j)$ (6.12) and $f(\lambda_j)$ (6.3) into the relation $Kv(\lambda) = f(\lambda)$.

The resulting system of fifteen equations is used to determine the components of the matrix K. These equations are split into three systems of the form $V_1 \cdot \begin{pmatrix} k_{i1} \\ k_{i2} \\ k_{i3} \\ k_{i4} \\ k_{i5} \end{pmatrix} = \begin{pmatrix} f_i(\lambda_1) \\ f_i(\lambda_2) \\ f_i(\lambda_3) \\ f_i(\lambda_4) \\ f_i(\lambda_5) \end{pmatrix}$, $i = 1, 2, 3$. In the special case $\lambda_j = -\frac{1}{2}(j + 1)$ and $a = -\frac{1}{10}$, we have

$$V_1 = \begin{pmatrix} 63/100 & -133/100 & -77/100 & -171/50 & -27/100 \\ 189/50 & -249/50 & -147/100 & -581/50 & 119/50 \\ 323/25 & -731/50 & -119/50 & -817/25 & 551/50 \\ 33 & -176/5 & -143/40 & -384/5 & 153/5 \\ 7047/100 & -7317/100 & -513/100 & -7859/50 & 6757/100 \end{pmatrix}, \tag{6.13}$$

$f_1 = (-252/125 - 4z,\ -2457/125 - 4z,\ -11\,628/125 - 4z,\ -1518/5 - 4z,\ -98\,658/125 - 4z),$
$f_2 = (9z,\ 9z\, 9z\, 9z\, 9z),$
$f_3 = (-65\,201/2 - 50z,\ -136\,255/8 - 25z,\ 17\,431,\ -15\,687/4,\ 62\,181/8 + 25z).$

As a result, $V_1^{-1} f_i^T$ are the elements of the ith row of the matrix K. Finally,

$$K = \begin{pmatrix} 128\,394/5 - 40z_1 & 134\,253/10 - 20z_2 & -68\,724/5 & 15\,437/5 & -61\,163/10 + 20z_3 \\ 90z_1 & 45z_2 & 0 & 0 & -45z_3 \\ -65\,201/2 - 50z_1 & -136\,255/8 - 25z_2 & 17\,431 & -15\,687/4 & 62\,181/8 + 25z_3 \end{pmatrix}$$

with arbitrary values z_1, z_2, and z_3. Substituting the above matrix K into the expression $A + BK - \lambda \cdot I$ yields

$$\det(A + BK - \lambda I) = \begin{vmatrix} 12\,839/2 - \lambda & 134\,253/40 & -17\,181/5 & 15\,437/20 & -61\,163/40 \\ -65\,201/10 & -136\,259/40 - \lambda & 17\,431/5 & -15\,687/20 & 62\,181/40 \\ -502/5 & -1001/20 & 499/10 - \lambda & -25/2 & 509/20 \\ -65\,201/5 & -27\,251/4 & 34\,862/2 & -1569 - \lambda & 62\,181/20 \\ 63\,193/10 & 132\,251/40 & -16\,931/5 & 15\,187/20 & -60\,157/40 - \lambda \end{vmatrix}$$
$$= -\frac{45}{2} - \frac{261}{4}\lambda - \frac{145}{2}\lambda^2 - \frac{155}{4}\lambda^3 - 10\lambda^4 - \lambda^5$$
$$= (\lambda + 1)\left(\lambda + \frac{3}{2}\right)(\lambda + 2)\left(\lambda + \frac{5}{2}\right)(\lambda + 3).$$

Note that in this example, the variability of the gain matrix depends on the choice of arbitrary values z and on the values of w_{13} and w_{21} such that the vectors $v(\lambda_j)$ (6.11) are linearly independent.

The case of a multiple spectrum

It is required to construct a matrix K such that $\lambda = -2$ is an eigenvalue of multiplicity 5 of the matrix $A + BK$, where A and B are the matrices appearing in system (6.1).

The vector $v(-2) = \tau_0$ is constructed by formula (6.12) with $a = -\frac{1}{10}$ and $\lambda = -2$; the components of the vector $v(-2) = \tau_0$ are those of the third row of the matrix V_1 (6.13). The vectors τ_j, $j = 1, 2, \dots, 4$, are obtained from the well-posedness conditions of the equations $BK\tau_j = (\lambda I - A)\tau_j + \tau_{j-1}$.

The components of the vectors τ_j are those of the jth rows of the matrix V_2, where

$$V_2 = \begin{pmatrix} 323/25 & -731/50 & -119/50 & -817/25 & 551/50 \\ -72/5 & 77/5 & 7/5 & 162/5 & -67/5 \\ 4 & -57/10 & -17/10 & -59/5 & 21/10 \\ 0 & 1 & 1 & 2 & 1 \\ 0 & -17/10 & -17/10 & -38/10 & -19/10 \end{pmatrix}.$$

The equations $Bg_j = (\lambda I - A)\tau_j + \tau_{j-1}$ are used to find the solutions g_j; in this case, for a particular choice of arbitrary constants, we obtain

$g_1 = (-11\,628/125 - 4z,\quad 3884/25 - 4z,\quad -432/5 - 4z,\quad 16 - 4z,\quad -4z),$
$g_2 = (9z,\quad 9z,\quad 9z,\quad 9z,\quad 9z),$
$g_3 = (13\,889/100 - 5z,\quad -1097/5 - 5z,\quad 2623/20 - 5z,\quad -38 - 5z,\quad 423/20 - 5z)$

with an arbitrary value z.

Solving the system

$$V_2 \cdot \begin{pmatrix} k_{i1} \\ k_{i2} \\ k_{i3} \\ k_{i4} \\ k_{i5} \end{pmatrix} = g_i^{\mathrm{T}}$$

yields k_{ij}, $i = 1, 2, 3$, $j = 1, 2, \ldots, 5$.

Computations in Mathcad produced the result

$$K = \begin{pmatrix} 16\,704/5 - 112z & 9519/5 - 58z & -8759/5 & 6613/20 & -7973/10 + 54z \\ 252z & 261/2z & 0 & 0 & -243/2z \\ -4163 - 140z & -4765/2 - 145/2z & 8509/4 & -6363/16 & 8101/8 + 135/2z \end{pmatrix}.$$

Direct check shows that

$$\det(A + BK - \lambda \cdot I)$$
$$= \begin{vmatrix} 835 - \lambda & 9519/20 & -8759/20 & 6613/80 & -7973/40 \\ -4163/5 & -2383/5 - \lambda & 8509/20 & -6363/80 & 8101/40 \\ 13/5 & -11/20 & -63/5 - \lambda & 25/8 & 16/5 \\ -8326/5 & -953 & 8509/10 & -1275/8 - \lambda & 8101/20 \\ 4189/5 & 2377/5 & -9009/20 & 6863/80 & -785\,740 - \lambda \end{vmatrix}$$
$$= -32 - 80\lambda - 80\lambda^2 - 40\lambda^3 - 10\lambda^4 - \lambda^5 = -(\lambda + 2)^5.$$

Remark 1. In the case of a nonsingular matrix B, system (1.1) is controllable, and K is determined from the equations

$$Kv_j = B^{-1}(\lambda_j I - A)v_j, \qquad j = 1, \ldots, n,$$

or

$$Kv_s = B^{-1}(\lambda_j I - A)v_s,$$
$$K\tau_j = B^{-1}(\lambda_j I - A)\tau_j + B^{-1}\tau_{j-1}, \quad j = 1, \ldots, k-1, \quad \tau_0 = v_s,$$

where arbitrary linearly independent vectors can be taken for v_j or $v_1, \ldots, v_s, \tau_1, \ldots, \tau_{k-1}, v_{k+1}, \ldots, v_n$.

7. CONCLUSIONS

This paper has proposed a new algorithm for constructing gain matrices in the spectrum assignment problem of a continuous-time linear dynamic system without any constraints on its matrix coefficients, except for the controllability of the pair (A, B).

The algorithm is based on the cascade decomposition method developed in [8]. In contrast to [8] (with expanding spaces into subspaces, constructing projectors onto them, and using semi-inverse matrices), the algorithm in this paper involves elementary algebraic operations: solving linear algebraic equations, making changes of variables, and checking the linear independence of vectors.

Such a solution of the problem significantly simplifies the computational process and allows creating simple computing programs.

In addition, the dependence of the gain matrix on a certain number of arbitrary or conditionally related scalar parameters has been revealed, and the complete set of such matrices for each problem has been determined.

The case of a unique gain matrix has been identified as well.

Illustrative examples of constructing gain matrices have been provided.

Finally, various cases of constructing such a matrix for a dynamic system describing the operation of a multi-chamber heating furnace have been considered.

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APPENDIX

JUSTIFICATION FOR THE ALGORITHM FOR CONSTRUCTING LINEARLY INDEPENDENT VECTORS (SUBSECTIONS 3.4 AND 4.2)

Section 2 has presented the solution of a linear algebraic equation of the form $Cx = y$ with a singular matrix $C : \mathbb{R}^s \to \mathbb{R}^l$, $y \in \mathbb{R}^l$. This solution is explained as follows: using a mapping C, one can expand spaces R^s and $\mathbb{R}^l$ into the direct sums of subspaces:

$$R^s = \operatorname{Coim} C \dot{+} \operatorname{Ker} C, \qquad R^l = \operatorname{Im} C \dot{+} \operatorname{Coker}\ C, \tag{A.1}$$

where Ker C, Im C, and Coker C stand for the kernel, image, and defect (defective) subspace of C, respectively, and Coim C is the direct complement to Ker C in R^s. (Mappings and the corresponding matrices are indicated identically.) Here, the narrowing of $\widetilde{C}$ to Coim C has an inverse mapping $\widetilde{C}^-$ [11]. Let P and Q denote projectors onto Ker C and Coker C, respectively; then the mapping $\widetilde{C}^-(I-Q)$ is called semi-inverse [12, p. 164] and is denoted by C^-. Here, I means an identity mapping in any subspace.

The following result is well-known [8, 13–17].

Lemma. *The relation $Cx = y$ is equivalent to the system*

$$\begin{cases} Qy = 0 \\ x = C^- y + z \end{cases} \tag{A.2}$$

for any z from Ker C.

Note that $z = Px$. The relation $Qy = 0$ in (A.2) is the well-posedness condition for the equality $Cx = y$, and the second relation in (A.2) is the solution of this equation.

By this lemma, equation (3.3) with omitted j, $Bf = (\lambda I - A)v$, has the solution

$f = B^-(\lambda I - A)v + z$ (3.4) for any $z \in$ Ker B under the well-posedness condition (3.5):

$$Q(\lambda I - A)v = 0. \tag{A.3}$$

The first step. Denoting

$$Qv = w_1, \quad (I - Q)v = v_1, \quad QAQ = A_1, \quad QA(I - Q) = B_1, \tag{A.4}$$

we have

$$v = v_1 + w_1, \tag{A.5}$$

and (A.3) can be written as

$$B_1 v_1 = (\lambda I - A_1) w_1. \tag{A.6}$$

Note that the designations $Qv = w_1$ in (A.4) and (3.6) are identical.

Next we study (A.6) with $B_1 : \mathrm{Im}\ B \to \mathrm{Coker}\ B$. Based on the decomposition

$$\mathrm{Im}\, B = \mathrm{Coim}\, B_1 \dot{+} \mathrm{Ker}\, B_1, \qquad \mathrm{Coker}\, B = \mathrm{Im}\ B_1 \dot{+} \mathrm{Coker}\ \ B_1, \tag{A.7}$$

equation (A.6) (see the lemma) has the solution

$$v_1 = B_1^{-}(\lambda I - A_1)w_1 + \tilde{z}_1, \quad \forall \tilde{z}_1 \in \mathrm{Ker}\, B_1, \tag{A.8}$$

under the condition

$$Q_1(\lambda I - A_1)w_1 = 0, \tag{A.9}$$

where P_1 and Q_1 are projectors onto Ker B_1 and Coker B_1, respectively, corresponding to the decomposition (A.7), and B_1^{-} is the semi-inverse of B_1. Formulas (A.5), (A.8), and (A.9) correspond to formulas (3.7) and (3.8).

The second step. Denoting

$$Qw_1 = w_2, \quad (I - Q_1)w_1 = v_2, \quad Q_1 A_1 Q_1 = A_2, \quad Q_1 A_1 (I - Q_1) = B_2,$$

we have

$$w_1 = v_2 + w_2, \tag{A.10}$$

and (A.9) is

$$B_2 v_2 = (\lambda I - A_2)w_2.$$

Note that the designation $Qw_1 = w_2$ appears in (3.9).

Applying the lemma to the last equation, we express v_2 through w_2 and an arbitrary vector $\tilde{z}_2 \in \mathrm{Ker}\, B_2$; using (A.10), we express w_1 through λ, w_2, and $\tilde{z}_2$, this is (3.11) with $i = 2$. And so on ....

As a result, the relation (A.3) is equivalent to the system consisting of the relations (A.5), (A.8), (A.10), and

$$w_i = v_{i+1} + w_{i+1}, \tag{A.11}$$

$$v_{i+1} = B_{i+1}^{-}(\lambda I - A_{i+1})w_{i+1} + \tilde{z}_{i+1}, \quad \forall \tilde{z}_{i+1} \in \mathrm{Ker}\, B_{i+1}, \tag{A.12}$$

$$B_p v_p = (\lambda I - A_p)w_p, \tag{A.13}$$

where $B_{i+1}^{-} = Q_i A_i (I - Q_i)$, $A_{i+1} = Q_i A_i Q_i$, $w_i = Q_{i-1} w_{i-1}$, $v_{i+1} = (I - Q_i)w_i$, Q_i and P_i are projectors onto Coker B_i and Ker B_i, respectively, corresponding to the decomposition

$$\mathrm{Im}\, B_{i-1} = \mathrm{Coim}\, B_i \dot{+} \mathrm{Ker}\, B_i, \qquad \mathrm{Coker}\, B_{i-1} = \mathrm{Im}\ B_i \dot{+} \mathrm{Coker}\ \ B_i,$$

$B_{i-1}^{-} = \tilde{B}_{i-1}^{-1}(I - Q_{i-1})$, $\tilde{B}_{i-1}$ is the narrowing of B_{i-1} to Coim B_{i-1}, $i = 1, \ldots, p$, $B_0 = B$, and $A_0 = A$.

Due to the controllability of the pair (A, B), equation (A.13) is solvable with respect to v_p for any $w_p \in$ Coker B_{p-1} [8, 13–17]. From (A.11) with $i = p - 1$ we find w_{p-1}; next, from (A.12) with $i = p - 2$ we determine v_{p-1}; then using formula (A.11) with $i = p - 2$ we construct w_{p-2}, and so on .... Finally, using formula (A.5) we obtain $v = v(\lambda, w_p, \tilde{z}_1, \tilde{z}_2, \ldots \tilde{z}_p)$ with arbitrary vectors $w_p \in$ Coker B_{p-1} and $\tilde{z}_s \in$ Ker B_s, $s = 1, \ldots, p$.

Obviously, constructing v by the method proposed in this work—using linear changes of unknown vectors and solving linear algebraic systems—significantly simplifies the design of gain matrices in the spectrum assignment problem.

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