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Relay Controller Design in Self-Oscillating Control Systems Based on Training Examples

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Abstract—The problem of designing relay controllers in a self-oscillating control system with a linear plant is considered. It is required to ensure self-oscillations in the system with given parameters and to make its behavior maximally close to the desired one, defined by a set of time-varying system output laws (training examples). Controller's structural constraints and the requirement for the degree of stability of self-oscillations are taken into account. Explicit-form relations are obtained and applied to develop an iterative method for solving the above problem. This method yields relay controllers having a simple structure. Some examples of implementing the method are provided.

Keywords: relay controller, self-oscillations, training examples

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1. INTRODUCTION

Relay feedback control, as well as analysis and design of self-oscillating control systems based on its application, are classical themes of control theory [1–9].

The topicality of the problem of designing self-oscillating control systems with a relay controller is confirmed by numerous examples of their practical use for various purposes. In particular, we mention self-oscillating systems for regulating various physical quantities [1], servosystems [1, 2, 4, 5, 8], self-tuning and adaptive systems using self-oscillations in the controller auto-tuning mode [6–8], self-oscillating voltage converters and signal auto-generators [1, 10], vibratory gyroscopes and accelerometers [11, 12], sensors based on nanoresonators, resonistors and cantilevers [13, 14], vibration machines [15], control systems for underactuated multilink mechanisms and robots using self-oscillations [16], and many other devices. In these systems, self-oscillations are an operating mode where a controlled variable oscillates with a given frequency within an admissible range.

For a long time, the predominant approach to the analysis and design of self-oscillating systems was based on frequency-domain methods [1–3, 8]. Among them, the harmonic linearization method [2, 3] became the most widespread due to its simple application. A significantly more accurate and universal frequency-domain method for analyzing self-oscillations was developed by Ya.Z. Tsypkin [1], with his original concept of the hodograph (also called locus or plot in the literature) of a relay system. The development of state-space analysis methods for relay systems ensured higher efficiency of their analysis and design procedures, as well as the deeper consideration of possible operating modes of relay systems compared to frequency-domain methods. A method for designing self-oscillating systems based on the concept of a phase plot was proposed in [4, 5]. A phase plot is a line in the state space of a plant in which each point x^* corresponds to relay switching from minus to plus for some self-oscillation period inherent to it. The phase plot remains unchanged when choosing feedback coefficients (gains), which significantly simplifies the system design and allows achieving the assigned self-oscillation parameters with high accuracy. An analytical dependence of the vector x^* on the self-oscillation period and the matrices and vectors of parameters appearing in the state-space description of a linear plant was obtained, and algebraic conditions for the existence of self-oscillations of a given frequency, including the condition of their local stability, were presented in [6, 7].

For a control system with a linear plant, relay controllers ensuring self-oscillations of given frequency and amplitude and making its behavior maximally close to the desired one, defined by a given transfer function, were designed in [9]. The design method proposed therein is based on a polynomial representation of the harmonically linearized system and reduces the design procedure to solving systems of linear algebraic equations and inequalities. Due to the use of harmonic linearization in the method [9], in some cases, the self-oscillation frequency significantly deviates from its given value, sometimes leading to an appreciable difference between the behavior of the system designed and the desired one.

In this paper, to overcome the limitations of the method [9] when designing a relay self-oscillating system, we utilize the results of the studies [4–7], with their higher accuracy in predicting self-oscillation parameters compared to the harmonic linearization method. The design procedure proposed below is based on explicit-form relations obtained using the state-space description of a plant and is the result of developing and extending the algorithm [17] to the class of relay self-oscillating control systems. The desired behavior of the system is defined by a set of its time-varying output laws, acting as training examples, as well as by the requirement for its degree of stability. The controller implements static feedback with a relay output. We propose a method for finding an initial approximation of the desired gains and refining them iteratively. In the general case, when the plant output contains several measurable and linearly independent variables, controllers with a simple structure [18, 19] are designed. In such controllers, only those gains are non-zero that are necessary and sufficient to give the system the desired properties.

2. PROBLEM STATEMENT

Consider a plant described by the system of equations

$$\dot{x} = Ax + Bu, \tag{1}$$

$$y = Cx \tag{2}$$

with the following notation: x is the state vector; $\dot{x} = dx/dt$; t indicates time; u is the scalar control input; y is the output vector, all its components are measurable and available for control; the matrices A and C and the vector B are given, their elements, as well as the values of u and all components of the vectors x and y, are real numbers. By assumption, the pairs (A, B) and (A, C) are controllable and observable, respectively.

The mathematical description of the controller has the form

$$\tilde{u} = K^\top y, \tag{3}$$

$$u = \operatorname{sgn}(\tilde{u}), \tag{4}$$

where the scalar variable $\tilde{u}$ is the output of the linear part (3) of the controller; the vector K of the gains (real numbers) has to be determined; the function $\operatorname{sgn}(\cdot)$ describes the relay forming the control input u, its value is -1, $+1$, or a real number from the closed interval $[-1, 1]$ for the negative, positive, zero value of its argument, respectively. The amplitude values of u being equal to 1 do not restrict the generality of the problem under consideration: the real amplitudes of the control input can be taken into account in (1) when determining the vector B.

Let us impose structural constraints on the linear part of the controller, which usually reduce [20] to the zero-value requirement for some components of the vector $K = (k_i)$. Therefore, we adopt the condition

$$k_i = 0,\ \forall i \notin S, \tag{5}$$

where S is the set of the serial numbers of those gains k_i that are not required to be zero. The set S in the controller description (3)–(5) defines the list of its feedback signals and, in view of (3)–(5), its structure. Assume that when solving this problem, the controller structure can change only as a result of changing the set S. From this point onwards, the choice of the controller structure is hence identified with the choice of the set S, and the terms "controller structure" and "set S" are used as synonyms.

The solution of the discontinuous system (1)–(5) is understood in the Filippov sense [21].

In system (1)–(5), the steady-state operating mode should be an asymptotically orbitally stable periodic motion representing self-oscillations with a given frequency ω_0 and a given amplitude $\tilde{U}$ of oscillations of the signal $\tilde{u}$. The self-oscillations are associated with an isolated closed trajectory in the phase space, i.e., a limit cycle. As a result of solving the controller design problem, the limit cycle L must be stable, symmetric (with each point $x \in L$ the point $-x \in L$ must be associated), and simple (the signal $\tilde{u}$ must change its sign only twice per self-oscillation period).

We describe the stability margin requirement for self-oscillations by the condition

$$\eta(K) \geqslant \check{\eta}, \tag{6}$$

where $\eta(K)$ is the stability margin of self-oscillations and $\check{\eta}$ is a given lower limit of its admissible values. The mathematical description of the function $\eta(K)$ will be presented in Section 3.

Let the desired behavior of system (1)–(5) be defined by specifying, for initial conditions x_0^γ, $\gamma \in \{\overline{1,q}\}$, the corresponding desired trajectories of the output (2) of system (1)–(5), represented by the vectors $Y_\gamma = (y_k^\gamma)$, $k \in \{\overline{1,N}\}$, of its values at discrete time instants $t = k\Delta t$, where Δt is a sampling step and k is a natural number. The values of N and Δt determine the time interval $[0, N\Delta t]$ on which the desired trajectories are specified. The initial states x_0^γ, $\gamma \in \{\overline{1,q}\}$, must be sufficiently distant from the steady-state trajectory so that the transient has a duration significantly exceeding the self-oscillation period. In addition, the initial states must differ significantly from each other. It is reasonable to assign $q \in [1, n]$. The set $Q = \{(x_0^\gamma, Y_\gamma)\}$, $\gamma \in \{\overline{1,q}\}$, is a set of training examples that defines the desired behavior of the system designed during its transition from a disturbed to steady-state motion mode. The requirement for system (1)–(5) to match the desired behavior is written as

$$\varepsilon_k^{\gamma-} \leqslant y(x_0^\gamma, K)_k - y_k^\gamma \leqslant \varepsilon_k^{\gamma+},\ \forall k \in \{\overline{1,N}\},\ \forall \gamma \in \{\overline{1,q}\}, \tag{7}$$

where $\varepsilon_k^{\gamma-}$ and $\varepsilon_k^{\gamma+}$ are given constant vectors; $y(x_0^\gamma, K)_k$ is the output of system (1)–(5) at the kth time instant that corresponds to the initial condition $x(0) = x_0^\gamma$ and the chosen vector K; when applied to vectors, the symbol $\leqslant$ indicates their component-wise inequality. The vectors $\varepsilon_k^{\gamma-}$ and $\varepsilon_k^{\gamma+}$ can be specified by calculating their coordinates $\varepsilon_{ki}^{\gamma-}$, $\varepsilon_{ki}^{\gamma+}$, e.g., using the formulas $\varepsilon_{ki}^{\gamma-} = -\delta_i m_i^\gamma$ and $\varepsilon_{ki}^{\gamma+} = +\delta_i m_i^\gamma$, where δ_i is an admissible value of the absolute relative error $(y(x_0^\gamma, K)_{ki} - y_{ki}^\gamma)/m_i^\gamma$ of the reproducing the ith coordinate of the desired trajectory in the system; m_i^γ is the maximum absolute value of the ith coordinate of the desired trajectory y^γ. In a more general case, particular values of δ_i for different time instants $k\Delta t$ and trajectories y^γ can be used when calculating $\varepsilon_{ki}^{\gamma-}$ and $\varepsilon_{ki}^{\gamma+}$. It is reasonable to assign $\delta_i \in [1, 20]10^{-2}$.

The vector K will be chosen appropriately for making the behavior of system (1)–(5) maximally close to the desired one in the sense of minimizing the Euclidean norm of the vector $\Delta y(K)$

composed of the differences $y(x_0^\gamma, K)_k - y_k^\gamma$, i.e., from the condition

$$|\Delta y(K)| \to \min_K, \tag{8}$$

where $|\cdot|$ denotes the Euclidean norm [22].

In the case of a given controller structure (a specified set S, defining together with (3)–(5) this structure), the problem is to find a vector K under which the control system (1)–(5) will fulfill the requirements (6)–(8) for the given values of ω_0 and $\tilde{U}$.

In the general case, when the set S is not specified and the output vector y contains several measurable linearly independent variables, we will solve the structural design problem, i.e., given ω_0 and $\tilde{U}$, determine the sets S and the corresponding vectors K for which conditions (6)–(8) are satisfied and the structure of the controller (3), (5) is simple. (According to [18, 19], simplicity means that only those gains k_i are non-zero that are necessary and sufficient to give system (1)–(5) the desired properties.) A structure S' is said to be simpler than another structure S if $S' \subset S$. Formally, the problem of obtaining the set Ω of simple controller structures is to find admissible structures $S \in \zeta$ for which no less complex admissible structure $S' \in \zeta$ can be specified. In other words, it is required to find

$$\Omega = \{S \in \zeta \mid \{S' \in \zeta \mid S' \subset S\} = \varnothing\}, \tag{9}$$

where ζ stands for the set of admissible structures, i.e., those for which there exists a vector K under which the control system (1)–(5) will fulfill the requirements (6)–(8). The formula $\{S' \in \zeta \mid S' \subset S\} = \varnothing$ indicates the absence of an admissible structure S' that is simpler than a structure $S \in \Omega$.

3. PROBLEM ANALYSIS

The following conditions [5, 6] are necessary for system (1)–(5) to have a simple symmetric limit cycle with period $2h = 2\pi/\omega_0$:

$$x^* = -(I + \mathrm{e}^{Ah})^{-1} \int_0^h \mathrm{e}^{A(h-\tau)} B \, d\tau, \tag{10}$$

$$K^\top C x^* = 0, \tag{11}$$

$$K^\top C \left(\mathrm{e}^{At} x^* + \int_0^t \mathrm{e}^{A(t-\tau)} B \, d\tau \right) > 0, \quad 0 < t < h, \tag{12}$$

where x^* is the limit cycle point corresponding to the change of sign of $\tilde{u}$ from minus to plus and, consequently, to the switching of the relay (4) from -1 to $+1$, and I denotes an identity matrix of compatible dimensions.

A limit cycle is locally stable [5–7] if and only if the absolute values of all the eigenvalues of the matrix

$$W = (I - \upsilon K^\top C/(K^\top C \upsilon))\mathrm{e}^{Ah}, \tag{13}$$

$$\upsilon = -Ax^* + B, \tag{14}$$

are less than unity. This statement was proved in [6, 7] for systems of the form (1)–(5) using the classical approach to the stability analysis of periodic motions based on the Poincaré maps.

The stability margin requirement (6) for self-oscillations will be considered fulfilled if

$$\eta(K) = 1 - \rho(W(K)) \geqslant \check{\eta}, \tag{15}$$

where $\rho(W(K))$ denotes the spectral radius of the matrix $W(K)$; it is reasonable to assign $\check{\eta} \in [1, 10]h/(N\Delta t)$.

Processes in relay systems that evolve much more slowly compared to a self-oscillation process are conventionally called slow processes [1, 2, 5]. In the case where the signal range at the relay input slightly exceeds the amplitude $\tilde{U}$, such processes are described by an approximate model in which the relay (4) with a stable simple symmetric limit cycle is replaced by its linearized representation [1, 2, 5]

$$u = R\tilde{u}, \qquad R = 4/(\pi\tilde{U}), \tag{16}$$

where R is the linearization coefficient. Equations (1)–(3), (5), and (16) will be used as a simplified model of slow processes in the self-oscillating system (1)–(5).

Assume that the set of training examples contains descriptions of processes that can be considered slow for given parameters ω_0 and $\tilde{U}$. Then the problem of training system (1)–(5) to the desired behavior can be replaced by that of training system (1)–(3), (5), and (16). To apply the training method [17], we replace system (1)–(3) and (16) with its discrete-time analog

$$x_{k+1} = \widehat{A}x_k + \widehat{B}u_k, \quad y_k = Cx_k, \quad u_k = RKy_k, \tag{17}$$

where k is discrete time (a natural number); $t = k\Delta t$; the time discretization step Δt is chosen from the condition $\Delta t \ll h$; the vectors x_k, y_k, and u_k are the discrete approximations of the vectors x, y, and u, respectively; finally, $\widehat{A} = \mathrm{e}^{A\Delta t}$ and $\widehat{B} = \int_0^{\Delta t} \mathrm{e}^{At} B\, dt$.

According to [17], the conditions (7), (8) for making the behavior of system (17) maximally close to the desired one, defined by the set of training examples Q, are equivalent to

$$\sum_{\gamma=1}^{q} \mid G_\gamma(K)_S K_S - \hat{Y}_\gamma \mid^2 \to \min_K, \tag{18}$$

$$\hat{Y}_\gamma + \varepsilon_\gamma^- \leqslant G_\gamma(K)_S K_S \leqslant \hat{Y}_\gamma + \varepsilon_\gamma^+, \quad \gamma \in \{\overline{1, q}\}, \tag{19}$$

where $\hat{Y}_\gamma = Y_\gamma - Y_{0\gamma}$, with the columns Y_γ, $Y_{0\gamma}$, and $G_\gamma(K)$ being composed of the blocks y_k^γ, $C\widehat{A}^k x_0^\gamma$, and $G_{k\gamma}(K) = C\sum_{i=0}^{k-1}\left(y(x_0^\gamma, K)_i^\top \otimes \widehat{A}^{k-i-1}\widehat{B}R\right)$, $k \in \{\overline{1, N}\}$, respectively; $\otimes$ indicates the Kronecker product [23, p. 83]; finally, the matrix $G_\gamma(K)_S$ and the vector K_S are composed of the columns of the matrix $G_\gamma(K)$ and the coordinates of the vector K, respectively, with the numbers specified in the set S.

System (17) differs from the control system considered in [17] only by the factor R in the expression for u_k. This difference is taken into account in the above expression for $G_{k\gamma}(K)$.

In view of (5), condition (11) becomes the linear equation

$$(Cx^*)_S^\top K_S = 0, \tag{20}$$

where the vector $(Cx^*)_S$ is composed of the coordinates of the vector Cx^* with the numbers specified in the set S.

Due to (5), the discrete analog of the requirement (12) is the system of linear inequalities

$$H_{kS}K_S > 0, \ \ \forall k \in \{\overline{1, \hat{h}}\}, \tag{21}$$

where the matrix H_{kS} is composed of the columns of the matrix

$$H_k = \left(C\mathrm{e}^{Ak\Delta t}x^* + C \int\limits_0^{k\Delta t} \mathrm{e}^{A(k\Delta t-\tau)}B\,d\tau \right)^{\top}$$

with the numbers specified in S and $\hat{h} = \mathrm{fix}(h/\Delta t - 1)$, where $\mathrm{fix}(\cdot)$ means rounding to the nearest integer.

For the self-oscillation mode, the left-hand side of inequality (12) determines the values of the signal $\tilde{u}$ within the half-wave of its oscillations, so the amplitude $\tilde{U}$ of the signal $\tilde{u}$ is close to the value of $H_{kS}K_S$ when k lies at the middle of the closed interval $[1, \hat{h}]$. To realize the specified amplitude $\tilde{U}$ of the signal $\tilde{u}$ in the mode of simple symmetric self-oscillations, we impose the condition

$$H_{kS}K_S = \tilde{U}, \quad k = \mathrm{fix}((\hat{h}+1)/2). \tag{22}$$

Thus, the above problem of finding the vector K under which the control system (1)–(5) will fulfill the requirements (6)–(8) for the given values of ω_0 and $\tilde{U}$ is reduced to problem (18)–(22), (15). Here, the existence conditions (20)–(22) of self-oscillations with given parameters and their stability condition (15) have been obtained from the nonlinear model of the relay system (1)–(5) without simplifying assumptions: only the conditions (18), (19) for making its behavior maximally close to the desired one are based on its simplified linearized model (17).

4. SOLUTION METHOD

Let the controller structure be given, i.e., the set S is specified. As shown above, the desired vector K can be determined by solving problem (18)–(22) and (15), which is equivalent to the problem of training a static controller considered in [17]. For this problem, we apply the method proposed in [17]: at each iteration, it is necessary to solve a constrained linear least-squares (CLLS) problem [24, p. 225], belonging to the class of convex programming problems [25]. For such problems, effective optimization procedures have been developed that guarantee a solution or claim its absence. (In MATLAB, the lsqlin function is intended for solving CLLS problems.) At each iteration of the method [17], when solving the original problem, the transition to the CLLS problem is based on replacing the matrices $G_\gamma(K)_S$ in (18) and (19) with the matrix $G_\gamma(\acute{K})_S$ (fixed during this iteration) that corresponds to the vector $\acute{K}$ found at the previous iteration, as well as on replacing the function $\rho(W(K))$ in (15) with its linear approximation $r_0 + r_1\acute{K}_S$ in the neighborhood of $\acute{K}$, i.e., on replacing condition (15) with the linear inequality

$$r_0 + r_1\acute{K}_S \leqslant 1 - \check{\eta}, \tag{23}$$

where r_0 and r_1 are linearization coefficients.

To determine an initial approximation for the solution of the problem under consideration, we separate its important special case with conditions (18), (20), and (22) only. Here, we simplify condition (18) by assuming that all desired trajectories in the set Q belong to the solution set of the system designed. Then [17] the matrix $G_\gamma(K)_S$ in (18) can be replaced by the matrix $\bar{G}_{\gamma S}$, independent of K, and condition (18) can be therefore written as

$$\sum_{\gamma=1}^{q} \mid \bar{G}_{\gamma S}K_S - \hat{Y}_\gamma \mid^2 \to \min_K, \tag{24}$$

where the matrix $\bar{G}_{\gamma S}$ consists of the columns of the matrix $\bar{G}_\gamma$ specified in S, and the latter matrix is the column of the blocks $\bar{G}_{k\gamma} = C\sum_{i=0}^{k-1}\left(y_i^{\gamma\top} \otimes \widehat{A}^{k-i-1}\widehat{B}R\right)$, $k \in \{\overline{1,N}\}$.

As is well known [24], a quadratic programming problem with linear equality constraints (and problem (20), (22), and (24) as its particular case) is reduced to a system of linear equations. In the case under study, this system takes the form

$$\begin{pmatrix} F & L^\top \\ L & 0 \end{pmatrix} \begin{pmatrix} K_S \\ \lambda \end{pmatrix} = \begin{pmatrix} c \\ d \end{pmatrix}, \tag{25}$$

where $F = \bar{G}_S^\top \bar{G}_S$, $\bar{G}_S$ is the matrix of the system of equations $\bar{G}_{\gamma S} K_S = \hat{Y}_\gamma$, $\forall \gamma \in \{\overline{1,q}\}$, $L = ((Cx^*)^\top,\ H_{kS})^\top$, $c = \bar{G}_S^\top \hat{Y}_\gamma$, $d = (0, \tilde{U})^\top$, and λ is the vector of Lagrange multipliers.

The vector K_S obtained by solving system (25) can serve as an initial approximation for the solution of problem (18), (22), and (15).

Suppose that with the initial approximation given by (25), problem (18)–(22), (15) has not been solved successfully. Then it is possible to use the initial approximation calculated under the requirements described by the system of equalities (20), (22), and (24) and inequalities (19), (21), and (23). In this case, replacing (19) with its approximate analog, i.e., the condition

$$\hat{Y}_\gamma + \varepsilon_\gamma^- \leqslant \bar{G}_{\gamma S} K_S \leqslant \hat{Y}_\gamma + \varepsilon_\gamma^+, \quad \gamma \in \{\overline{1,q}\}, \tag{26}$$

one finds the initial approximation by solving the CLLS problem (20)–(24) and (26).

The above initial approximation is refined at subsequent iterations of the method without assuming that the trajectories constituting the set Q belong to the solution set of the system designed.

The set Ω of simple controller structures can be found by a method intended for designing simple structures of a general form [19]. In this method, the above procedure for solving problem (18)–(22) and (15) can be used to assess the admissibility of the controller structure and determine the corresponding vector K.

5. EXAMPLES

Example 1. Consider the problem of designing a relay controller for a self-oscillating control system of an electric drive. The plant consists of a DC motor with a constant excitation flux, a gearbox, and an inertial load. This plant is described by equations (1)–(2) with

$$A = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & C_m/J_m \\ 0 & -C_e/L_e & -R_e/L_e \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 0 \\ U/L_e \end{pmatrix}, \quad C = \begin{pmatrix} 1/k_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

where R_e and L_e are the resistance and inductance of the motor armature winding; C_e and C_m are the back electromotive force (EMF) coefficient and the torque coefficient of the motor; U is the supply voltage; k_r is the gear ratio; finally, J_m is the moment of inertia of the moving parts reduced to the motor shaft. Let $R_e = 0.475$, $L_e = 5.7 \times 10^{-4}$, $C_e = C_m = 6.83 \times 10^{-2}$, $U = 27$, $k_r = 2$, and $J_m = 9.43 \times 10^{-5}$ in the international system of units. Direct verification shows that the pairs (A, B) and (A, C) are controllable and observable, respectively. The state vector has the form $x = (x_1,\ x_2,\ x_3)$, where x_1 and x_2 are the angular position and velocity of the motor shaft, respectively, and x_3 is the current in the armature winding. The output has the form $y = (y_1,\ y_2,\ y_3) = (x_1/k_r,\ x_2,\ x_3)$, where y_1 is the controlled variable (the angular position of the output shaft of the electric drive), and $S = \{1, 2, 3\}$. Let us assign $\tilde{U} = 0.01$ V and suppose that the amplitude of self-oscillations of the controlled variable y_1 should not exceed 0.1 mrad. To fulfill this requirement, according to the analysis of the plant's frequency response, the self-oscillation frequency must be at least 1820 Hz, so we set $f = 2000$ Hz, with $\omega_0 = 2\pi f = 12\,566$ rad/s.

It is required to find the gains constituting the desired vector K.

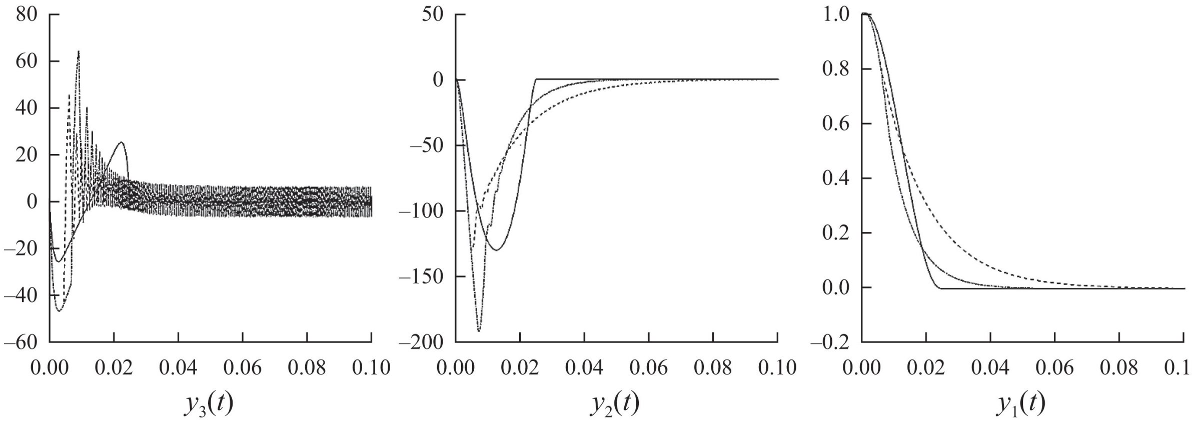

Fig. 1. System transients for $x(0) = x_0^1$. (The desired process and processes obtained in Examples 1 and 2 are indicated by solid, dotted, and dash-dotted lines, respectively.)

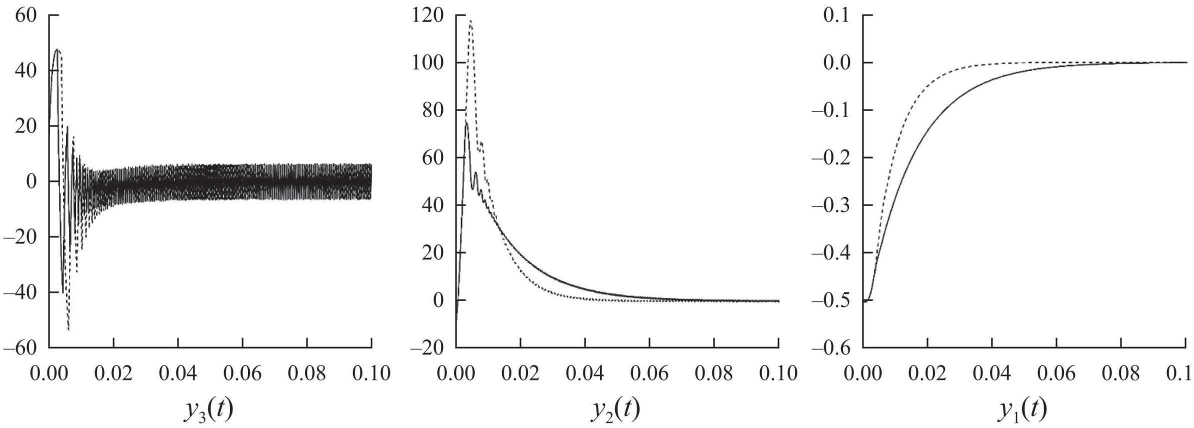

Fig. 2. System transients for $x(0) = x_0^3$. (The desired process and processes obtained in Examples 1 and 2 are indicated by solid and dotted lines, respectively.)

We form a set of training examples defining the desired behavior of the system designed during its transition from a disturbed to a steady-state motion mode. Following the above recommendation $q \in [1, n]$ (see Section 2), $q = 2$ training examples are taken. The desired system behavior is the trajectories of the plant (1) and (2) from the significantly different initial states $x_0^1 = (1; 0; 0)$ and $x_0^2 = (0; 10; -10)$ to the origin in a time of 0.025 s. These trajectories can be calculated using known dependencies [22, p. 128]. The specified trajectories Y_1 and Y_2, together with the corresponding initial conditions on the time interval from 0 to 0.05 s, constitute the set of training examples $Q = \{(x_0^1, Y_1), (x_0^2, Y_2)\}$. When calculating the trajectories Y_1 and Y_2, the time discretization step $\Delta t = 2.5 \times 10^{-5} \ll h = 2.5 \times 10^{-4}$ is chosen according to the recommendation of Section 3; the corresponding value is $N = 2000$. The problem of calculating these trajectories has a solution due to the controllability and observability of the plant.

First, we solve the design problem without the constraints (6), (7), which is equivalent to problem (20), (22), and (24). Let the initial approximation be the solution of system (25). Three iterations of the method presented in subsection 4.1 yielded $K = (-5.11;\ -3.73 \times 10^{-2};\ 1.06 \times 10^{-4})$, with the stability margin $\eta(K) = 0.017$ and 2.1×10^{-6} as the corresponding value of the objective function (18). The resulting values of $\tilde{U}$ and ω_0 were 0.010 V and 12 566 rad/s, respectively. The transients in system (1)–(5), corresponding to the found vector K, are presented in Figs. 1 and 2 for the initial states $x(0) = x_0^1$ and $x(0) = x_0^3 = (-0.5;\ -10;\ +10)$, respectively. Figure 2 demonstrates the system behavior corresponding to the initial state x_0^3, which significantly differs from the initial states x_0^1 and x_0^2 specified in this example.

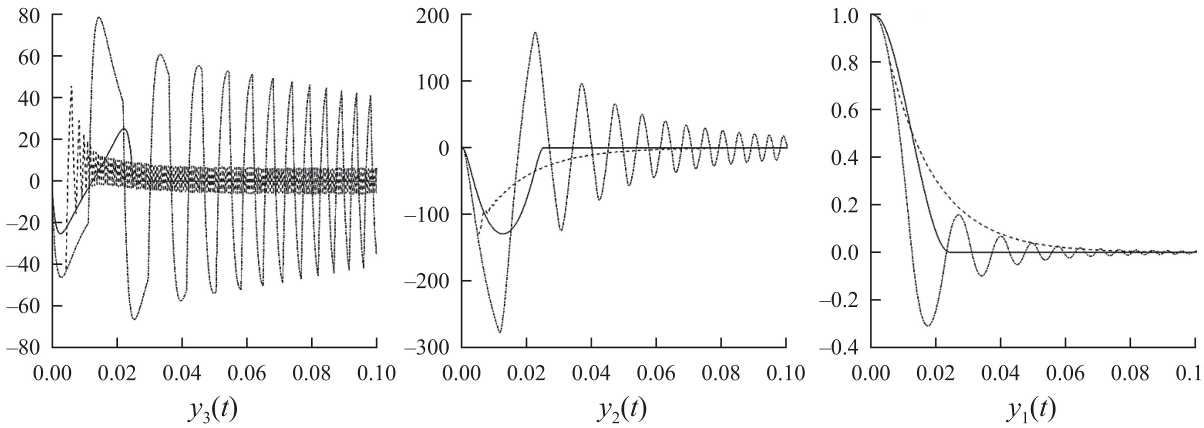

Fig. 3. System transients for $x(0) = x_0^1$. (The desired process and processes obtained in Examples 1 and 4 are indicated by solid, dotted, and dash-dotted lines, respectively.)

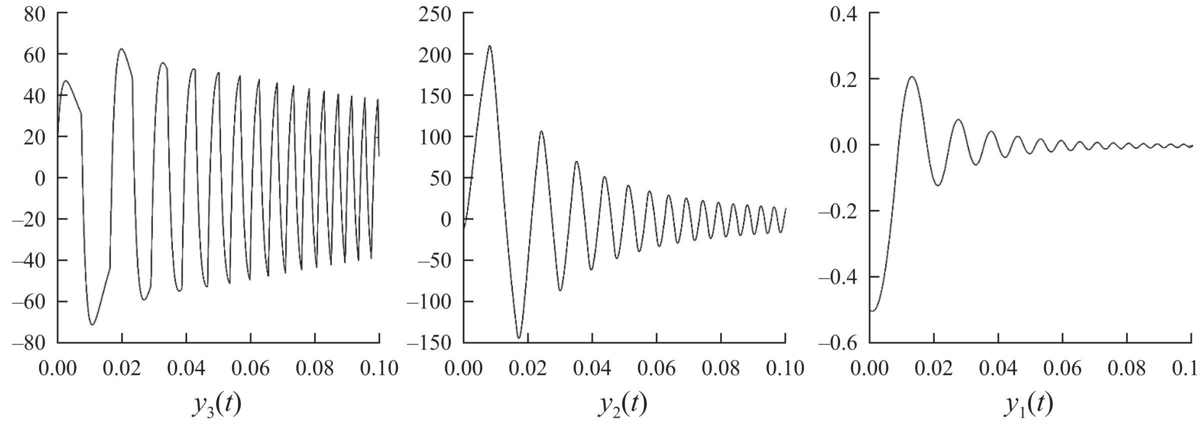

Fig. 4. System transients for $x(0) = x_0^3$, obtained in Example 4.

Example 2. Next, we solve the design problem of Example 1 with the following modifications: increase the stability margin $\eta(K)$ to at least $\check{\eta} = 0.030$ and reduce the admissible deviation of the output coordinates y_1 and y_2 from their desired values, assigning for them $\delta_1 = 0.05$ and $\delta_2 = 0.15$ on the time interval from 0.025 to 0.05 s and $\delta_1 = 0.1$ and $\delta_2 = 0.2$ on the time interval from 0 to 0.025 s. The vectors $\varepsilon_k^{\gamma-}$ and $\varepsilon_k^{\gamma+}$ corresponding to the above values of δ_1 and δ_2 are calculated according to the explanations to (7). The value $\check{\eta} = 0.030$ is chosen according to the explanations to (15) by the formula $\check{\eta} = 6h/(N\Delta t)$, where $h = 1/(2f) = 1/4000$ s and $N\Delta t = 0.05$ s.

Under the specified requirements, we solve problem (18)–(22), and (15), with the solution of the CLLS problem (20)–(24), and (26) taken as the initial approximation.

Three iterations of the method from subsection 4.1 yielded

$$K = (-9.81;\ \ -3.76 \times 10^{-2};\ \ 9.75 \times 10^{-5}).$$

In this case, $\eta(K) = 0.0323$, conditions (6) and (7) hold, and the value of the objective function (18) is 4.1×10^{-6}.

The transients corresponding to the found vector K are presented in Figs. 1 and 2 for the initial states $x(0) = x_0^1$ and $x(0) = x_0^3$, respectively.

Example 3. Let us change the problem statement of Example 1, i.e., solve the structural design problem: given ω_0 and $\tilde{U}$, it is required to find the sets S and the corresponding vectors K under which the controller structure will be simple according to (9). The desired set Ω can be obtained by the design method of simple structures presented in [19, p. 19] for "general-form problems." At the first step, it is necessary to check the admissibility of all structures $S \subseteq \{1,2,3\}$. Recall that for an admissible structure S, it possible to find a vector K under which the control system (1)–(5) will fulfill the requirements (6)–(8). (See the explanations to (9).)

We check the existence of such a vector K using the solution procedure of problem (18)–(22), and (15), described in Section 4. According to the checking results, the structure $S = \{1,2,3\}$ is admissible, whereas all structures $S \subseteq \{1,2,3\}$ corresponding to two-component sets (i.e., $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$) are inadmissible. As a consequence, the sets $\{1\}$, $\{2\}$, $\{3\}$, representing the subsets of the inadmissible sets $\{1,2\}$, $\{1,3\}$, and $\{2,3\}$, are surely inadmissible and do not require checking [19, Proposition 8]. Thus, in this example, the structure $S = \{1,2,3\}$ is admissible and simple.

Example 4. Now we solve the problem of Example 3, with the lower limit of the stability margin reduced to $\check{\eta} = 2.5 \times 10^{-3}$. The resulting set Ω contains the single set $S = \{1,2\}$, which defines the controller structure using feedback signals for the angular position y_1 and velocity y_2 of the output shaft of the drive. In this case, the structure $S = \{1,2,3\}$ is redundant, and the other structures are inadmissible. The found simple structure $S = \{1,2\}$ is associated with $K = (-61.45;\ -3.73 \times 10^{-2};\ 0)$. The transients in system (1)–(5) corresponding to the found vector K are presented in Figs. 3 and 4 for the initial states $x(0) = x_0^1$ and $x(0) = x_0^3$, respectively.

6. CONCLUSIONS

In this paper, we have posed and solved the problem of designing relay controllers for a self-oscillating system with a linear plant. The novelty of this problem statement lies in the joint fulfillment of existence conditions for asymptotically stable self-oscillations with given parameters in the system, controller's structural constraints, and requirements for making the system behavior maximally close to a desired one. The desired system behavior has been specified by a set of time-varying system output laws (training examples). A method for solving the problem has been proposed, which further develops and extends the algorithm outlined in [17] to the class of relay self-oscillating control systems. The mathematical model of the controller uses the function of an ideal relay. The impact of the differences between the characteristics of real and ideal relays on the system behavior can be assessed via modeling by replacing the function of an ideal relay in (1)–(5) with a more complete and detailed mathematical description of a real relay.

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