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NONLINEAR SYSTEMS

Pseudo-Optimal Solution of a Variational Problem with a Free Right End and a Specified Time of Ending the Transient Process

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Abstract—The problem of optimal control of the final state of the system in a sense is the core of any other optimization problem. The formulation of such problems includes a description of the dynamic object itself, constraints imposed on the controls and states of the object, and a quality functional, in general, the Bolz's functional. The necessary optimality conditions in the problem of synthesizing the corresponding controls have written in the form of the canonical Euler–Lagrange system with the assignment of the corresponding boundary conditions. The synthesis of the corresponding controls faces the problem of the need to find solutions to boundary value problems, which has usually realized by numerical methods. The paper proposes an alternative to such methods for solving two-point boundary value problems based on the assumption of the validity of R. Bellman's inverse optimality principle, which consists in preserving the functional relationship between the components of a two-point boundary value problem in the entire control interval. The obtained theoretical results have confirmed by modeling the control system with synthesized control.

Keywords: optimal control, Hamiltonian of the system, canonical Euler–Lagrange system, boundary conditions of the canonical system

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1. INTRODUCTION

Control systems described by ordinary differential equations quite fully reflect many real processes and therefore are the most common objects to which mathematical methods of constructing controls are applied. The task of constructing an optimal dynamic control system with complete information in relation to a set of goals, a quality functional, a set of admissible controls, a set of states and the initial state of the object at the start of control is to find a control belonging to an admissible set of controls that minimizes a given quality functional on the solutions of the object equation [1–5]. The solution to the problem has carried out using variation methods [4–7].

The synthesis of an optimal control system has carried out using necessary and sufficient conditions for the minimum of the quality functional [2–5]. It should be note that the existence of optimal control is not necessary: the set of admissible controls may not include controls at all that transfer the object from the initial state to a given set of goals.

The necessary conditions for the existence of optimal control are described by a two-point boundary value problem and the condition for choosing the control itself in the form of a certain function depending on the behavior of the Hamiltonian on the optimal trajectory. The main problem of finding optimal control is associated with finding a solution to a two-point boundary value problem. Such boundary value problems for systems of differential equations have rarely solved analytically and require the use of numerical methods, which are divided into two types — iterative and non-iterative. For linear problems, a solution can be obtained without iterations, while iterative methods are indispensable when solving nonlinear problems [8–12]. Such methods include: Euler's method, linear interpolation method, finite difference method, shooting method [10]. The essence of this method is to reduce the boundary value problem to a multiple solution of the Cauchy problem. A certain development of the shooting method is the differential sweep method, in which auxiliary Cauchy problems have solved not for the original differential equation, but for other equations of a lower order.

In general, the problem of solving two-point problems is relevant, and methods for solving it have proposed today, some of them based on the use of neural networks [13–19].

One of the methods for solving boundary value problems is the method of successive approximations. This method, which has not yet received wide application, reduces the original problem to a certain sequence of linear-quadratic problems. The paper [15] presents a method for synthesizing optimal control with feedback for one class of nonlinear systems using a quadratic criterion. This method has based on a special method of successive approximations, the convergence of which allows one to prove the existence of optimal control and obtain a procedure for constructing it. The paper considers an analytical and numerical study of this method and its implementation in the MathCloud system.

To solve variation problems, two non-classical methods has intensively developed since the end of the 20th century. The first method has based on the Differential Transform Method (DTM), which seeks an analytical solution in the form of a certain functional series [18–20]. The second method is based on a neural network based on mathematical models of natural sciences (Physics-Informed Neural Network, PINN), where artificial intelligence in the form of a neural network is used to solve a differential equation [19, 20].

The first method, DTM, has characterized by its flexibility both in the form of the differential equation and in the boundary conditions. One of its strengths is its scalability to handle approximate solutions of various orders, and sometimes it even allows predicting the exact solution based on the form of the found coefficients of the original equation. The disadvantage of this method is the difficulty of automating the process of solving a given problem.

The second method (PINN) uses neural networks to solve the corresponding differential equation. One of the advantages of this method is its relatively simple implementation and flexibility. Once a differential equation model has trained, it can provide solutions for different grids without recalculating the problem each time.

In this paper, an alternative to numerical methods for solving two-point boundary value problems is proposed, based on the assumption of the validity of R. Bellman's inverse optimality principle [21], which consists in preserving the functional connection between the components of a two-point boundary value problem in the entire control interval.

2. TERMINAL OPTIMAL CONTROL PROBLEM

2.1. Statement of the Problem

The article considers an object that is described by an ordinary differential equation

$$\begin{aligned} &\frac{d}{dt}x(t) = f(t, x(t)) + g(t)u(t), \quad x(t_0) = x_0, \\ &f, g : T \times \Omega_s \to \mathbb{R}^n, \quad (t, x) \to f(t, x),\, g(t). \end{aligned} \tag{1}$$

Here T is the control interval $[t_0, t_f]$; $u \in \{u(t) \in \mathbb{R}^r \subset U, t \in [t_0, t_f]\}$ is the control to be found, the matrices $f(t,x)$, $g(t)$ are real and continuous; U is a compact set of admissible controls; Ω_x is the set of trajectories $x(\cdot) : [t_0, t_f] \to \mathbb{R}^n$, that satisfy the initial condition of $x(t_0) = x_0$ and the differential inclusion $\{dx(t)/dt\} \in \operatorname{co}\{[f(t, x(t)) + g(t)u(t)] : u \in U\}$.

It is assumed that for all (t, x) the pair of $\{f(t, x(t)), g(t)\}$ is controllable. In addition, we will assume that the functions $f(t, x(t))$, $g(t)$ are smooth enough so that one and only one solution (1) of the $x(t, t_0, x_0) \in \Omega_x$ passes through any $(t_0, x_0) \in T \times X_0$.

The set of goals in this problem is defined as $S \in \mathbb{R}^n \times [t_0, t_f]$. The elements of the set of goals are pairs (t, x), consisting of a state X and a point t from the interval of definition of the system $[t_0, t_f]$.

Considering the problem of synthesis of the control law, we introduce the Boltz's functional

$$J(x(\cdot), u(\cdot)) = K(x(t_f)) + \int_{t_0}^{t_f} \{L(t, x(t), u(t))\}dt. \tag{2}$$

Let be $L(t, x(t), u(t))$ a continuous real function defined on $\mathbb{R}^n \times \mathbb{R}^r \times [t_0, t_f]$, and $K(x(t_f))$ be a real function on $\mathbb{R}^n \times t_f$.

Proposition 1. *On the properties of a function $f(t,x)$ and a functional $L(t,x,u)$* [2–4].

1. *Functions $f(t,x)$, $L(t,x,u)$ and $K(x(t_f))$ are continuous and satisfy the constraints*

$$\|f(t,x)\| \le (1 + \|x\|)R_f, \quad |L(t,x,u)| \le (1 + \|x\|)R_L, \quad K(x(t)) \le (1 + \|x\|)R_K$$

for all $(t,x,u) \in [t_0, t_f] \times \mathbb{R}^n \times U$ (here R_f, R_L, R_K are positive numbers).

2. *We will assume that the functions $f(t, x(t))$ and $L(t,x,u)$ satisfy the Lipschitz's condition for the variable x:*

$$\|f(t, x + y) - f(t,x)\| + |L(t, x + y, u) - L(t, x, u)| \le \lambda_{up}\|y\|$$

for all $(t,x) \in [t_0, t_f] \times \mathbb{R}^n$, $y \in \mathbb{R}^n$.

3. *Functions $f(t, x(t))$, $L(t, x(t), u(t))$ and their partial derivatives with respect to t, x, u, i.e.*

$$\begin{aligned} &f_i(t, x(t)), \quad \frac{\partial f_i(t, x(t))}{\partial t}, \quad \frac{\partial f_i(t, x(t))}{\partial x}, \quad i = 1, \ldots, n, \\ &L(t, x(t), u(t)), \quad \frac{\partial L(t, x(t), u(t))}{\partial t}, \quad \frac{\partial L(t, x(t), u(t))}{\partial x}, \quad \frac{\partial L(t, x(t), u(t))}{\partial u}, \end{aligned}$$

are continuous on $\mathbb{R}^n \times \mathbb{R}^r \times [t_0, t_f]$.
Note that for a number of problems the existence of continuous partial derivatives $L(t, x(t), u(t))$ with respect to u is not required.

4. *The finite cost function $K(x(t_f))$, defined on $\mathbb{R}^n \times t_f$, is a real function such that*

$$K(x(t)), \quad \frac{\partial K(x(t))}{\partial x}, \quad \frac{\partial^2 K(x(t))}{\partial x^2}$$

are continuous on $\mathbb{R}^n \times [t_0, t_f]$.

Additional assumptions about the properties of the vector $f(t, x(t))$ will be made below (Section 4 of the article).

Let the element $\xi = (x(t), u(t), t_0, t_f)$, for which all the specified conditions and constraints of the problem are met, be an admissible controlled process. We will consider the admissible elements $\xi = (x(t), u(t), t_0, t_f)$ in the stated problem to be functions of the class $x(\cdot) \in C^1([t_0, t_f], \mathbb{R}^n)$, $u(\cdot) \in C([t_0, t_f], \mathbb{R}^r)$. The type of restrictions imposed on control are specified below.

The control problem consists of constructing an optimal strategy, i.e. finding an admissible controlled process $\xi^0 = (x^0(t), u^0(t), t_f)$, that minimizes a functional of the form (2) on an object (1), where the control objective is specified in the form $S \in \mathbb{R}^n \times [t_0, t_f]$.

2.2. Necessary Optimality Conditions

Let us write the Hamiltonian of the system $H(t, x(t), u(t), \lambda(t))$:

$$H(t, x(t), u(t), \lambda(t)) = L(t, x(t), u(t)) + \lambda^{\mathrm{T}}(t)[f(t, x) + g(t)u(t)]. \tag{3}$$

Here $\lambda(t)$ is the Lagrange function.

The necessary optimality conditions have the form [1, 2, 7]

$$\frac{d}{dt}x(t) = \left\{\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial \lambda}\right\}^{\mathrm{T}}, \quad x(t_0) = x_0, \tag{4}$$

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial x}\right\}^{\mathrm{T}}, \quad \lambda(t_f) = \left\{\frac{\partial K(x(t_f))}{\partial x}\right\}^{\mathrm{T}}. \tag{5}$$

It is known [1–4] that the optimal control $u(t)$, the synthesis of which is performed using the Hamiltonian (3), leads to the need to solve the two-point boundary value problem (4), (5). However, it should be noted that the Hamiltonian does not contain any information about the functional relationship of the processes $x(t)$ and $\lambda(t)$.

In the problem under consideration, taking into account the assumptions made above, the necessary condition must be met

$$H^0(t, x(t), u(t), \lambda(t)) = \min_{u(t)} H(t, x(t), u(t), \lambda(t)), \quad t \in [t_0, t_f], \tag{6}$$

where $\lambda(t)$ is the solution of system (5). In the case where the set of admissible controls U coincides with the entire space $\mathbb{R}^n$, i.e. $U = \mathbb{R}^n$, then condition (6) can be satisfied only at a stationary point, i.e.

$$\frac{\partial H(t, x(t), u(t), \lambda(t))}{\partial u} = \frac{\partial L(t, x(t), u(t))}{\partial u} + g^{\top}(t)\lambda(t) = 0, \quad t \in [t_0, t_f]. \tag{7}$$

However, if the set U is closed and $U \neq \mathbb{R}^n$, then relation (7) is not satisfied in the general case and Pontryagin's principle [2–4] should be applied to synthesize optimal control.

Let us assume that there exists an optimal control satisfying the necessary conditions (6) or (7), which we write in the form

$$u(t) = -\varphi(x(t))\lambda(t), \tag{8}$$

where is a matrix function $\varphi(x(t)) \in \mathbb{R}^{r\times n}$ such that

$$g(t)\varphi(x(t)) \leq \Phi \in \mathbb{R}^{n\times n}. \tag{9}$$

Here Φ is a parametrically specified matrix that determines the set of possible values of the parameters of the matrix $\varphi(x(t))$ given the known matrix $g(t)$; which means that the restrictions imposed on the control $u(t) \subset U$, are specified in the form of condition (9). The issue of the connection between the established restrictions on the parameters of the regulator will be considered in the fourth section of the article when specifying the type of functions $L(t, x(t), u(t))$ and $K(x(t_f))$ the quality criterion (2).

Thus, the synthesis of the optimal admissible process $\xi = \{x(t), u(t), t_0, t_f\}$ is replaced by the search for a matrix $\varphi(x(t))$, in which control $u(t) = -\varphi(x(t))\lambda(t) \subset U$ ensures the fulfillment of condition (9) and minimizes the functional (2).

Let us rewrite conditions (4), (5) taking into account (8):

$$\frac{d}{dt}x(t) = f(t, x(t)) - g(t)\varphi(x(t))\lambda(t), \quad x(t_0) = x_0, \tag{10}$$

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial f(t, x(t))}{\partial x}\right\}^{\mathrm{T}} \lambda(t) - \left\{\frac{\partial L(t, x(t), u(t))}{\partial x}\right\}^{\mathrm{T}}, \quad \lambda(t_f) = \left\{\frac{\partial K(x(t_f))}{\partial x}\right\}^{\mathrm{T}}. \tag{11}$$

Thus, the successful solution of the optimal control synthesis problem in the form $u(t) = -\varphi(x(t))\lambda(t)$ depends on the possibility of successfully solving the two-point boundary value problem (10), (11).

3. PSEUDO-OPTIMAL SOLUTION OF THE SYNTHESIS PROBLEM

It should be noted that the above stages of constructing optimal control are based on the use of the properties of the Hamiltonian (3), however, the Hamiltonian does not contain information regarding the boundary conditions of the control system. Only the condition $K(x(t_f))$ in the functional (2), that is $\lambda(t_f) = \{\partial K(x(t_f))/\partial x\}^{\mathrm{T}}$ in the boundary condition of equation (11), can connect the variables $x(t)$ and $\lambda(t)$ at the moment t_f. In addition, it should be noted that since system (1) is described by a nonlinear differential equation, then constraint (9), imposed on the optimal control, in the general case, may not contain control actions that achieve the control objective for given initial conditions $x_0 \in X_0$ and, moreover, provide uniform. asymptotic stability to the system [1–4].

To justify the proposed method for solving the problem of constructing pseudo-optimal control, we use the inverse Bellman optimality principle [4, 21], which assumes that optimal control has the property that for any initial state and the initial control used, the value of the criterion on the final interval is affected by the control on this interval and the value of the phase vector at the end of the interval. For the problem considered in this paper, we will clarify this definition.

Definition 1. *Inverse principle of optimality.* For the optimality of the admissible process $\xi^0 = (x^0(t), u^0(t), t_0, t_f)$ in problem (1), (2) it is necessary that for any of the subintervals $[t_0, \tau] \subset [t_0, t_f]$, $\tau \le t_f$ the admissible process starting at time t_0, be optimal with respect to the control on this interval and the value of the phase vector at the end of the interval $x(\tau)$, which determines the value of the function of the auxiliary variable $\lambda(\tau)$, i.e. $\xi^0 = (x^0(t), u^0(t), t_0, \tau)$.

Based on this definition, we make the following proposition:

Proposition 2. *Noting that the condition connecting the variables $x(t)$ and $\lambda(t)$ at the moment t_f, is defined as $\lambda(t_f) = \{\partial K(x(t_f))/\partial x\}^{\mathrm{T}}$, we will assume, based on the inverse principle of optimality, that the relation $\tilde{\lambda}(\tau) = \{\partial K(\tilde{x}(\tau))/\partial \tilde{x}\}^{\mathrm{T}}$ is valid for all $\tau \in [t_0, t_f]$.*

The fulfillment of Proposition 2 means that for all $t \in [t_0, t_f]$ the variable $\tilde{\lambda}(t)$ is a function of the state $\tilde{x}(t)$.

Definition 2. The control $\tilde{u}(t) = -\varphi(\tilde{x}(t))\tilde{\lambda}(t)$, whose synthesis is based on the adoption of Proposition 2 is called **pseudo-optimal control**.

Let us rewrite equations (10), (11) with admissible control $\tilde{u}(t) = -\varphi(\tilde{x}(t))\tilde{\lambda}(t)$ in the form

$$\frac{d}{dt}\tilde{x}(t) = f(t, \tilde{x}(t)) - g(t)\varphi(\tilde{x}(t))\tilde{\lambda}(t), \quad \tilde{x}(t_0) = x_0, \tag{12}$$

$$\frac{d}{dt}\tilde{\lambda}(t) = -\left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} \tilde{\lambda}(t) - \left\{\frac{\partial L(t, \tilde{x}(t), u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}, \quad \tilde{\lambda}(t_f) = \lambda(t_f) = \left\{\frac{\partial K(x(t_f))}{\partial x}\right\}^{\mathrm{T}}. \tag{13}$$

When Proposition 2 is fulfilled, the first full derivatives of the main and auxiliary equations are related by the relation

$$\frac{d}{dt}\tilde{\lambda}(t) = \left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\frac{d}{dt}\tilde{x}(t), \quad t \in [t_0, t_f]. \tag{14}$$

Substituting $dx(t)/dt$ and $d\lambda(t)/dt$, defined in (12) and (13), we get

$$\begin{aligned} &-\left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\tilde{\lambda}(t) - \left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} \\ &= \left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\} f(t,\tilde{x}(t)) - \left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\} g(t)\varphi(\tilde{x}(t))\tilde{\lambda}(t). \end{aligned}$$

From where, bringing similar terms and taking into account (9), we obtain

$$\begin{aligned} \tilde{\lambda}(t) &= \left[-\left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} + \left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi\right]^{-1} \\ &\times \left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\} f(t,\tilde{x}(t)) + \left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]. \end{aligned} \tag{15}$$

It is obvious that the expression $\{\partial^2 K(\tilde{x}(t))/\partial \tilde{x}^2\}\,\Phi - \{\partial f(t,\tilde{x}(t))/\partial \tilde{x}\}^{\mathrm{T}}$ must not be negative or equal to zero. This condition can be ensured by choosing the appropriate elements of the terminal penalty function of the functional (2). Let us establish for definiteness that the matrix

$$\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1}$$

is positive definite, which will be taken into account when analyzing the stability of a system with synthesized control.

Pseudo-optimal control of the form (8) will be determined by the following expression

$$\begin{aligned} \tilde{u}(t) &= -\varphi(\tilde{x}(t))\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1} \\ &\times \left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\} f(t,\tilde{x}(t)) + \left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]. \end{aligned} \tag{16}$$

To establish the uniform asymptotic requirements that system (12) with control (16), must meet, we introduce the Lyapunov function

$$V_L(t) = \tilde{x}^{\mathrm{T}}(t)\tilde{x}(t). \tag{17}$$

The total derivative of the Lyapunov function (17) has the form

$$\frac{d}{dt}V_L(t) = f^{\mathrm{T}}(t,\tilde{x}(t))\tilde{x}(t) - \tilde{\lambda}^{\mathrm{T}}(t)\Phi^{\mathrm{T}}\tilde{x}(t) + \tilde{x}^{\mathrm{T}}(t)f(t,\tilde{x}(t)) - \tilde{x}^{\mathrm{T}}(t)\Phi\tilde{\lambda}(t) \le 0, \tag{18}$$

where $\tilde{\lambda}(t)$ is determined by equation (15). Thus, system (12) is uniformly asymptotically stable if the condition

$$f^{\mathrm{T}}(t,\tilde{x}(t))\tilde{x}(t) + \tilde{x}^{\mathrm{T}}(t)f(t,\tilde{x}(t)) \le \tilde{\lambda}^{\mathrm{T}}(t)\Phi^{\mathrm{T}}\tilde{x}(t) + \tilde{x}^{\mathrm{T}}(t)\Phi\tilde{\lambda}(t) \tag{19}$$

is satisfied.

It can be noted that the fulfillment of condition (19) depends on the matrix Φ, which limits the control capabilities.

Let us formulate the theorem.

Theorem 1. *A pseudo-optimal solution to the control problem of a nonlinear dynamic object* (1) *with functional* (2) *exists if and only if*

$$\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} > 0, \quad \forall (t,\tilde{x}) \in [t_0,t_f]\times\Omega_x. \tag{20}$$

In this case, the trajectory $\tilde{x}^0(t)$ of system (1)*, originating from $\tilde{x}(t_0) = x(t_0)$ and corresponding to pseudo-optimal control $\tilde{u}^0(t)$, is a solution to the equation*

$$\begin{aligned}\frac{d}{dt}\tilde{x}(t) = f(t,\tilde{x}(t)) - g(t)\varphi(\tilde{x}(t))&\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1}\\ \times&\left[\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\} f(t,\tilde{x}(t)) + \left\{\frac{\partial L(t,\tilde{x}(t),u(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right].\end{aligned} \tag{21}$$

Satisfaction of condition (20) *taking into account the restrictions* (9) *imposed on the parameters of the control action allows us to determine the region of initial conditions of the system* (21)*, under which pseudo-optimal control* (16) *will provide the nonlinear system with uniform asymptotic stability:*

$$\left\{\frac{\partial^2 K(\tilde{x}(t))}{\partial \tilde{x}^2}\right\}_{t=t_0}\Phi - \left\{\frac{\partial f(t,\tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}_{t=t_0} \succ 0. \tag{22}$$

4. PROBLEM WITH A QUADRATIC QUALITY FUNCTIONAL

The problem of dynamic system control by a quadratic criterion is a classical problem of modern control theory. For linear systems, this problem has been completely solved and its solution has been fully and thoroughly described in many books (see, for example, [1–3]). For nonlinear systems, one of its first considerations has given in book [4]. It describes a certain scheme of successive approximations, which in some cases provides optimal control in the form of a law with feedback. A feature of the method is that the system operator changes from iteration to iteration. This complicates the analytical study of the problem and greatly complicates the computational procedure. Therefore, this method has not found wide application. In works [6–8], various aspects of the problem has considered (from constructing optimal control to proving the existence of its solution). In work [17], a modified scheme of successive approximations for the problem of optimal control of a nonlinear system by a quadratic quality functional has described. This scheme provides a solution to the problem in many important cases. However, the question of the convergence of the method on an arbitrary finite horizon remains open and is a topic for further research.

This article considers the problem of controlling an object of the form (1)

$$\begin{aligned}&\frac{d}{dt}x(t) = f(t,x(t)) + g(t)u(t), \quad x(t_0) = x_s,\\ &y(t) = Cx(t),\\ &f,g: T\times\Omega_x \to \mathbb{R}^n, \quad (t,x)\to f(t,x(t)),\, g(t),\end{aligned} \tag{23}$$

with the functional

$$J(x(\cdot),u(\cdot)) = \frac{1}{2}x^{\mathrm{T}}(t_f)Fx(t_f) + \frac{1}{2}\int\limits_{t_0}^{t_f}\left\{x^{\mathrm{T}}(t)C^{\mathrm{T}}QCx(t) + u^{\mathrm{T}}(t)Ru(t)\right\}dt. \tag{24}$$

The matrices F, Q, R are positive definite.

The control problem consists of constructing an optimal strategy, i.e. finding an admissible controlled process $\xi^0 = (x^0(t), u^0(t), t_0, t_f)$, that minimizes a functional of the form (24) on the object (23), where the control objective is specified in the form $S \in \mathbb{R}^n \times [t_0, t_f]$.

Let us assume that there exists an optimal control satisfying the necessary conditions (6) or (7), which we write in the form

$$u(t) = -\varphi(x(t))\lambda(t), \tag{25}$$

where the matrix function $\varphi(x(t)) \in \mathbb{R}^{r\times n}$, taking into account (10), is such that

$$g(t)R^{-1}g^{\mathrm{T}}(t) = g(t)\varphi(x(t)) \leq \Phi \in \mathbb{R}^{n\times n}. \tag{26}$$

Here Φ is a parametrically specified matrix that elementwise determines the set of possible values of the parameters of the matrix $\varphi(x(t))$ with a known matrix $g(t)$, which means that the restrictions imposed on the control of $u(t) \subset U$, are specified in the form of condition (26).

Proposition 3. *By assigning the penalty matrices F, Q, R of the quality functional* (24) *it is possible, when determining the optimal control, to ensure that condition* (26) *is satisfied.*

The two-point boundary value problem in this subproblem, taking into account (26) has the form

$$\frac{d}{dt}x(t) = f(t, x(t)) - \Phi\lambda(t), \quad x(t_0) = x_0, \tag{27}$$

$$\frac{d}{dt}\lambda(t) = -\left\{\frac{\partial f(t, x(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} \lambda(t) - C^{\mathrm{T}}QCx(t), \quad \lambda(t_f) = Fx(t_f). \tag{28}$$

Thus, the successful solution of the optimal control synthesis problem in the form $u(t) = -\varphi(x(t))\lambda(t)$ depends on the possibility of successfully solving the two-point boundary value problem (27), (28).

When Proposition 2 made above is fulfilled in the problem under consideration with a quadratic quality functional, the functions $\tilde{\lambda}(t)$ and $\tilde{x}(t)$ are linearly related by the relation $\tilde{\lambda}(t) = F\tilde{x}(t)$. Thus, the full derivatives of the main and auxiliary equations have the form

$$\frac{d}{dt}\tilde{\lambda}(t) = F\frac{d}{dt}\tilde{x}(t), \quad t \in [t_0, t_f]. \tag{29}$$

Substituting $d\tilde{x}(t)/dt$ and $d\tilde{\lambda}(t)/dt$, defined in (28), we have

$$-\left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} \tilde{\lambda}(t) + F\Phi\tilde{\lambda}(t) = Ff(t, \tilde{x}(t)) + C^{\mathrm{T}}QC\tilde{x}(t).$$

Whence, by reducing similar terms, we obtain

$$\tilde{\lambda}(t) = \left[F\Phi - \left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}}\right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}}QC\tilde{x}(t)\right]. \tag{30}$$

Here we assume that

$$F\Phi - \left\{\frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}}\right\}^{\mathrm{T}} > 0, \quad \forall (t, \tilde{x}) \in [t_0, t_f] \times \Omega_x. \tag{31}$$

As can be seen, the fulfillment of condition (31) under the given constraints (26) depends on the purpose of the matrix F in the functional (24).

Control (27) taking into account (30) takes the form

$$\tilde{u}(t) = -\varphi(x(t)) \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}} QC\tilde{x}(t) \right]. \tag{32}$$

Let us write the original system (23) with control (32) (taking into account (25) and (26)):

$$\begin{gathered} \frac{d}{dt}\tilde{x}(t) = f(t, \tilde{x}(t)) - \Phi \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}} QC\tilde{x}(t) \right], \\ \tilde{x}(t_0) = x_0. \end{gathered} \tag{33}$$

To check the stability of the system (33), let's first introduce some notations: let

$$S(\tilde{x}(t)) = \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} > 0. \tag{34}$$

Let us rewrite (33) taking into account the notation made:

$$\frac{d}{dt}\tilde{x}(t) = [I - \Phi S(\tilde{x}(t))F] f(t, \tilde{x}(t)) - \Phi S(\tilde{x}(t)) C^{\mathrm{T}} QC\tilde{x}(t), \quad \tilde{x}(t_0) = x_0. \tag{35}$$

Let us introduce the Lyapunov function [22] in the form

$$V_L\tilde{x}(t) = \tilde{x}^{\mathrm{T}}(t)\tilde{x}(t). \tag{36}$$

The total derivative of the Lyapunov function is determined by the expression

$$\begin{aligned} \frac{d}{dt}V_L\tilde{x}(t) = {} & f^{\mathrm{T}}(t, \tilde{x}(t))[I - \Phi S(\tilde{x}(t))F]^{\mathrm{T}}\tilde{x}(t) - C^{\mathrm{T}}QC\tilde{x}(t)S^{\mathrm{T}}(\tilde{x}(t))\Phi^{\mathrm{T}}\tilde{x}(t) \\ & + \tilde{x}^{\mathrm{T}}(t)[I - \Phi S(\tilde{x}(t))F]f(t, \tilde{x}(t)) - \tilde{x}^{\mathrm{T}}(t)\Phi S(\tilde{x}(t))C^{\mathrm{T}}QC\tilde{x}(t) \leq 0. \end{aligned} \tag{37}$$

By assigning the matrices Q and F in the functional (24) accordingly, it is possible to ensure that inequality (37) is satisfied, under which the dynamic system (33) has the property of uniform asymptotic stability.

Generalizing the result obtained above, we formulate Theorem 2.

Theorem 2. *A pseudo-optimal solution to the control problem of a nonlinear dynamic object* (23) *with functional* (24) *exists if and only if*

$$F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} > 0, \quad \forall (t, \tilde{x}) \in [t_0, t_f] \times \Omega_x. \tag{38}$$

In this case, the trajectory $\tilde{x}^0(t)$ of system (23)*, originating from $\tilde{x}(t_0) = x(t_0)$ and corresponding to pseudo-optimal control $\tilde{u}^0(t)$, is a solution to the equation*

$$\frac{d}{dt}\tilde{x}(t) = f(t, \tilde{x}(t)) - \Phi \left[F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \left[Ff(t, \tilde{x}(t)) + C^{\mathrm{T}} QC\tilde{x}(t) \right]. \tag{39}$$

Satisfaction of condition (31) *allows us to determine the region of initial conditions of system* (33)*, under which pseudo-optimal control* (32) *will provide the nonlinear system with uniform asymptotic stability:*

$$\left\{ F\Phi - \left\{ \frac{\partial f(t, \tilde{x}(t))}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right\}_{t=t_0} \succ 0. \tag{40}$$

The result presented above will be extended to a certain class of nonlinear systems represented using the SDC-parameterization method (State Dependent Coefficient, [23, 24]). To do this, we will make several assumptions.

Proposition 4. *The vector function $f = f(t, \tilde{x}(t))$ is continuous and differentiable by $x \in \Omega_x$, i.e. $f(\cdot) \in C^1(\Omega_x)$.*

Proposition 5. *Without loss of generality, we assume that the condition $\tilde{x} = 0 \subset \Omega_x$ is an equilibrium point of the system such that $f(t, 0) = 0$.*

Proposition 6. *We assume* [23] *that*

$$\frac{|f(t, \tilde{x}(t))|}{|\tilde{x}|} \to 0 \quad for \quad |x| \to 0. \tag{41}$$

Taking into account the assumptions made regarding the property $f(t, \tilde{x}(t))$, we move from the description of the original system (23) to its SDC representation [23]. Writing $f(t, \tilde{x}(t))$ in the form

$$f(t, \tilde{x}(t)) = [A(t) + A(\tilde{x}(t))]x(t) = A(t, \tilde{x})\tilde{x}(t), \tag{42}$$

we have

$$\begin{aligned} &\frac{d}{dt}\tilde{x}(t) = A(t, \tilde{x})\tilde{x}(t) + g(t)u(t), \quad \tilde{x}(t_0) = x_0, \\ &y(t) = C\tilde{x}(t), \\ &A(t, \tilde{x})\tilde{x}(t), g(t) : T \times \Omega_x \to \mathbb{R}^n, \quad (t, x) \to f(t, x), \, g(t). \end{aligned} \tag{43}$$

Proposition 7. *Let us assume that the pair $\{A(t, \tilde{x}), g(t)\}$ is controlled, $\{A(t, \tilde{x}), C\}$ is observable.*

Let us write the equation of the object (39) with control takes the form

$$\begin{aligned} &\frac{d}{dt}\tilde{x}(t) = A(t, \tilde{x})\tilde{x}(t) - \Phi \left[F\Phi - \left\{ \frac{\partial A(t, \tilde{x})\tilde{x}(t)}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} \times \left[FA(t, \tilde{x}) + C^{\mathrm{T}}QC \right] \tilde{x}(t), \\ &y(t) = C\tilde{x}(t), \quad x(t_0) = x_0. \end{aligned} \tag{44}$$

Obviously, a solution to this equation exists if and only if a condition similar to condition (31) is satisfied

$$F\Phi - \left\{ \frac{\partial A(t, \tilde{x})\tilde{x}(t)}{\partial \tilde{x}} \right\}^{\mathrm{T}} > 0, \quad \forall (t, \tilde{x}) \in [t_0, t_f] \times \Omega_x. \tag{45}$$

To check the stability of system (44) we first introduce some notations: let

$$S(\tilde{x}(t)) = \left[F\Phi - \left\{ \frac{\partial A(t, \tilde{x})\tilde{x}(t)}{\partial \tilde{x}} \right\}^{\mathrm{T}} \right]^{-1} > 0, \tag{46}$$

i.e. the controlled system (44), taking into account the notations made, takes the form (35).

Taking into account (45) and taking into account (37), we write the total derivative of the Lyapunov function

$$\begin{aligned} \frac{d}{dt}V_L(t, \tilde{x}) = \tilde{x}^{\mathrm{T}}(t)A^{\mathrm{T}}(t, \tilde{x}) \left[I - \Phi S(\tilde{x}(t))F \right]^{\mathrm{T}} \tilde{x}(t) - \tilde{x}^{\mathrm{T}}(t)C^{\mathrm{T}}QC\tilde{x}(t)S^{\mathrm{T}}(\tilde{x}(t))\Phi^{\mathrm{T}}\tilde{x}(t) \\ + \tilde{x}^{\mathrm{T}}(t) \left[I - \Phi S(\tilde{x}(t))F \right] A(t, \tilde{x})\tilde{x}(t) - \tilde{x}^{\mathrm{T}}(t)\Phi S(\tilde{x}(t))C^{\mathrm{T}}QC\tilde{x}(t) \le 0. \end{aligned} \tag{47}$$

Thus, the asymptotic stability of the controlled system (44) must ensure that the condition is met

$$\begin{aligned} A^{\mathrm{T}}(t, \tilde{x}) \left[I - \Phi S(\tilde{x}(t))F \right]^{\mathrm{T}} + \left[I - \Phi S(\tilde{x}(t))F \right] A(t, \tilde{x}) \\ \le C^{\mathrm{T}}QCS^{\mathrm{T}}(\tilde{x}(t))\Phi^{\mathrm{T}} + \Phi S(\tilde{x}(t))C^{\mathrm{T}}QC. \end{aligned} \tag{48}$$

The fulfillment of conditions (26) and (48) for a given matrix Φ can be ensured by appropriately assigning the penalty matrices F, Q, R of the quality functional (24).

5. EXAMPLE

To illustrate the obtained theoretical results, let us consider an example [25] of the synthesis of pseudo-optimal control for a system of the form (23)

$$\begin{aligned} \frac{d}{dt}x_1(t) &= x_2(t) + \left[x_1^5(t) - x_1^3(t) - x_1(t) + x_1(t)x_2^4(t)\right] + x_1(t)u_1(t), \\ \frac{d}{dt}x_2(t) &= -x_1(t) - x_2(t) + \left[x_2^5(t) - x_2^3(t) - x_2(t) + x_2(t)x_1^4(t)\right] + x_2(t)u_2(t). \end{aligned} \tag{49}$$

Here, in accordance with Section 4 of the article,

$$A = \begin{pmatrix} 0 & 1 \\ -1 & -1 \end{pmatrix}, \quad \begin{pmatrix} f_1(x(t)) \\ f_2(x(t)) \end{pmatrix} = \begin{pmatrix} x_1^5(t) - x_1^3(t) - x_1(t) + x_1(t)x_2^4(t) \\ x_2^5(t) - x_2^3(t) - x_2(t) + x_2(t)x_1^4(t) \end{pmatrix},$$

$$g(x(t))u(t) = \begin{pmatrix} g_1(x(t))u_1(t) \\ g_2(x(t))u_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t)u_1(t) \\ x_2(t)u_2(t) \end{pmatrix} = \begin{pmatrix} x_1(t) & 0 \\ 0 & x_2(t) \end{pmatrix} \begin{pmatrix} u_1(t) \\ u_2(t) \end{pmatrix}.$$

A quadratic quality functional of the form (24) has given with parameters

$$F = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad Q = \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}, \quad R = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Control interval $[t_0, t_f] = [0, 2]$.

Constraints imposed on controls for a given matrix F and

$$\Phi = \begin{pmatrix} 0.5 & 0 \\ 0 & 0.5 \end{pmatrix}, \tag{50}$$

are defined as follows:

$$g(x(t))R^{-1}g^{\mathrm{T}}(x(t)) = \begin{pmatrix} x_1^2(t) & 0 \\ 0 & x_2^2(t) \end{pmatrix} \le \Phi. \tag{51}$$

The synthesized pseudo-optimal control, according to (32), has the form

$$\tilde{u}(t) = -\varphi(x(t)) \left[F\Phi - \left\{\frac{\partial f(t, x(t))}{\partial x}\right\}^{\mathrm{T}}\right]^{-1} \left[Ff(t, x(t)) + C^{\mathrm{T}}QCx(t)\right], \tag{52}$$

where the matrix $\varphi(x(t))$ satisfies condition (50), and $C = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

The constraints imposed on the controls for a given matrix F will be such that condition (40), is satisfied, in this example:

$$\begin{pmatrix} -5x_1^4(t) + 4x_1^2(t) + 1 - x_2^4(t) & -4x_2(t)x_1^3(t) + 1 \\ -4x_1(t)x_2^3(t) - 1 & -5x_2^4(t) + 4x_2^2 + 2 - x_1^4(t) \end{pmatrix} \succ 0, \forall \tilde{x}(t) \in \Omega. \tag{53}$$

System (49) with control (52) has the form

$$\begin{aligned} \frac{d}{dt}x(t) &= [Ax(t) + f(t, x(t))] \\ &\quad - \Phi \left[F\Phi - \left\{\frac{\partial f(t, x(t))}{\partial x}\right\}^{\mathrm{T}}\right]^{-1} F\left[Ax(t) + f(t, x(t)) + Qx(t)\right], \\ x(t_0) &= x_0. \end{aligned} \tag{54}$$

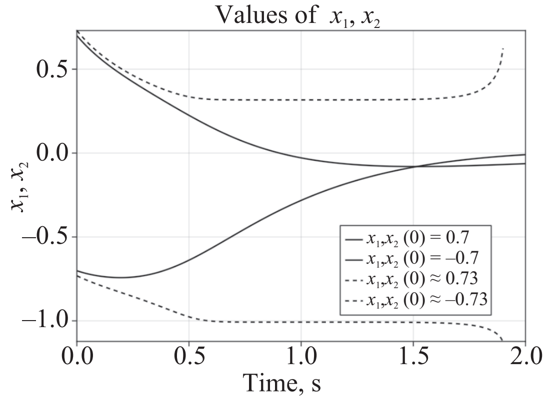

Fig. 1. Graphs of transient processes.

According to the condition (53), which defines the control capabilities of (52), the system is stable, as shown in the graphs (Figure), with initial conditions $|x_1(t_0)| \leq 0.72$; $|x_2(t_0)| \leq 0.72$ The system becomes unstable under the initial conditions $|x_1(t_0)| = 0.73$; $|x_2(t_0)| = 0.73$.

This is consistent with the conclusions of Theorem 2 ((40)) and the condition (53).

6. CONCLUSION

The Euler-Lagrange canonical system with the assignment of the corresponding boundary conditions is the basis of the necessary optimality conditions in the problem of synthesizing optimal controls for a dynamic object, while the synthesis of these controls has carried out based on the analysis of the Hamiltonian behavior on the optimal trajectory. However, the Hamiltonian does not contain any information on the relationship between the processes included in the canonical system. The success of synthesizing optimal control as a whole depends on the possibility of solving the canonical system with the specified boundary conditions. It should be noted that the functional relationship between the processes of a two-point boundary value problem exists only in the partial derivative with respect to the first term of the Bolza quality functional. In this paper, an alternative to numerical methods for solving two-point boundary value problems is proposed, based on the assumption of the validity of R. Bellman's inverse optimality principle, which consists in preserving the functional relationship between the components of a two-point boundary value problem in the entire control interval.

In this paper, an alternative to numerical methods for solving two-point boundary value problems is proposed, based on the assumption of the validity of R. Bellman's inverse optimality principle, which consists in preserving the functional connection between the components of a two-point boundary value problem in the entire control interval.

Because of the study of the control synthesis problem for a nonlinear dynamic system described by an ordinary differential equation and the Bolza functional, an analytical expression for pseudo-optimal control has obtained, and a theorem on the necessary conditions for the optimality of this control has formulated. For a problem with a quadratic quality functional, SDC representation of a nonlinear dynamic object and the corresponding pseudo-optimal control, a theorem on the asymptotic stability of the control system has proved. The conditions that must be met by the penalty matrices of the quality functional are obtained, under which the required quality of transient processes of the controlled nonlinear system is ensured when the established constraints on the control are met. The theoretical results obtained have confirmed by modeling the control system with synthesized control.

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NONLINEAR SYSTEMS

Design of Hybrid Nonlinear Control Systems Based on a Quasilinear Approach

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Abstract—A method for designing hybrid nonlinear control systems for plants with differentiable nonlinearities and a measurable state vector is developed based on continuous quasilinear models and quasilinear discretization. The hybrid system is designed with an increased control discretization period and zero static error for a reference signal. A solution of the control design problem exists if the nonlinear plant satisfies state and output controllability criteria and some additional conditions. The stability of the hybrid system is proven using the Aizerman–Pyatnitsky "technical" approach and the Lyapunov function method. The effectiveness of the design method proposed for hybrid control systems is illustrated by a numerical example. This method can be applied to create hybrid control systems for different-purpose nonlinear plants.

Keywords: differentiable nonlinearity, quasilinear model, state controllability criterion, output controllability criterion, quasilinear discretization, hybrid system, stability, static error

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1. INTRODUCTION

Recently, significant attention has been paid to the development of methods for designing hybrid nonlinear control systems, which are characterized by continuous and discrete nonlinear dynamics, requiring the use of differential and difference equations [1–3]. In reality, such systems represent a combination of continuous (hardware) and digital (programmable) elements [4]. In known publications, a wide variety of systems are referred to as hybrid.

Hybrid systems of the first class include either one nonlinear plant operating in switched modes or several plants that must be switched on in a certain sequence. In this case, difference equations describe the digital part, which ensures the switching of the continuous part elements [1, 5–8]. To create hybrid nonlinear optimal control systems, the Hybrid Necessary Principle was used in [1]. This principle allows considering the constraints due to the switching strategy. In [5], the same goal was achieved by applying the Hamilton–Jacobi–Bellman equation and the spectral Galerkin method.

The hybrid nature of the control system considered in [6] is due to the switching of several continuous subsystems, while that of the one created in [7] is due to the switching of its operating modes. A quadcopter with in-flight switchable morphology was considered in [8]. Its hybrid control system includes a nonlinear PID controller and a discrete controller that stabilize the control system for all possible quadcopter configurations.

The second class of hybrid systems seems to be more defined [2, 3]. Here, a controlled continuous plant is equipped with a discrete (digital) controller. The main problem is ensuring the stable operation of the hybrid system with a relatively large period of the digital controller's operation.

This necessity arises when creating control systems for inertial plants, such as baking ovens, incubators, greenhouses, etc., as well as plants operating under harsh temperature conditions where controller cooling is difficult. Due to the large period, the conditions of the Kotelnikov theorem fail, violating the stability of the system, which holds in classical cases with a sufficiently small discretization period. Therefore, various methods are employed to create hybrid control systems of this class [2, 3, 9–12]. For example, predictive control based on quadratic and integer programming was used in [2]. The effectiveness of the approach was illustrated by an example of designing a hybrid control system for a plant with three spherical tanks.

A hybrid terminal control method with an identified model of the controlled plant was developed in [3]. A two-layer artificial neural network was proposed for identifying nonlinear plants. For nonlinear plants with delay and parametric uncertainty, hybrid control systems were designed in [9–11] using the hyperstability criterion, the L-dissipativity condition, and a filter-corrector. The discrete control was obtained by discretizing a continuous one. The problem of tracking a given trajectory by a quadcopter under uncertainty was considered in [12]. A hybrid autopilot implementing predictive and fuzzy control was applied in the tracking system.

This paper develops a new method for designing hybrid nonlinear control systems for single-input single-output (SISO) plants with differentiable nonlinearities and a measurable state vector. Quasilinear models of nonlinear plants and the quasilinear discretization method [13–17] are used. By assumption, the plant satisfies state and output controllability criteria and some additional conditions. Hybrid systems of the second class with an increased discretization period are designed, which significantly reduces the performance requirements for computing resources. The main limitation of the developed method is the differentiability of the plant's nonlinearities. If the state vector is unmeasurable, a state observer can be used.

2. PROBLEM STATEMENT

Consider control-affine nonlinear SISO plants described by the following equations in deviations:

$$\dot{x} = \varphi(x, u, f), \quad y = \xi(x, u), \tag{1}$$

where $x = [\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n]^{\mathrm{T}} \in \mathrm{R}^n$ denotes the vector of state variables; $u, f, y \in \mathrm{R}$ are scalar control signal, disturbance, and controlled output, respectively; $\varphi(x, u, f)$ is a nonlinear n-dimensional vector function, and $\xi(x, u)$ is a scalar nonlinear function. These functions are bounded and differentiable in all arguments; moreover, $\varphi(\mathbf{0}, 0, 0) = \mathbf{0}$ and $\xi(\mathbf{0}, 0) = 0$. The state vector x and the output y or the deviation $\varepsilon = g - y$ are measured. Here, $\mathbf{0} \in \mathrm{R}^n$ stands for the zero vector, $g = g(t) \in \mathrm{R}$ and $f = f(t) \in \mathrm{R}$ are a reference signal and disturbance, representing arbitrary time-varying functions bounded by absolute value, and $f(t)$ is not measured.

Since the nonlinearities $\varphi(x, u, f)$ and $\xi(x, u)$ in (1) are differentiable, the method described in [15, 16] yields a quasilinear model (QLM) of the form

$$\dot{x} = A(x)x + b(x)u + h(x)f, \quad y = c^{\mathrm{T}}(x)x + d(x)u, \tag{2}$$

where $A(x) \in \mathrm{R}^{n\times n}$ and $b(x), h(x), c(x) \in \mathrm{R}^n$ are functional matrix and vectors, respectively, whose all elements, as well as $d(x) \in \mathrm{R}$, are known bounded differentiable nonlinear functions or numbers. Let us emphasize that QLMs describe the corresponding plants with differentiable nonlinearities with the same accuracy as equations (1). In other words, the properties of equations (2) fully match those of (1). Various methods for building QLMs have long been known. For example, the equation $\dot{x} = D(x)x$ was used by N.N. Krasovskii et al. to construct Lyapunov functions for nonlinear systems as early as the middle of the previous century [18].

By assumption, the QLM (2) satisfies the state controllability criterion

$$\det U_s(x) = \det[b(x) \;\; A(x)b(x) \;\; \ldots \;\; A^{n-1}(x)b(x)] \neq 0, \;\; \forall\, x \in \Omega_{Us}, \tag{3}$$

as well as the output controllability criterion

$$\gamma_{\text{pl}}(x) \neq 0, \;\; \forall\, x \in \Omega_{Uo}, \tag{4}$$

where $\gamma_{\text{pl}}(x)$ is the output controllability index of the plant (1), defined by the expression

$$\gamma_{\text{pl}}(x) = d(x) \det A(x) - c^{\mathrm{T}}(x) \mathrm{adj} A(x)\, b(x). \tag{5}$$

In (3)–(5), $\Omega_{Us} = \{x \in \mathrm{R}^n : \det U_s(x) \neq 0\}$; $\Omega_{Uo} = \{x \in \mathrm{R}^n : \gamma_{\text{pl}}(x) \neq 0\}$; $\mathrm{adj}A(x)$ is the adjoint matrix for $A(x)$ [19]; $\Omega_{Uu} = \Omega_{Uo} \cap \Omega_{Us}$ is the set of vectors $x \in \mathrm{R}^n$ for which conditions (3) and (4) hold; moreover, both Ω_{Us} and Ω_{Uo} include the vector $x = \mathbf{0}$.

The objective of this work is to develop a method for designing second-class hybrid control systems for nonlinear plants of the form (1). The discretization periods of these systems must be significantly larger than those of discrete control systems created by conventional methods. To solve this problem, we apply a piecewise-constant control obtained by the quasilinear discretization method of nonlinear plants [17].

3. THE QUASILINEAR DISCRETIZATION METHOD

In this method, not the equations of nonlinear plants but their quasilinear models are discretized using the trapezoidal method. It is possible due to the boundedness of the right-hand sides of the QLM equations (2) for bounded x, u, g, and f.

Let T be a certain discretization period for the solutions $x = x(t) \in \Omega_{Uu}$ of the differential equation (2). With each time instant $t = kT$, $k = 0, 1, 2, \ldots$, a discrete value $x_k = x(kT)$ of this solution is associated. The exact value $x_{k+1} = x(kT + T)$ is given by the expression

$$x_{k+1} = x_k + \int\limits_{kT}^{kT+T} F(t)dt, \tag{6}$$

where $F(t) = A(x)x + b(x)u + h(x)f(t\,)|_{x=x(t)}$ is the right-hand side of the first equation in (2). Assume that the input $u = u_k$ is a bounded and piecewise constant function. Nowadays, exact methods for computing the integrals (6) are unknown, so based on the modified trapezoidal method, the integrand in (6) is replaced by

$$\bar{F} = 0.5[A_k x_k + A_k x_{k+1} + 2b_k u_k + 2h_k f_k] + \Delta_k,$$

where $\Delta_k = 0.5[(A_{k+1} - A_k)x_{k+1} + b_{k+1}u_{k+1} - b_k u_k + h_{k+1}f_{k+1} - h_k f_k]$. (For brevity, $A_k = A(x_k)$, $b_k = b(x_k)$, and $h_k = h(x_k)$.) Replacing $F(t)$ in (6) with $\bar{F}$ for $\Delta_k = \mathbf{0}$ and integrating, we obtain the difference equation

$$[E - 0.5TA(x_k)]x_{k+1} = [E + 0.5TA(x_k)]x_k + Tb(x_k)u_k + Th(x_k)f_k, \quad x_k \in \Omega_{Uu}. \tag{7}$$

Note that the modification of the trapezoidal method consists in adding and subtracting the sum $A_k x_{k+1} + b_k u_k + h_k f_k$ when deriving the expression for $\bar{F}(kT)$ from $F(t)$.

Equation (7) can be solved for x_{k+1} if the matrix $[E - 0.5TA(x_k)]$ has an inverse, i.e., under the following condition imposed on the choice of the period T :

$$\det[E - 0.5TA(x)] \neq 0, \quad x \in \Omega_{Uu}. \tag{8}$$

To find T, we determine the roots η_i of the auxiliary equation $\det[E - 0.5\eta A(x)] = 0$. Let this equation for $x \in \Omega_{Uu}$ have $0 < m(x) \le n$ [20] positive real roots, of which $m_1(x)$ are independent of x and $m_2(x) = m(x) - m_1(x)$ depend on x. Then

$$0 < T < \min\{\eta_{\min,1},\ \eta_{\min,2}\}, \tag{9}$$

where $\eta_{\min,1} = \min\{\eta_i, i = \overline{1, m_1(x)}, x \in \Omega_{Uu}\}$, $\eta_{\min,2} = \inf\{\eta > 0 : \eta_i = \eta_i(x), i = \overline{1, m_2(x)}, x \in \Omega_{Uu}\}$.

If $m(x) \equiv 0$ (i.e., the equation $\det[E - 0.5\eta A(x)] = 0$ has no positive real roots η_i), the condition on the matrix $[E - 0.5TA(x_k)]$ will not imply any constraints on T.

In this case, the value of T in (8) is taken arbitrarily, based on constructive constraints; and the value of T can be refined later.

If the period T is chosen according to (8), then from (7) and the second equation in (2) it follows that

$$\begin{aligned} x_{k+1} &= A_d(x_k)x_k + b_d(x_k)u_k + h_d(x_k)f_k, \\ y_k &= c^{\mathrm{T}}(x_k)x_k + d(x_k)u_k, \quad x_k \in \Omega_{Uu}, \end{aligned} \tag{10}$$

where

$$A_d(x_k) = [E - 0.5TA(x_k)]^{-1}[E + 0.5TA(x_k)], \tag{11}$$

$$\begin{aligned} b_d(x_k) &= [E - 0.5TA(x_k)]^{-1}Tb(x_k), \\ h_d(x_k) &= [E - 0.5TA(x_k)]^{-1}Th(x_k). \end{aligned} \tag{12}$$

The relations (6)–(12) represent the quasilinear discretization method, and the expressions (10)–(12) are the discrete quasilinear model (DQLM) of the plant (1) [17]. In contrast to the exact QLM (2), this model is approximate. However, as shown below under certain conditions, some control signal u_k stabilizing the equilibrium of the DQLM (10) also ensures the stability of the equilibrium of the control system for the plant (1). In this sense, quasilinear discretization is analogous to classical linearization in the continuous case, where the control law based on first-approximation equations stabilizes the equilibrium of the nonlinear system in the small.

The application of the DQLM (10)–(12) allows hybrid systems to have a significantly larger discretization period compared to conventional approaches, thereby substantially reducing the performance requirements for system controllers.

4. STABILIZING CONTROL

This control is constructed by the algebraic polynomial-matrix (APM) method [21, 22]. Let the period T be chosen so that condition (8) holds, and let the corresponding DQLM (10)–(12) satisfy the state controllability criterion[1] for nonlinear discrete plants:

$$\det U_d(x_k) = \det[b_d(x_k)\ \ A_d(x_k)b_d(x_k)\ \ \ldots\ \ A_d^{n-1}(x_k)b(x_k)] \ne 0, \quad x_k \in \Omega_{Ud}, \tag{13}$$

where $\Omega_{Ud} = \{x_k \in \Omega_{Uu} : \det U_d(x_k) \ne 0\}$. In other words, $\Omega_{Ud} \subset \Omega_{Uu}$ is the domain where conditions (3), (4), (8), and (13) hold, and it contains the point $x = \mathbf{0}$.

The discrete control law stabilizing system (10)–(12) has the form

$$u_k(x_k) = -l^{\mathrm{T}}(x_k)x_k = -[l_1(x_k)\ l_2(x_k)\ \ldots\ l_n(x_k)]x_k. \tag{14}$$

The gains $l_i(x_k)$ are determined by the algorithm with the following steps [21].

[1] The question of whether condition (13) holds under conditions (3) and (8) is open.

1) Using (11) and (12), it is necessary to find the functional polynomials

$$A_d(z, x_k) = \det[zE - A_d(x_k)] = z^n + \alpha_{n-1}(x_k)z^{n-1} + \ldots + \alpha_1(x_k)z + \alpha_0(x_k), \qquad (15)$$

$$V_{d,i}(z, x_k) = e_i^{\mathrm{T}} \operatorname{adj}[zE - A_d(x_k)]b_d(x_k) = v_{i,n-1}(x_k)z^{n-1} + v_{i,n-2}(x_k)z^{n-2} + \ldots + v_{i,0}(x_k), \quad (16)$$

where e_i is the ith column of the identity matrix E of dimensions $n \times n$; $\alpha_j(x_k)$ and $v_{i,j}(x_k)$ are functional or numerical coefficients, $i = 1, \ldots, n$, and $j = 0, 1, \ldots, n-1$.

2) This step is to form the polynomial

$$D^*(z) = \prod_{i=1}^{n}(z - \sigma_i^*) = z^n + \delta_{n-1}^* z^{n-1} + \ldots + \delta_1^* z + \delta_0^*, \qquad (17)$$

where σ_i^* are real numbers for which there exist $0 < \varsigma_1 < 1$ and $0 < \varsigma_2$, independent of i and κ, such that

$$|\sigma_i^*| \leq 1 - \varsigma_1, \quad \varsigma_2 < |\sigma_i^* - \sigma_\kappa^*|, \quad i \neq \kappa, \quad i, \kappa = 1, \ldots, n. \qquad (18)$$

3) One determines the coefficients of the difference

$$D^*(z) - A_d(z, x_k) = \rho_{n-1}(x_k)z^{n-1} + \ldots + \rho_1(x_k)z + \rho_0(x_k), \qquad (19)$$

where $\rho_j(x_k) = \delta_j^* - \alpha_j(x_k)$, $j = 0, 1, \ldots, n-1$. Next, it is necessary to equate the coefficients of the sum of the products of the polynomials $V_{d,i}(z, x_k)$ (16) by the coefficients $l_i(x_k)$ (14), $i = 1, \ldots, n$, to those of the polynomial (19) at the same powers of z. The resulting equations, written in the vector-matrix form, constitute the system of linear algebraic equations (SLAE)

$$\begin{bmatrix} v_{10}(x_k) & v_{20}(x_k) & \cdots & v_{n0}(x_k) \\ v_{11}(x_k) & v_{21}(x_k) & \cdots & v_{n1}(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,n-1}(x_k) & v_{2,n-1}(x_k) & \cdots & v_{n,n-1}(x_k) \end{bmatrix} \begin{bmatrix} l_1(x_k) \\ l_2(x_k) \\ \vdots \\ l_n(x_k) \end{bmatrix} = \begin{bmatrix} \rho_0(x_k) \\ \rho_1(x_k) \\ \vdots \\ \rho_{n-1}(x_k) \end{bmatrix}. \qquad (20)$$

The SLAE (20) has a unique solution due to condition (13). Its solution—the vector $l(x_k)$—is substituted into (14), and the resulting control law u_k is then substituted into the DQLM equation (10). Thus, one arrives at the following equation of the virtual discrete system:

$$x_{k+1} = D_d(x_k)x_k + h_d(x_k)f_k, \quad x_k \in \Omega_{Ud}, \quad = 0, 1, 2, \ldots, \qquad (21)$$

where

$$D_d(x_k) = A_d(x_k) - b_d(x_k)l^{\mathrm{T}}(x_k). \qquad (22)$$

The lemma below establishes the effectiveness of the APM method.

Lemma 1. *Under condition* (13), *the SLAE* (20) *has a unique solution* $l(x_k)$. *Moreover, for any* $x_k \in \Omega_{Ud}$, *the eigenvalues of the matrix* $D_d(x_k)$ (22) *coincide with the roots of the polynomial* $D^*(z)$ (17), *i.e., they do not depend on* x_k, *are real, distinct, and less than one by absolute value.*

The proof of Lemma 1 is postponed to the Appendix. We emphasize that the relations (8)–(22) can be used for designing discrete nonlinear control systems [17, 22].

5. HYBRID SYSTEM DESIGN

Proceeding to the solution of this problem, we introduce the matrix

$$\tilde{D}(x) = [E - 0.5Tb(x)l^{\mathrm{T}}(x)]H_g(x), \tag{23}$$

with $H_g(x) = [D_d(x) - E][D_d(x) + E]^{-1}$, the matrix $D_d(x)$ (22), and the vector $l(x) = l(x_k)$ for $x_k = x$.

Let $\lambda_i^{\tilde{D}(x)}$ be the eigenvalues of the matrix $\tilde{D}(x) \in \mathrm{R}^{n\times n}$. Assume that the period T in (10)–(20) and (23) satisfies conditions (8) and (13) and the inequalities

$$\mathrm{Re}\lambda_i^{\tilde{D}(x)} < 0, \quad i = 1, \dots, n, \quad x \in \Omega_{sys}, \tag{24}$$

where $\Omega_{sys} = \{x \in \Omega_{Ud} : \mathrm{Re}\lambda_i^{\tilde{D}(x)} < 0,\ i = 1, \dots, n\}$. In other words, the eigenvalues of the matrix $\tilde{D}(x)$ can be either real or complex conjugate but with negative real parts (i.e., the matrix $\tilde{D}(x)$ is Hurwitz in the domain Ω_{sys}). Note that the choice of T can be iterative: if condition (24) fails for some value of the period T, then this value in (8)–(20) and (23) is decreased.

Under conditions (3), (4), (8), (13), and (24), the control law of the hybrid system is a discrete, piecewise-constant function of the form

$$u = u_{\mathrm{hyb}}(x_k, g_k) = l_g(x_k)g_k - l^{\mathrm{T}}(x_k)x_k, \quad k = 0, 1, 2, \dots, \tag{25}$$

where $x_k \in \Omega_{sys}$; $g_k = g(t\,)|_{t=kT}$ are the values of the reference signal $g(t)$; the gain $l_g(x)$ and the matrix $D_{\mathrm{hyb}}(x) \in \mathrm{R}^{n\times n}$ are given by

$$l_g(x) = \det D_{\mathrm{hyb}}(x)/\gamma_{\mathrm{pl}}(x) \tag{26}$$

and

$$D_{\mathrm{hyb}}(x) = A(x) - b(x)l^{\mathrm{T}}(x), \tag{27}$$

respectively. From now on, the vector $x = x(t) \in \mathrm{R}^n$ is the solution of system (1), (25) or (2), (25).

From the expressions (2) and (25) we derive the following QLM equations of the hybrid system:

$$\dot{x} = A(x)x - b(x)l^{\mathrm{T}}(x_k)x_k + b(x)l_g(x_k)g_k + h(x)f, \quad kT \le t < (k+1)T, \tag{28}$$

$$y = c^{\mathrm{T}}(x)x - d(x)l^{\mathrm{T}}(x_k)x_k + d(x)l_g(x_k)g_k, \quad kT \le t < (k+1)T, \quad k = 0, 1, 2, \dots. \tag{29}$$

According to the definition (25), on the surfaces $\varphi(t, x) = t - kT = 0$, $k = 1, 2, 3, \dots$, the control signal $u = u(t)$ undergoes discontinuities of the first kind, i.e., instantaneously changes its value [23]. Such instantaneously changing controls were used as admissible in [24–26]. In the case under consideration, for each $k \ge 1$, the above discontinuity surfaces form two continuity sectors for both the control signal u (25) and the right-hand side of equation (28) [23]. By formula (25), in the left continuity sectors $kT - T < t < kT$, the control law is given by the expression $u^-(t) = l_g(x_{k-1})g_{k-1} - l^T(x_{k-1})x_{k-1}$; in the right continuity sectors $kT < t < kT + T$, by the expression $u^+(t) = l_g(x_k)g_k - l^T(x_k)x_k$. Here, x_k are the values of the solution of the differential system (28) on the discontinuity surfaces, i.e., at $t = kT$, $k = 1, 2, 3, \dots$. Following [24] or [25], we assume the existence of finite right and left limits in each continuity sector.

However, the values x_k are not determined by the differential system (28) in the classical sense due to the discontinuities of its right-hand side. There are several approaches to overcome this problem [26]. The so-called "technical" [23] (or "physical" [26]) one was proposed by M.A. Aizerman and E.S. Pyatnitsky: the idea is to consider the physical meaning of the problem, using "additional information about the 'original system' to narrow the domain of possible solutions" on the discontinuity surfaces [23, p. 39]. Recall that x_0 is given, and the subsequent values of x_k, $k = 1, 2, 3, \ldots$, are measured in the hybrid system of the above type. Having this in mind, we assume that the solution of equation (28) in the left continuity sectors is given by

$$x^-(t) = x_{k-1} + \int\limits_{kT-T}^{t} [A(x(\tau))x(\tau) - b(x(\tau))l^T(x_{k-1})x_{k-1} + \upsilon^-(\tau)]d\tau, \quad kT - T \le t < kT, \quad k = 1, 2, 3, \ldots, \tag{30}$$

where $x_0 = x(0)$, $\upsilon^-(\tau) = b(x(\tau))l_g(x_{k-1})g_{k-1} + h(x(\tau))f(\tau)$, and the integral is a Lebesgue integral [23]. Following [24], it seems convenient to define the measured values as $x_k = \lim_{t\to kT} x^-(t)$. Replacing the subscript k with $k+1$ in equality (30), we derive an explicit expression for $x(t)$ in the right continuity sectors:

$$x^+(t) = x_k + \int\limits_{kT}^{t} [A(x(\tau))x(\tau) - b(x(\tau))l^T(x_k)x_k + \upsilon^+(\tau)]d\tau, \quad kT \le t < kT + T, \quad k = 1, 2, 3, \ldots, \tag{31}$$

where $\upsilon^+(\tau) = b(x(\tau))l_g(x_k)g_k + h(x(\tau))f(\tau)$.

Due to the assumed existence of right and left limits in each continuity sector, both formulas (30) and (31) yield the same value: $x_k = \lim_{t\to kT} x^-(t) = \lim_{t\to kT} x^+(t)$. Thus, the Aizerman–Pyatnitsky approach allows obtaining the values of the continuous solution $x(t)$ of the differential system (28) for all t under the piecewise-constant control law (25) by utilizing the additional information about the properties of the hybrid system.

Let us formulate a theorem on the properties of system (1), (25).

Theorem. *Assume that conditions* (3), (4), (8), (13), (18), *and* (24) *hold, and the vector* $l(x_k)$ *in* (25) *is given by the solution of the SLAE* (20). *Then for* $g(t) = f(t) \equiv 0$ *and all* $t \ge 0$, *there exists a set of solutions* $x(t, x_0)$ *of equation* (28) *such that*

$$\lim_{t\to\infty} x(t, x_0) = \mathbf{0}, \quad x \in \Omega_{sys}. \tag{32}$$

If the gain $l_g(x_k)$ *in* (25) *is given by* (26), *then the static error of system* (1), (25) *with respect to the reference signal* $g(t)$ *is zero:*

$$\lim_{t\to\infty} \varepsilon_g(t) = \lim_{t\to\infty} [g(t) - y_g(t)] = 0 \tag{33}$$

for $g(t) = g_0 1(t)$ *and* $f(t) \equiv 0$.

Here, $1(t)$ indicates the unit step function (the Heaviside function); $y_g(t)$ is the response of system (28), (29), i.e., (1), (25) with $f(t) \equiv 0$, to the reference signal $g(t) = g_0 1(t)$ for some $x_0 = x(0)$ and g_0 such that $x(t, x_0, g_0) \in \Omega_{sys}$, $t \ge 0$.

The proof of this theorem is provided in the Appendix. The following lemma establishes that the controllability of the plant implies the "controllability" of the closed-loop system, i.e., the possibility of ensuring the necessary change in the system output by an appropriate reference signal.

Lemma 2. *If the matrices* $U_s(x)$ *and* $D_{\text{hyb}}(x)$ *are given by* (3) *and* (27), *then*

$$\det Q_{\text{hyb}}(x) = \det[b(x) \;\; D_{\text{hyb}}(x)b(x) \;\; \ldots \;\; D_{\text{hyb}}^{n-1}(x)b(x)] = \det U_s(x), \quad x \in \Omega_{sys}. \tag{34}$$

The proof of Lemma 2 can be found in the Appendix. The relations (2), (5), (9)–(20), (22), (23), and (25)–(29) constitute the mathematical foundation of the proposed method for designing hybrid nonlinear control systems; inequalities (3), (4), (8), (13), (18) and (24) express the solvability conditions of the design problem by this method. The effectiveness of the developed method is illustrated by a numerical example below.

6. A NUMERICAL EXAMPLE

It is required to design a hybrid pitch control system (HPCS) for an autonomous underwater vehicle (AUV). Pitch control is carried out using bow and stern tanks of variable volume [27] and is described by the system of equations

$$\ddot{\psi} = \alpha_1 U_\psi \cos\psi - \alpha_2 U_a \sin\psi - \beta \left|\dot{\psi}\right| \dot{\psi}, \quad \dot{U}_\psi = -k_v U_\psi + k_u u, \quad y = \psi, \tag{35}$$

with the following notation: ψ and $\dot{\psi}$ are the pitch angle and its rate of change, respectively; U_ψ stands for the difference in the volumes of the bow and stern tanks; U_a is the AUV displacement; α_1 and α_2 mean hydrodynamic coefficients; β is the pitch change resistance coefficient; k_v and k_u are parameters of the device changing U_ψ; u is the control signal of this device; y is the controlled output of the HPCS; finally, ψ, $\dot{\psi}$, and U_ψ are measured variables. The HPCS must have zero static error in pitch and transients of a duration not exceeding 5 s under zero initial conditions and the desired pitch $g(t) = \psi^*(t) = -0.5236 \times 1(t)$ rad.

Solution. Setting $\underset{\sim}{x}_1 = \psi$, $\underset{\sim}{x}_2 = \dot{\psi}$ and $\underset{\sim}{x}_3 = U_\psi$, we write equations (35) in the Cauchy form:

$$\begin{aligned}
&\dot{\underset{\sim}{x}}_1 = \underset{\sim}{x}_2,\\
&\dot{\underset{\sim}{x}}_2 = \alpha_1 \underset{\sim}{x}_3 \cos\underset{\sim}{x}_1 - \alpha_2 U_a \sin\underset{\sim}{x}_1 - \beta\left|\underset{\sim}{x}_2\right|\underset{\sim}{x}_2,\\
&\dot{\underset{\sim}{x}}_3 = -k_v \underset{\sim}{x}_3 + k_u u, y = \underset{\sim}{x}_1.
\end{aligned} \tag{36}$$

Since $\sin\underset{\sim}{x}_1 = \omega(\underset{\sim}{x}_1)\underset{\sim}{x}_1$, where $\omega(\underset{\sim}{x}_1) = (\sin\underset{\sim}{x}_1)/\underset{\sim}{x}_1$ is the QLM of the function $\sin\underset{\sim}{x}_1$ [16], the QLM of the nonlinear system of equations (36) has the form (2) with

$$\begin{gathered}
A(x) = \begin{bmatrix} 0 & 1 & 0 \\ -a_{21}(x) & -a_{22}(x) & a_{23}(x) \\ 0 & 0 & -a_{33}(x) \end{bmatrix},\\
b(x) = \begin{bmatrix} 0 \\ 0 \\ k_u \end{bmatrix}, \quad c(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad h(x) = \mathbf{0}, \quad d(x) = 0.
\end{gathered} \tag{37}$$

Here, $x = [\underset{\sim}{x}_1 \quad \underset{\sim}{x}_2 \quad \underset{\sim}{x}_3]^{\mathrm{T}}$, $a_{21}(x) = \alpha_2 U_a \omega(\underset{\sim}{x}_1)$, $a_{22}(x) = \beta\left|\underset{\sim}{x}_2\right|$, $a_{23}(x) = \alpha_1 \cos\underset{\sim}{x}_1$, and $a_{33}(x) = k_v$.

Consider the solution of the HPCS design problem for $g(t) = \psi^*(t)$ and the following model values of the coefficients in (37): $a_{21}(x) = 7.044\omega(\underset{\sim}{x}_1)$, $a_{22}(x) = 1.192\left|\underset{\sim}{x}_2\right|$, $a_{23}(x) = 6.48\cos\underset{\sim}{x}_1$, $a_{33}(x) = 1.326$, and $k_v = k_u = 0.12$. In this case, in view of (5) and (37), conditions (3) and (4) become $\det U_s(x) = 0.0933(\cos\underset{\sim}{x}_1)^2 \neq 0$ and $\gamma_{\mathrm{pl}}(x) = 0.7776\cos\underset{\sim}{x}_1 \neq 0$. In other words, the domain Ω_{Uu} is given by $\left|\underset{\sim}{x}_1\right| < \pi/2$, $\left|\underset{\sim}{x}_2\right| \leq \underset{\sim}{x}_{2,\max}$, and $\left|\underset{\sim}{x}_3\right| \leq \underset{\sim}{x}_{3,\max}$, where $\underset{\sim}{x}_{2,\max}$ and $\underset{\sim}{x}_{3,\max}$ are some design bounds. Let $\underset{\sim}{x}_{2,\max} = 3.5$ rad/s and $L(\eta, x) = [E - 0.5\eta A(x)]$; then, taking (37) into account, we arrive at the equation

$$\det L(\eta, x) = (1 + 0.663\eta)\left[1 + 0.596\left|\underset{\sim}{x}_2\right|\eta + 3.522\omega(\underset{\sim}{x}_1)\eta^2\right] = 0. \tag{38}$$

Equation (38) has no positive real roots η_i in the domain Ω_{Uu}. That is, condition (8) on the matrix $[E - 0.5TA(x_k)]$ does not yield any constraints on the period T. Therefore, we take $T_1 = 0.6$ s and

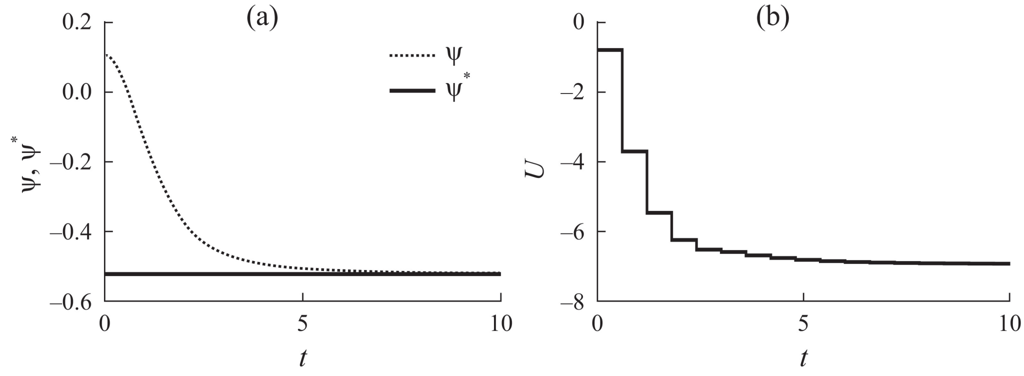

Fig. 1. The plots of the variables for $T = 0.6$ s: (a) pitch angles and (b) control input.

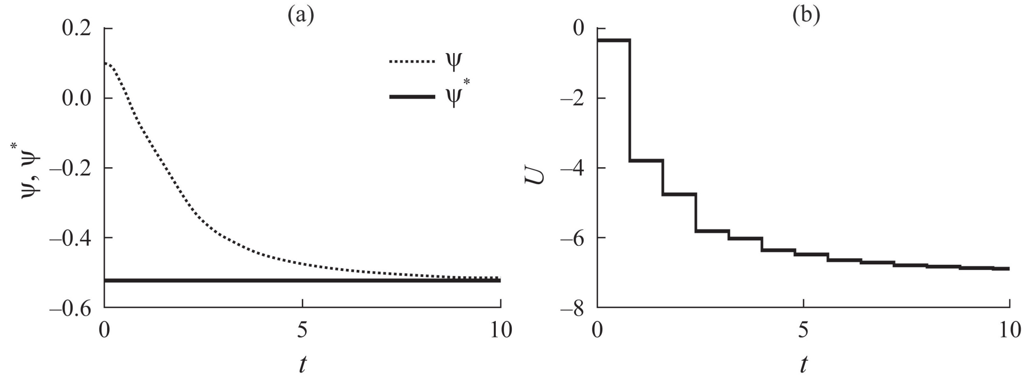

Fig. 2. The plots of the variables for $T = 0.8$ s: (a) pitch angles and (b) control input.

$T_2 = 0.8$ s based on constructive constraints. Let us determine the controller. The matrix $A_d(x_k)$ and the vector $b_d(x_k)$ are found from (11), (12), and (37); the fulfillment of condition (13) in the domain $\Omega_{Ud} = \Omega_{Uu}$ is verified numerically; the polynomials $A_d(z, x_k)$ and $V_{d,i}(z, x_k)$, $i = 1, 2, 3$, are constructed by (15) and (16).

The polynomial $D^*(z)$, found from (17) and (18), allows calculating the coefficients $\rho_j(x_k)$, $j = 0, 1, 2$, from (19) and compiling the SLAE (20). Its solution determines the three-dimensional vector $l(x_k)$. Next, the matrix $D_d(x_k)$ is obtained from (22) and the matrix $\tilde{D}(x)$ from (23). It is verified numerically that condition (24) holds both for $T = 0.6$ s and for $T = 0.8$ s in the domain $\Omega_{sys} = \Omega_{Uu}$. Finally, $D_{\mathrm{hyb}}(x)$, $l_g(x)$, and $u_{\mathrm{hyb}}(x_k, g_k)$ are determined by formulas (27), (26), and (25), respectively.

We emphasize that during the operation of the HPCS, almost all computations (first, the matrix $A(x)$ and the vector $b(x)$ (37), then $A_d(x_k)$ and $b_d(x_k)$, and, finally, the control input $u_{\mathrm{hyb}}(x_k, g_k)$) are performed by a digital control device for all $k = 0, 1, 2, \ldots$ with period T. (The only exception is the formation of the polynomial $D^*(z)$.) This is due to the nonlinear nature of the plant (35).

Analysis of the designed HPCS. For this purpose, we used MATLAB to compute the values of the discrete control signal $u_{\mathrm{hyb}}(x_k, g_k)$ for each $t = kT$ (see the description above) and integrate (via the ode45 function) the system of equations (36) with $u = u_{\mathrm{hyb}}(x_k, g_k)$ and the initial conditions $x_{0,k} = x(kT)$ and $\psi^*(t) = \psi_0 1(t)$, $k = 0, 1, 2, \ldots$, on each time interval $kT \le t < (k+1)T$. Figures 1 and 2 show the transients of the designed HPCS for $D^*(z) = z^3 - 0.8z^2 + 0.2032z - 0.01613$, $x_{0,0} = [0.1 \;\; 0.01 \;\; 0]^{\mathrm{T}}$, and $\psi_0 = -0.5236$ rad.

Clearly, the variables of the controlled plant are continuous functions, although the control signal changes with a significant period, which is characteristic of hybrid systems. The transients are similar under other conditions as well. A small increase in the discretization period just slightly extends the transients.

Table contains the eigenvalues $\lambda_i(x)$, $i = 1, 2, 3$, of the matrix $D_{\text{hyb}}(x_k)$ of the HPCS for two values of the period T and several values of k.

Table

	$T = 0.6$		$T = 0.8$	
k	λ_1	$\lambda_{2,3}$	λ_1	$\lambda_{2,3}$
0	-1.3463	$-0.7002 \pm 1.5983i$	-1.3250	$-0.2280 \pm 1.3446i$
1	-1.3484	$-0.8130 \pm 1.6057i$	-1.3249	$-0.3112 \pm 1.3637i$
5	-1.3468	$-0.7261 \pm 1.5810i$	-1.3249	$-0.2556 \pm 1.3322i$
10	-1.3466	$-0.7138 \pm 1.5753i$	-1.3249	$-0.2487 \pm 1.3249i$

According to this table, the eigenvalues of the matrix $D_{\text{hyb}}(x)$ have negative real parts, and increasing the period T reduces these parts by absolute value; for a large T, system stability is lost. Note that the real parts of the eigenvalues of the above matrix change insignificantly during the transient process of the hybrid system.

7. CONCLUSIONS

This paper has proposed a method for designing hybrid nonlinear control systems for continuous plants with differentiable nonlinearities and a measurable state vector. The problem has been solved using continuous and discrete quasilinear models, the algebraic polynomial-matrix method for designing nonlinear systems, and the Aizerman–Pyatnitsky solution approach to differential equations with a discontinuous right-hand side. The method proposed is applicable if the continuous and discrete quasilinear models of the nonlinear plant satisfy state and output controllability criteria and some additional conditions. The effectiveness of this method has been illustrated by a numerical example of designing a hybrid nonlinear pitch control system for an autonomous underwater vehicle. The method can be used to create hybrid control systems for nonlinear plants of industrial, social, and special purpose using moderate-performance computing means.

FUNDING

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APPENDIX

Proof of Lemma 1. Due to the expressions (6.3) and (6.55) [28, p. 145 and p. 169], the controllability conditions for the systems in the continuous and discrete cases coincide in form. Therefore, we use a theorem on the properties of controllable continuous systems to prove Lemma 1, formulated for the discrete system. In view of this remark, Theorems 1.1 and 1.2 [29, pp. 29, 31, 32] lead to the following assertion: under inequality (13), there is a unique gain vector $l^{\mathrm{T}}(x_k)$ for the control law $u_k = -l^{\mathrm{T}}(x_k)x_k$ (14) under which the eigenvalues of the matrix of the closed-loop discrete system (21) have a specified location on the complex plane z.

The matrix of the indicated closed-loop discrete system (21) is $D_d(x_k) = A_d(x_k) - b_d(x_k)l^{\mathrm{T}}(x_k)$ (22) with the characteristic polynomial

$$D_d(z, x_k) = \det[zE - A_d(x_k) + b_d(x_k)l^{\mathrm{T}}(x_k)], \quad x_k \in \Omega_{Ud}. \tag{A.1}$$

Thus, for a given polynomial $D_d(z, x_k) = D^*(z)$, the expression (A.1) is an equation with respect to the vector $l^{\mathrm{T}}(x_k)$; according to Theorems 1.1 and 1.2 from [29], this equation has a unique solution under condition (13). By applying to (A.1) equality (П.25) from [30, p. 233] for $\mu = 1$, we obtain

$$D_d(z, x_k) = \det[zE - A_d(x_k)] + l^{\mathrm{T}}(x_k)\mathrm{adj}\,[zE - A_d(x_k)]\, b_d(x_k).$$

(In particular, (П.25) is immediate from formulas (I) and (II) [19].) Hence, considering the notation (15), (16) and the vector $l^{\mathrm{T}}(x_k) = [l_1(x_k)\ l_2(x_k)\ \ldots\ l_n(x_k)]$, following [14, 15], we derive an equivalent representation of the same polynomial (A.1):

$$D_d(z, x_k) = A_d(z, x_k) + \sum_{i=1}^{n} l_i(x_k)V_{d,i}(z, x_k). \tag{A.2}$$

Moreover, by construction, system (20) is equivalent to the polynomial equation

$$\sum_{i=1}^{n} l_i(x_k)V_{d,i}(z, x_k) = \rho_{n-1}(x_k)z^{n-1} + \ldots + \rho_1(x_k)z + \rho_0(x_k).$$

In view of (19), it can be written as

$$\sum_{i=1}^{n} l_i(x_k)V_{d,i}(z, x_k) = D^*(z) - A_d(z, x_k), \quad x_k \in \Omega_{Ud}. \tag{A.3}$$

Based on (17), the roots of the polynomial $D^*(z)$ are the numbers σ_i^*, i.e., $D^*(\sigma_i^*) = 0$. Then according to (A.3), $A_d(\sigma_i^*, x_k) + \sum_{i=1}^{n} l_i(x_k)V_{d,i}(\sigma_i^*, x_k) = 0$. By (A.2), it follows that $D_d(\sigma_i^*, x_k) = 0$, $i = 1, 2, \ldots, n$, and the proof of Lemma 1 is complete.

Proof of Lemma 2. For this purpose, let us utilize a well-known property of determinants: if one column of the determinant's matrix, multiplied by a *number*, is added or subtracted from another column, the determinant value will not change [31, p. 143]. For brevity, we omit the arguments of the matrices $A(x)$, $D_{\mathrm{hyb}}(x)$, and $Q_{\mathrm{hyb}}(x)$ and vectors $b(x)$ and $l(x)$ from (3), (27), (34) and emphasize that $D_{\mathrm{hyb}} = A - bl^{\mathrm{T}}$ and $\det Q_{\mathrm{hyb}} = \det[b\ \ D_{\mathrm{hyb}}b\ \ \ldots\ \ D_{\mathrm{hyb}}^{n-1}b]$ for all n. Lemma 2 will be proved by induction. First, we show its validity for $n = 1$ and $n = 2$.

For $n = 1$, we have $A = a_1$, $b = b_1$, $l = l_1$, and $\det U_s = b_1$ by (3). Here, $D_{\mathrm{hyb}} = a_1 - b_1 l_1$, and $\det Q_{\mathrm{hyb}} = b_1$; obviously, $\det Q_{\mathrm{hyb}} = \det U_s$. Let $n = 2$; in this case, by (3), $\det U_s = \det[b\ \ Ab]$, and by (34), $\det Q_{\mathrm{hyb}} = \det[b\ D_{\mathrm{hyb}}b] = \det[b\ \ Ab - \breve{\beta} b]$, where $\breve{\beta} = l^{\mathrm{T}}b$ is a scalar number, since for each particular value of x the vectors $l(x)$ and $b(x)$ are numerical. Hence, due to the above property of determinants, $\det Q_{\mathrm{hyb}} = \det \left[\ b \quad Ab\ \right]$, i.e., $\det Q_{\mathrm{hyb}} = \det U_s$. Thus, Lemma 2 is valid for $n = 1$ and $n = 2$.

Now, under the inductive hypotheses $\det U_s = \det[b\ \ Ab\ \ldots\ A^{\mu-1}b]$ and $\det Q_{\mathrm{hyb}} = \det[b\ \ D_{\mathrm{hyb}}b\ \ \ldots\ \ D_{\mathrm{hyb}}^{\mu-1}b] = \det U_s$ (Lemma 2 for $n = \mu$), the validity of this lemma has to be shown for $n = \mu + 1$. To this end, we expand the left-hand side of the expression (34) with $n = \mu + 1$ as follows:

$$\det Q_{\mathrm{hyb}} = \det[b\ \ D_{\mathrm{hyb}}b\ \ D_{\mathrm{hyb}}^2 b\ \ \ldots\ \ D_{\mathrm{hyb}}^{\mu-3}b\ D_{\mathrm{hyb}}^{\mu-2}b\ D_{\mathrm{hyb}}^{\mu-1}b\ D_{\mathrm{hyb}}^{\mu}b]. \tag{A.4}$$

Further, we transform the columns of the matrix of the determinant (A.4) step by step, starting from $D^{\mu}_{\text{hyb}}b$, considering the above property of determinants and $D_{\text{hyb}} = A - bl^{\mathrm{T}}$.

Step 1.1. $D^{\mu}_{\text{hyb}}b = D^{\mu-1}_{\text{hyb}}(A - bl^{\mathrm{T}})b = D^{\mu-1}_{\text{hyb}}Ab + \beta_0 D^{\mu-1}_{\text{hyb}}b \succ D^{\mu-1}_{\text{hyb}}Ab$ since $\beta_0 = -l^{\mathrm{T}}b$ is a scalar number and the column $\beta_0 D^{\mu-1}_{\text{hyb}}b$ equals the column $D^{\mu-1}_{\text{hyb}}b$ of the matrix of the determinant (A.4) multiplied by β_0. From this point onwards, $\succ$ is the correspondence sign, indicating that the value of the determinant (A.4) will not change when replacing the column $D^{\mu}_{\text{hyb}}b$ in (A.4) with the column $D^{\mu-1}_{\text{hyb}}Ab$.

Step 1.2. $D^{\mu}_{\text{hyb}}b \succ D^{\mu-1}_{\text{hyb}}Ab = D^{\mu-2}_{\text{hyb}}(A - bl^{\mathrm{T}})Ab = D^{\mu-2}_{\text{hyb}}A^2b + \beta_1 D^{\mu-2}_{\text{hyb}}b \succ D^{\mu-2}_{\text{hyb}}A^2b$ due to the above property of determinants since $\beta_1 = -l^{\mathrm{T}}Ab$ is a scalar number and the column $\beta_1 D^{\mu-2}_{\text{hyb}}b$ equals to the column $D^{\mu-2}_{\text{hyb}}b$ of the matrix of the determinant (A.4) multiplied by β_1. (In other words, the column $\beta_1 D^{\mu-2}_{\text{hyb}}b$ is proportional to the column $D^{\mu-2}_{\text{hyb}}b$.)

Step 1.3. $D^{\mu}_{\text{hyb}}b \succ D^{\mu-2}_{\text{hyb}}A^2b = D^{\mu-3}_{\text{hyb}}(A - bl^{\mathrm{T}})A^2b = D^{\mu-3}_{\text{hyb}}A^3b + \beta_2 D^{\mu-3}_{\text{hyb}}b \succ D^{\mu-3}_{\text{hyb}}A^3b$ due to the above property of determinants since $\beta_2 = -l^{\mathrm{T}}A^2b$ is a scalar number and the column $\beta_2 D^{\mu-3}_{\text{hyb}}b$ is proportional to the column $D^{\mu-3}_{\text{hyb}}b$ of the matrix of the determinant (A.4). Continuing this process at Step 1.μ, we arrive at $D^{\mu}_{\text{hyb}}b \succ D^{\mu-\mu}_{\text{hyb}}A^{\mu}b = A^{\mu}b$.

Let us proceed to transforming the column $D^{\mu-1}_{\text{hyb}}b$ of the matrix of the determinant (A.4).

Step 2.1. $D^{\mu-1}_{\text{hyb}}b = D^{\mu-2}_{\text{hyb}}(A - bl^{\mathrm{T}})b = D^{\mu-2}_{\text{hyb}}Ab + \beta_0 D^{\mu-2}_{\text{hyb}}b \succ D^{\mu-2}_{\text{hyb}}Ab$.

Step 2.2. $D^{\mu-1}_{\text{hyb}}b \succ D^{\mu-2}_{\text{hyb}}Ab = D^{\mu-3}_{\text{hyb}}(A - bl^{\mathrm{T}})Ab = D^{\mu-3}_{\text{hyb}}A^2b + \beta_1 D^{\mu-3}_{\text{hyb}}b \succ D^{\mu-3}_{\text{hyb}}A^2b$.

Continuing the transformation, at Step 2.$(\mu - 1)$, we obtain $D^{\mu-1}_{\text{hyb}}b \succ A^{\mu-1}b$. Obviously, applying this transformation to each column $D^{j}_{\text{hyb}}b$ of the matrix of the determinant (A.4) yields the column A^jb, $j = 1, \ldots, \mu$. Based on the above property of determinants, this transformation does not change the value of (A.4); therefore, $\det Q_{\text{hyb}} = \det U_s$ for $n = \mu + 1$ as well.

So, Lemma 2 is valid for $n = 1, 2$, and its validity for $n = \mu$ implies the same for $n = \mu + 1$. By induction, Lemma 2 is valid for any positive integer n, and the proof is complete.

Proof of Theorem. As shown above, the continuous solution of equation (28) is defined for all $t \geq 0$ and $x \in \Omega_{sys}$. Moreover, its right-hand side depends on the time t, in addition to the vector $x(t)$, which is reflected in the additional expressions: $kT \leq t < kT + T$ and $k = 0, 1, 2, \ldots$. To make the dependence on t more explicit and eliminate k, we replace $x_k = x(kT)$ with $x(T\lfloor t/T \rfloor)$, where $\lfloor t/T \rfloor$ is the floor function of the ratio t/T. As a result, the state equation (28) of the hybrid system (1), (25) or, which is the same, (2) and (25), takes the form

$$\dot{x}(t) = D_{\text{hyb}}(x)x + \Upsilon_1(t, x) + b(x)l_g(x(T\lfloor t/T \rfloor))g(T\lfloor t/T \rfloor) + h(x)f(t), \tag{A.5}$$

$$\Upsilon_1(t, x) = b(x)[l^T(x)x - l^T(x(T\lfloor t/T \rfloor))x(T\lfloor t/T \rfloor)], \tag{A.6}$$

where $D_{\text{hyb}}(x)$ is the matrix given by (27), and still $x = x(t)$.

To prove the theorem, we first demonstrate that the eigenvalues of the matrix $D_{\text{hyb}}(x)$ have negative real parts. For this purpose, in view of (11) and (12), with $x_k = x$ for brevity, equality (22) can be written as follows:

$$\begin{gathered} {}^{-1}[E + 0.5TA(x)] \\ - [E - 0.5TA(x)]^{-1}Tb(x)l^{\mathrm{T}}(x) = D_d(x). \end{gathered} \tag{A.7}$$

Multiplying both sides of equality (A.7) by the matrix $[E - 0.5TA(x)]$ on the left, we expand the square brackets and factor the terms with the matrix $A(x)$ to the left-hand side. As a result,

$$0.5TA(x)[D_d(x) + E] = D_d(x) - E + Tb(x)l^{\mathrm{T}}(x). \tag{A.8}$$

By Lemma 1, all eigenvalues σ_i^* of the matrix $D_d(x)$ are such that $\sigma_i^* \neq \sigma_\kappa^*$, $i \neq \kappa$, and $|\sigma_i^*| < 1$. Therefore, the matrix $[D_d(x) + E]^{-1}$ exists, and (A.8) implies the equality

$$A(x) = 2T^{-1}[D_d(x) - E + Tb(x)l^{\mathrm{T}}(x)][D_d(x) + E]^{-1}. \tag{A.9}$$

Adding the term $-b(x)l^{\mathrm{T}}(x)$ to both sides of (A.9) and again factoring the matrix $[D_d(x) + E]^{-1}$ to the right, we obtain the expression

$$D_{\mathrm{hyb}}(x) = \left\{2T^{-1}\left[D_d(x) - E + Tb(x)l^{\mathrm{T}}(x)\right] - b(x)l^{\mathrm{T}}(x)[D_d(x) + E]\right\}[D_d(x) + E]^{-1}.$$

(Here, formula (27) is taken into account.) Expanding both bracketed expressions in the curly braces and collecting terms, we factor the matrix $[D_d(x) - E]$ out of the curly braces to the right and the term $2T^{-1}$ to the left. These manipulations yield

$$D_{\mathrm{hyb}}(x) = 2T^{-1}\left[E - 0.5Tb(x)l^{\mathrm{T}}(x)\right]H_g(x), \quad x \in \Omega_{sys}, \tag{A.10}$$

where $H_g(x) = [D_d(x) - E][D_d(x) + E]^{-1}$. Under the conditions of this theorem, the matrix $H_g(x)$ is Hurwitz and has distinct eigenvalues. This is easy to verify since the matrix $D_d(x)$ has distinct eigenvalues, i.e., it is similar to a diagonal matrix [20].

Comparing (A.10) with (23), we conclude that $D_{\mathrm{hyb}}(x) = 2T^{-1}\tilde{D}(x)$. Therefore, by formula (2.15.8) from [20] and condition (24), under the conditions of this theorem, the eigenvalues of the matrix $D_{\mathrm{hyb}}(x)$ (A.5) have negative real parts for $x \in \Omega_{sys}$.

Consider first the free motion of the hybrid system (28), (29) by letting $g(t) = f(t) \equiv 0$. Moreover, bearing in mind Lyapunov's theorem [32, p. 257] and (A.6), we represent equation (A.5) for $t \geq 0$ as follows:

$$\dot{x} = D_{\mathrm{hyb},0}x + \Upsilon(t, x), \tag{A.11}$$

where $D_{\mathrm{hyb},0} = D_{\mathrm{hyb}}(0)$, $\Upsilon(t, x) = \Upsilon_1(t, x) + \Upsilon_2(x)$, $\Upsilon_2(x) = [D_{\mathrm{hyb}}(x) - D_{\mathrm{hyb},0}]x$, and the vector $\Upsilon_1(t, x)$ is given by (A.6).

As established above, the matrix $D_{\mathrm{hyb}}(x)$ $\forall\, x \in \Omega_{sys}$ is Hurwitz; consequently, the constant matrix $D_{\mathrm{hyb},0}$ in (A.11) is also Hurwitz. Let us show that under the conditions of this theorem, the vector function $\Upsilon(t, x) = o(\|x\|)$ uniformly in t [32, p. 257]. For this purpose, we find the limits of the ratios $\Upsilon_1(x)/\|x\|^2$ and $\Upsilon_2(x)/\|x\|^2$ as $x(t) \to 0$. Obviously, for all $t \geq 0$,

$$\lim_{x \to \mathbf{0}}\left(\Upsilon_2(x)/\|x\|^2\right) = \lim_{x \to \mathbf{0}}\left(x^{\mathrm{T}}P[D_{\mathrm{hyb}}(x) - D_{\mathrm{hyb},0}]x/\|x\|^2\right) = 0. \tag{A.12}$$

Considering the limit of the ratio $\Upsilon_1(t, x)/\|x\|^2$, we observe that for all t, according to (31) and (32), $x \to 0$ implies $x(T\lfloor t/T\rfloor) \to 0$. Therefore, taking (A.6) into account,

$$\lim_{x \to \mathbf{0}}\left(\Upsilon_1(t, x)/\|x\|^2\right) = \lim_{x \to \mathbf{0}}\left(b(x)\left[l^T(x)x - l^T(x(T\lfloor t/T\rfloor))x(T\lfloor t/T\rfloor)\right]/\|x(t)\|^2\right) = 0 \tag{A.13}$$

since for all t, both vectors $l^T(x)$ and $l^T(x(T\lfloor t/T\rfloor))$ in the above expression are multiplied by those tending to zero. Thus, from (A.12) and (A.13) it follows that the vector function $\Upsilon(t, x) = o(\|x\|)$ uniformly in t, i.e.,

$$\frac{\Upsilon(t, x)}{\|x\|} \underset{t}{\rightrightarrows} 0 \text{ as } x \to 0. \tag{A.14}$$

The matrix $D_{\text{hyb},0}$ is Hurwitz; hence, due to (A.14), the differential system (A.11) satisfies the conditions of Lyapunov's theorem [32, p. 257], stating that the solution $x = \mathbf{0}$ of this system is asymptotically stable. In other words, condition (32) holds in the domain Ω_{sys}. Moreover, according to [32, pp. 258–260], there exists a Lyapunov function $V(x) = x^{\mathrm{T}} S_L x > 0$ with $\dot{V}(x) < 0$ along the trajectories of this system. Here, S_L is a real symmetric matrix.

On the other hand, equation (A.11) corresponds to equation (70.1) whereas equation (A.5) to equation (70.3) from the monograph [33]. Moreover, according to I.G. Malkin, equation (A.11) describes the perturbed motion of the Hurwitz system (1), (25) and (28), (29), and the term $b(x)l_g(x(T\underline{\lfloor t/T \rfloor}))g(T\underline{\lfloor t/T \rfloor}) + h(x)f(t)$ in (A.5) characterizes the constantly acting perturbations of this system. In addition, there exists a positive definite function $V(x) = x^{\mathrm{T}} S_L x$ for the differential system (A.11) whose total time derivative along the trajectories of this system is negative definite. In the domain $t \geq 0$, $x \in \Omega_{sys}$, the partial derivatives $(\partial V(x)/\partial x_i) = 2S_{Li}x$, where S_{Li} is the ith row of the matrix S_L, $i = 1, \ldots, n$, are obviously bounded. Therefore, by Malkin's theorem [33], the unperturbed motion of the hybrid system described by equations (A.5) and (28) is stable under the constantly acting perturbations. In other words, under sufficiently small initial conditions and external perturbations (the reference signal $g(t)$ and the disturbance $f(t)$) such that $x(t) \in \Omega_{sys}$, a steady-state regime arises in system (A.5) or, which is the same, in system (1), (25), whose QLM has the form (28), (29).

Consider this regime for $f(t) \equiv 0$, $g(t) = g_0 1(t)$, and sufficiently small $\|x_0\|$ and $|g_0|$. In this regime, as $t \to \infty$, we have $\dot{x}(t) \to \mathbf{0}$, $x(t) \to x^\circ$, $x(T\underline{\lfloor t/T \rfloor}) \to x^\circ$, and $y_g(t) \to y_g^\circ$, where x° and y_g° are the steady-state values of the variables $x(t)$, $x(T\underline{\lfloor t/T \rfloor})$ and $y_g(t)$, respectively, due to $g_0 1(t)$ (see Lemma 2). Then equations (28) and (29) take the form

$$\begin{aligned} \mathbf{0} &= D_{\text{hyb}}(x^\circ)x^\circ + b(x^\circ)l_g(x^\circ)g_0, \\ y_g^\circ &= [c^{\mathrm{T}}(x^\circ) - d(x^\circ)l^{\mathrm{T}}(x^\circ)]x^\circ + d(x^\circ)l_g(x^\circ)g_0, \end{aligned} \tag{A.15}$$

where the matrix $D_{\text{hyb}}(x^\circ)$ is given by (27) for $x = x^\circ$.

Since the matrix $D_{\text{hyb}}(x)$ is Hurwitz for $x \in \Omega_{sys}$, the matrix $D_{\text{hyb}}^{-1}(x^\circ)$ exists, so (A.15) implies the equalities $x^\circ = -D_{\text{hyb}}^{-1}(x^\circ)b(x^\circ)l_g(x^\circ)g_0$ and

$$y_g^\circ = \Big\{[d(x^\circ)l^{\mathrm{T}}(x^\circ) - c^{\mathrm{T}}(x^\circ)]D_{\text{hyb}}^{-1}(x^\circ)b(x^\circ) + d(x^\circ)\Big\}\, l_g(x^\circ)g_0. \tag{A.16}$$

Equality (33) will obviously be satisfied if $y_g^\circ = g_0$. Therefore, from (A.16) we obtain the following necessary and sufficient condition for this: $\{[d(x^\circ)l^{\mathrm{T}}(x^\circ) - c^{\mathrm{T}}(x^\circ)]D_{\text{hyb}}^{-1}(x^\circ)b(x^\circ) + d(x^\circ)\}l_g(x^\circ) = 1$. However, the value of x° is unknown in advance, so this condition is replaced, in view of the formula $D_{\text{hyb}}^{-1}(x) = \mathrm{adj}D_{\text{hyb}}(x)/\det D_{\text{hyb}}(x)$, by the equality

$$\Big\{[d(x)l^{\mathrm{T}}(x) - c^{\mathrm{T}}(x)]\mathrm{adj}D_{\text{hyb}}(x)b(x) + d(x)\det D_{\text{hyb}}(x)\Big\}\, l_g(x) = \det D_{\text{hyb}}(x). \tag{A.17}$$

The equality $y_g^\circ = g_0$ is immediate from (A.17) and (A.16) by Malkin's theorem (see above). Based on the definition (27) and formulas (П.25) and (П.26) from [30, p. 233], we have the equalities

$$\mathrm{adj}D_{\text{hyb}}(x)b(x) = \mathrm{adj}\Big[A(x) - b(x)l^{\mathrm{T}}(x)\Big]\, b(x) = \mathrm{adj}A(x)b(x) \quad \text{and}$$

$$\det D_{\text{hyb}}(x) = \det\Big[A(x) - b(x)l^{\mathrm{T}}(x)\Big] = \det A(x) - l^{\mathrm{T}}(x)\mathrm{adj}A(x)b(x).$$

Substituting them into (A.17) yields, after trivial simplifications, the relation

$$\Big\{d(x)\det A(x) - c^{\mathrm{T}}(x)\mathrm{adj}A(x)b(x)\Big\}\, l_g(x) = \det D_{\text{hyb}}(x).$$

Taking (5) into account, this result finally leads to equality (33) under condition (4) and the gain $l_g(x)$ (26). The proof of the theorem is complete.

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NONLINEAR SYSTEMS

The Green's Function Method in the Problem of Fuzzy Signal Transformation by a Linear Dynamic System

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Abstract—In this paper, the problem of transforming a fuzzy signal by a linear dynamic system is reduced to studying the problem of bounded solutions for a high-order linear differential equation with constant coefficients and a fuzzy-valued inhomogeneity on the right-hand side. To solve the latter, a modification of the Green's function method for fuzzy problems is developed. A class of equations with positive coefficients and a nonnegative Green's function is identified, and some results on the existence and smoothness of a fuzzy-valued bounded solution on the entire axis are established for this class of equations. As shown, in the case of a triangular right-hand side, the solution will also be triangular. Applications to radio circuits with fuzzy input signals are considered. A relationship between the modal values of fuzzy input and output signals for a linear dynamic system is derived.

Keywords: fuzzy numbers, dynamic systems with constant coefficients and fuzzy input signals, Green's function method

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1. INTRODUCTION

Dynamic models of many applied processes are characterized by uncertainty in input data. When the probabilistic characteristics of an input signal can be estimated during modeling, the theory of stochastic processes is used [1].

On the other hand, in recent decades, the interval approach has been widely applied: expert assessments serve to indicate the bounds of intervals for input variables [2].

Along with the interval approach, researchers actively employ methods of fuzzy mathematics [3, 4]. If a formal probability distribution is a priori unknown, but it is still possible to specify some possibilistic estimates, then fuzzy methods are often used. They are characterized by a rather simple apparatus producing an intuitively understandable result. Furthermore, approaches based on fuzzy methods allow transforming the possibilistic estimate of the initial data into that of the results.

Within the fuzzy approach, the membership function of a fuzzy number is used, which characterizes the possibility that the fuzzy number will take a given real value. Thus, the fuzzy approach contains more information about the existing uncertainty than the interval counterpart. Accordingly, fuzzy modeling yields more meaningful results compared to interval modeling. In practice, membership functions are constructed using expert assessments.

This paper considers fuzzy dynamic systems described by linear differential equations of order n with constant coefficients and fuzzy-valued right-hand sides. Such systems are encountered in automatic control theory, signal processing (radio engineering), and other applications.

The foundations of the theory of fuzzy differential equations were laid in [5–7] and further developed in [8–11]. Various applications were reflected in [12, 13]. Among recent works, we mention [14–17].

The literature considers various definitions of differentiability for fuzzy-valued functions. In this paper, we use the classical definition of a Hukuhara derivative [5] and the related one of a Seikkala derivative [7].

As a rule [18], when dealing with linear fuzzy differential equations, the system of equations for the corresponding α-levels is written and solved. Then it is necessary to check whether the derivatives of the resulting α-levels define the derivative of the fuzzy-valued function. According to illustrative examples [18, 19], this is not always the case. Note that the so-called operator method was developed in [19]: differential equations for α-levels are reduced to integral equations, which are then solved.

In recent years, the method of the fuzzy Laplace transform has become widespread for solving linear fuzzy differential equations of high order [20–22]. However, this method does not determine in advance whether the resulting functions will be smooth (and, accordingly, the desired solutions).

In contrast to conventional approaches, the one proposed below rests on a development of the Green's function method, widely used in the theory of ordinary differential equations [23, Chs. 1 and 2; 24], to the case of fuzzy differential equations.

The Green's function method is fruitful as it gives formulas for the α-levels of fuzzy solutions. Hence, conditions can be provided under which the Seikkala derivatives of the α-levels will define the Seikkala derivative of the fuzzy-valued solution. In this paper, such conditions are the positivity of the coefficients of the dynamic system, the nonnegativity of the corresponding Green's function, and the Seikkala differentiability of the fuzzy input signal. These conditions are natural for several applications. In particular, the positivity of the coefficients is a necessary condition for the stability of the characteristic polynomial corresponding to the linear dynamic system. Nonnegativity conditions for Green's functions are well-known; for example, see [23]. They have been studied in connection with various applications.

Let us clarify a significant aspect: this paper involves the concept of an ultra-weak fuzzy solution as a fuzzy-valued function whose α-levels satisfy the equations for the α-levels derived from a given fuzzy differential equation. Note the following fact established below: the j-times differentiability of a fuzzy-valued inhomogeneity implies the j-times differentiability of the ultra-weak fuzzy-valued solution (in the Seikkala sense).

Furthermore, an important issue concerns the form of the output fuzzy signal of a linear dynamic system receiving a fuzzy input signal of a given type (e.g., triangular). As shown below, under definite conditions (the positivity of the system coefficients and the nonnegativity of the corresponding Green's function), the output is also a triangular fuzzy signal.

Let us add that fuzzy differential equations with the generalized (Bede–Gal) derivatives have been recently investigated by several authors [10, 18–22]. The approach developed in this paper is applicable to this case as well.

As applications, this paper considers models of radio circuits with fuzzy input signals. The relationship between the modal values of the triangular fuzzy input and output signals of a linear dynamic system is obtained. The concept of a possibilistic confidence interval is introduced and used.

2. FUZZY NUMBERS AND FUZZY-VALUED FUNCTIONS

Let $\mathbb{R}$ be the set of all real numbers. A fuzzy number $\tilde{u}$ is a subset of $\mathbb{R}$ defined by its membership (possibility) function $\mu_{\tilde{u}} : \mathbb{R} \to [0, 1]$ (e.g., see [3, Ch. 5; 4, Chs. 2 and 3]), which assigns to each number $x \in \mathbb{R}$ a number $\mu_{\tilde{u}}(x)$ from the interval $[0, 1]$ characterizing the grade of membership of the element x in the set $\tilde{u}$. Here, 0 and 1 represent the lowest and highest grades of membership of an element in a given set, respectively. Thus, a fuzzy number $\tilde{u}$ can be treated as a pair $\{x, \mu_{\tilde{u}}(x) : x \in \mathbb{R}\}$.

Note that the concept of a membership function is introduced due to the insufficiency of the probabilistic approach for describing problems with uncertainty. In particular, in many applications, it is difficult to determine an a priori probability distribution. Here, the membership function is a certain analog of the probability distribution of a random variable in probability theory. Let us clarify that the number $\mu_{\tilde{u}}(x)$ is interpreted as the possibility of taking the value x for $\tilde{u}$. We emphasize that, as a rule, expert assessments are used to construct membership functions.

Also note that the membership function of a fuzzy number conceptually generalizes the characteristic function of a set, which can take only two values: 0 and 1 (0 when the element does not belong to the set and 1 otherwise).

The number x_M for which $\mu(x_M) = \max_{x\in\mathbb{R}} \mu(x)$ is called the modal value (or mode) of the fuzzy number. It is interpreted as the most possible value.

By a common assumption, the support of a fuzzy number $\tilde{u}$ (i.e., the set $\{x : \mu_{\tilde{u}}(x) > 0\}$) is bounded, and its membership function is convex, upper semicontinuous, and normal (i.e., $\sup_x \mu_{\tilde{u}}(x) = 1$). Let J denote the set of such fuzzy numbers.

Below, the interval representation of fuzzy numbers will be considered.

As is well known [3, Ch. 5], the α-level intervals (α-levels) of a fuzzy number $\tilde{u} \in J$ with a membership function $\mu_{\tilde{u}}(x)$ are defined by the relations

$$u_\alpha = \{x|\mu_{\tilde{u}}(x) \geqslant \alpha\}, \quad (\alpha \in (0, 1]), \quad z_0 = cl\{x|\mu_{\tilde{u}}(x) > 0\},$$

where cl indicates the closure of an appropriate set. According to the accepted assumptions, all α-levels of a fuzzy number are closed and bounded intervals of the real axis.

We denote by u^-_α and u^+_α the left and right bounds of an α-interval, respectively: $u_\alpha = [u^-_\alpha, u^+_\alpha]$. The expressions u^-_α and u^+_α are called the left and right α-indices (or, simply, indices) of the fuzzy number, respectively.

The α-indices of a fuzzy number $\tilde{u} \in J$ have the following properties:

1. $u^-_\alpha \leqslant u^+_\alpha \ \forall\, \alpha \in [0, 1]$.
2. The function u^-_α is bounded, nondecreasing, left continuous on the interval $(0, 1]$, and right continuous at the point 0.
3. The function u^+_α is bounded, nonincreasing, left continuous on the interval $(0, 1]$, and right continuous at the point 0.

Conversely, a pair of functions on the interval $[0, 1]$ satisfying conditions 1–3 defines a fuzzy number whose α-interval has the form $[u^-_\alpha, u^+_\alpha]$.

The sum of fuzzy numbers with indices u^-_α, u^+_α and v^-_α, v^+_α is understood as a fuzzy number with the α-level intervals $[u^-_\alpha + v^-_\alpha, u^+_\alpha + v^+_\alpha]$. Multiplication by a positive real number c is characterized by the α-level intervals $[cu^-_\alpha, cu^+_\alpha]$ whereas multiplication by a negative real number c by the α-level intervals $[cu^+_\alpha, cu^-_\alpha]$. Equality for fuzzy numbers is understood as equality for all the corresponding α-indices $\forall\, \alpha \in [0, 1]$.

Triangular numbers, for which membership functions have a triangular shape, are widely used in applications.

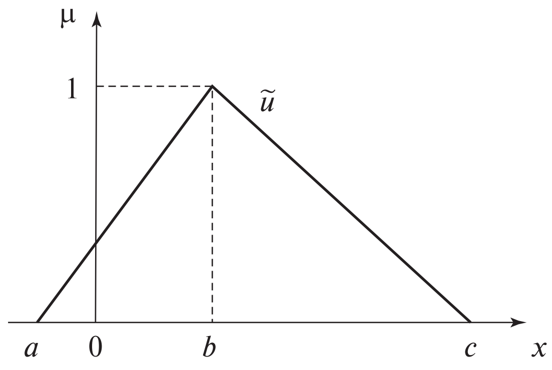

Fig. 1. A triangular fuzzy number.

Example 1. A triangular fuzzy number $\tilde{u}$ characterized by a triple of real numbers (a, b, c) with $a < b < c$ is defined by the membership function

$$\mu_{\tilde{u}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if } x \in [a,b] \\ \dfrac{x-c}{b-c} & \text{if } x \in [b,c] \\ 0 & \text{otherwise.} \end{cases}$$

Its graph generates the triangle shown in Fig. 1.

Note that the number b is the modal value (or mode) of the considered fuzzy number. For this number, we have $\mu_{\tilde{u}}(b) = 1$. The triangular fuzzy number in Fig. 1 is interpreted as the one near b.

As is well known, in the case of a triangular number, the lower and upper bounds of the α-interval have the form

$$u_\alpha^- = (b-a)\alpha + a, \quad u_\alpha^+ = -(c-b)\alpha + c.$$

Distances between fuzzy numbers can be defined in different ways. Within the interval approach, the Hausdorff distance between the α-level sets of fuzzy numbers is often used: for fuzzy numbers $\tilde{u}$ and $\tilde{v}$ with α-levels u_α and v_α, this metric [25] is given by

$$\rho(\tilde{u}, \tilde{v}) = \sup_{0 \leqslant \alpha \leqslant 1} \max\{|u_\alpha^- - v_\alpha^-|,\ |u_\alpha^+ - v_\alpha^+|\}. \tag{1}$$

Here, $[u_\alpha^-, u_\alpha^+]$ and $[v_\alpha^-, v_\alpha^+]$ are the α-level intervals of the fuzzy numbers $\tilde{u}$ and $\tilde{v}$, respectively.

Note that by (1), the condition $\rho(\tilde{u}, \tilde{v}) = 0$ is equivalent to the equality of fuzzy numbers $\tilde{u}$ and $\tilde{v}$ (see the definition above).

We fix an interval T of the real axis. A mapping $\tilde{z} : T \to J$ will be called a fuzzy-valued function.

Let a fuzzy-valued function $\tilde{z}(t)$ $\forall\, t \in T$ be characterized by a membership function $\mu_{\tilde{z}(t)}(x)$. For a fixed number $\alpha \in (0, 1]$, we consider the α-interval $z_\alpha(t) = \{x \in \mathbb{R} : \mu_{\tilde{z}(t)}(x) \geqslant \alpha\}$ and $z_0(\alpha) = cl\{x \in \mathbb{R} : \mu_{\tilde{z}(t)}(x) > 0\}$. The left and right bounds of the α-interval will be denoted by $z_\alpha^-(t)$ and $z_\alpha^+(t)$, respectively: $z_\alpha(t) = [z_\alpha^-(t), z_\alpha^+(t)]$.

The continuity of a function $\tilde{z}(t)$ in t will be understood in terms of the metric (1) whereas its boundedness in the following sense: there exists a constant $C > 0$ such that, for all $t \in T$,

$$\rho(\tilde{z}(t), \tilde{0}) = \sup_{0 \leqslant \alpha \leqslant 1} \max\{|z_\alpha^-(t)|, |z_\alpha^+(t)|\} \leqslant C.$$

Here, $\tilde{0}$ is the fuzzy number with the α-indices $0_\alpha^\pm = 0$ $\forall\, \alpha \in [0, 1]$.

Remark 1. The indices $z_\alpha^-(t)$ and $z_\alpha^+(t)$ of a continuous fuzzy-valued function $\tilde{z}(t)$ are continuous in t for any $\alpha \in [0,1]$. And if $\tilde{z}(t)$ is bounded for $t \in T$, then $z_\alpha^\pm(t)$ is bounded in $t \in T$ for any $\alpha \in [0,1]$.

The integral of a continuous fuzzy-valued function $\tilde{z}(t)$ over an interval T is defined as the fuzzy number $\tilde{g}$ with the α-level intervals $g_\alpha = \int_T z_\alpha(t)dt$ for any $\alpha \in [0,1]$ [6]. The integral is denoted by $\int_T \tilde{z}(t)dt$.

Essentially, this is the Aumann integral [26] of a multivalued mapping $z_\alpha(t)$. In fact, we have the interval representation [27]

$$\int\limits_T \tilde{z}(t)dt = \left[\int\limits_T z_\alpha^-(t)dt, \int\limits_T z_\alpha^+(t)dt\right].$$

Let us proceed to the derivatives of fuzzy-valued functions. Various definitions are used in the literature. One of the most common rests on the definition of the Hukuhara difference [28]: a set C is called the Hukuhara difference of sets A and B if $A = B + C$, and is denoted by $A \ominus B$.

A function $\tilde{z} : T \to J$ is said to be Hukuhara differentiable (H-differentiable) at a point $t \in T$ [5] if, for all sufficiently small $h > 0$, there exist Hukuhara differences $\tilde{z}(t+h) \ominus \tilde{z}(t)$ and $\tilde{z}(t) \ominus \tilde{z}(t-h)$ and an element $\tilde{z}'(t) \in J$ such that

$$\lim_{h\to 0^+} \rho\left(\frac{\tilde{z}(t+h) \ominus \tilde{z}(t)}{h}, \tilde{z}'(t)\right) = \lim_{h\to 0^+} \rho\left(\frac{\tilde{z}(t) \ominus \tilde{z}(t-h)}{h}, \tilde{z}'(t)\right) = 0,$$

where the distance ρ is given by (1). In this case, the element $\tilde{z}'(t)$ is called the H-derivative at the point t.

A fuzzy-valued function $\tilde{z} : T \to J$ is said to be Seikkala differentiable (S-differentiable) at a point $t \in T$ [7] if its α-indices $z_\alpha^-(t)$ and $z_\alpha^+(t)$ are differentiable and their derivatives $(z_\alpha^-)'(t)$ and $(z_\alpha^+)'(t)$ $\forall\, \alpha \in [0,1]$ form a fuzzy number with the α-interval $[\tilde{z}'(t)]_\alpha = [(z_\alpha^-)'(t), (z_\alpha^+)'(t)]$.

Proposition 1 [8]. *Let a fuzzy-valued function $\tilde{z}(t)$ be H-differentiable at a point $t \in T$. Then it is S-differentiable at the point $t \in T$.*

For example, a fuzzy-valued function $\tilde{z}(t)$ of the form $\tilde{z}(t) = g(t)\tilde{r}$, where $g(t)$ is a real-valued differentiable function and $\tilde{r} \in J$ is a given fuzzy number, is H-differentiable (hence, S-differentiable) at a point t provided that $g(t) \cdot g'(t) \geqslant 0$ [9].

Remark 2. By definition, the fuzzy S-derivative is additive and positively homogeneous, i.e., for S-differentiable fuzzy-valued functions $\tilde{z}(t)$ and $\tilde{w}(t)$, we have $(\tilde{z}(t) + \tilde{w}(t))' = \tilde{z}'(t) + \tilde{w}'(t)$ and $(c\tilde{z}(t))' = c\tilde{z}'(t)$ for any real constant $c \geqslant 0$.

The second Seikkala derivative $\tilde{z}''(t)$ at a point $t \in T$ is defined as the S-derivative of the first derivative, i.e., as the fuzzy number $\tilde{z}''(t)$ with the left $(z_\alpha^-)''(t)$ and right $(z_\alpha^+)''(t)$ α-index $\forall\, \alpha \in [0,1]$.

Higher-order S-derivatives are defined by analogy.

3. TRANSFORMATION OF A CONTINUOUS FUZZY SIGNAL BY A LINEAR DYNAMIC SYSTEM

A device is called a linear dynamic system if the relationship between its input and output is described by a differential equation of order n with constant coefficients. If fuzzy signals $\tilde{f}(t)$ and $\tilde{z}(t)$ ($t \in T$) are observed at the input and output, respectively, then the linear dynamic system is described by a fuzzy differential equation of the form

$$a_n\tilde{z}^{(n)}(t) + a_{n-1}\tilde{z}^{(n-1)}(t) + \cdots + a_1\tilde{z}'(t) + a_0\tilde{z}(t) = \tilde{f}(t). \tag{2}$$

Here, the coefficients a_i $(i = 0, \ldots, n)$ are real numbers, $\tilde{f}(t)$ is an input fuzzy-valued function, and the derivatives of a fuzzy-valued function $\tilde{z}(t)$ are understood as S-derivatives.

Below, the interval T is taken as $T = (-\infty, \infty)$.

Consider the problem of bounded solutions for a real differential equation with constant coefficients of the form

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \cdots + a_1 x'(t) + a_0 x(t) = f(t), \quad t \in (-\infty, \infty). \tag{3}$$

A function $G(t)$ is called a Green's function in the problem of bounded solutions of equation (3) if it has the following properties (for details, e.g., see [23, Ch. 1, § 4; 24]):

1) $G(t)$ is continuously differentiable $(n-2)$ times for all t, the nth and $(n-1)$th derivatives are continuously differentiable for all t except $t = 0$, and

$$G^{(n-1)}(+0) - G^{(n-1)}(-0) = \frac{1}{a_n}.$$

2) At all points except $t = 0$, the function $G(t)$ satisfies the homogeneous differential equation corresponding to (3) (with $f(t) \equiv 0$).

3) The Green's function and its derivatives are estimated as

$$|G^{(i)}(t)| \leqslant M e^{-\gamma|t|} \quad (i = 0, 1, \ldots, n, \ -\infty < t < +\infty),$$

where M and γ are some positive constants.

Proposition 2 [23, Ch. 1; 24]. *Let the roots of the characteristic equation $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$ contain no points on the imaginary axis. Then for any continuous function $f(t)$ bounded on the entire real axis, equation* (3) *has a unique bounded solution on the entire real axis given by*

$$x(t) = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds, \tag{4}$$

where $G(t)$ is the Green's function in the problem of bounded solutions of equation (3).

Moreover,

$$x^{(j)}(t) = \int_{-\infty}^{\infty} G_t^{(j)}(t-s) f(s)\, ds, \quad j = (0, 1, \ldots, n-1),$$

$$x^{(n)}(t) = f(t) + \int_{-\infty}^{\infty} G_t^{(n)}(t-s) f(s)\, ds.$$

We emphasize that the convergence of the improper integral (4) and the corresponding integrals for the derivatives is ensured by the exponential estimates of the Green's function and its derivatives, as well as by the continuity and boundedness of the function $f(t)$ on the entire real axis.

Note that in the problem of bounded solutions of equation (3), the Green's function has a known general form; for example, see [23, Ch. 2, § 8].

Proposition 3 [23, Ch. 2]. *Under the hypotheses of Proposition 2, let all roots of the characteristic equation lie in the left half-plane ($Re\lambda_i < 0$, $i = 1, \ldots, n$). Then the bounded solution of equation* (3) *is asymptotically Lyapunov stable. Moreover, the solution* (4) *takes the form*

$$x(t) = \int_{-\infty}^{t} G(t-s) f(s)\, ds. \tag{5}$$

Here, the Green's function is $G(\sigma) = \begin{cases} K(\sigma) & \text{for } \sigma \geqslant 0 \\ 0 & \text{for } \sigma < 0, \end{cases}$ *with* $K(\sigma)$ *being the Cauchy function that represents the solution of the homogeneous differential equation corresponding to equation* (3) (*with* $f(t) \equiv 0$) *and satisfies the initial conditions*

$$K^{(j)}(0) = 0, \quad j = 0, 1, \dots, n-2, \quad K^{(n-1)}(0) = 1.$$

Now we address the case of fuzzy input and output signals. In some cases, the representation (4) can be used to write explicitly the α-indices of the fuzzy signal at the output of the dynamic system (2).

A strong solution of the fuzzy differential equation (2) is an n times continuously S-differentiable fuzzy-valued function satisfying (2) on the corresponding interval.

Lemma 1. *Let the coefficients of the fuzzy differential equation* (2) *be positive:* $a_i > 0$, $i = 0, \dots, n$. *If a fuzzy-valued function* $\tilde{z}(t)$ *is a strong solution of equation* (2) *on the interval* T, *then the corresponding* α*-indices* $z^{\pm}_{\alpha}(t)$ *for all* $\alpha \in [0,1]$ *and* $t \in T$ *satisfy the ordinary differential equations*

$$a_n(z^-_\alpha)^{(n)}(t) + a_{n-1}(z^-_\alpha)^{(n-1)}(t) + \dots + a_1(z^-_\alpha)'(t) + a_0 z^-_\alpha(t) = f^-_\alpha(t), \tag{6}$$
$$a_n(z^+_\alpha)^{(n)}(t) + a_{n-1}(z^+_\alpha)^{(n-1)}(t) + \dots + a_1(z^+_\alpha)'(t) + a_0 z^+_\alpha(t) = f^+_\alpha(t). \tag{7}$$

Indeed, we substitute the strong solution $\tilde{z}(t)$ of the fuzzy differential equation into (2). Recall that the equality of fuzzy numbers means the equality of all the corresponding α-indices. Hence, by the rules of interval arithmetic, the positivity of the coefficients a_i, the definition of a fuzzy Seikkala derivative, and Remark 2, for all $\alpha \in [0,1]$ and $t \in (-\infty, \infty)$ equation (2) implies equalities (6) and (7).

Lemma 2. *Let the coefficients of the fuzzy differential equation* (2) *be positive,* ($a_i > 0$, $i = 0, \dots, n$), *and let the roots of the characteristic equation* $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \dots + a_1\lambda + a_0 = 0$ *contain no points on the imaginary axis. In addition, let the fuzzy-valued function* $\tilde{f}(t)$ *on the right-hand side of equation* (2) *be continuous and bounded in the metric* (1) *for* $t \in (-\infty, \infty)$. *Then for all* $\alpha \in [0,1]$ *there exists a unique bounded solution for each of the two equations* (6), (7) *on the entire real axis, and they can be represented as*

$$z^-_\alpha(t) = \int_{-\infty}^{\infty} G(t-s) f^-_\alpha(s) ds, \quad z^+_\alpha(t) = \int_{-\infty}^{\infty} G(t-s) f^+_\alpha(s) ds. \tag{8}$$

Indeed, due to the hypotheses and Remark 1, for all $\alpha \in [0,1]$ the functions $f^{\pm}_\alpha(t)$ are continuous and bounded on the entire real axis. Then, by Proposition 2, for all $\alpha \in [0,1]$ the solutions of equations (6), (7) exist, are unique, and equalities (8) hold.

Lemma 3. *Under the hypotheses of Lemma 2, let the Green's function* G *in problem* (3) *be nonnegative. Then for all* $t \in (-\infty, \infty)$ *the expressions* (8) *satisfy conditions 1–3 for the indices of fuzzy numbers (see Section 2).*

Proof. We fix $t \in (-\infty, \infty)$. By the hypothesis, $f^-_\alpha(s) \leqslant f^+_\alpha(s)\ \forall\, s \in (-\infty, \infty)$. Then, due to the nonnegativity of the Green's function,

$$G(t-s) f^-_\alpha(s) \leqslant G(t-s) f^+_\alpha(s).$$

Therefore, based on (8) and the monotonicity of the integral, we have $z^-_\alpha(t) \leqslant z^+_\alpha(t)$. That is, for all $t \in (-\infty, \infty)$ the expressions (8) satisfy condition 1 for the α-indices of fuzzy numbers (see Section 2).

Now we fix an arbitrary $t \in (-\infty, \infty)$ and show that the function $z_\alpha^-(t)$ is nondecreasing in α. Let $\alpha_1, \alpha_2 \in [0,1]$ and $\alpha_1 < \alpha_2$. By the hypothesis, the condition $f_{\alpha_1}^-(s) \leqslant f_{\alpha_2}^-(s)$ holds for all $s \in (-\infty, \infty)$. Then, due to the nonnegativity of the Green's function, $G(t-s)f_{\alpha_1}^-(s) \leqslant G(t-s)f_{\alpha_2}^-(s)$. Consequently, in view of the monotonic property of the integral, we obtain $z_{\alpha_1}^-(t) \leqslant z_{\alpha_2}^-(t)$, i.e., the function $z_\alpha^-(t)$ is monotonically nondecreasing in α. The monotonic nonincrease of the function $z_\alpha^+(t)$ in α can be established by analogy.

By the hypothesis, the function $\tilde{f}(t)$ is bounded in t for $t \in (-\infty, \infty)$ in the metric (1). This means the existence of a constant $C > 0$ such that $\rho(\tilde{f}(t), \tilde{0}) \leqslant C$ for any $t \in (-\infty, \infty)$; consequently, $\sup_{0 \leqslant \alpha \leqslant 1} |f_\alpha^\pm(t)| \leqslant C$. Then according to (8), for fixed t and all $\alpha \in [0,1]$, we have

$$|z_\alpha^\pm(t)| \leqslant \int\limits_{-\infty}^{\infty} |G(t-s)||f_\alpha^\pm(s)| ds \leqslant C \int\limits_{-\infty}^{\infty} |G(t-s)| ds,$$

which ensures the boundedness of the expressions (8) in $\alpha \in [0,1]$ for each $t \in (-\infty, \infty)$.

For fixed t, the left continuity of the functions $z_\alpha^\pm(t)$ in $\alpha \in (0,1]$ is immediate from the following consideration. Let us fix $\alpha_0 \in [0,1)$ and consider the equality

$$\lim_{\alpha \to \alpha_0 - 0} z_\alpha^\pm(t) = \int\limits_{-\infty}^{\infty} G(t-s) \lim_{\alpha \to \alpha_0 - 0} f_\alpha^\pm(s) ds = \int\limits_{-\infty}^{\infty} G(t-s) f_{\alpha_0}^\pm(s) ds = z_{\alpha_0}^\pm(t).$$

Here, we take advantage of the representation (8), the possibility of passing to the limit under the sign of an absolutely convergent improper integral, and the left continuity of the α-indices $f_\alpha^\pm(s)$ of the fuzzy-valued function $\tilde{f}(s)$ in $\alpha \in (0,1]$ for arbitrary $s \in (-\infty, \infty)$. The right continuity of the functions $z_\alpha^\pm(t)$ in α at $\alpha = 0$ is verified similarly. In other words, $z_\alpha^\pm(t)$ satisfy conditions 2 and 3 of Section 2 as well.

Thus, the expressions $z_\alpha^\pm(t)$ (8) satisfy all the conditions for the α-levels of fuzzy numbers (see Section 2).

Let us emphasize the significance of Lemma 3. According to illustrative examples [18, 19], the solutions of systems for the α-indices of linear fuzzy differential equations are not always the α-indices of some fuzzy-valued function.

Theorem 1. *Under the hypotheses of Lemma 3, the fuzzy-valued function generated by the α-indices* (8) $\forall\, t \in (-\infty, \infty)$ *is characterized by the representation*

$$\tilde{z}(t) = \int\limits_{-\infty}^{\infty} G(t-s)\tilde{f}(s) ds. \tag{9}$$

Indeed, by Lemmas 1–3 and the definition of the integral of a fuzzy-valued function, for all $\alpha \in [0,1]$ the α-indices satisfy the relations

$$\left(\int\limits_{-\infty}^{\infty} G(t-s)\tilde{f}(s) ds \right)_\alpha^- = \int\limits_{-\infty}^{\infty} G(t-s) f_\alpha^-(s) ds,$$

$$\left(\int\limits_{-\infty}^{\infty} G(t-s)\tilde{f}(s) ds \right)_\alpha^+ = \int\limits_{-\infty}^{\infty} G(t-s) f_\alpha^+(s) ds.$$

In view of (8), these relations imply the representation (9).

Theorem 2. *Under the hypotheses of Theorem 1, the fuzzy-valued function* $\tilde{z}(t)$ (9) *is continuous and bounded in* t *on the entire real axis.*

Proof. Let us fix real numbers t_1 and t_2. Due to (8),

$$z_\alpha^\pm(t_1) - z_\alpha^\pm(t_2) = \int\limits_{-\infty}^{\infty} (G(t_1 - s) - G(t_2 - s)) f_\alpha^\pm(s) ds$$

and, consequently,

$$|z_\alpha^\pm(t_1) - z_\alpha^\pm(t_2)| \leqslant \int\limits_{-\infty}^{\infty} |G(t_1 - s) - G(t_2 - s)||f_\alpha^\pm(s)| ds.$$

Note that $G(t_1 - s) - G(t_2 - s) = G_t'(\tau - s)(t_1 - t_2)$, where $\tau \in ((1-\theta)t_1 + \theta t_2)$ and $\theta \in (0, 1)$. Therefore, by the previous considerations,

$$|z_\alpha^\pm(t_1) - z_\alpha^\pm(t_2)| \leqslant \left(\int\limits_{-\infty}^{\infty} |G_t'(\tau - s)||f_\alpha^\pm(s)| ds \right) |t_1 - t_2| \leqslant C|t_1 - t_2| \int\limits_{-\infty}^{\infty} |G_t'(\tau - s)| ds,$$

where the constant $C > 0$ characterizes the boundedness condition: $\rho(\tilde{f}(t), \tilde{0}) \leqslant C \ \forall t \in (-\infty, \infty)$.

Next, we utilize the exponential estimate of the derivative of the Green's function $|G_t'(t)| \leqslant Me^{-\gamma|t|}$ (see property 3 of Green's functions in Section 3). As a result,

$$|z_\alpha^\pm(t_1) - z_\alpha^\pm(t_2)| \leqslant MC|t_1 - t_2| \int\limits_{-\infty}^{\infty} e^{-\gamma|\tau - s|} ds = 2\frac{MC}{\gamma}|t_1 - t_2|.$$

Therefore, by the definition of the metric (1), we obtain $\rho\left(z_\alpha^\pm(t_1), z_\alpha^\pm(t_2)\right) \leqslant 2\frac{MC}{\gamma}|t_1 - t_2|$, which implies the continuity of the fuzzy-valued function $\tilde{z}(t)$ (9).

Let us show the boundedness of the fuzzy-valued function $\tilde{z}(t)$ (9) for $t \in (-\infty, \infty)$. By definition,

$$\rho(\tilde{z}(t), \tilde{0}) = \sup_{0 \leqslant \alpha \leqslant 1} |z_\alpha^\pm(t)|.$$

Moreover, according to (8),

$$|z_\alpha^\pm(t)| \leqslant \int\limits_{-\infty}^{\infty} |G(t - s)||f_\alpha^\pm(s)| ds \leqslant \left(\int\limits_{-\infty}^{\infty} |G(t - s)| ds \right) \rho(\tilde{f}(t), \tilde{0}) \leqslant 2\frac{M\rho(\tilde{f}(t), \tilde{0})}{\gamma}.$$

Here, the exponential estimate of the Green's function is used again. Thus, $\rho(\tilde{z}(t), \tilde{0}) \leqslant 2\frac{M\rho(\tilde{f}(t), \tilde{0})}{\gamma} \leqslant 2\frac{MC}{\gamma} \ \forall t \in (-\infty, \infty)$.

We call a continuous fuzzy-valued function $\tilde{z}(t)$ an ultra-weak solution of the fuzzy differential equation (2) if its α-indices $z_\alpha^\pm(t) \ \forall \alpha \in [0, 1]$ are n times continuously differentiable with respect to t and satisfy equations (6) and (7) on the corresponding time interval. A weak solution of the fuzzy differential equation (2) is a fuzzy-valued function satisfying the integro-differential or integral fuzzy equation corresponding to (2) (for example, see [19]). Such solutions will not be considered in this paper. The following result is true.

Theorem 3. *Under the hypotheses of Theorem 1, there exists a unique ultra-weak solution of the fuzzy differential equation* (2) *that is bounded on the entire real axis.*

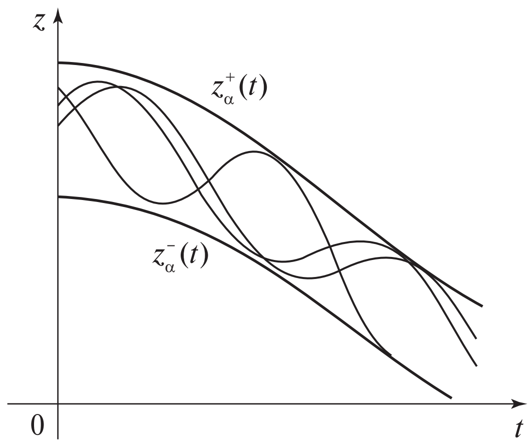

Fig. 2. The graphical representation of an ultra-weak fuzzy-valued solution in interval form.

Existence is ensured by Theorems 1 and 2. Let us establish uniqueness. Assume on the contrary that $\tilde{z}(t)$ and $\tilde{w}(t)$ are two different ultra-weak solutions of equation (2) that have boundedness on the entire real axis. Then the indices $z_\alpha^-(t)$ and $w_\alpha^-(t)$ are bounded solutions of equation (6). Therefore, by Proposition 2, $z_\alpha^-(t) = w_\alpha^-(t)$ for all $\alpha \in [0,1]$ and $t \in (-\infty, \infty)$. Similarly, $z_\alpha^+(t) = w_\alpha^+(t)$ for all $\alpha \in [0,1]$ and $t \in (-\infty, \infty)$. But in this case, according to the equality criterion of fuzzy numbers, we have $\tilde{z}(t) = \tilde{w}(t)$ for all $t \in (-\infty, \infty)$. This obvious contradiction concludes the proof of the theorem.

Theorem 4. *Under the hypotheses of Theorem 1, let all roots of the characteristic equation lie in the left half-plane. Then there exists a unique ultra-weak solution of the fuzzy differential equation* (2) *that is bounded on the entire real axis, and this solution is given by*

$$\tilde{z}(t) = \int_{-\infty}^{t} G(t-s)\tilde{f}(s)ds. \tag{10}$$

This fact is immediate from Theorem 1 and Proposition 3.

Figure 2 provides the geometric illustration of a fuzzy-valued solution.

We say that a fuzzy-valued function $\tilde{f} : \mathbb{R} \to J$ is triangular, or has the triangular type $(a(t), b(t), c(t))$, if there exist continuous real-valued functions $a(t)$, $b(t)$, and $c(t)$ on the entire real axis such that $a(t) < b(t) < c(t)\ \forall t \in \mathbb{R}$ and the membership functions $\mu_{\tilde{f}(t)}(x)\ \forall t \in \mathbb{R}$ have the triangular form

$$\mu_{\tilde{f}(t)}(x) = \begin{cases} \dfrac{x - a(t)}{b(t) - a(t)} & \text{if } x \in [a(t), b(t)] \\[2ex] \dfrac{x - c(t)}{b(t) - c(t)} & \text{if } x \in [b(t), c(t)] \\[2ex] 0 & \text{otherwise.} \end{cases}$$

(Also, see Example 1.)

In particular, a fuzzy-valued function of the form $\tilde{f}(t) = g(t)\tilde{r}$, where $g(t)$ is a continuous and nonnegative real-valued function and $\tilde{r} \in J$ is a given triangular fuzzy number, has triangular type as well.

Consider a special case when a triangular fuzzy signal is supplied to the input of a dynamic system described by equation (2). The following result is valid.

Theorem 5. *Under the hypotheses of Theorem 1, let the right-hand side of equation* (2), *i.e., the fuzzy-valued function* $\tilde{f}(t)\ \forall\, t \in (-\infty, \infty)$, *have the triangular type* $(a(t), b(t), c(t))$. *Then the bounded ultra-weak solution* (9) *of equation* (2) *on the entire real axis has the triangular type* $(\int_{-\infty}^{\infty} G(t-s)a(s)ds,\ \int_{-\infty}^{\infty} G(t-s)b(s)ds,\ \int_{-\infty}^{\infty} G(t-s)c(s)ds)\ \forall\, t \in (-\infty, \infty)$.

Indeed, by the hypothesis, $\tilde{f}(t)$ is generated by a triple of continuous and bounded functions $a(t)$, $b(t)$, $c(t)$ on the entire axis, with $a(t) < b(t) < c(t)\ \forall\, t \in (-\infty, \infty)$. Then, according to Example 1, for all $\alpha \in [0, 1]$ and $t \in (-\infty, \infty)$ we have

$$f_\alpha^-(t) = (b(t) - a(t))\alpha + a(t), \quad f_\alpha^+(t) = (b(t) - c(t))\alpha + c(t).$$

Therefore, due to (8), it follows that

$$z_\alpha^-(t) = (B(t) - A(t))\alpha + A(t), \quad z_\alpha^+(t) = (B(t) - C(t))\alpha + C(t),$$

where

$$A(t) = \int\limits_{-\infty}^{\infty} G(t-s)a(s)ds, \quad B(t) = \int\limits_{-\infty}^{\infty} G(t-s)b(s)ds, \quad C(t) = \int\limits_{-\infty}^{\infty} G(t-s)c(s)ds.$$

Thus, $\tilde{z}(t)\ \forall\, t \in (-\infty, \infty)$ is a fuzzy number of the triangular type $(A(t),\ B(t),\ C(t))$.

A similar result holds for trapezoidal fuzzy-valued functions $\tilde{f}(t)$ [3, Ch. 5]: the solution $\tilde{z}(t)$ will also have the trapezoidal type.

Let us find conditions under which the bounded ultra-weak solution (10) of the fuzzy differential equation (2) will be S-differentiable with respect to t.

Under the hypotheses of Proposition 3, the following representation will be used below for the bounded solution of the scalar differential equation (3):

$$x(t) = \int\limits_0^{\infty} G(\sigma) f(t-\sigma) d\sigma. \tag{11}$$

It is obtained from (5) by substituting $t - s = \sigma$.

Theorem 6. *Under the hypotheses of Theorem 4, let the fuzzy-valued function* $\tilde{f}(t)$ *be Seikkala differentiable for all* $t \in (-\infty, \infty)$, *and let its* S*-derivative* $\tilde{f}'(t)$ *be continuous and bounded for* $t \in (-\infty, \infty)$. *Then the bounded ultra-weak solution* (10) *of the fuzzy differential equation* (2) *is* S*-differentiable for* $t \in (-\infty, \infty)$ *and*

$$\tilde{z}'(t) = \int\limits_0^{\infty} G(\sigma) \tilde{f}_t'(t-\sigma) d\sigma. \tag{12}$$

Indeed, due to (6), (7), and (11), for the α-indices of the ultra-weak solution $\tilde{z}(t)$ of the fuzzy differential equation (2) we write

$$z_\alpha^\pm(t) = \int\limits_0^{\infty} G(\sigma) f_\alpha^\pm(t-\sigma) d\sigma. \tag{13}$$

Similar to Lemma 3, they generate the fuzzy-valued function

$$\tilde{z}(t) = \int\limits_0^{\infty} G(\sigma) \tilde{f}(t-\sigma) d\sigma. \tag{14}$$

By the hypothesis, for all $\alpha \in [0,1]$ the α-indices $f_\alpha^\pm(t)$ are differentiable with respect to $t \in (-\infty, \infty)$, and the derivatives $(f_\alpha^\pm)'_t(t)$ are continuous and bounded scalar functions on the entire real axis. Then, in view of (13), for the derivatives of the α-indices $z_\alpha^\pm(t)$ with respect to t we have

$$(z_\alpha^\pm)'(t) = \int\limits_0^\infty G(\sigma)(f_\alpha^\pm)'_t(t-\sigma)d\sigma. \tag{15}$$

(Here, differentiation is performed with respect to the parameter under the sign of an absolutely convergent integral.)

By the nonnegativity of the Green's function (similar to the proof of Lemma 3), for each t the expressions (15) are the α-indices of the S-derivative $\tilde{z}'(t)$. Therefore, used jointly with Theorem 1, formulas (14) and (15) imply (12).

Then, by Theorem 3 on the uniqueness of the bounded ultra-weak solution of equation (2) on the entire real axis, based on (10) and (14) we obtain

$$\left(\int\limits_{-\infty}^t G(t-s)\tilde{f}(s)ds\right)'_t = \left(\int\limits_0^\infty G(\sigma)\tilde{f}(t-\sigma)d\sigma\right)'_t = \int\limits_0^\infty G(\sigma)\tilde{f}'_t(t-\sigma)d\sigma.$$

And the desired conclusion follows.

Corollary 1. *Under the hypotheses of Theorem 6, the fuzzy derivative* (12) *is continuous and bounded for* $t \in (-\infty, \infty)$.

This fact is established by analogy with Theorem 2.

Proposition 4. *Under the hypotheses of Theorem 4, let the fuzzy-valued function* $\tilde{f}(t)$ *be twice* S*-differentiable, and let the first* $\tilde{f}'_t(t)$ *and second* $\tilde{f}''_t(t)$ S*-derivatives be continuous and bounded for* $t \in (-\infty, \infty)$*. Then the bounded ultra-weak solution* $\tilde{z}(t)$ (14) *on the entire real axis is twice* S*-differentiable for* $t \in (-\infty, \infty)$.

Indeed, due to equality (15), for all $\alpha \in [0,1]$ we write the following relation for the α-indices $z_\alpha^\pm(t)$:

$$(z_\alpha^\pm)''(t) = \left(\int\limits_0^\infty G(\sigma)(f_\alpha^\pm)'_t(t-\sigma)d\sigma\right)' = \int\limits_0^\infty G(\sigma)(f_\alpha^\pm)''_t(t-\sigma)d\sigma.$$

Therefore, based on the nonnegativity of the Green's function, for all $t \in (-\infty, \infty)$ the integrals on the right-hand side generate a fuzzy number $\tilde{z}''(t)$, i.e., the second S-derivative of $\tilde{z}(t)$.

Corollary 2. *Under the hypotheses of Proposition 4, the derivatives* $\tilde{z}'(t)$ *and* $\tilde{z}''(t)$ *are continuous and bounded for* $t \in (-\infty, \infty)$.

Corollary 3. *Under the hypotheses of Theorem 4, let the fuzzy-valued function* $\tilde{f}(t)$ *be continuously* S*-differentiable* $j > 2$ *times, and let all its derivatives up to order* j *be bounded on the entire real axis. Then the bounded ultra-weak solution* (14) *of the fuzzy differential equation* (2) *is a continuously* j *times* S*-differentiable fuzzy-valued function.*

Remark 3. According to Proposition 2, the derivatives of the α-indices $z_\alpha^\pm(t)$ (8) are represented as

$$(z_\alpha^\pm)_t^{(j)}(t) = \int\limits_{-\infty}^\infty G_t^{(j)}(t-s)f_\alpha^\pm(s)ds \quad (j = 1, \ldots, n-1).$$

However, the derivatives $G_t^{(j)}(t-s)$ of the Green's function generally do not preserve sign (see Examples 2 and 3 below). So, it is more convenient to prove the S-differentiability of the fuzzy-valued solution using the representation (13).

In many applications, the fuzzy-valued solution $\tilde{z}(t)$ must be nonnegative, which means $z_\alpha^-(t) \geqslant 0$ $\forall\, \alpha \in [0,1]$, $\forall\, t \geqslant 0$. The following result is true.

Corollary 4. *Under the hypotheses of Theorem 1, let the fuzzy-valued inhomogeneity $\tilde{f}(t)$ of equation* (2) *be nonnegative, i.e., $f_\alpha^-(t) \geqslant 0$ $\forall\, \alpha \in [0,1]$, $\forall\, t \in (-\infty, \infty)$. Then the bounded ultra-weak solution of equation* (2) *is nonnegative on the entire real axis.*

This fact is immediate from the representation (9) due to the assumed nonnegativity of the Green's function $G(t-s)$ and $f_\alpha^-(s)$.

In particular, for a triangular fuzzy number (a, b, c), nonnegativity means that $a \geqslant 0$.

4. EXAMPLES OF RADIO CIRCUITS WITH FUZZY INPUT SIGNALS

Here, we consider some applications of the results of Section 3 to elementary radio circuits (e.g., see [29]) with fuzzy input signals.

Example 2. Consider an RC filter, i.e., a radio circuit shown in Fig. 3, where R and C are resistance and capacitance, respectively.

This filter is a dynamic system described by the first-order differential equation with constant coefficients

$$\tilde{z}'(t) + \beta \tilde{z}(t) = \tilde{y}(t), \quad \beta = \frac{1}{RC} > 0. \tag{16}$$

Let a continuous fuzzy signal $\tilde{y}(t)$ bounded on the entire real axis be supplied to the system input.

We determine the characteristics of the bounded fuzzy output signal $\tilde{z}(t)$ of the RC filter. Note that in the problem of bounded solutions of the scalar equation $x' + \beta x = y(t)$ with $\beta > 0$, the Green's function can be represented as

$$G_1(t) = \begin{cases} e^{-\beta t} & \text{for } \ t \geqslant 0 \\ 0 & \text{for } \ t < 0. \end{cases}$$

Thus, $G_1(t) \geqslant 0$ for all $t \in (-\infty, \infty)$.

Proposition 5. *Let the coefficient β of the fuzzy differential equation* (16) *be positive, and let the right-hand side $\tilde{y}(t)$ be a continuous fuzzy-valued function bounded for $t \in (-\infty, \infty)$. Then the continuous fuzzy ultra-weak signal at the output of the dynamic system* (16) *is bounded on the entire*

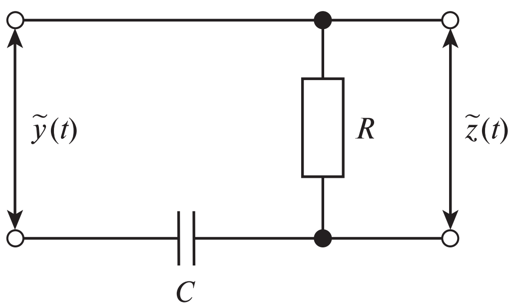

Fig. 3. An RC filter.

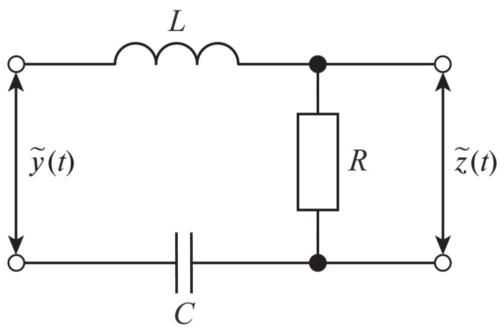

Fig. 4. A series oscillatory circuit.

real axis and has the form

$$\tilde{z}(t) = \int_{-\infty}^{t} e^{-\beta(t-s)} \tilde{y}(s) ds. \tag{17}$$

This fact follows from Theorem 4.

Proposition 6. *Under the hypotheses of Proposition 5, let the fuzzy-valued function $\tilde{y}(t)$ on the right-hand side of equation* (16) *have triangular type for $t \geqslant 0$. Then the solution* (17) *is also triangular for all $t \geqslant 0$.*

This fact is immediate from the representation (17) and Theorem 5.

The following case was considered in [9]: the right-hand side of equation (16) has the form $\tilde{y}(t) = \tilde{r} f(t)$, where $\tilde{r} \in J$ is a fuzzy number, and the function $f : \mathbb{R} \to \mathbb{R}$ is almost periodic and $f(t) \geqslant 0 \ \forall t \in \mathbb{R}$. As established in this case, the condition $f(t) > \int_{-\infty}^{t} e^{-\beta(t-s)} f(s) ds$ ensures the H-differentiability of the solution (17) of the fuzzy differential equation (16).

This condition imposes an additional constraint on either the range of t or the relationship between the parameters β and $f(t)$ of equation (16).

The next proposition, following from Theorem 6, ensures the S-differentiability of the fuzzy-valued solution of equation (16).

Proposition 7. *Under the hypotheses of Proposition 5, let the fuzzy-valued right-hand side $\tilde{y}(t)$ of equation* (16) *be continuously S-differentiable for all $t \in (-\infty, \infty)$, and let the S-derivative $\tilde{y}'(t)$ be bounded on the entire real axis. Then the bounded ultra-weak solution* (17) *is continuously S-differentiable for all $t \in (-\infty, \infty)$ and satisfies the fuzzy differential equation* (16) *for all $t \in (-\infty, \infty)$.*

Thus, under the hypotheses of Proposition 7, formula (17) gives a bounded strong solution of the fuzzy equation (16).

Example 3. Consider a series oscillatory circuit, i.e., a radio circuit in Fig. 4, where R, C, and L are resistance, capacitance, and inductance, respectively.

This oscillatory circuit is a dynamic system described by the second-order differential equation with constant coefficients

$$a_2 \tilde{z}''(t) + a_1 \tilde{z}'(t) + a_0 \tilde{z}(t) = \tilde{y}(t), \quad a_2 = L \geqslant 0, \quad a_1 = R > 0, \quad a_0 = \frac{1}{RC} > 0. \tag{18}$$

Let a continuous fuzzy signal $\tilde{y}(t)$ bounded on the entire real axis be supplied to the system input (see Fig. 4).

We determine the characteristics of the bounded fuzzy output signal $\tilde{z}(t)$ of this circuit.

Theorem 7. *Let the coefficients of the fuzzy differential equation* (18) *satisfy the conditions* $a_i > 0$ $(i = 0, 1, 2)$ *and* $a_1^2 - 4a_0a_2 > 0$. *In addition, let the input signal* $\tilde{y}(t)$ *be a continuous fuzzy-valued function bounded for* $t \in (-\infty, \infty)$. *Then the fuzzy ultra-weak signal* $\tilde{z}(t)$ *at the output of the dynamic system* (18) *is represented as*

$$\tilde{z}(t) = \int\limits_{-\infty}^{t} G_2(t-s)\tilde{y}(s)ds, \tag{19}$$

where $G_2(t)$ *is the Green's function in the problem of bounded solutions of the real differential equation* $a_2x'' + a_1x' + a_0x = f(t)$ *with the continuous real-valued function* $f(t)$ *bounded on the entire real axis, which has the form*

$$G_2(t) = \begin{cases} (e^{\lambda_2 t} - e^{\lambda_1 t})(\lambda_2 - \lambda_1)^{-1} & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0. \end{cases}$$

In this formula, λ_1 *and* λ_2 *are distinct negative real-valued roots of the characteristic equation* $a_2\lambda^2 + a_1\lambda + a_0 = 0$ $(\lambda_1 < \lambda_2 < 0)$.

Indeed, note that $G_2(t) \geqslant 0$ $\forall\, t \in (-\infty, \infty)$. Then, by Theorem 4, equality (19) holds for the fuzzy-valued signal $\tilde{z}(t)$ at the system output.

Moreover, the expression (19) is an ultra-weak solution of the fuzzy differential equation (18).

Proposition 8. *Under the hypotheses of Theorem 7, let the right-hand side* $\tilde{y}(t)$ *of* (18) *be a triangular fuzzy number for all* $t \in (-\infty, \infty)$. *Then the bounded ultra-weak solution* (19) *of the fuzzy differential equation* (18) *on the entire real axis is also a triangular fuzzy number for* $t \in (-\infty, \infty)$.

This fact follows from the representation (19) and Theorem 5.

Let us find conditions under which the bounded ultra-weak solution (19) of the fuzzy differential equation (18) will be an S-differentiable fuzzy-valued function.

Theorem 8. *Under the hypotheses of Theorem 7, let the fuzzy-valued function* $\tilde{y}(t)$ *be continuously* S*-differentiable for all* $t \in (-\infty, \infty)$, *and let the* S*-derivative* $\tilde{y}'(t)$ *be bounded for* $t \in (-\infty, \infty)$. *Then the bounded ultra-weak solution* $\tilde{z}(t)$ (19) *of the fuzzy differential equation* (18) *is* S*-differentiable for all* $t \in (-\infty, \infty)$.

This fact is immediate from Theorem 6.

Moreover, another result is valid.

Theorem 9. *Under the hypotheses of Theorem 7, let the right-hand side* $\tilde{y}(s)$ *of equation* (18) *be twice continuously* S*-differentiable, and let the* S*-derivatives* $\tilde{y}'(t)$ *and* $\tilde{y}''(t)$ *be bounded for* $t \in (-\infty, \infty)$. *Then the bounded ultra-weak solution* $\tilde{z}(t)$ (19) *is also twice continuously* S*-differentiable and satisfies the fuzzy differential equation* (18).

This fact follows from Proposition 4.

Thus, under the hypotheses of Theorem 9, formula (19) provides a bounded strong solution of the fuzzy equation (18).

5. ON THE RELATIONSHIP BETWEEN MODAL VALUES OF FUZZY INPUT AND OUTPUT SIGNALS

The peculiarity of considering fuzzy input signals in radio engineering (when the a priori probability distribution of the input signal is unknown) consists in identifying the most possible input signal. After that, the system parameters are used to characterize the most possible fuzzy output signal.

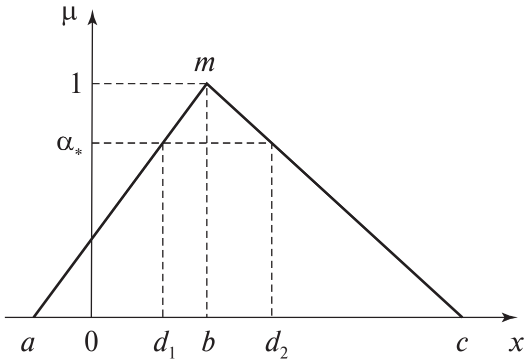

Fig. 5. A possibilistic confidence interval for the modal value of a fuzzy input signal.

In probability theory, for the corresponding problem, the term "possible" would have to be replaced by the term "probable." This is especially clear for triangular fuzzy input and output signals (see Example 1). For instance, let a triangular fuzzy input signal $(a(t), b(t), c(t))$ be supplied to the input of a radio circuit described by the fuzzy differential equation (2). According to Example 1, for each t, $b(t)$ is its modal value. Based on Theorem 5, we arrive at the following result.

Proposition 9. *Under the hypotheses of Theorem 5, the modal value of the triangular fuzzy signal at the output of the linear dynamic system* (2) *is determined from the modal value* $b(t)$ *of its triangular fuzzy input signal* $(a(t), b(t), c(t))$ *by the formula* $B(t) = \int_{-\infty}^{\infty} G(t-s)b(s)ds$, *with the designations of Theorem 5.*

An important branch of mathematical statistics is the theory of confidence intervals; for example, see [30, Ch. 2]. By analogy, let us discuss the problem of possibilistic confidence intervals for the values of fuzzy input and output signals.

Suppose that for an arbitrary $t \geqslant 0$, experts have modeled a fuzzy input signal with a possibilistic confidence level $\alpha_* \in [0.7, 1)$ as triangular with a membership function $\mu_t(x)$ having support $(a(t), c(t))$. Moreover, for a given $t \geqslant 0$, let the modal value lie in a possibilistic confidence interval $[d_1(t), d_2(t)]$ with the possibilistic confidence level α_*, where $[d_1(t), d_2(t)] \subset (a(t), c(t))$ (Fig. 5).

Remark 4. Under the above assumptions, the modal value of the triangular fuzzy signal with the possibilistic confidence level α_* has the form

$$b(t) = \frac{1}{\alpha_*}(d_1(t) - (1-\alpha_*)a(t)).$$

Indeed, according to Fig. 5, the point m with coordinates $(b, 1)$ lies on the straight line passing through the points with coordinates $(a, 0)$ and (d_1, α_*); hence, this line is described by the equation $\frac{\mu}{\alpha_*} = \frac{x-a}{d_1-a}$.

Consider the issue regarding the possibilistic confidence interval of the modal value of the fuzzy output signal with the same possibilistic confidence level α_*.

Proposition 10. *Under the hypotheses of Theorem 5, let a triangular fuzzy signal with support* $(a(t), c(t))$ *be supplied, with a possibilistic confidence level* $\alpha_* \in [0.7, 1)$, *to the input of the dynamic system described by differential equation* (2), *and let its modal value for an arbitrary* $t \geqslant 0$ *and this possibilistic confidence level lie in a possibilistic confidence interval* $[d_1(t), d_2(t)]$. *Then the possibilistic confidence interval for the modal value of the fuzzy output signal has the form* $[D_1(t), D_2(t)]$, *where* $D_1(t) = (1-\alpha_*)A(t) + \alpha_* B(t)$, $D_2(t) = \alpha_* B(t) + (1-\alpha_*)C(t)$, *and* $A(t)$, $B(t)$, *and* $C(t)$ *are given by Theorem 5. Moreover, the modal value* $B(t)$ *is determined using the value* $b(t)$ *from Remark 4.*

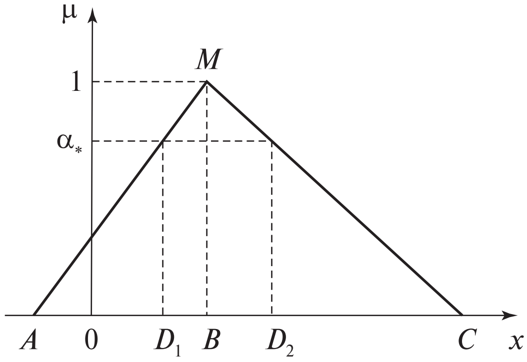

Fig. 6. A possibilistic confidence interval for the modal value of a fuzzy output signal.

Indeed, by Theorem 5 and Remark 4, the fuzzy output signal has the triangular type $(A(t), B(t), C(t))$. Then, according to Fig. 6, we obtain the possibilistic confidence interval $[D_1(t), D_2(t)]$ for the modal value of the fuzzy output signal for fixed $t > 0$ with a possibilistic confidence level α_*.

According to Fig. 6, $D_1 = (1 - \alpha_*)A + \alpha_* B$ is the abscissa of the intersection point of the line $[A, M]$, described by the equation $\mu = (x - A)(B - A)^{-1}$ (see Example 1), and the line $\mu = \alpha_*$. In turn, the point $D_2 = \alpha_* B + (1 - \alpha_*)C$ is the abscissa of the intersection point of the line $[M, C]$, described by the equation $\mu = (x - C)(B - C)^{-1}$, and the line $\mu = \alpha_*$.

Note that for practical use of the results of Sections 4 and 5, it is convenient to model the fuzzy input signal in the form $\tilde{y}(t) = \tilde{r}g(t)$, where $\tilde{r} \in J$ and $g(t)$ are a triangular fuzzy number and a real-valued function, respectively. Alternatively, one should choose a fuzzy signal model with $\tilde{y}(t)$ for $t \geqslant 0$ and $\tilde{y}(t) = \tilde{y}(0)$ for $t \leqslant 0$.

6. CONCLUSIONS

The main results of this paper concern fuzzy dynamic systems described by linear differential equations of order n with constant coefficients under the assumption of the continuity and boundedness of a fuzzy input signal (Section 3). They are based on a development of the Green's function method for the case of fuzzy differential equations. An important part of the research has been devoted to the smoothness of solutions. Also note the above result on the triangular type of the fuzzy output signal of a dynamic system receiving a fuzzy signal at its input.

The applications in Section 4 have illustrated the use of the general theoretical constructs in radio circuits with fuzzy input signals. They further refine some results of Section 3 for the case of dynamic systems described by first- and second-order differential equations. Section 5 plays a significant role as well: by analogy with confidence intervals of mathematical statistics, the concept of a possibilistic confidence interval has been introduced and used therein.

The approach presented in this paper is an alternative to the conventional analysis of linear dynamic systems with constant coefficients, which involves the frequency response, the direct and inverse Fourier transform, and the Laplace transform.

Note that the developed approach can be extended to the case of periodic and almost periodic signals and can be fruitful in the study of boundary value problems for fuzzy differential equations.

Many results can be modified to the case of generalized fuzzy derivatives.

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The Asymptotic Behavior of Anisotropic DOF Controller at Infinitesimal Values of Upper Bound of Input's Mean Anisotropy

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Abstract—An asymptotic formula is obtained for the optimal anisotropic controller for linear discrete time-invariant system driven by random disturbance with infinitesimal mean anisotropy. The result is accompanied with the asymptotic formula for the anisotropic norm of the closed-loop system. An upper bound is computed for the mean anisotropy at which the optimal anisotropic controller can be approximated by the $\mathcal{H}_2$-optimal controller with loss in performance gain less than a given threshold.

Keywords: anisotropy-based theory, linear discrete time invariant systems, optimal controller design, asymptotic behavior

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1. INTRODUCTION

The optimal control design problem remains to be one of the most relevant problems in control theory. To design the optimal controller, one needs to determine a plant to control, a family of control laws with parameters to be adjusted, and a certain function that quantitatively specifies the quality of the closed-loop system performance. Such a criterion is chosen based on the control objectives and the operating conditions of the system. The dynamic output feedback (DOF) control, in both of its versions – strictly proper (causal) and non-strictly proper, is frequently used to solve the linear quadratic control problems. In practice, the measurements used to define a control actions contain random noise, in most cases with somewhat uncertain statistical parameters. When the disturbances driven the linear system are Gaussian white noise, and the quadratic loss function is defined as the performance gain, the corresponding control problem is referred to as the linear-quadratic Gaussian (LQG) problem. A significant amount of publications exists on this topic [1–4]. Nevertheless, usually, external disturbance is rarely happen to be white noise, for which case the LQG controller loses its efficiency.

In the period from the late 60-s to the early 80-s, the cornerstone of the $\mathcal{H}_\infty$-theory has been developed [5–8]. This theory addresses the optimal control design under assumption that the disturbances are the square-integrable signals, and the L_2 operator norm is used as performance gain of the system. However, the optimal $\mathcal{H}_\infty$-controller is too conservative in the sense that it only performs in the best manner when the inputs are of the worst case corresponding to the maximum value of the closed-loop system performance gain.

In the 1990-s, the so-called anisotropy-based control theory has been developed by I.G. Vladimirov as an attempt to generalize $\mathcal{H}_2$- and $\mathcal{H}_\infty$-control approaches to optimal control design [9]. The fundamental concepts of the anisotropy of a random vector, the mean anisotropy of a sequence of random vectors, and the anisotropic norm of a system were introduced [10, 11]. The anisotropy of random vector is defined as a measure of the divergence (in informational sense) of the distribution of this vector with respect to a uniform distribution on the unit sphere. Later, it was re-defined as the divergence of the vector's distribution from isotropic Gaussian distributions [12]. Subsequently, the anisotropy-based theory apparatus has been also developed to solve the analysis problems, optimal control and filtering problems [13–16].

In [17], the problem of the asymptotic representation of the anisotropic norm of a linear discrete time invariant (LDTI) system has been solved when mean anisotropy upper bound was infinitesimal (the so-called left asymptotic) or infinitely large (the right asymptotic). Based on the aforementioned results, the asymptotic formula of the optimal anisotropy-based filter in terms of its deviation from the $\mathcal{H}_2$-optimal filter for infinitesimal values of mean anisotropy was presented in [18]. The formula of the mean anisotropy maximum upper bound threshold at which the $\mathcal{H}_2$-optimal filter approximates the anisotropy-based filter with a given accuracy is also obtained. In the subsequent study [19], a solution for the special case of the similar anisotropy-based control problem is presented. All the results obtained for the left asymptotic of anisotropic filters and controllers form the basis for the present study.

This paper presents a solution for the general case of the left asymptotic representation for the optimal anisotropic controller for an LDTI system. The first section provides a brief overview of the object of study, anisotropy-based control theory, and addresses the methods for the optimal $\mathcal{H}_2$ and anisotropic controllers design. In the second section, a solution of the general optimal anisotropic control problem is given. The third section of the article addresses the asymptotic representation of the DOF anisotropic controller in a general form.

2. BACKGROUND

2.1. Fundamental Notations

The following notations are used in the paper: $\mathbb{R}^n$ – a set of n-dimensional real vectors; $\mathbb{R}^{n\times m}$ – a set of $(n\times m)$-dimensional real matrices; $\mathbb{C}$ – a set of complex numbers; $\mathbb{L}_2^n$ – a set of n-dimensional real valued square integrated random vectors; $\mathcal{H}_\infty^{p\times m}$ – the Hardy space of $(n \times m)$-dimensional complex-valued matrix functions, which are analytical in a unit circle $\mathbb{C}_\odot = \{z \in \mathbb{C} : |z| < 1\}$ and have limited $\mathcal{H}_\infty$–norm $\|F\|_\infty = \sup_{|z|<1} \overline{\sigma}(F(z))$; $\overline{\sigma}(X) = \max \sqrt{\lambda(X^*X)}$ – maximum singular number of the matrix X; $\lambda(X)$ – eigenvalue of matrix X; $X^* = \overline{X}^T$ – Hermitian conjugate matrix to X; $\mathcal{H}_2^{p\times m}$ – the Hardy space of analytical in a unit circle $\mathbb{C}_\odot$ complex-valued matrix functions $F(z) = \sum_{k=0}^{+\infty} f_k z^k$ with limited $\mathcal{H}_2$-norm $\|F\|_2 = \left(\sum_{k=0}^{+\infty} \operatorname{tr}(f_k f_k^T)\right)^{1/2}$, where $f_k \in \mathbb{R}^{p\times m}$.

2.2. Research Object

The research object of the paper is linear discrete time invariant system F with state space realization

$$x_{k+1} = Ax_k + B_w w_k + B_u u_k, \quad k = 0, 1, \dots, \tag{1}$$

where $x_k \in \mathbb{L}_2^{n_x}$ – state vector, $x_0 = 0$; $w_k \in \mathbb{L}_2^{n_w}$ is the input disturbance vector; $u_k \in \mathbb{L}_2^{n_u}$ is the input control vector. Controlled output of system (1) denoted as a vector $z_k \in \mathbb{L}_2^{n_z}$ is determined by

$$z_k = C_z x_k + D_z u_k. \tag{2}$$

Sensor measurements are used to determine the control input u_k of the system F. This data is represented as a sequence of vectors $y_k \in \mathbb{L}_2^{n_y}$ described as follows:

$$y_k = C_y x_k + D_y w_k. \tag{3}$$

Matrices A, B_w, B_u, C_z, D_z, C_y, D_y are known real matrices of corresponding dimensions. The system of equations (1)–(3) is associated with transfer function $T_{yw}(z) = D_y + C_y(zI_{n_x} - A)^{-1}B_w$, which is described by four matrices

$$T_{yw} \sim (A, B_w, C_y, D_y), \tag{4}$$

and transfer function $T_{zu}(z) = D_z + C_z(zI_{n_x} - A)^{-1}B_u$ with quadruple of matrices

$$T_{zu} \sim (A, B_u, C_z, D_z). \tag{5}$$

The general formulation of the DOF control problem is to find a controller K of the form

$$K \sim \begin{cases} h_{k+1} = \widehat{A}h_k + \widehat{B}y_k, \\ \quad u_k = \widehat{C}h_k + \widehat{D}y_k \end{cases} \tag{6}$$

with state vector $h_k \in \mathbb{L}_2^{n_x}$, input vector $y_k \in \mathbb{L}_2^{n_y}$ and output vector $u_k \in \mathbb{L}_2^{n_u}$, which provides the fulfillment of some quality criterion. In (6), the matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{D}$ are to be derived. The following section presents basic information regarding two controller types based on its quality criteria: the $\mathcal{H}_2$ controller which minimizes the trace of the state or controlled output covariance matrix of the closed-loop system and the anisotropy-based controller, which minimizes the anisotropic norm of the linear operator mapping external disturbances to the controlled output of the closed-loop system.

2.3. $\mathcal{H}_2$-Optimal Control

To facilitate further expositions, we introduce the following matrices :

$$U_L = (D_z^{\mathrm{T}}D_z + B_u^{\mathrm{T}}\widehat{P}_\star B_u)^{-1}, \quad U_R = -(D_z^{\mathrm{T}}C_z + B_u^{\mathrm{T}}\widehat{P}_\star A), \quad U_\star = U_L U_R, \tag{7}$$

$$V_L = -(A\widehat{Q}_\star C_y^{\mathrm{T}} + B_w D_y^{\mathrm{T}}), \quad V_R = (D_y D_y^{\mathrm{T}} + C_y\widehat{Q}_\star C_y^{\mathrm{T}})^{-1}, \quad V_\star = V_L V_R. \tag{8}$$

The optimal $\mathcal{H}_2$-control problem is to find a controller that minimizes the $\mathcal{H}_2$ norm of the closed-loop system. Consider the linear discrete time invariant system (1), controlled output (2) and measured output (3) with external random disturbance w_k distributed normally with zero mean $\mathbf{E}[w_k] = 0$ and an identity covariance matrix $\mathbf{E}[w_k w_k^{\mathrm{T}}] = I_{n_w}$. Consider the problem of designing $\mathcal{H}_2$-optimal controller of the form (6). Thus, we have the following solution to the stated optimal $\mathcal{H}_2$-optimal control problem [2]:

$$\begin{aligned}
\widehat{A}_\star &= A + B_u U_\star + V_\star C_y - B_u\widehat{D}_\star C_y, \\
\widehat{B}_\star &= B_u\widehat{D}_\star - V_\star, \\
\widehat{C}_\star &= U_\star - \widehat{D}_\star C_y, \\
\widehat{D}_\star &= -U_L(D_z^{\mathrm{T}}C_z\widehat{Q}_\star C_y^{\mathrm{T}} + B_u^{\mathrm{T}}\widehat{P}_\star A\widehat{Q}_\star C_y^{\mathrm{T}} + B_u^{\mathrm{T}}\widehat{P}_\star B_w D_y^{\mathrm{T}})V_R,
\end{aligned}$$

where $\widehat{P}_\star$ and $\widehat{Q}_\star$ are the stabilizing solutions of algebraic Riccati equations (for control and filtering, respectively):

$$\begin{aligned}
\widehat{P}_\star &= A^{\mathrm{T}}\widehat{P}_\star A + C_z^{\mathrm{T}}C_z - U_R^{\mathrm{T}}U_\star, \\
\widehat{Q}_\star &= A\widehat{Q}_\star A^{\mathrm{T}} + B_w B_w^{\mathrm{T}} - V_\star V_L^{\mathrm{T}}.
\end{aligned}$$

The matrix V_* is related to the coefficient matrix of the Kalman filter (as part of the $\mathcal{H}_2$-controller) with respect to the update sequence, while U_* is responsible for forming the control action based on the filter's estimate of the current state of the plant (due to the separation principle inherent in Linear-Quadratic-Gaussian (LQG) control).

Next, we consider the fundamental concepts and principles of anisotropy-based theory upon which the solution to the problem presented in the article is based.

2.4. Anisotropic Norm

The synthesis of an $\mathcal{H}_2$-optimal controller typically assumes that the input of the system under research is a Gaussian white noise. In practice, external disturbances affecting systems are frequently correlated (and not necessarily Gaussian) noise, and its statistical characteristics are often imprecisely known.

Let us assume that the input of the system (1) is a random disturbance in the form of a stationary sequence of mutually independent random vectors $W = (w_k)_{0\leqslant k<+\infty}$, $w_k \in \mathbb{L}_2^{n_w}$, whose properties deviate from the standard normal distribution. To characterize the deviation of the random vector's distribution from the normal distribution, the concepts of anisotropy of the random vector and the mean anisotropy of a sequence of random vectors is used within the framework of anisotropy-based theory.

Definition 1 [12]. Anisotropy $\mathbf{A}(W)$ of n_w-dimensional random vector W is a nonnegative function defined by the following expression:

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{D}(f\|p_{n_w,\lambda}),$$

where $\mathbf{D}(f\|p_{n_w,\lambda})$ is the relative entropy (Kulback-Leibler information divergence) of probability density function (pdf) f regarding to the Gaussian pdf $p_{n_w,\lambda}$ with zero mean and scalar covariance matrix λI_{n_w}, $\lambda > 0$, and $\mathbf{h}(W) = -\int_{\mathbb{R}^{n_w}} f(w)\ln f(w)dw$ is the differential entropy of W.

Characterizing a sequence of random vectors using the concept of anisotropy of random vector defined above is not feasible, as it tends towards infinity with an increasing number of sequence elements. Therefore, the concept of mean anisotropy for a sequence of random vectors was introduced.

Definition 2 [12]. The mean anisotropy of a (stationary ergodic) sequence $W = (w_k)_{0\leqslant k<+\infty}$ is defined as limit

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N},$$

where $W_{s:t} = (w_s^{\mathrm{T}},\ w_{s+1}^{\mathrm{T}},\ \ldots,\ w_t^{\mathrm{T}})^{\mathrm{T}}$ is the vector formed by the vectors of the sequence fragment $(w_k)_{s\leqslant k\leqslant t}$.

As is known [20], the vectors of a stationary Gaussian sequence of random disturbances $W = (w_k)_{0\leqslant k<+\infty}$ can be represented as

$$w_j = \sum_{k=0}^{+\infty} g_k v_{j-k},$$

where $V = (v_k)_{0\leqslant k<+\infty}$ is a sequence of independent n_w-dimensional random vectors with a standard normal distribution; g_k is the impulse response of the generating filter, and $G(z) \in \mathcal{H}_2^{n_w\times n_w}$ is the transfer function of the generating filter with the sequence of vectors V as input and the sequence W as output. Since the sequence of vectors W is generated by the filter G, the notation $\overline{\mathbf{A}}(G)$ can be used to denote the mean anisotropy $\overline{\mathbf{A}}(W)$ of the sequence. It has been shown (see [11, formula (4) and Lemma 1]) that the mean anisotropy $\overline{\mathbf{A}}(G)$ of a sequence of random vectors W generated by the shaping filter can be computed using the following formula:

$$\overline{\mathbf{A}}(G) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\left(\frac{n_w}{\|G\|_2^2}\widehat{G}(w)(\widehat{G}(w))^{\star}\right)dw,$$

where $\widehat{G}(w) = \lim_{r\to 1-0} G(re^{iw})$, $w \in [-\pi,\pi)$, $i^2 = -1$.

One of the system response measures for system F of the form (4) in case of input disturbance represented as a sequence of vectors W with mean anisotropy $\overline{\mathbf{A}}(G) \leqslant a$ is the anisotropic norm of the system [11], defined as follows:

$$F|||F|||_a = \sup_{G \in \mathbf{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{9}$$

where $\mathbf{G}_a = \{G \in \mathcal{H}_2^{n_w \times n_w} : \overline{\mathbf{A}}(G) \leqslant a\}$ is a set of generating filters with a bounded mean anisotropy of the sequence W.

To compute the anisotropic norm, it is necessary to determine the parameters of the generating filter G that provides the supremum in the expression (9). This filter is called the worst-case shaping filter and has the representation [11, formulas (32), (33)]

$$G \sim \left[\begin{array}{c|c} A + BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right] \tag{10}$$

with state vector x_k, input vector v_k and output vector w_k. Next, the formulation of the lemma concerning the computation of the anisotropic norm for a linear discrete-time invariant system is presented.

Lemma 1 [11, Lemma 3]. *Given a stable linear discrete time-invariant system F of the form (4), defined by the matrix quadruple A, B, C, D. For any $a > 0$, there exists a unique pair (q, R), where $q \in (0, \|F\|_\infty^{-2})$ is a scalar parameter that satisfies the equation*

$$-\frac{1}{2} \ln\det \frac{n_w \Sigma}{\operatorname{tr}(LPL^{\mathrm{T}} + \Sigma)} = a, \tag{11}$$

and $R \in \mathbb{R}^{n_x \times n_x}$ is a matrix that is a stabilizing solution of the Riccati equation

$$\begin{aligned} R &= A^{\mathrm{T}} R A + q C^{\mathrm{T}} C + L^{\mathrm{T}} \Sigma^{-1} L, \\ \Sigma &= (I_{n_w} - q D^{\mathrm{T}} D - B^{\mathrm{T}} R B)^{-1}, \\ L &= \Sigma (B^{\mathrm{T}} R A + q D^{\mathrm{T}} C). \end{aligned}$$

Furthermore, the anisotropic norm of the system F is computed as

$$F|||F|||_a = \left(\frac{1}{q} \left(1 - \frac{n_w}{\operatorname{tr}(LPL^{\mathrm{T}} + \Sigma)} \right) \right)^{1/2}, \tag{12}$$

where matrix $P \in \mathbb{R}^{n_x \times n_x}$ satisfies Lyapunov equation

$$P = (A + BL) P (A + BL)^{\mathrm{T}} + B \Sigma B^{\mathrm{T}}. \tag{13}$$

The aforementioned concepts and principles of anisotropy-based control theory will be subsequently used in addressing the problem of determining the asymptotic representation of a general anisotropy-based controller and the maximum anisotropy threshold below which the anisotropy-based controller can be approximated by an $\mathcal{H}_2$-controller with a specified accuracy.

3. OPTIMAL ANISOTROPIC CONTROLLER

The optimal anisotropy-based control problem (6) for a linear discrete-time invariant system (5) with a measured output (3) is considered in this section. In [19], a solution is presented for the asymptotic representation problem with small values of the mean anisotropy a for a static state controller $u_k = Kx_k$. In a similar way, asymptotic representation problem for a dynamic anisotropy-based output controller is solved.

Initially, the representation of the original system with a dynamic controller is expressed as the result of substituting the controller's expression (6) into the system (1)–(3):

$$\mathcal{L}(F,K) \sim \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right], \tag{14}$$

where matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$ have the form

$$\overline{A} = \begin{pmatrix} A + B_u\widehat{D}C_y & B_u\widehat{C} \\ \widehat{B}C_y & \widehat{A} \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} B_w + B_u\widehat{D}D_y \\ \widehat{B}D_y \end{pmatrix},$$
$$\overline{C} = \begin{pmatrix} C_z + D_z\widehat{D}C_y & D_z\widehat{C} \end{pmatrix}, \quad \overline{D} = D_z\widehat{D}D_y.$$

It is assumed that the input disturbance vectors, denoted as w_k, of the system under consideration are the output of a worst-case generating filter of the form (10) and can be represented as

$$w_k = L_x x_k + L_h h_k + \Sigma^{1/2} v_k.$$

The Riccati equation from the lemma no.1 of calculating the anisotropic norm (11)–(13) for the system (14) has the form

$$R = \overline{A}^{\mathrm{T}} R \overline{A} + q\overline{C}^{\mathrm{T}}\overline{C} + L^{\mathrm{T}}\Sigma^{-1}L, \tag{15}$$
$$\Sigma = (I_{n_w} - q\overline{D}^{\mathrm{T}}\overline{D} - \overline{B}^{\mathrm{T}}R\overline{B})^{-1}, \tag{16}$$
$$L = (L_x \quad L_h) = \Sigma(\overline{B}^{\mathrm{T}}R\overline{A} + q\overline{D}^{\mathrm{T}}\overline{C}). \tag{17}$$

Thus, the control problem is decomposed into two subproblems: determining the worst-case generating filter for the closed-loop system (14), and synthesizing an optimal dynamic anisotropy-based controller in the form of an LQG controller that minimizes the trace of the covariance matrix of the regulated output of the closed-loop system (14) when affected by the worst-case noise. In [16], a solution to a similar control problem is presented for the case $\widehat{D} = 0$. After performing a similar analysis for the controller case (6), one has that the matrices $\widehat{A}$ and $\widehat{B}$ satisfy the following formulas:

$$\widehat{A} = A + B_wM + B_u\widehat{C} + (B_u\widehat{D} - \Lambda)(C_y + D_yM), \quad \widehat{B} = \Lambda, \tag{18}$$

where

$$M = L_x + L_h, \tag{19}$$
$$\begin{aligned} S = {} & (A + B_wL_x + B_u\widehat{D}D_yL_x)S(A + B_wL_x + B_u\widehat{D}D_yL_x)^{\mathrm{T}} \\ & + (B_w + B_u\widehat{D}D_y)\Sigma(B_w + B_u\widehat{D}D_y)^{\mathrm{T}} - \Lambda\Theta\Lambda^{\mathrm{T}}, \end{aligned} \tag{20}$$
$$\Theta = (C_y + D_yL_x)S(C_y + D_yL_x)^{\mathrm{T}} + D_y\Sigma D_y^{\mathrm{T}}, \tag{21}$$
$$\begin{aligned} \Lambda = {} & ((A + B_wL_x + B_u\widehat{D}C_y + B_u\widehat{D}D_yL_x)S(C_y + D_yL_x) \\ & + (B_w + B_u\widehat{D}D_y)\Sigma D_y^{\mathrm{T}})\Theta^{-1}. \end{aligned} \tag{22}$$

To determine the unknown matrices $\widehat{C}$ and $\widehat{D}$ of the controller, the methodology for solving synthesis problems of dynamic $\mathcal{H}_2$-optimal output controllers presented in [2] should be employed.

Therefore, one expresses the system (1)–(3) with the dynamic controller (6) and the worst-case generating filter (10) in the form

$$\begin{cases} \widetilde{x}_{k+1} = \widetilde{A}\widetilde{x}_k + \widetilde{B}_w v_k + \widetilde{B}_u u_k, \\ \widetilde{z}_k = \widetilde{C}_z \widetilde{x}_k + \widetilde{D}_z u_k, \\ \widetilde{y}_k = \widetilde{C}_y \widetilde{x}_k + \widetilde{D}_y v_k, \end{cases} \tag{23}$$

where state vector $\widetilde{x}$ includes the state vector x_k of the initial system (1) and state vector h_k of controller (6), i.e. $\widetilde{x} = (x_k^{\mathrm{T}} \quad h_k^{\mathrm{T}})^{\mathrm{T}}$, $\widetilde{z}_k = z_k$, $\widetilde{y} = (y_k^{\mathrm{T}} \quad h_k^{\mathrm{T}})^{\mathrm{T}}$, and system matrices have the form

$$\widetilde{A} = \begin{pmatrix} A + B_w L_x & B_w L_h \\ \widehat{B} C_y + \widehat{B} D_y L_x & \widehat{A} + \widehat{B} D_y L_h \end{pmatrix}, \ \widetilde{B}_w = \begin{pmatrix} B_w \Sigma^{1/2} \\ \widehat{B} D_y \Sigma^{1/2} \end{pmatrix}, \ \widetilde{B}_u = \begin{pmatrix} B_u \\ 0 \end{pmatrix}, \tag{24}$$

$$\widetilde{C}_z = (C_z \quad 0), \quad \widetilde{D}_z = D_z, \tag{25}$$

$$\widetilde{C}_y = \begin{pmatrix} C_y + D_y L_x & D_y L_h \\ 0 & I_{n_x} \end{pmatrix}, \quad \widetilde{D}_y = \begin{pmatrix} D_y \Sigma^{1/2} \\ 0 \end{pmatrix}. \tag{26}$$

Consequently, the desired control u_k is determined by the following formula:

$$u_k = \widetilde{N}\widetilde{y}_k,$$

where $\widetilde{N} = (\widehat{D} \quad \widehat{C})$.

Applying the $\mathcal{H}_2$-optimal control method for the system (23) yields

$$\widetilde{N} = -\widetilde{U}_L(\widetilde{D}_z^{\mathrm{T}}\widetilde{C}_z Q_\star \widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}} P_\star \widetilde{A} Q_\star \widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}} P_\star \widetilde{B}_w \widetilde{D}_y^{\mathrm{T}})\widetilde{V}_R, \tag{27}$$

where matrices $\widetilde{U}_L$ and $\widetilde{V}_R$ are introduced analogously to (7) and (8) by replacing the corresponding matrices with similar matrices marked with a tilde $\widehat{P}_\star$, $\widehat{Q}_\star$ to $P_\star$, $Q_\star$, and the matrices $P_\star$ and $Q_\star$ satisfy equations

$$P_\star = \widetilde{A}^{\mathrm{T}} P_\star \widetilde{A} + \widetilde{C}_z^{\mathrm{T}} \widetilde{C}_z - \widetilde{U}_R^{\mathrm{T}} \widetilde{U}_\star, \tag{28}$$

$$Q_\star = \widetilde{A} Q_\star \widetilde{A}^{\mathrm{T}} + \widetilde{B}_w \widetilde{B}_w^{\mathrm{T}} - \widetilde{V}_\star \widetilde{V}_L^{\mathrm{T}}. \tag{29}$$

From (27), it follows that desired controller matrices $\widehat{C}$ and $\widehat{D}$ are expressed as follows:

$$\widehat{C} = \widetilde{N}\begin{pmatrix} 0 \\ I_{n_x} \end{pmatrix}, \quad \widehat{D} = \widetilde{N}\begin{pmatrix} I_{n_y} \\ 0 \end{pmatrix}.$$

Thus, the matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and $\widehat{D}$ of the desired dynamical anisotropy-based output controller are uniquely determined by the system of equations (18), (27)–(29).

The subsequent section details a solution to the problem of determining an asymptotic representation for the derived optimal anisotropy-based controller as $a \to 0^+$.

4. ASYMPTOTIC REPRESENTATION OF CONTROLLER

The next step in solving the stated problem is to derive the formulas for the asymptotic representation of the obtained anisotropy-based dynamical controller. To achieve this goal, it is necessary to determine the components of the matrix decomposition for the controller, the system (23), and all related matrices. Let us express the matrices of the system (23) as the following series:

$$X(a) = \sum_{k=0}^{n} X_k a^{k/2} + o(a^{n/2}), \quad a \to 0 + 0, \tag{30}$$

where X denotes *any* variable, except for the matrices A, B_w, B_u, C_z, C_y, D_z, D_y of the initial system, which, by the problem statement, are independent of a (for example, the matrix Σ depends on a, so the representation (30) applies to it; i.e. $\Sigma(a) = \Sigma_0 + \Sigma_1\sqrt{a} + \Sigma_2 a + o(a)$, if we set $n = 2$). Note that $\widetilde{X}(\sqrt{a}) \doteq X(a)$ has to be a sufficiently smooth function of its argument $\sqrt{a}$. All matrices obtained from sums and products of individual matrices that can be represented in the form of (30) also have a similar form.

In similar way as in case of the static control problem, to determine the zero components of the expansions of matrix functions, it is necessary to determine the function values when $a = 0$. This case corresponds to the matrices of the $\mathcal{H}_2$-controller. For convenience, let us introduce the auxiliary matrix $\Upsilon = -\widetilde{U}_L^{-1}\widetilde{N}\widetilde{V}_R^{-1}$. All variables X_0 corresponding to the case $a = 0$ are not presented here, as they are trivially obtained by substituting the values $q = 0$, $L = 0$, and $\Sigma = I_{n_w}$ into all the necessary formulas.

Based on the results presented in [18, 19], the second-order terms in the expansions of the matrix functions $R(a)$, $\Sigma(a)$, $L(a)$, and $q(a)$ are expressed as follows:

$$\begin{aligned} &q_1^2 = 4n_w/\left(2n_w\mathrm{tr}(\overline{B}_0^{\mathrm{T}}\mathcal{Q}\overline{A}_0\overline{P}_0\overline{A}_0^{\mathrm{T}}\mathcal{Q}\overline{B}_0 + n_w(\overline{B}_0^{\mathrm{T}}\mathcal{Q}\overline{B}_0)^2) - \mathrm{tr}^2(\overline{B}_0^{\mathrm{T}}\mathcal{Q}\overline{B}_0)\right), \\ &R_1 = q_1\mathcal{Q}, \quad \Sigma_1 = \overline{B}_0^{\mathrm{T}}R_1\overline{B}^0, \quad L_1 = \overline{B}_0^{\mathrm{T}}R_1\overline{A}_0, \end{aligned} \tag{31}$$

where matrices $\mathcal{Q}$ $\overline{P}_0$ satisfy equations

$$\mathcal{Q} = \overline{A}_0^{\mathrm{T}}\mathcal{Q}\overline{A}_0 + \overline{C}_0^{\mathrm{T}}\overline{C}_0, \quad \overline{P}_0 = \overline{A}_0\overline{P}_0\overline{A}_0^{\mathrm{T}} + \overline{B}_0\overline{B}_0^{\mathrm{T}}.$$

With that, one obtains the following expressions for the first components of the non-zero matrices in the closed system (note the dependence of these matrices on various X_0 and X_1):

$$\widetilde{A}_1 = \begin{pmatrix} B_wL_{x,1} & B_wL_{h,1} \\ \widehat{B}_1C_y + \widehat{B}_0D_yL_{x,1} & \widehat{B}_0D_yL_{h,1} \end{pmatrix}, \quad \widetilde{B}_{w,1} = \begin{pmatrix} B_w\Sigma_1^{1/2} \\ \widehat{B}_1D_y + \widehat{B}_0D_y\Sigma_1^{1/2} \end{pmatrix},$$

$$\widetilde{C}_{y,1} = \begin{pmatrix} D_yL_{x,1} & D_yL_{h,1} \\ 0 & 0 \end{pmatrix}, \quad \widetilde{D}_{y,1} = \begin{pmatrix} D_y\Sigma_1^{1/2} \\ 0 \end{pmatrix}.$$

Having derived derivations of the anisotropy-based controller matrices, one obtains equations for the first components of the anisotropy-based controller matrices:

$$\widehat{A}_1 = B_wM_1 + B_u\widehat{C}_1 + B_u\widehat{D}_1C_y - \Lambda_1C_y + (B_u\widehat{D}_0 - \Lambda_0)D_yM_1, \quad \widehat{B}_1 = \Lambda_1,$$

$$\widehat{C}_1 = \widetilde{N}_1\begin{pmatrix} 0 \\ I_{n_x} \end{pmatrix}, \quad \widehat{D}_1 = \widetilde{N}_1\begin{pmatrix} I_{n_y} \\ 0 \end{pmatrix}.$$

Although the expressions for the second terms in the expansion (30) of different matrix variables are quite complex, they are all obtained in a similar manner and share a similar structure. Therefore, to conserve space, we will present only the general principle of their derivation, using the matrix Υ as an illustrative example. According to the established notation,

$$\Upsilon = \widetilde{D}_z^{\mathrm{T}}\widetilde{C}_zQ_\star\widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}}P_\star\widetilde{A}Q_\star\widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}}P_\star\widetilde{B}_w\widetilde{D}_y^{\mathrm{T}}, \tag{32}$$

where all forming matrices depend on a. Therefore, for the first term in its decomposition according to formula (30), we have the following representation:

$$\Upsilon_0 = \widetilde{D}_{z,0}^{\mathrm{T}}\widetilde{C}_{z,0}Q_{\star,0}\widetilde{C}_{y,0}^{\mathrm{T}} + \widetilde{B}_{u,0}^{\mathrm{T}}P_{\star,0}\widetilde{A}_0Q_{\star,0}\widetilde{C}_{y,0}^{\mathrm{T}} + \widetilde{B}_{u,0}^{\mathrm{T}}P_{\star,0}\widetilde{B}_{w,0}\widetilde{D}_{y,0}^{\mathrm{T}}, \tag{33}$$

and for the second term – as follows:

$$\Upsilon_1 = \sum_{\substack{i,j,k,l\geqslant 0\\ i+j+k+l=1}} \widetilde{D}_{z,i}^{\mathrm{T}}\widetilde{C}_{z,j}Q_{\star,k}\widetilde{C}_{y,l}^{\mathrm{T}} + \sum_{\substack{i,j,k,l,m\geqslant 0\\ i+j+k+l+m=1}} \widetilde{B}_{u,i}^{\mathrm{T}}P_{\star,j}\widetilde{A}_kQ_{\star,l}\widetilde{C}_{y,m}^{\mathrm{T}} + \sum_{\substack{i,j,k,l\geqslant 0\\ i+j+k+l=1}} \widetilde{B}_{u,i}^{\mathrm{T}}P_{\star,j}\widetilde{B}_{w,k}\widetilde{D}_{y,l}^{\mathrm{T}}. \tag{34}$$

It is easy to notice the general principle behind the formation of the matrix Υ_1: among all possible index combinations forming its matrices, only one index in each matrix product takes the value of 1. Similarly, to write out the third term Υ_2, it is necessary to consider all possible combinations of indices whose sum equals 2 (the total number of terms in this case will be 35). Therefore, one can assume that all the necessary matrices in the representation (30) have been written out; i.e. the asymptotic representation of the dynamic anisotropy-based controller has been determined with the specified accuracy as $a \to 0+0$. The obtained results are summarized in the following statement:

Theorem 1. *Consider a linear time-invariant system of the form* (1)–(3) *and a dynamical controller of the form* (6) *in an output feedback configuration. For small values of the mean anisotropy* $a \to 0+0$ *of the input disturbances, the following asymptotic expansions, given by equation* (30), *are valid for the matrices* $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, *and* $\widehat{D}$ *of the controller. The terms of the series are determined analogously to equations* (32)–(34) *for the matrix* Υ, *and its dependence on* a *is given by equation* (11).

The following section presents a solution to the problem of the asymptotic representation of the anisotropic norm for a closed-loop system with an obtained controller.

5. ASYMPTOTIC REPRESENTATION OF ANISOTROPIC NORM

The next step in problem solving is to obtain an asymptotic representation of the anisotropic norm of the closed-loop system with the obtained controller and to determine the maximum mean anisotropy level $a_{\max}$ at which the corresponding optimal anisotropy-based controller can be approximated by an $\mathcal{H}_2$-optimal controller with a specified accuracy level ε. Therefore, it is necessary to determine the first components of the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$. By determining the partial derivatives of these matrix functions with respect to $\sqrt{a}$ and substituting $a = 0$, we readily obtain the required first components of the expansions of the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$.

To obtain the asymptotic representation of the anisotropic norm, it is necessary to determine the second components of the matrix functions $R(a)$ and $\Sigma(a)$. Having determined the second partial derivatives of the matrices (15)–(17) with respect to $\sqrt{a}$, and substituting the zero value of the mean anisotropy a into them, one have

$$\begin{aligned} R_2 &= \overline{A}_0^{\mathrm{T}}R_2\overline{A}_0 + Y_{R_2} + Y_{R_2}^{\mathrm{T}},\\ \Sigma_2 &= \overline{B}_0^{\mathrm{T}}R_2\overline{B}_0 + Y_{\Sigma_2} + Y_{\Sigma_2}^{\mathrm{T}}, \end{aligned} \tag{35}$$

where $Y_{R_2} = q_1\left(\overline{A}_1^{\mathrm{T}}\mathcal{Q}\overline{A}_0 + \overline{C}_1^{\mathrm{T}}\overline{C}_0\right)$, $Y_{\Sigma_2} = q_1\left(\overline{B}_1^{\mathrm{T}}\mathcal{Q}\overline{B}_0 + \overline{D}_1^{\mathrm{T}}\overline{D}_0\right)$. Substituting the derived expansions into the series of matrix functions R, Σ, L, and P in formula (12) for the anisotropic norm, the asymptotic representation of the anisotropic norm for the system (14) as $a \to 0+$ is expressed as follows:

$$\mathcal{L}(F,K_\star)|\!|\!|\mathcal{L}(F,K_\star)|\!|\!|_a = \frac{\|\mathcal{L}(F,K_{\star,0})\|_2}{\sqrt{n_w}}\left(1 + \left(\sqrt{\frac{\Xi}{n_w}} + \frac{\mathrm{tr}(\Sigma_2)}{2q_1\|\mathcal{L}(F,K_{\star,0})\|_2^2}\right)\sqrt{a}\right) + o(\sqrt{a}), \tag{36}$$

where $\mathcal{L}(F, K_{\star,0})$ represents a system of the form (14), closed by the optimal controller at the mean anisotropy level $a = 0$, and Ξ is of the form

$$\Xi = \frac{n_w \|\mathcal{L}(F, K_{\star,0})\|_4^4 - \|\mathcal{L}(F, K_{\star,0})\|_2^4}{\|\mathcal{L}(F, K_{\star,0})\|_2^4}. \tag{37}$$

The formulas for $\|\cdot\|_4^4$ and $\|\cdot\|_2^4$ are known and can be found in [17].

The final step is to determine the maximum level of mean anisotropy for a specified accuracy level $\varepsilon = \overline{o}(\|\mathcal{L}(F, K_{\star,0})\|_2)$, with which the $\mathcal{H}_2$-optimal controller approximates the anisotropy-based controller. This condition takes the form $a \leqslant a_{\max}$, where $a_{\max}$ satisfies the inequality:

$$\left| \mathcal{L}(F, K_\star) \| \mathcal{L}(F, K_\star) \|_{a_{\max}} - \frac{\|\mathcal{L}(F, K_{\star,0})\|_2}{\sqrt{n_w}} \right| < \varepsilon \frac{\|\mathcal{L}(F, K_{\star,0})\|_2}{\sqrt{n_w}}. \tag{38}$$

Substituting the asymptotic representation formula (36) for the anisotropic norm into inequality (38), one obtain

$$a \leqslant a_{\max} = \varepsilon^2 \left(\sqrt{\frac{\Xi}{n_w}} + \frac{\operatorname{tr}(\Sigma_2)}{2q_1 \|\mathcal{L}(F, K_{\star,0})\|_2^2} \right)^{-2}. \tag{39}$$

The described above results of solving the problem of the asymptotic representation of the anisotropic norm are presented as the following theorem.

Theorem 2. *Consider a linear time-invariant system of the form* (1)–(3) *and a dynamical controller of the form* (6) *in an output feedback configuration. For small values of the mean anisotropy* $a \to 0 + 0$ *of input disturbances, the anisotropic norm of the system closed by the controller* (6) *admits the asymptotic representation* (36)*, and the maximum level of mean anisotropy at which the relative deviation of the anisotropic norm* $\mathcal{L}(F, K_\star) \| \mathcal{L}(F, K_\star) \|_a$ *from the scaled* $\mathcal{H}_2$*-norm of the closed-loop system does not exceed a specified threshold* ε*, is determined by formula* (39)*, where* q_1*,* Σ_2*, and* Ξ *are defined according to formulas* (31)*,* (35)*, and* (37)*.*

Obviously, the maximum mean anisotropy level is determined by the matrices of the original system. Earlier papers devoted to asymptotic representation of the anisotropy-based filter [18] and the static anisotropy-based controller [19] have clearly shown that its $\mathcal{H}_2$-optimal analogues sufficiently effectively approximate the anisotropy-based filter and controller, respectively, when the mean anisotropy of the input disturbance is small.

6. CONCLUSION

The paper addresses the problems of synthesis of a dynamic optimal anisotropy-based controller for linear discrete stationary systems and the determination of the maximum mean anisotropy threshold below which the anisotropy-based controller can be approximated by an $\mathcal{H}_2$-optimal controller with a specified level of accuracy. As a result of solving these problems, asymptotic representations were derived for all matrices of the anisotropy-based controller, the matrices of the closed-loop system, and its anisotropic norm for small values of mean anisotropy. Future research may address a similar anisotropy-based control problem for the right asymptotics, deriving asymptotic representations for the anisotropy-based controller and the closed-loop system norm as the mean anisotropy tends to infinity.

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Application of a Linear Pseudomeasurement Filter to Tracking and Positioning Based on Observations with Random Delays

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Abstract—The possibility of adaptation and effectiveness of a linear pseudomeasurement filter in a stochastic observation system model with random time delays between arriving observations and the factual state of a moving object are investigated. The method of pseudomeasurements is modified to combine the results of observations performed by several measuring devices located at different distances from the object and having different time delays. The filter is realized in a model that considers measurements of direction angles and range. Experimental computations are carried out for a model example describing the motion of an autonomous underwater vehicle that uses two stationary acoustic beacons for positioning.

Keywords: stochastic system with random observation delays, linear pseudomeasurements, extended Kalman filter (EKF), positioning, target tracking, sonars

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1. INTRODUCTION

State filtering methods in stochastic dynamic systems find application in various fields, including the control of autonomous underwater vehicles (AUVs) [1]. Along with unmanned aerial vehicles [2] and autonomous cars [3], this field is currently a topical source of research problems. The aquatic environment itself has some features unnecessary to be considered in surface motion problems. For instance, these are such factors as variable water temperature, salinity, and pressure [4]; flows [5] are quite interesting as well. In addition to affecting the moving object, water creates significant challenges for measuring devices. Let us consider only external observers, without discussing the onboard accelerometers and gyroscopes of AUVs. Then all available measuring devices are based on general physical laws and use acoustic signals, i.e., belong to acoustic sensors or sonars [6]. A fundamental feature of such devices is the significant effect of random delays in arriving data about the observed AUV state on the measurement accuracy. This effect also occurs in measuring devices using electromagnetic radiation. For instance, a radar observing an object at a distance of 1 km will receive its coordinates with a delay of about 10^{-7} s. Such values can be neglected. For a sonar with the sound velocity in water being 1500 m/s, the delay in determining the object's coordinates at a distance of 1 km will reach about 0.7 s; at a distance of 10 km, 7 (!) s. Even if the object is not moving fast, such values cannot be neglected. This effect must be taken into account in models oriented to high-velocity AUVs.

A stochastic dynamic observation system model incorporating the delay factor of the acoustic signal was proposed in [7, 8] and extended to the identification of unknown motion model parameters in [9, 10]. The relations for optimal Bayesian filtering [11] were derived for state and parameter estimation.

As in most applications, it is impractical to use universal filtering methods, such as the extended Kalman filter (EKF) [12], particle filters [13], and various types of sigma-point filters [14], or conditionally optimal and minimax Pugachev–Pankov filters [15, 16] of standard structure, or (even more so) optimal Bayesian filters: either the realization turns out to be very costly in computational sense or suboptimal algorithms exhibit a tendency to diverge. An exception could be the method of linear pseudomeasurements, which occupies an intermediate position between universal methods applicable to any model and special ones (i.e., those intended exclusively for a particular model). Although the idea of this method is quite universal, it should be applied to particular measurements, linearizing them. In underwater navigation problems, various sensors with indirect information about the object's position are used. Among them, note direction angle and range sensors [17].

The idea of pseudomeasurements itself has been known for a long time and seems to be a logical supplement or development of the EKF as the most popular suboptimal filtering method [12]. The EKF reproduces the structure of the linear Kalman filter [18], which is optimal for state filtering in a linear Gaussian observation system and also possesses a series of outstanding properties in various problems of robust and adaptive estimation and control. Formal adherence to the linear filter structure implies linearization. In the case of the EKF, this is linearization around the state prediction to obtain heuristic estimates of the state prediction covariance and the filtering estimate. The linearization of observations improved through some functional transformations, making combinations of observations more linear, was apparently first demonstrated for direction angle measurements in [19]. A modern practical setting was presented in [20]. By updating the model, a new quality was attributed to the pseudomeasurement filter in a series of research works initiated in [21].

This paper aims to adapt the EKF based on linear pseudomeasurements for a model with time delays. To this end, Section 2 proposes a more universal pseudomeasurement model, developing the classical method [19, 20]. In Section 3, this model is used to derive the filtering equations based on the EKF for the stochastic observation system model with time delays. Section 4 is devoted to a computational experiment of tracking the motion of an AUV, observed by two stationary acoustic beacons, towards a given target. In the Conclusions, we summarize the results, including possible shortcomings of the EKF-based method of linear pseudomeasurements and some ways to eliminate them.

2. FILTERING BY THE METHOD OF LINEAR PSEUDOMEASUREMENTS

2.1. System Model and Definition of Pseudomeasurements

In this paper, the following notation is adopted: $\mathsf{E}\{X\}$ means the mathematical expectation of a random vector X; $\operatorname{cov}(X, Y)$ is the covariance of X and Y; X' stands for the transpose of X.

By assumption, the motion of an autonomous AUV (denoted by $\mathcal{A}$) is described in a reference frame $Oxyz$ where the plane Oxy coincides with the sea surface and the axis Oz is directed downward and corresponds to depth (Fig. 1).

Let the coordinates of $\mathcal{A}$ at some fixed (e.g., initial) time instant form the vector $(X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}})'$ and, when considered as time-varying functions, the vector $(X(t), Y(t), Z(t))'$.

First, suppose that the motion is observed by one measuring complex ($\mathcal{M}$) with coordinates $(X_{\mathcal{M}}, Y_{\mathcal{M}}, Z_{\mathcal{M}})$. This could be a passive acoustic device for estimating the direction of movement [22], allowing measurement aboard the AUV, or an active hydroacoustic beacon [23], forming measurements for an external observer.

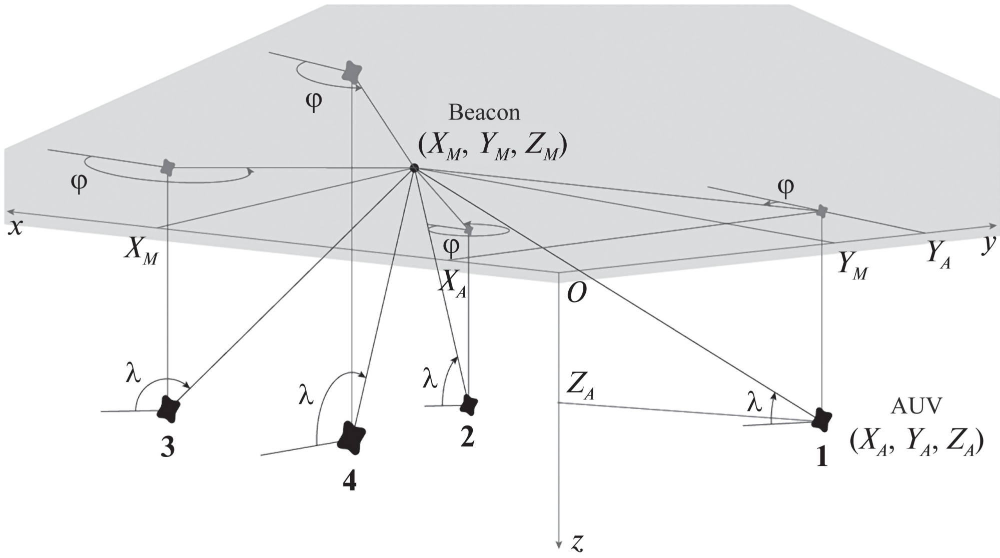

Fig. 1. Possible relative positions of AUVs and the observer.

The type of measuring device depends on the navigation task being solved. If the AUV interacts with the measuring device (the cooperative scenario), then positioning is performed aboard the AUV. In the case of opposing interests, an external device tracks a target.

Regardless of the task, it is necessary to measure the direction angle φ (azimuth or bearing) in the plane Oxy, the elevation angle λ (the inclination of the acoustic ray relative to the straight line Oz), and the range r (the distance between $\mathcal{A}$ and $\mathcal{M}$). Figure 1 illustrates possible relative positions of AUVs and the observer and the angle measurement rules.

The proposed approach to form linear pseudomeasurements from the measurements of φ, λ, and r combines the idea of a classical filter [19] and the model with tangent observations [20].

First, consider the measurement $y_\varphi = \varphi + v_\varphi$ of the bearing φ, where the error v_φ has a distribution with zero mean ($\mathsf{E}\{v_\varphi\} = 0$) and a standard deviation σ_φ ($\operatorname{cov}(v_\varphi, v_\varphi) = \sigma_\varphi^2$). Assuming that sonar errors in angle measurements are about 1–2°, we represent measurements in radians; then the value σ_φ can be set to $\sigma_\varphi = \frac{\pi}{180} \approx 0.0175$ and, consequently, $v_\varphi \ll 1$.

For the measurement y_φ, we write the sine and cosine and approximate them with the corresponding linear parts of the Taylor expansion for small v_φ :

$$y_\varphi^{\sin} = \sin(y_\varphi) = \sin(\varphi + v_\varphi) \approx \sin(\varphi) + \cos(\varphi) v_\varphi,$$
$$y_\varphi^{\cos} = \cos(y_\varphi) = \cos(\varphi + v_\varphi) \approx \cos(\varphi) - \sin(\varphi) v_\varphi.$$

Assume further that for the distribution of v_φ, only the moments $\mathsf{E}\{v_\varphi\} = 0$ and $\mathsf{E}\{v_\varphi^2\} = \sigma_\varphi^2$ are given, and the variables φ and v_φ are independent. In this case, we have $\mathsf{E}\{(\cos(\varphi)v_\varphi)^2\} \leqslant \sigma_\varphi^2$ and $\mathsf{E}\{(\sin(\varphi)v_\varphi)^2\} \leqslant \sigma_\varphi^2$. Moreover, $\mathsf{E}\{\cos(\varphi)v_\varphi \sin(\varphi)v_\varphi\} = \frac{1}{2}\sigma_\varphi^2 \mathsf{E}\{\sin(2\varphi)\} = 0$ if φ is distributed symmetrically about zero. In view of the physical meaning of φ, the latter assumption seems quite realistic. Recall the well-known minimax property of the Gaussian distribution, which maximizes the variance of a random variable in the class of distributions with known mean and bounded variance [24]. Therefore, we arrive at an approximation of the form

$$y_\varphi^{\sin} \approx \sin(\varphi) + v_1, \quad y_\varphi^{\cos} \approx \cos(\varphi) + v_2,$$

where v_1 and v_2 are independent Gaussian random variables with $\mathsf{E}\{v_1\} = \mathsf{E}\{v_2\} = 0$ and $\mathsf{E}\{v_1^2\} = \mathsf{E}\{v_2^2\} = \sigma_\varphi^2$. Here, the measurement error $(v_1, v_2)'$ is interpreted as the worst-case one.

Such an interpretation is expected to be excessively rough, which is characteristic of minimax estimates. Thus, it makes sense to focus on another approximation of $\mathsf{E}\{(\cos(\varphi)v_\varphi)^2\}$ and $\mathsf{E}\{(\sin(\varphi)v_\varphi)^2\}$. These moments cannot be computed without knowing the distribution of φ. But it can be supposed that φ takes any value with equal probability, i.e., has a "nearly" uniform distribution. This reflects the assumption that the target can appear anywhere; then

$$\mathsf{E}\{(\cos(\varphi)v_\varphi)^2\} = \mathsf{E}\{(\sin(\varphi)v_\varphi)^2\} \approx \frac{1}{4}\sigma_\varphi^2.$$

Which approximation is better, $\mathsf{E}\{v_{1,2}^2\} = \sigma_\varphi^2$ or $\mathsf{E}\{v_{1,2}^2\} = \frac{1}{4}\sigma_\varphi^2$? It is possible to test them experimentally.

Continuing the considerations, we obtain

$$\sin(\varphi) \approx y_\varphi^{\sin} - v_1, \quad \cos(\varphi) \approx y_\varphi^{\cos} - v_2,$$

$$\tan(\varphi) = \frac{Y_\mathcal{A} - Y_\mathcal{M}}{X_\mathcal{A} - X_\mathcal{M}} \approx \frac{y_\varphi^{\sin} - v_1}{y_\varphi^{\cos} - v_2},$$

$$(Y_\mathcal{A} - Y_\mathcal{M})y_\varphi^{\cos} - (X_\mathcal{A} - X_\mathcal{M})y_\varphi^{\sin} \approx (Y_\mathcal{A} - Y_\mathcal{M})v_2 - (X_\mathcal{A} - X_\mathcal{M})v_1.$$

By replacing the exact coordinates $(X_\mathcal{A}, Y_\mathcal{A})$ in the last expression with their estimates, one derives the pseudomeasurement residual figuring in the filtering equations. The pseudomeasurements themselves can be written as

$$-Y_\mathcal{M}y_\varphi^{\cos} + X_\mathcal{M}y_\varphi^{\sin} \approx (y_\varphi^{\sin}, -y_\varphi^{\cos})\begin{pmatrix} X_\mathcal{A} \\ Y_\mathcal{A} \end{pmatrix} + (Y_\mathcal{A} - Y_\mathcal{M})v_2 - (X_\mathcal{A} - X_\mathcal{M})v_1,$$

which explains the meaning of the above transformations: the pseudomeasurements $-Y_\mathcal{M}y_\varphi^{\cos} + X_\mathcal{M}y_\varphi^{\sin}$ approximate the measurements of a linear combination of the estimated coordinates $(X_\mathcal{A}, Y_\mathcal{A})$ under an additive noise with known covariance.

Thus, for the measurement $y_\varphi = \varphi + v_\varphi$ of the bearing φ, the pseudomeasurement Y_φ is formed as follows:

$$Y_\varphi = -Y_\mathcal{M}y_\varphi^{\cos} + X_\mathcal{M}y_\varphi^{\sin}, \quad y_\varphi^{\sin} = \sin(y_\varphi), \quad y_\varphi^{\cos} = \cos(y_\varphi); \tag{1}$$

the filtering algorithm uses the observation model

$$Y_\varphi = (y_\varphi^{\sin}, -y_\varphi^{\cos})\begin{pmatrix} X_\mathcal{A} \\ Y_\mathcal{A} \end{pmatrix} + (X_\mathcal{M} - X_\mathcal{A}, Y_\mathcal{A} - Y_\mathcal{M})\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2}$$

Next, consider the measurement $y_\lambda = \lambda + v_\lambda$ of the elevation angle λ. Similarly to the bearing, we approximate the sine and cosine based on the same assumptions about the measurement error v_λ:

$$y_\lambda^{\sin} = \sin(y_\lambda) \approx \sin(\lambda) + \cos(\lambda)v_\lambda \approx \sin(\lambda) + v_3,$$
$$y_\lambda^{\cos} = \cos(y_\lambda) \approx \cos(\lambda) - \sin(\lambda)v_\lambda \approx \cos(\lambda) + v_4,$$

where v_3 and v_4 are independent Gaussian random variables with $\mathsf{E}\{v_3\} = \mathsf{E}\{v_4\} = 0$ and $\mathsf{E}\{v_3^2\} = \mathsf{E}\{v_4^2\} = \sigma_\lambda^2$ (in the alternative approximation, $\mathsf{E}\{v_{3,4}^2\} = \frac{1}{4}\sigma_\lambda^2$). Hence,

$$\sin(\lambda) \approx y_\lambda^{\sin} - v_3, \quad \cos(\lambda) \approx y_\lambda^{\cos} - v_4,$$

$$\tan(\lambda) = \frac{Z_\mathcal{A} - Z_\mathcal{M}}{|X_\mathcal{A} - X_\mathcal{M}|}\cos(\varphi) \approx \frac{y_\lambda^{\sin} - v_3}{y_\lambda^{\cos} - v_4}.$$

To simplify manipulations with the measurement of λ, let the reference frame be chosen so that, for the relative position of $\mathcal{A}$ and $\mathcal{M}$, $X_{\mathcal{A}} > X_{\mathcal{M}}$ and $X(t) > X_{\mathcal{M}}$ during the further motion. Using the available bearing approximation, we replace $\cos(\varphi)$ with $y_\varphi^{\cos} - v_2$ to get

$$\begin{gathered} y_\varphi^{\cos} y_\lambda^{\cos}(Z_{\mathcal{A}} - Z_{\mathcal{M}}) - y_\lambda^{\sin}(X_{\mathcal{A}} - X_{\mathcal{M}}) \\ \approx (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} v_4 - (Z_{\mathcal{A}} - Z_{\mathcal{M}}) v_2 v_4. \end{gathered}$$

By the above assumption, all v_i are independent and centered, so the variance of the right-hand side of this expression (the pseudomeasurement error) has the form

$$\begin{gathered} \mathsf{E}\Big\{\big((Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} v_4 - (Z_{\mathcal{A}} - Z_{\mathcal{M}}) v_2 v_4\big)^2\Big\} \\ = \mathsf{E}\left\{(Z_{\mathcal{A}} - Z_{\mathcal{M}})^2 (y_\lambda^{\cos})^2\right\} \sigma_\varphi^2 + \mathsf{E}\left\{(X_{\mathcal{A}} - X_{\mathcal{M}})^2\right\} \sigma_\lambda^2 \\ + \mathsf{E}\left\{(Z_{\mathcal{A}} - Z_{\mathcal{M}})^2 (y_\varphi^{\cos})^2\right\} \sigma_\lambda^2 + \mathsf{E}\left\{(Z_{\mathcal{A}} - Z_{\mathcal{M}})^2\right\} \sigma_\varphi^2 \sigma_\lambda^2. \end{gathered}$$

Appealing to the same arguments about the small values of v_i, σ_φ, and σ_λ, we neglect the last term and represent the pseudomeasurement residual as

$$\begin{gathered} y_\varphi^{\cos} y_\lambda^{\cos}(Z_{\mathcal{A}} - Z_{\mathcal{M}}) - y_\lambda^{\sin}(X_{\mathcal{A}} - X_{\mathcal{M}}) \\ \approx (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_A - Z_M) y_\varphi^{\cos} v_4. \end{gathered}$$

(Here, the estimates are substituted for the exact coordinates $(X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}})$.) The pseudomeasurements themselves become

$$\begin{gathered} -y_\varphi^{\cos} y_\lambda^{\cos} Z_{\mathcal{M}} + y_\lambda^{\sin} X_{\mathcal{M}} \\ \approx -y_\varphi^{\cos} y_\lambda^{\cos} Z_{\mathcal{A}} + y_\lambda^{\sin} X_{\mathcal{A}} + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} v_4. \end{gathered}$$

Thus, for the measurement $y_\lambda = \lambda + v_\lambda$ of the elevation angle λ, the pseudomeasurement Y_λ is formed as follows:

$$\begin{gathered} Y_\lambda = -y_\varphi^{\cos} y_\lambda^{\cos} Z_{\mathcal{M}} + y_\lambda^{\sin} X_{\mathcal{M}}, \\ y_\lambda^{\sin} = \sin(y_\lambda), \quad y_\lambda^{\cos} = \cos(y_\lambda), \quad y_\varphi^{\cos} = \cos(y_\varphi); \end{gathered} \tag{3}$$

the filtering algorithm uses the observation model

$$Y_\lambda = \left(y_\lambda^{\sin}, \ \ -y_\varphi^{\cos} y_\lambda^{\cos}\right) \begin{pmatrix} X_{\mathcal{A}} \\ Z_{\mathcal{A}} \end{pmatrix} + \left((Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos}, \ \ X_{\mathcal{M}} - X_{\mathcal{A}}, \ \ (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos}\right) \begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{4}$$

Finally, consider the measurement $y_r = r + v_5$ of the range r with an error v_5 independent of the previous ones v_i: $\mathsf{E}\{v_5\} = 0$ and $\mathsf{E}\{v_5^2\} = \sigma_r^2$. Using the measurement of the elevation angle λ and the approximation $\sin(\lambda) \approx y_\lambda^{\sin} - v_3$, we write

$$r = \frac{Z_{\mathcal{A}} - Z_{\mathcal{M}}}{\sin(\lambda)} \quad \Rightarrow \quad y_r - v_5 \approx \frac{Z_{\mathcal{A}} - Z_{\mathcal{M}}}{y_\lambda^{\sin} - v_3}.$$

Similarly to the angle transformations, it follows that

$$Z_{\mathcal{A}} - Z_{\mathcal{M}} - y_r y_\lambda^{\sin} \approx -y_r v_3 - y_\lambda^{\sin} v_5 + v_3 v_5.$$

The centered error on the right has the variance

$$\mathsf{E}\{(-y_r v_3 - y_\lambda^{\sin} v_5 + v_3 v_5)^2\} = \mathsf{E}\left\{y_r^2\right\}\sigma_\lambda^2 + \mathsf{E}\left\{(y_\lambda^{\sin})^2\right\}\sigma_r^2 + \sigma_\lambda^2\sigma_r^2.$$

Here, the third term can be neglected compared to the first two, and the pseudomeasurement residual becomes

$$Z_{\mathcal{A}} - Z_{\mathcal{M}} - y_r y_\lambda^{\sin} \approx -y_r v_3 - y_\lambda^{\sin} v_5.$$

(Here, the estimate is substituted for the exact coordinate $Z_{\mathcal{A}}$.)

We write the pseudomeasurements themselves as

$$Z_{\mathcal{M}} + y_r y_\lambda^{\sin} \approx Z_{\mathcal{A}} + y_r v_3 + y_\lambda^{\sin} v_5.$$

Thus, for the measurement $y_r = d + v_5$ of the range, the pseudomeasurement Y_r is formed as follows:

$$Y_r = Z_{\mathcal{M}} + y_r y_\lambda^{\sin}, \quad y_\lambda^{\sin} = \sin(y_\lambda); \tag{5}$$

the filtering algorithm uses the observation model

$$Y_r = Z_{\mathcal{A}} + \begin{pmatrix} y_r, & y_\lambda^{\sin} \end{pmatrix} \begin{pmatrix} v_3 \\ v_5 \end{pmatrix}. \tag{6}$$

Now we combine all the three models (2), (4), and (6) into the single observation vector $Y = (Y_\varphi, Y_\lambda, Y_r)'$:

$$\begin{gathered} Y = \begin{pmatrix} y_\varphi^{\sin} & -y_\varphi^{\cos} & 0 \\ y_\lambda^{\sin} & 0 & -y_\varphi^{\cos} y_\lambda^{\cos} \\ 0 & 0 & 1 \end{pmatrix} X \\ + \begin{pmatrix} X_{\mathcal{M}} - X_{\mathcal{A}} & Y_{\mathcal{A}} - Y_{\mathcal{M}} & 0 \\ 0 & (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} & X_{\mathcal{M}} - X_{\mathcal{A}} \\ 0 & 0 & y_r \end{pmatrix} \begin{pmatrix} 0 \\ (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y_\lambda^{\sin} \end{pmatrix} V, \end{gathered} \tag{7}$$

where

$$X = (X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}})' \quad \text{and} \quad V = (v_1, v_2, v_3, v_4, v_5)'.$$

2.2. Application of the Extended Kalman Filter

Let the observation vector $y_t \in \mathbb{R}^{q_y}$ of the AUV be formed from the measurements y_φ, y_λ, and y_r at a time instant t. (For one observer, the dimension is $q_y = 3$.) The AUV has the state vector $X_t \in \mathbb{R}^{p_X}$; without loss of generality, assume that the state X_t is determined by the AUV coordinates in the $Oxyz$ system, denoted by $X_t = (X(t), Y(t), Z(t))'$, $p_X = 3$. Below, the motion model will be supplemented with other variables, but the objective is still to estimate the AUV position.

The estimation of X_t begins at the time instant $t = 0$ and is performed at discrete time instants $1, 2, \ldots, t, \ldots,$ corresponding to the partition of the observation interval with a step δ s: $\delta, 2\delta, \ldots, t\delta, \ldots$ The AUV initial position is given by the vector $X_0 = \eta = (\eta_X, \eta_Y, \eta_Z)' = (X(0), Y(0), Z(0))'$.

The vectors X_t and y_t are described by a discrete stochastic dynamic system of general form:

$$\begin{gathered} X_t = \Phi_t^{(1)}(X_{t-1}) + \Phi_t^{(2)}(X_{t-1})W_t, \quad t = 1, 2, \ldots, \quad X_0 = \eta, \\ y_t = \psi_t^{(1)}(X_t) + \psi_t^{(2)}(X_t)v_t. \end{gathered} \tag{8}$$

By assumption, the random sequences of X_t and y_t have finite covariances, and the disturbances $W_t \in \mathbb{R}^{p_W}$ and the measurement errors $v_t \in \mathbb{R}^{q_v}$ are independent discrete white noises of the second order; the initial condition vector $\eta \in \mathbb{R}^{p_X}$ is independent of W_t and v_t and has finite covariance. The corresponding central moments are denoted, e.g., for W_t, by $m_W(t)$ and $D_W(t)$.

We supplement system (8) with an equation for the pseudomeasurements $Y_t \in \mathbb{R}^{q_Y}$:

$$Y_t = \boldsymbol{\Psi}_t^{(1)}(X_t, y_t) + \Psi_t^{(2)}(X_t, y_t)V_t. \tag{9}$$

(In all the available examples, exactly one pseudomeasurement is formed for one measurement and, accordingly, $q_Y = q_y = 3$; in the general case, the dimensions may differ.) Here, the matrix functions $\boldsymbol{\Psi}_t^{(1)}(X, y)$ and $\Psi_t^{(2)}(X, y)$ are given by (7), and due to its linearity, we have $\boldsymbol{\Psi}_t^{(1)}(X, y) = \Psi_t^{(1)}(y)X$.

Filtering by the method of linear pseudomeasurements [19] consists in applying the EKF [12] to system (8), with the observations y_t replaced by the pseudomeasurements (9). In the current notation, such a filter has the form

$$\begin{gathered} \widetilde{X}_t = \Phi_t^{(1)}(\widehat{X}_{t-1}) + \Phi_t^{(2)}(\widehat{X}_{t-1})m_W(t), \\ \widetilde{K}_t = \widetilde{\Phi}_t^{(1)}\widehat{K}_{t-1}\left(\widetilde{\Phi}_t^{(1)}\right)' + \widetilde{\Phi}_t^{(2)}D_W(t)\left(\widetilde{\Phi}_t^{(2)}\right)', \\ \widetilde{\Phi}_t^{(1)} = \left.\frac{\partial \Phi_t^{(1)}(X)}{\partial X}\right|_{X=\widetilde{X}_t}, \quad \widetilde{\Phi}_t^{(2)} = \Phi_t^{(2)}(\widetilde{X}_t), \\ \widehat{X}_t = \widetilde{X}_t + K_t\left(Y_t - \boldsymbol{\Psi}_t^{(1)}(\widetilde{X}_t, y_t) - \Psi_t^{(2)}(\widetilde{X}_t, y_t)m_V(t)\right), \\ K_t = \widetilde{K}_t\left(\widetilde{\Psi}_t^{(1)}\right)'\left(\widetilde{\Psi}_t^{(1)}\widetilde{K}_t\left(\widetilde{\Psi}_t^{(1)}\right)' + \widetilde{\Psi}_t^{(2)}D_V(t)\left(\widetilde{\Psi}_t^{(2)}\right)'\right)^{-1}, \\ \widetilde{\Psi}_t^{(1)} = \left.\frac{\partial \boldsymbol{\Psi}_t^{(1)}(X, y_t)}{\partial X}\right|_{X=\widetilde{X}_t} = \Psi_t^{(1)}(y_t), \quad \widetilde{\Psi}_t^{(2)} = \Psi_t^{(2)}(\widetilde{X}_t, y_t), \\ \widehat{K}_t = \widetilde{K}_t - K_t\widetilde{\Psi}_t^{(1)}\widetilde{K}_t. \end{gathered} \tag{10}$$

The difference from the classical EKF here is the mandatory linearity of the function $\boldsymbol{\Psi}_t^{(1)}$ in the estimated state, so that $\frac{\partial \boldsymbol{\Psi}_t^{(1)}(X, y_t)}{\partial X} = \Psi_t^{(1)}(y_t)$, which is determined by the pseudomeasurement equation (7) itself. The other elements are the same as in the standard EKF: the prediction $\widetilde{X}_t$ is constructed along the system trajectories; the heuristic prediction error covariance $\widetilde{K}_t$ is the result of linearizing the state equation in the neighborhood of the prediction; the correction is the observation residual with the Kalman gain K_t; the heuristic estimation error covariance $\widehat{K}_t$ is the result of linearizing the observation equation. Also, a feature of the EKF by the method of linear pseudomeasurements is the dependence of $\widetilde{\Psi}_t^{(1)}$ and $\widetilde{\Psi}_t^{(2)}$ on the "real" observations y_t. Indeed, according to (7), the values of y_t are used to compute not only the pseudomeasurements (1), (3), and (5) but also the approximate model matrices $\Psi_t^{(1)}$ and $\Psi_t^{(2)}$.

2.3. Pseudomeasurements with Time Delay

The dependence of observations y_t and pseudomeasurements Y_t (in (8) and (9), respectively) on the state X_t varies fundamentally if the information exchange time between the observed object and the observer cannot be neglected. This is the situation with sonars, see the discussion above. Accordingly, the current measurements turn out to match the position of A for some previous time instant $s < t$. This instant is determined as follows.

Let $v_s = \text{const}$ be the sound velocity in water. Regardless of the sonar type and the source of measurements (aboard the AUV or an external measuring complex), there is a difference between the time when the observer $\mathcal{M}$ receives the measurement and the time when $\mathcal{A}$ had the "measured" position. This difference is the time taken by the acoustic signal to travel the distance between $\mathcal{A}$ and $\mathcal{M}$, i.e., $\tau = t - s = r/(\delta v_s)$. Following the idea of pseudomeasurements, this random value can be approximated by $\widetilde{\tau} = y_r/(\delta v_s)$. Considering that the typical value is $v_s = 5400$ km/h (1500 m/s), the introduced error can be neglected, and the pseudomeasurement model (7) can be supplemented with the relation

$$X = (X(t-\widetilde{\tau}), Y(t-\widetilde{\tau}), Z(t-\widetilde{\tau}))', \quad \widetilde{\tau} = y_r/\delta v_s. \tag{11}$$

The general form of the observation–pseudomeasurement system is

$$\begin{gathered} X_t = \Phi_t^{(1)}(X_{t-1}) + \Phi_t^{(2)}(X_{t-1})W_t, \quad t = -T, -T+1, \ldots, 1, 2, \ldots, \quad X_{-T-1} = \eta, \\ y_t = \psi_t^{(1)}(X_{t-\tau_t}) + \psi_t^{(2)}(X_{t-\tau_t})v_t, \quad \tau_t = \tau_t(X_t), \\ Y_t = \Psi_t^{(1)}(y_t)X_{t-\widetilde{\tau}_t} + \Psi_t^{(2)}(X_{t-\widetilde{\tau}_t}, y_t)V_t, \quad \widetilde{\tau}_t = \widetilde{\tau}_t(y_t). \end{gathered} \tag{12}$$

This model rests on the following assumptions. First, the maximum possible time delay of observations, i.e., the value $T\delta > 0$, is known. (Essentially, this is the maximum detection range of the moving object.) Second, the motion of $\mathcal{A}$ starts at the time instant $-T\delta$, i.e., $t = -T$, so that the observer $\mathcal{M}$ will surely perform a measurement at the time instant $t = 0$. The initial position of $\mathcal{A}$ is given by the vector $\eta = (\eta_X, \eta_Y, \eta_Z)' = (X(-T-1), Y(-T-1), Z(-T-1))'$. The time delay τ_t is a function of the state X_t, i.e., the time required for the sound wave to travel the distance between $\mathcal{A}$ and $\mathcal{M}$. (Precisely from this consideration, the estimate $\widetilde{\tau}$ is included in the pseudomeasurements (11).)

The functions $\tau_t(X)$ and $\widetilde{\tau}_t(y)$ in (12) must take integer values from the set $\{0, 1, \ldots, T\}$. For "real" states and observations, they have the form

$$\begin{gathered} \tau_t = \min\left\{T, \left[\frac{\sqrt{(X(t) - X_{\mathcal{M}})^2 + (Y(t) - Y_{\mathcal{M}})^2 + (Z(t) - Z_{\mathcal{M}})^2}}{\delta v_s}\right]\right\}, \\ \widetilde{\tau}_t = \min\left\{T, \left[\frac{y_r}{\delta v_s}\right]\right\}, \end{gathered} \tag{13}$$

where $[\cdot]$ denotes the floor function.

2.4. Filtering in the Model with Several Observers

Now, let the observation vector y_t in (12) combine measurements of angles and range coming from q observers, i.e., $y_t = (y_{\varphi_t}^{(1)}, y_{\lambda_t}^{(1)}, y_{r_t}^{(1)}, \ldots, y_{\varphi_t}^{(q)}, y_{\lambda_t}^{(q)}, y_{r_t}^{(q)})'$. Since $y_t \in \mathbb{R}^{q_y}$, $q_y = 3q$. For each ith observer, we define a time delay $\tau_t^{(i)}$, $i = 1, \ldots, q$, with values in the set $\{0, 1, \ldots, T\}$.

The delays $\tau_t^{(i)}$ are combined into the vector $\tau_t = (\tau_t^{(1)}, \ldots, \tau_t^{(q)})' \in \mathbb{R}^q$, which is a function of X_t just like τ_t in (12). Thus, the measurements $y_{\varphi_t}^{(i)}, y_{\lambda_t}^{(i)}$, and $y_{r_t}^{(i)}$ in each group can be represented as functions of the position $X_{t-\tau_t^{(i)}}$. The observation system takes the form

$$\begin{gathered} X_t = \Phi_t^{(1)}(X_{t-1}) + \Phi_t^{(2)}(X_{t-1})W_t, \\ t = -T, -T+1, \ldots, 0, 1, \ldots, \quad X_{-T-1} = \eta, \\ y_t^{(i)} = \psi_t^{(i,1)}\left(X_{t-\tau_t^{(i)}}\right) + \psi_t^{(i,2)}\left(X_{t-\tau_t^{(i)}}\right) v_t^{(i)}, \quad i = 1, \ldots, q, \\ Y_t^{(i)} = \Psi_t^{(i,1)}\left(y_t^{(i)}\right) X_{t-\widetilde{\tau}_t^{(i)}} + \Psi_t^{(i,2)}\left(X_{t-\widetilde{\tau}_t^{(i)}}, y_t^{(i)}\right) V_t^{(i)}. \end{gathered} \tag{14}$$

For (14) to correctly reflect the above assumptions and turn into (8), (9) under $T = 0$ (no time delays), we introduce the following designations:

$y_t = \left((y_t^{(1)})', \ldots, (y_t^{(q)})'\right)'$ is the observation vector composed of q groups of measurements $y_t^{(i)} = \left(y_{\varphi_t}^{(i)}, y_{\lambda_t}^{(i)}, y_{r_t}^{(i)}\right)'$;

$v_t^{(i)}$ is the vector of measurement errors in this group;

$Y_t = \left((Y_t^{(1)})', \ldots, (Y_t^{(q)})'\right)'$ is the vector of q groups of pseudomeasurements;

$Y_t^{(i)} = \left(Y_t^{(i)}, Y_t^{(i)}, Y_t^{(i)}\right)'$, which is associated with the corresponding group $y_t^{(i)}$, and $V_t^{(i)}$ is the vector of measurement errors in this group.

The vector functions $\psi_t^{(i,1)}$, as well as the matrices $\psi_t^{(i,2)}$, $\Psi_t^{(i,1)}$, and $\Psi_t^{(i,2)}$, $i = 1, \ldots, q$, are defined for each group of observations and pseudomeasurements. Their presence implies the independence of each observer forming the measurements of the group, i.e.,

$$\psi_t^{(1)} = \left((\psi_t^{(1,1)})', \ldots, (\psi_t^{(q,1)})'\right)', \quad \psi_t^{(2)} = \operatorname{diag}\left(\psi_t^{(1,2)}, \ldots, \psi_t^{(q,2)}\right),$$

$$\Psi_t^{(1)} = \begin{pmatrix} \Psi_t^{(1,1)} \\ \vdots \\ \Psi_t^{(q,1)} \end{pmatrix}, \quad \Psi_t^{(2)} = \operatorname{diag}\left(\Psi_t^{(1,2)}, \ldots, \Psi_t^{(q,2)}\right).$$

Then the basic EKF equations (10) can be refined for the model with time delays (14) as follows:

$$\begin{gathered} \widetilde{X}_t = \Phi_t^{(1)}(\widehat{X}_{t-1}) + \Phi_t^{(2)}(\widehat{X}_{t-1}) m_W(t), \\ \widetilde{K}_t = \widetilde{\Phi}_t^{(1)} \widehat{K}_{t-1} (\widetilde{\Phi}_t^{(1)})' + \widetilde{\Phi}_t^{(2)} D_W(t) (\widetilde{\Phi}_t^{(2)})', \\ \widetilde{\Phi}_t^{(1)} = \left.\frac{\partial \Phi_t^{(1)}(X)}{\partial X}\right|_{X=\widetilde{X}_t}, \quad \widetilde{\Phi}_t^{(2)} = \Phi_t^{(2)}(\widetilde{X}_t), \\ \widehat{X}_t = \widetilde{X}_t + K_t \Delta\widetilde{Y}_t, \\ \Delta\widetilde{Y}_t = \left(Y_t^{(1)} - \Psi_t^{(1,1)}(y_t^{(1)})\widetilde{X}_{t-\widetilde{\tau}_t^{(1)}}, \ldots, Y_t^{(q)} - \Psi_t^{(q,1)}(y_t^{(q)})\widetilde{X}_{t-\widetilde{\tau}_t^{(q)}}\right), \\ K_t = \widetilde{K}_t (\Psi_t^{(1)})' \left(\Psi_t^{(1)} \widetilde{K}_t (\Psi_t^{(1)})' + \widetilde{\Psi}_t^{(2)} D_V(t) (\widetilde{\Psi}_t^{(2)})'\right)^{-1}, \\ \Psi_t^{(1)} = \Psi_t^{(1)}(y_t), \widetilde{\Psi}_t^{(2)} = \operatorname{diag}\left\{\Psi_t^{(1,2)}(\widetilde{X}_{t-\widetilde{\tau}_t^{(1)}}, y_t), \ldots, \Psi_t^{(q,2)}(\widetilde{X}_{t-\widetilde{\tau}_t^{(q)}}, y_t)\right\}, \\ \widehat{K}_t = \widetilde{K}_t - K_t \Psi_t^{(1)} \widetilde{K}_t. \end{gathered} \tag{15}$$

Essentially, compared to the filter (10), the filter (15) simply incorporates the estimates $\widetilde{\tau}_t^{(i)}$, $i = 1, \ldots, q$, of the time delays. The observation residual $\Delta\widetilde{Y}_t$ and the matrix of measurement error deviations $\Psi_t^{(2)}$ are composed of the values of the position predictions $\widetilde{X}_{t-\widetilde{\tau}_t^{(i)}}$ corresponding to the time instants for which the current observations y_t were performed and the pseudomeasurements Y_t were composed. For this purpose, we use the position predictions shifted relative to the current time instant by the value of the estimate $\widetilde{\tau}_t^{(i)}$ of the time delay $\tau_t^{(i)}$ for the corresponding ith observer.

3. TRACKING OF AUV'S APPROACH USING ACOUSTIC BEACON MEASUREMENTS

3.1. Observation System Model

To apply the filtering algorithm (15), we adopt the same model as in [9, 10], with slight meaningful modifications for the tracking problem of an approaching unknown object. (In the papers cited, this model was studied for parameter identification.) Following Fig. 1, let the origin O of the reference frame $Oxyz$ define a stationary object ($\mathcal{O}$) located on the sea surface, to which an AUV is approaching. $\mathcal{A}$ is detected in the initial position $\eta = (\eta_X, \eta_Y, \eta_Z)'$, whose random elements are independent and have a uniform distribution: $\eta_X \sim R[10, 20]$, $\eta_Y \sim R[10, 20]$, and $\eta_Z \sim R[0.5, 1.5]$. Thus, the initial position of $\mathcal{A}$ is characterized by the mean $\mathsf{E}\{\eta\} = (15, 15, 1)'$ and covariance $\operatorname{cov}(\eta, \eta) \approx \operatorname{diag}\{2.9^2; 2.9^2; 0.29^2\}$. All distances are given in kilometers (km). By assumption, the detected AUV moves towards $\mathcal{O}$ with chaotic maneuvering but an average constant velocity of about 21 km/h.

There are two complexes ($\mathcal{F}$, first) and ($\mathcal{S}$, second) for observing the AUV. In the reference frame $Oxyz$, the z axis is directed vertically downward to the water surface (as in Fig. 1, corresponding to the AUV depth) whereas the y and x axes are directed from the object to the first and second observers, respectively ($\mathcal{O} \to \mathcal{F}$ and $\mathcal{O} \to \mathcal{S}$). The observers are considered to be stationary on the water surface, i.e., at zero depth. Thus, their coordinates are $\mathcal{F}(X_\mathcal{F}, Y_\mathcal{F}, Z_\mathcal{F}) = (0, Y_\mathcal{F}, 0)$ and $\mathcal{S}(X_\mathcal{S}, Y_\mathcal{S}, Z_\mathcal{S}) = (X_\mathcal{S}, 0, 0)$, where $X_\mathcal{S} = -2$ km and $Y_\mathcal{F} = -1$ km. Furthermore, assume that throughout the observation time, the coordinates $\mathcal{A}(X(t), Y(t), Z(t))$ are such that the AUV remains at depth without surfacing (i.e., $Z(t) > 0$), and the conditions $X(t) > X_\mathcal{M}$ used for the pseudomeasurement (3) are valid for both observers, i.e., $X(t) > 0$. The intersection of the motion trajectory with the Ox axis does not affect the pseudomeasurements, and a possible intersection

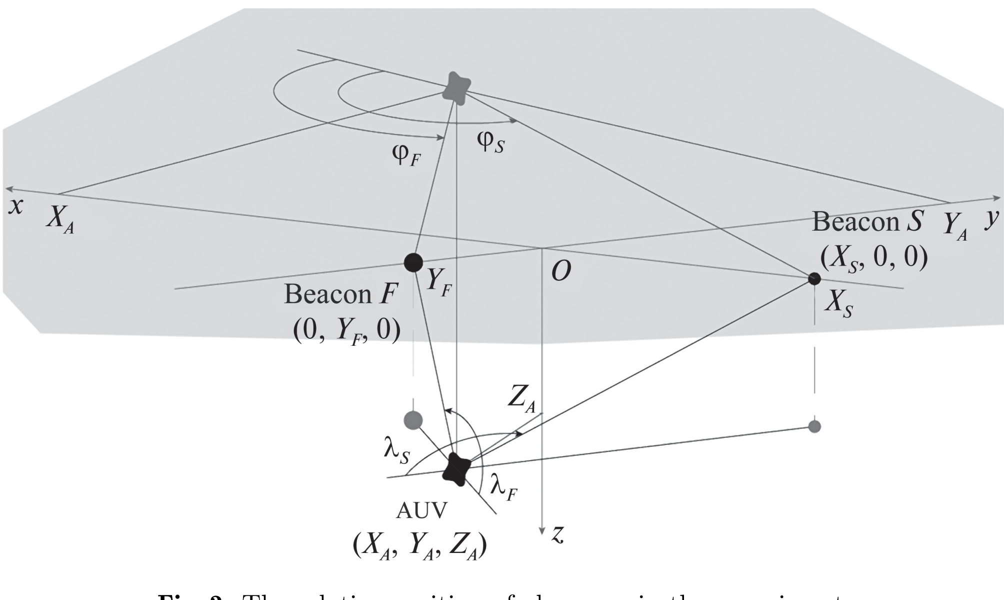

Fig. 2. The relative position of observers in the experiment.

with the Oy axis can be easily considered by using the cotangent instead of the tangent for the pseudomeasurements (3). The experiment is schematically illustrated in Fig. 2.

The vector $(X(t), Y(t), Z(t))'$ describes the position of $\mathcal{A}$ at discrete time instants $t = 0, \dots, 1000$, which correspond to the partition of the observation time interval with a discretization step of $\delta = 0.0001$ h. Taking the maximum time delay T into account, the navigation task is thus solved in 0.1 h = 6 min. Measurements are performed at the same time instants, i.e., about three measurements per second by each complex. With an absolute constant average velocity of 21 km/h, during this time the AUV travels on average a distance of about 2.1 km, approaching $\mathcal{O}$. The maximum distance from $\mathcal{A}$ to $\mathcal{O}$ and to $\mathcal{F}$ or $\mathcal{S}$ is about 28 and 30 km, respectively; the minimum distances are 14 and 16 km, respectively. Hence, the maximum possible time delay at the AUV detection instant is $T = 56$ (i.e., 0.0056 h or about 20 s).

By assumption, the AUV moves with a constant average velocity $(s_x, s_y, s_z)'$, and the deviations from this velocity are described by the vector of additive disturbances $(w_x(t), w_y(t), w_z(t))'$:

$$\begin{aligned} X(t) &= X(t-1) + \delta S_x(t), \quad S_x(t) = s_x + \sigma_{s_x} w_x(t), \\ Y(t) &= Y(t-1) + \delta S_y(t), \quad S_y(t) = s_y + \sigma_{s_y} w_y(t), \\ Z(t) &= Z(t-1) + \delta S_z(t), \quad S_z(t) = s_z + \sigma_{s_z} w_z(t). \end{aligned} \tag{16}$$

On each trajectory, the average velocity $(s_x, s_y, s_z)'$ is specified by a random vector of independent uniformly distributed variables: $s_x \sim R[-20, -10]$, $s_y \sim R[-20, -10]$, and $s_z \sim R[-2, 0]$. Thus, the average velocity of $\mathcal{A}$ is characterized by the mean $\mathsf{E}\{S(t)\} = (-15, -15, -1)'$ (hence, the average velocity has an absolute value of about 21 km/h and the direction of motion is towards $\mathcal{O}(0, 0, 0)$) and the covariance $\operatorname{diag}\{D_{s_x}; D_{s_y}; D_{s_z}\} \approx \operatorname{diag}\{2.9^2; 2.9^2; 0.4^2\}$. The standard deviations of the additive velocity disturbance vector $W_t = (w_x(t), w_y(t), w_z(t))'$ are $\sigma_{s_x} = 15$, $\sigma_{s_y} = 15$, and $\sigma_{s_z} = 1$. As a result, the velocity covariance is $\operatorname{cov}(S(t), S(t)) \approx \operatorname{diag}\{15.3^2; 15.3^2; 1.1^2\}$.

In addition to (16), we model abrupt changes (jumps) in the average velocity. Consider a standard Poisson process $P(u)$ independent of the position of $\mathcal{A}$ and a known intensity λ_u of changes in the constant average velocity of the AUV (i.e., the average time between velocity jumps). The discrete time t is related to the continuous time u via the discretization step: $u = t\delta$. The state vector $X_t \in \mathbb{R}^{p_X}$ can be augmented by the $(p_X + 1)$th element, so that $X_{(p_X+1)_t} = P(\lambda_{t\delta} t\delta)$.

The constant, or rather piecewise constant, average velocity is described by a sequence $(s_x^p(t), s_y^p(t), s_z^p(t))'$, whose cross-section at $t = 0$ has the same distribution as $(s_x, s_y, s_z)'$. Thus, the motion model takes the form

$$\begin{aligned} X(t) &= X(t-1) + \delta S_x(t), \quad S_x(t) = s_x^p(t) + \sigma_{s_x} w_x(t), \\ Y(t) &= Y(t-1) + \delta S_y(t), \quad S_y(t) = s_y^p(t) + \sigma_{s_y} w_y(t), \\ Z(t) &= Z(t-1) + \delta S_z(t), \quad S_z(t) = s_z^p(t) + \sigma_{s_z} w_z(t). \end{aligned} \tag{17}$$

To determine the sequence $s^p(t) = (s_x^p(t), s_y^p(t), s_z^p(t))'$ for $t > 0$, we define $p(t) = X_{(p_X+1)_t} - X_{(p_X+1)_{t-1}}$ as the indicator of jumps of the process $P(\lambda_{t\delta} t\delta)$ on the current discretization interval. Assume that $s^p(t) = s^p(t-1)$ if $p(t) = 0$, i.e., the constant average velocity remains invariable without jumps. For $p(t) = 1$, $s^p(t)$ becomes a new random variable. To find its distribution, we use the same idea as for the distribution of the initial velocity $(s_x, s_y, s_z)'$, which means motion on average towards the object $\mathcal{O}$ (i.e., the origin) while preserving the variance. To this end, the new value of the average velocity $s^p(t)$ is modeled by the uniform distribution with mean $-X_t$ and the same covariance as in the previous model. Specifically, if we denote $s_x \sim R[a_x, b_x]$, then $s_x^p(t) \sim R[a_x, b_x] - (a_x + b_x)/2 - X(t-1)$, i.e., the conditional distribution of $s_x^p(t)$ given $X(t-1)$ has the mean $-X(t-1)$ (preserves on average the direction of $\mathcal{A}$'s motion towards the object $\mathcal{O}$) and the variance $\mathsf{D}[s_x^p(t) \,|\, X(t-1)] = D_{s_x}$. Similar expressions describe $s_y^p(t)$ and $s_z^p(t)$.

The process $P(u)$ used in the experiment has an intensity $\lambda_u = \frac{3}{6\text{ min}}$, i.e., during the observation time, three changes in the constant average velocity $(s_x, s_y, s_z)'$ occur on average. (In other words, the average time between jumps is 2 min.) In other respects, model (17) retains the same parameters.

It remains to specify the parameters of the observers. According to the aforesaid, there are two observers with chosen coordinates. Hence, we have to set the parameters of the measurement accuracy of y_t. The used values can be represented as

$$\begin{gathered}\operatorname{cov}(v_t^F, v_t^F) = \operatorname{cov}(v_t^S, v_t^S) = \operatorname{diag}\{\sigma_\varphi, \sigma_\lambda, \sigma_r\}, \\ \sigma_\varphi = \sigma_\lambda = \frac{\pi}{180}\text{ rad }(1^\circ), \quad \sigma_r = 0.1\text{ km }(100\text{ m}).\end{gathered} \tag{18}$$

The distributions of the errors v_t^F and v_t^S are Gaussian.

3.2. Numerical Experiments

Using computer simulations of $N = 10,000$ motion trajectories of the form (16) and (17) and observations $y_t^{(\mathcal{F})} = (y_{\varphi_t}^{(\mathcal{F})}, y_{\lambda_t}^{(\mathcal{F})}, y_{r_t}^{(\mathcal{F})})'$ and $y_t^{(\mathcal{S})} = (y_{\varphi_t}^{(\mathcal{S})}, y_{\lambda_t}^{(\mathcal{S})}, y_{r_t}^{(\mathcal{S})})'$, we computed the position estimates $\widehat{X}_t = (\widehat{X}(t), \widehat{Y}(t), \widehat{Z}(t))'$ by formulas (15) for the observation modes with time delays ($T = 56$) and without them ($T = 0$) and the approximations of angular pseudomeasurements with the parameters $\mathsf{E}\{v_i^2\} = \sigma_{\varphi,\lambda}^2$ and $\mathsf{E}\{v_i^2\} = 1/4\,\sigma_{\varphi,\lambda}^2$. The estimation accuracy was determined by the root-mean-square deviations $\sigma_{\widehat{X}}(t), \sigma_{\widehat{Y}}(t)$, and $\sigma_{\widehat{Z}}(t)$ (indicated in meters in the figures below), computed by averaging the estimation errors over the simulated pencil.

Figure 3 illustrates the experiment with a typical example of the AUV trajectory: the coordinates $X(t)$ and $Y(t)$ with their estimates $\widehat{X}(t)$ and $\widehat{Y}(t)$ (Fig. 3a) and the velocities $S_x(t)$ and $S_y(t)$ (Fig. 3b). This example corresponds to model (17) and the approximation of angular pseudomeasurements with $\mathsf{E}\{v_i^2\} = 1/4\,\sigma_{\varphi,\lambda}^2$. The motion trajectories (16) differ by a more rectilinear form, as there are no changes in direction and velocity magnitude; the dynamics in depth $Z(t)$ are an order of magnitude smoother. Note that despite the quite chaotic velocity values, the general direction of $\mathcal{A}$ towards $\mathcal{O}$ is maintained both along the trajectory and when the velocity direction changes. For the presented trajectory, the time delays varied from 35 to 32; among all the simulated trajectories, from 55 to 25.

Note that in Fig. 3a, the beginning of the motion is accompanied by a group of inaccurate estimates. It corresponds to the first 56 steps (the initial period) without EKF estimation by the algorithm (15). At these steps, direct measurements were estimated: assuming the error-free nature of the two available measurements $y_t^{(\mathcal{F})}$ and $y_t^{(\mathcal{S})}$, the coordinates were computed from each set of angles and range, and the final position was estimated as their average value. Further, the root-mean-square deviations of this estimate are denoted by $\Sigma_{\widehat{X}}(t), \Sigma_{\widehat{Y}}(t)$, and $\Sigma_{\widehat{Z}}(t)$. The estimation accuracy is illustrated in Fig. 4. We pay the reader's attention to the initial period where $\sigma_{\widehat{X}}(t) = \Sigma_{\widehat{X}}(t), \sigma_{\widehat{Y}}(t) = \Sigma_{\widehat{Y}}(t)$, and $\sigma_{\widehat{Z}}(t) = \Sigma_{\widehat{Z}}(t)$. The EKF estimate was computed starting from $t = 57$.

Other variants of the computations (the motion model (16), no time delays ($T = 0$), and the approximation of angular pseudomeasurements with $\mathsf{E}\{v_i^2\} = \sigma_{\varphi,\lambda}^2$) have some differences. For instance, model (16) gives a more rectilinear trajectory, observations with $T = 0$ lead to no transition period with the direct measurement filter, and the parameters of the pseudomeasurement noises change the accuracy of the resulting estimates. These figures illustrate a qualitative picture of the effectiveness of the linear pseudomeasurement filter in the most complex model. A formal comparison in all models is given in the table below. To characterize the accuracy, the root-mean-square

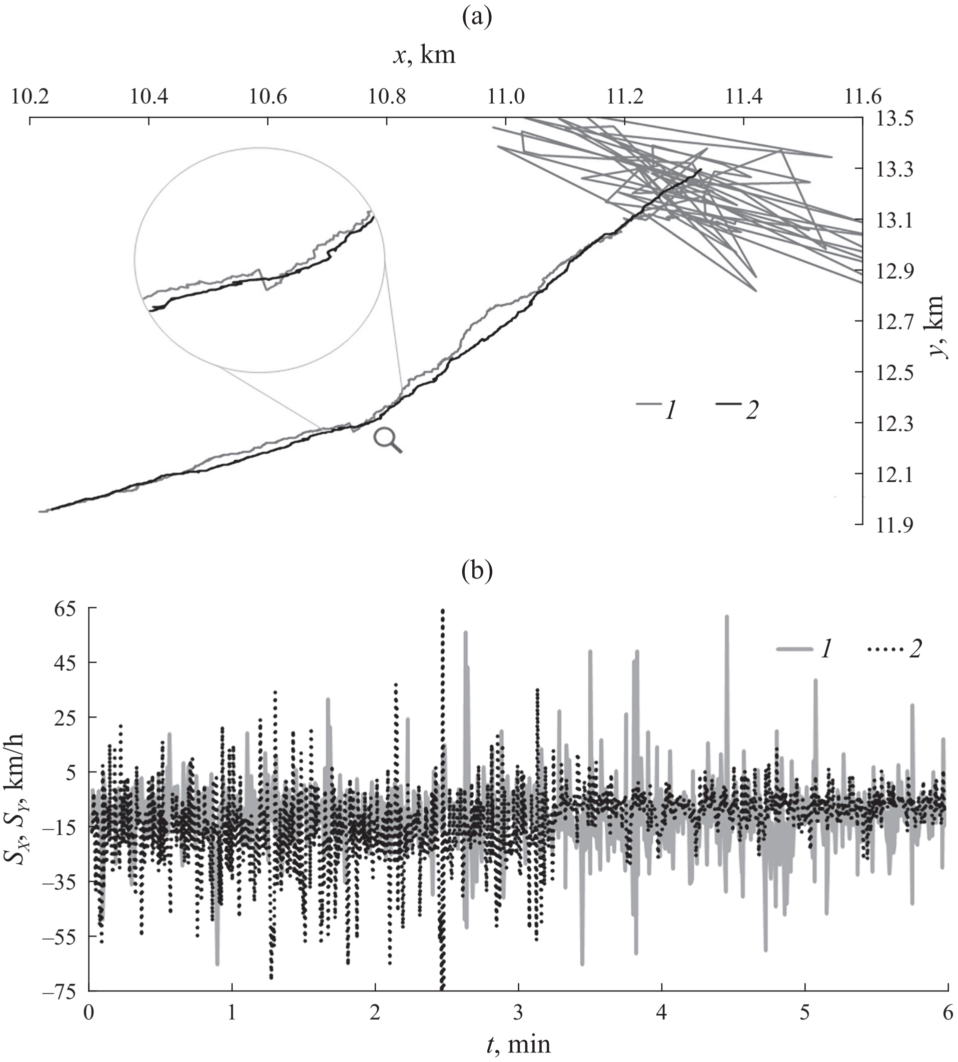

Fig. 3. A typical example of the AUV trajectory: (a) (*1*) the coordinates $X(t)$ and $Y(t)$ and (*2*) their estimates $\widehat{X}(t)$ and $\widehat{Y}(t)$; (b) the velocities (*1*) $S_x(t)$ and (*2*) $S_y(t)$.

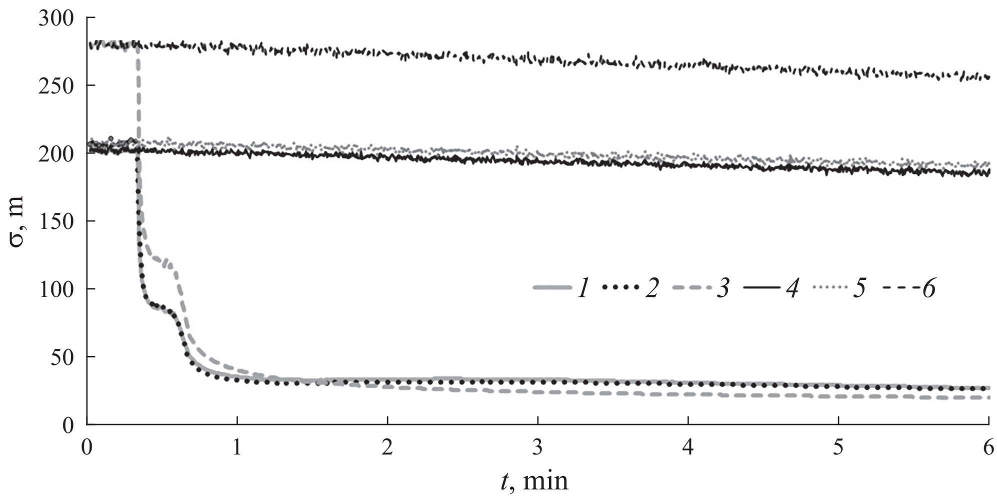

Fig. 4. Root-mean-square deviations: (*1*) $\sigma_{\widehat{X}}(t)$, (*2*) $\sigma_{\widehat{Y}}(t)$, (*3*) $\sigma_{\widehat{Z}}(t)$, (*4*) $\Sigma_{\widehat{X}}(t)$, (*5*) $\Sigma_{\widehat{Y}}(t)$, and (*6*) $\Sigma_{\widehat{Z}}(t)$.

deviations of the estimation errors were averaged over the trajectories: for $\widehat{X}(t)$ as an example, the values $\widehat{\sigma}_{\widehat{X}} = \frac{1}{1000}\sum_{t=1}^{1000}\sigma_{\widehat{X}}(t)$ and $\widehat{\Sigma}_{\widehat{X}} = \frac{1}{1000}\sum_{t=1}^{1000}\Sigma_{\widehat{X}}(t)$ were computed, etc. All deviations are presented in meters.

Comparison of estimation quality

Model	$\widehat{\sigma}_{\widehat{X}}$	$\widehat{\sigma}_{\widehat{Y}}$	$\widehat{\sigma}_{\widehat{Z}}$	$\widehat{\Sigma}_{\widehat{X}}$	$\widehat{\Sigma}_{\widehat{Y}}$	$\widehat{\Sigma}_{\widehat{Z}}$
(16), $T=0$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	24.01	22.33	27.04	192.54	198.35	266.86
(16), $T=0$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	**21.96**	**22.07**	**22.69**			
(16), $T=56$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	37.82	37.09	44.76	193.42	199.23	267.89
(16), $T=56$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	36.47	37.32	41.31			
(17), $T=0$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	24.78	23.34	26.55	193.04	198.56	267.44
(17), $T=0$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	22.73	22.72	24.55			
(17), $T=56$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	50.46	47.37	49.23	193.95	199.48	268.49
(17), $T=56$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	**44.46**	**43.37**	**45.63**			

4. CONCLUSIONS

The experiment has confirmed the ability of the linear pseudomeasurements filter to estimate the system state in the model with time delays. Compared to direct estimation, the results have demonstrated the effectiveness of this filter and a twofold deterioration in estimation quality in the case of time delays. Regarding the absolute error values of tens of meters, we emphasize extreme estimation conditions: a large distance to the object and external disturbances of the same magnitude as the object's velocity. Moreover, from the standpoint of tracking, tens of meters is a quite satisfactory order of errors for an object located farther than 10 km. More accurate results are needed when solving the positioning task aboard the AUV. But in this case, onboard measurements (e.g., velocity) can be utilized besides external observers. This significantly increases the accuracy [25].

Furthermore, note the superiority of the filter with $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$ (i.e., the model with softer assumptions regarding the error in pseudomeasurements). This parameter gives the greatest advantage in the last (most complex) model; in the others, the difference is small. Here, we should mention the results not included in the table, namely, the experiments with other values of $\mathsf{E}\{v_i^2\}$. According to the results in the table, the filter seems to be insensitive to this value, since the filtering quality estimates change little for different $\mathsf{E}\{v_i^2\}$. However, this is true only for the values of $\mathsf{E}\{v_i^2\}$ in the range $[\frac{1}{4}, 1]\sigma^2_{\varphi,\lambda}$. Additional calculations (not included in the table) have shown that the filtering estimate deteriorates significantly for $\mathsf{E}\{v_i^2\}$ beyond this range (on the left or right).

Also, we underline that the results are in good agreement with computations performed for other similar models. For example, in [7–10], the same motion model was used together with observations of direction angle tangent, and the conditionally minimax nonlinear filtering method [15, 16] was applied for estimation.

Finally, it is crucial that in the experiments presented here, only the constant average velocity has been assumed to be known among the motion model parameters (including the model with abrupt velocity changes). Due to this feature, in particular, the accuracy of the direct measurement estimate does not deteriorate too much when passing from the model with $T = 0$ to the one with $T = 56$. According to the previous results, this parameter can be identified, which is the foundation of the approach.

While listing the positive aspects, some negative ones cannot be ignored. Despite the demonstrated effectiveness of the method of linear pseudomeasurements, the EKF has retained its worst features, primarily the tendency to diverge. Such an effect has manifested itself for the nonlinear motion model (due to the unknown parameter), when direct measurements are not used to set the initial condition for the EKF estimate, when the velocity increases and trajectories could approach coordinate planes. Therefore, although the method of linear pseudomeasurements is very good, it should be tried not only in the EKF but also in other, more reliable and stable, filtering schemes.

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STOCHASTIC SYSTEMS

Batch Quasi-Poisson Models in the Queue Analysis of Packet Telecommunication Traffic

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Abstract—A new type of input flow for queuing systems, a relative of a batch Poisson process, is proposed. As shown below, this flow is a more adequate model of modern traffic than a batch Poisson process when determining the dependence of the mean queue length in a receiving buffer on the load of an outgoing data transmission channel (using H.264 video traffic as an illustrative example). An analytical formula for the mean queue length in a G/D/1 queuing system with such an input flow is derived; therefore, the model's parameters can be estimated to approximate the mean queue length of real traffic using the least squares method. The feasibility of determining the model parameters using a neural network is also demonstrated.

Keywords: non-ordinary input flow, queuing system, telecommunication traffic, simulation modeling, neural network

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1. INTRODUCTION

In the analysis and modeling of telecommunication network traffic, a common problem is to determine the characteristics of the queue created by this traffic in the receiving buffer of a network node. In this case, the coincidence of the statistical characteristics of the model traffic with its real prototype is not as important as the coincidence of the statistical characteristics of the corresponding queues.

For packet-switched data transmission networks, classical Poisson flow models are not adequate. At the same time, the description of complex correlated flows using self-similar processes, which has attracted much attention from the scientific community in the last two decades, allows reflecting the complex correlation properties of the traffic itself, but is inconvenient for analyzing queue characteristics.

A well-proven alternative to this approach is the class of flows governed by a Markov chain and similar models. The development stages of these models were presented in the review [1]. From versatile flows, through Neuts flows (N-flows) [2], they have evolved to Markov Arrival Processes (MAPs) and their generalization known as Batch Markov Arrival Processes (BMAPs); for example, see [3–5].

In this paper, the batch Poisson flow model, which can be considered a relative of BMAP, is generalized to a batch quasi-Poisson flow in which the dependence of the mean queue length in the receiving buffer on the server load, obtained in a simulation experiment, well approximates this dependence for H.264 video traffic.

To determine the parameters of the proposed model, we apply an extension of the Pollaczek–Khinchin formula for a G/D/1 queuing system, obtained by the interval method [6]. For the quasi-Poisson flow proposed, this formula is used to derive a semi-empirical formula for the dependence of the mean queue length in the G/D/1 queuing system on the server load that agrees well with the simulation results.

The feasibility of determining the model parameters using a neural network is also demonstrated.

2. PROBLEM STATEMENT AND BRIEF DESCRIPTION OF PREVIOUS RESULTS

Let us present the previous results that will be utilized below.

Consider a Poisson flow of events with a parameter λ in which each event represents the simultaneous arrival of several requests for service (request packets). The numbers of requests in different packets are independent identically distributed discrete random variables. Such a flow is called a *batch* (*non-ordinary*) *Poisson flow* [7]. It obviously possesses the properties of stationarity and no aftereffect, but does not possess the property of ordinariness.

We denote by B_k the size of the kth request packet. Assume that its distribution is given:

$$Pr\{B_k = n\} = b_n \; \forall k.$$

Consider a random interval of length τ on the time axis. It is required to determine the moments of the random variable $m(\tau)$ that represents the number of requests of the batch Poisson flow arriving in such an interval. To solve this problem, we find the generating function of the random variable $m(\tau)$. If the generating function of the number of requests in a packet is $G_B(z) = \sum_{n=0}^{\infty} b_n z^n$, then for the generating function $G_{m(\tau)}(z)$ we have

$$G_{m(\tau)}(z) = \sum_{k=0}^{\infty} e^{-\lambda\tau} \frac{(\lambda\tau)^k}{k!} \left(G_B(z)\right)^k = e^{\lambda\tau(G_B(z)-1)}.$$

(For details, see [8].) Therefore, the mean and variance of $m(\tau)$ are given by

$$M(m(\tau)) = \lambda\tau\overline{B}, \quad D(m(\tau)) = \lambda\tau\overline{B^2}.$$

In the case of a constant size B of all packets (it will be needed below), we obtain

$$M(m(\tau)) = \lambda\tau B, \quad D(m(\tau)) = \lambda\tau B^2.$$

Let this batch Poisson flow be the input of a G/D/1 queuing system with a service time τ_S of one request. In such a system, the moments of the queue length (in principle, of any order) can be found by the interval method. (The formulas for the first and second moments were derived in [8]; a generalization to the case where the service time of one request in the queuing system is a discrete random variable with a finite number of values was presented in [9].) Here, we need only the expression for the mean under a deterministic service time of one request, which is provided by the generalized Pollaczek–Khinchin formula

$$\overline{Q}(\rho) = \frac{D(m(\tau_S)) + 2R\left(Q_i, m_{i+1}(\tau_S)\right)}{2(1-\rho)} - \frac{\rho}{2} \tag{1}$$

with the following notation: ρ is the server load; $D(m(\tau_S))$ is the variance of the number of requests arriving during the service time of one request; finally, $R\left(Q_i, m_{i+1}(\tau_S)\right)$ is the correlation moment between the queue length at the time of completing the service of a certain request and the number of requests arriving during the next service interval (all values are taken for a given ρ).

In the general case, this formula is difficult to use due to the need to compute $R\left(Q_i, m_{i+1}(\tau_S)\right)$, but for some important special cases it is possible. In particular, for the batch Poisson flow $R\left(Q_i, m_{i+1}(\tau_S)\right) = 0$, due to no aftereffect and the relations

$$M(m(\tau_S)) = \lambda\tau_S B = \rho, \quad D(m(\tau_S)) = \lambda\tau_S B^2 = \rho B$$

for packets of constant size, we obtain

$$\overline{Q}(\rho) = \frac{\rho B}{2(1-\rho)} - \frac{\rho}{2} = \frac{\rho B - \rho(1-\rho)}{2(1-\rho)}. \tag{2}$$

It is easy to see that for $B = 1$, the classical Pollaczek–Khinchin formula for an M/D/1 queuing system arises here.

In [10], the batch Poisson flow was used to approximate the first two statistical moments of the queue length of H.264 video traffic by the least squares method and showed better results than the ordinary Poisson flow. However, this approximation (obtained by the least squares method relative to the packet size as a model parameter) still cannot be considered sufficiently good. Therefore, in this paper, we present a new flow model that will be used for the same purpose.

Consider the following request flow: requests arrive in packets, and the numbers of requests in different packets are independent identically distributed discrete random variables B_k. The intervals between the arrivals of sequential packets are equal to the sum of a certain constant value T and an independent exponentially distributed random variable with a parameter λ. We will call such a flow a *quasi-Poisson flow with identical pauses of duration* T (for brevity, simply a *quasi-Poisson flow*).

The problem is to use this flow as the input of a G/D/1 queuing system to approximate the mean queue length created by real traffic (as mentioned above, H.264 video traffic).

Since we are interested in the dependence of the mean queue length on the server load, and the time scale is not important, the quasi-Poisson flow will be characterized by two parameters: B (the constant packet size) and $\alpha = T\lambda$ (the ratio of the pause between arrivals, i.e., the constant part of the inter-arrival interval, to the mean length of the exponentially distributed part of this interval). These parameters are chosen by minimizing, under the same server load, the difference between the mean queue lengths of the real traffic and the quasi-Poisson flow.

An exact solution of this problem has not been obtained; but in the next section, we present an approximate formula for the mean queue length of such a flow.

3. APPROXIMATION FOR THE MEAN QUEUE LENGTH OF THE QUASI-POISSON FLOW

First of all, note the following empirically established property of a quasi-Poisson flow: for any finite α and constant packet size B, the sample variance of the number of requests arriving in a random interval τ is close to the sample variance of a batch Poisson flow with the same B provided that they create the same server load ρ in a G/D/1 queuing system.

This property was established using a software simulation model of the flows, which was also engaged in other simulation experiments of the work. The model generated a sequence of arrival times for request packets according to the definition of the corresponding flow. In other experiments (see below), that flow of request packets was the input for the simulation model of a G/D/1 queuing system. In the former experiment, for an available flow, it was necessary to determine a constant service time of one request such that the server load would have a given value. (Values from the range [0.1, 0.9], most interesting in practice, were selected.)

In the experiments where the above closeness of sample variances for batch Poisson and quasi-Poisson flows was observed, $\tau = \tau_S$ was taken. In other words, the interval to determine the number of arriving requests was set equal to the service time of one request in the G/D/1 queuing system for a given ρ. These intervals densely covered the entire time axis, and the sample variance $\hat{D}(\rho)$ was computed from the obtained sample of the number of requests arriving in them using the formula

$$\hat{D}(\rho) = \frac{1}{N}\sum_{n=1}^{N} \hat{m}_n^2(\tau_S) - \left(\frac{1}{N}\sum_{n=1}^{N} \hat{m}_n(\tau_S)\right)^2,$$

where the sample element $\hat{m}_n(\tau_S)$, $n = 1, \dots, N$, is the number of requests arriving in the nth interval τ_S during the simulation run.

As an example, Table provides the sample variances of the number of requests arriving during the service time of one request for batch Poisson and batch quasi-Poisson flows with different α, obtained through simulation modeling.

The sample variances of arrivals for various batch flows. In all cases, $B = 20$

Load	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9
Poisson	2.00	4.00	6.00	8.00	10.03	12.05	14.05	16.06	17.96
Quasi-Poisson, $\alpha = 0.5$	1.99	3.97	5.91	7.84	9.75	11.64	13.51	15.36	17.19
Quasi-Poisson, $\alpha = 1$	1.99	3.96	5.91	7.84	9.74	11.63	13.50	15.35	17.18
Quasi-Poisson, $\alpha = 2$	1.99	3.96	5.91	7.84	9.749	11.64	13.51	15.36	17.19

The observed closeness can be explained by the following (non-rigorous) reasoning: by the definition of the quasi-Poisson flow, the interval between sequential packet arrivals always exceeds the pause T, and if T is subtracted from each of these intervals, the remainders will be independent exponentially distributed variables. That is, the flow with these remainders as the inter-arrival intervals will be a batch Poisson flow. Let us call it the *embedded flow*. Thus, the original quasi-Poisson flow is obtained from the embedded one by inserting a *pause* (an interval of constant length T during which no requests arrive) after the arrival of each packet.

On any sufficiently large time interval, the embedded Poisson flow will occupy only the $(1/(1+\alpha))$th part of this interval on average, and the rest will be occupied by pauses. Therefore, for the embedded flow to create the same server load over the entire interval as the batch Poisson flow with the same packet size, the rate of packet arrivals in the embedded flow must be proportionally higher.

For a Poisson flow, increasing the rate also means increasing the variance of the number of arriving packets over a certain time interval by the same factor. As the experiment showed, alternating the realization segments of the embedded flow with pause intervals introduces no significant changes in the sample variance.

Based on the closeness of the values of these sample variances, we will consider the theoretical variances of the number of requests arriving during the service interval of one request in the G/D/1 queuing system to be equal for Poisson and quasi-Poisson flows creating the same server load.

Next, let us estimate the mean queue length from the quasi-Poisson flow in the G/D/1 queuing system under low server loads. What is meant by *low load*? Each packet in the quasi-Poisson flow arrives before the start of a pause interval of length T, and the server (if free) begins to serve it in this interval. A load ρ is low if, under it, the service of the packet completely ends within this interval:

$$B\,\tau_S \leqslant T, \tag{3}$$

where τ_S denotes the service time of one request from the packet. The boundary of the interval for low ρ, denoted by ρ_0, can be calculated as follows: by the definition of a quasi-Poisson flow, B requests arrive on average during the time $T + 1/\lambda$ (i.e., one request per $(T+1/\lambda)/B$ units of time). Under a server load ρ, the service time of one request should accordingly be

$$\tau_S = \rho\,\frac{T + 1/\lambda}{B}. \tag{4}$$

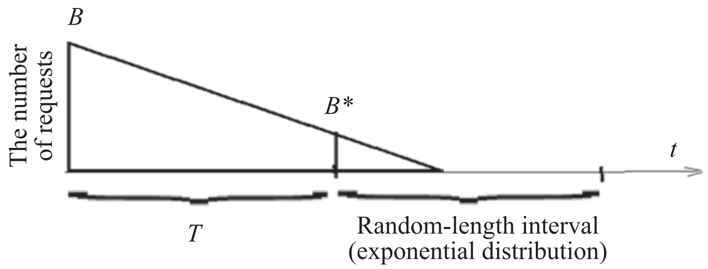

Fig. 1. Change in the number of arrived packet requests during a pause in the system with high load.

In view of (3), we obtain

$$\rho_0 = \frac{T}{T + 1/\lambda} = \frac{\alpha}{\alpha + 1}. \tag{5}$$

Clearly, under low loads, the situation is very similar to the case of a batch deterministic flow (where packets always arrive at equal intervals). Therefore, the generalized Pollaczek–Khinchin formula can be applied to this situation, as to the deterministic flow. (This is another special case where it can be easily done.) Here, we utilize the following consideration:

$$R\left(Q_i, m_{i+1}(\tau_S)\right) = M\left(Q_i\, m_{i+1}(\tau_S)\right) - M\left(Q_i\right)\ M(m_{i+1}(\tau_S)),$$

but Q_i is nonzero only during the pause interval, in which $m_{i+1}(\tau) = 0$, i.e.,

$$R\left(Q_i, m_{i+1}(\tau_S)\right) = -M\left(Q_i\right)\ M(m_{i+1}(\tau_S)) = -\,\overline{Q}(\rho)\ \rho.$$

Substituting this relation into (1) yields

$$\overline{Q}(\rho) = \frac{D(m(\tau)) - 2\overline{Q}(\rho)\rho}{2(1-\rho)} - \frac{\rho}{2}$$

and, after straightforward transformations,

$$\overline{Q}(\rho) = \frac{\rho B - \rho(1-\rho)}{2}. \tag{6}$$

Note that for large B, the dependence of $\overline{Q}$ on ρ is close to linear.

Now consider an approximate estimate of the queue length under high loads ($\rho > \rho_0$).

In this case, $B\tau_S > T$, and at the start of the exponentially distributed part of the inter-arrival interval, the number of packet requests still remaining in the system can be estimated as

$$B^*(\rho) = \left\lceil \frac{B\tau_S - T}{\tau} \right\rceil = \left\lceil B - \frac{T}{\tau_S} \right\rceil = \left\lceil B\left(1 - \frac{\rho_0}{\rho}\right) \right\rceil, \tag{7}$$

where formulas (4) and (5) have been used and the brackets $\lceil \ldots \rceil$ stand for the ceiling of an appropriate number (the smallest integer greater than or equal to this number). For illustration, Fig. 1 shows the dependence of the number of packet requests remaining in the system during the corresponding pause at the start of which this packet has arrived. For a request being served, its currently uncompleted part is taken into account.

In other words, at the end of the pause, the size of the unserved packet will be less than the initial one and will depend on ρ. By removing all pauses from the quasi-Poisson flow (“gluing” the exponentially distributed intervals), we get a batch Poisson flow with the packet size B^*. For estimation, assume that the queue formed by such a Poisson flow will be added to the queue from (6). This additional queue can be found using formula (2), but with ρ replaced by

$$\rho^* = \frac{\rho - \rho_0}{1 - \rho_0}. \tag{8}$$

Indeed, this special batch Poisson flow appears only if $\rho > \rho_0$; for $\rho = 1$, the equality $\rho^* = 1$ must hold for the queue to go to infinity.

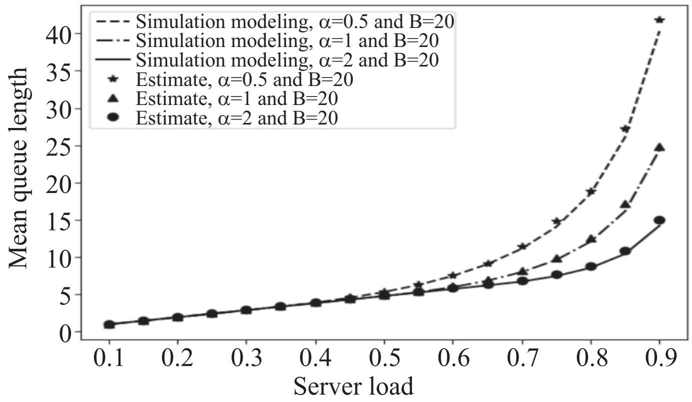

Fig. 2. Comparison of the dependence of the mean queue length on the server load: the simulation experiment vs. formula (9). For all flows, $B = 20$.

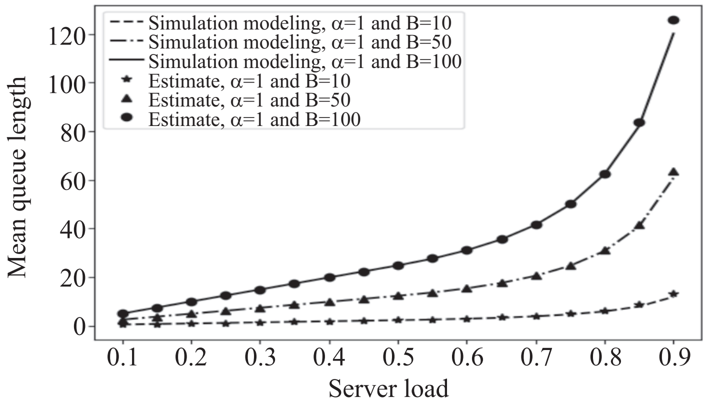

Fig. 3. Comparison of the dependence of the mean queue length on the server load: the simulation experiment vs. formula (9). For all flows, $\alpha = 1$.

In the queue expression, it is also necessary to correct the term related to the segment of the pause T, i.e., the one similar to the expression (6). Now the "tail" of the packet of size B is served beyond the interval T and is accounted for in the expression for the Poisson part of the queue. Therefore, it must be excluded from (6). Since the mean queue length is proportional to the area of the figure in Fig. 1, the correction factor can be found as the ratio of the areas of the trapezoid with bases B and B^* and the triangle with leg B :

$$\frac{S^*}{S} = \frac{\tau(B^2 - (B^*)^2)}{\tau B^2} = 1 - \frac{(B^*)^2}{B^2} \approx 1 - \left(1 - \frac{\rho_0}{\rho}\right)^2 = \left(2 - \frac{\rho_0}{\rho}\right)\frac{\rho_0}{\rho}.$$

Thus, we arrive at the following formula for the queue length:

$$\overline{Q}(\rho) = \begin{cases} \dfrac{\rho B - \rho(1-\rho)}{2}, & \rho \leqslant \rho_0, \\ \dfrac{\rho_0}{\rho}\left(2 - \dfrac{\rho_0}{\rho}\right)\left(\dfrac{\rho B - \rho(1-\rho)}{2}\right) + \dfrac{\rho^* B^* - \rho^*(1-\rho^*)}{2(1-\rho^*)}, & \rho > \rho_0, \end{cases} \tag{9}$$

where B^* and ρ^* are given by (7) and (8), respectively.

Finally, we compare the mean queue length calculated by formula (9) with the results of a simulation experiment on the passage of a quasi-Poisson flow through the G/D/1 queuing system, see Figs. 2 and 3. Here, examples are given for different values of the quasi-Poisson flow parameters: the lines correspond to the simulation results and various markers to the estimates (9). The approximation turns out to be good.

4. APPROXIMATION OF A REAL VIDEO TRAFFIC QUEUE

For experiments, we take traces of H.264 video traffic under different values of the video buffer size used for compressing video frames. In this case, a trace is a sequence of times when information packets exit the video codec, and each such time is considered to be the arrival of a service request in a G/D/1 queuing system. The service time of one request is taken so that the server load ρ under this input flow equals a given value. During the passage of this flow through the queuing system, the empirical time-averaged queue length for a given ρ is calculated.

For each trace of the real traffic, the mean queue lengths were calculated thereby for several values of the server load ρ, denoted by $\hat{Q}(\rho_i),\ i = 1, \ldots, N$. They were used to estimate, by the least squares method, the parameters of the quasi-Poisson flow for which the dependence of the mean queue length in the G/D/1 queuing system on ρ approximates the dependence of the mean queue length of the real traffic on ρ found in the above simulation experiment.

The estimates $\hat{\alpha}$ and $\hat{B}$ of the parameters α and B of the approximating quasi-Poisson flow were found as

$$\hat{\alpha}, \hat{B} = \arg\min_{\alpha, B} \sum_{i=1}^{N} \left(\hat{Q}(\rho_i) - \overline{Q}(\rho_i) \right)^2,$$

where $\overline{Q}(\rho_i)$ was calculated by formula (9).

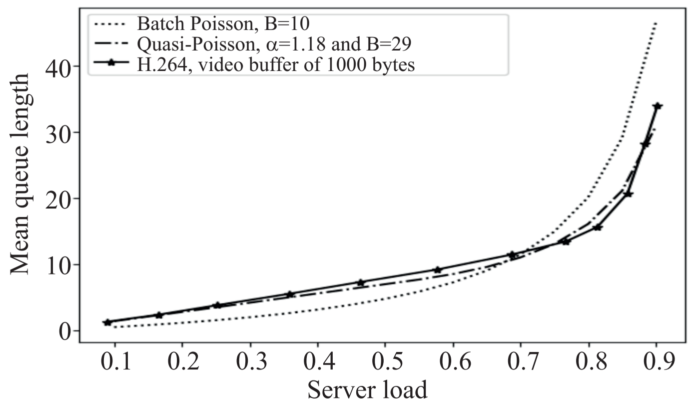

Fig. 4. Approximation of the dependence of the mean queue length on the server load for H.264 video traffic. Video buffer size is 1000 bytes.

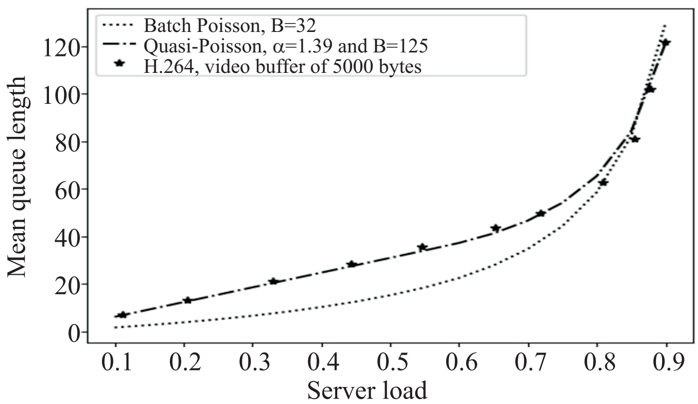

Fig. 5. Approximation of the dependence of the mean queue length on the server load for H.264 video traffic. Video buffer size is 5000 bytes.

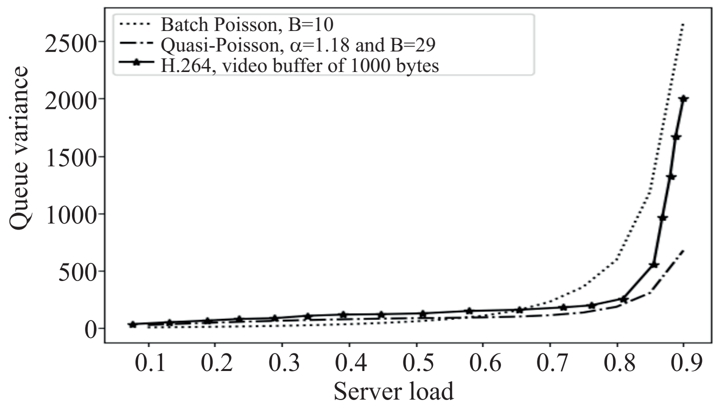

Fig. 6. Approximation of the dependence of the queue variance on the server load for H.264 video traffic. Video buffer size is 1000 bytes.

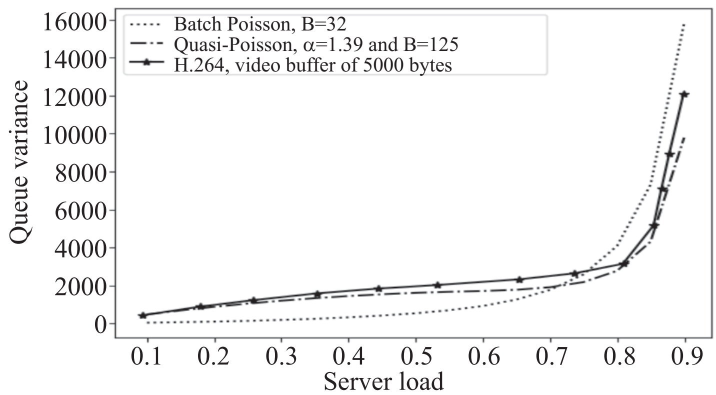

Fig. 7. Approximation of the dependence of the queue variance on the server load for H.264 video traffic. Video buffer size is 5000 bytes.

Figures 4 and 5 show the results, including a similar approximation by a batch Poisson flow for comparison; its packet size was also found by the least squares method. (The real traffic is indicated by the solid line, its quasi-Poisson flow approximation by the dashed line with markers, and its Poisson flow approximation by the dotted line.) Obviously, the quasi-Poisson flow provides a significantly more accurate approximation of the mean queue length of this real traffic.

For a complete picture, it is interesting to see the quality of approximation for the variances of the queue lengths of the real traffic. They are demonstrated in Figs. 6 and 7. Although the difference between the real traffic and the approximating model traffic is greater than for the mean queue length, the quasi-Poisson flow approximation is still good and much better than the batch Poisson flow counterpart.

Note that from a practical viewpoint, the closeness of the first two moments of the distributions of two random variables allows speaking of a considerable closeness of their distributions as well. Therefore, the estimates presented above have practical value.

5. ESTIMATION OF THE PARAMETERS OF THE APPROXIMATING MODEL USING A NEURAL NETWORK

Above, we have derived a good approximating formula for the mean queue length. Here, let us present another way to estimate the traffic model parameters, suitable for the case of no approximating formula. It consists in training a neural network to determine the parameters of the quasi-Poisson flow.

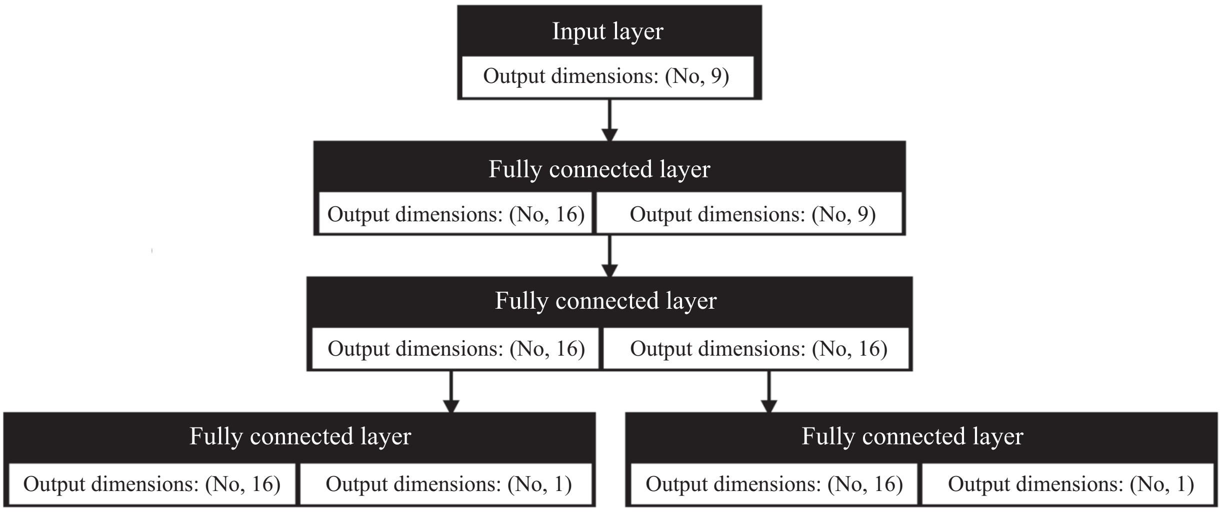

Fig. 8. The neural network architecture (the standard graphical representation in the Keras package of the Python language).

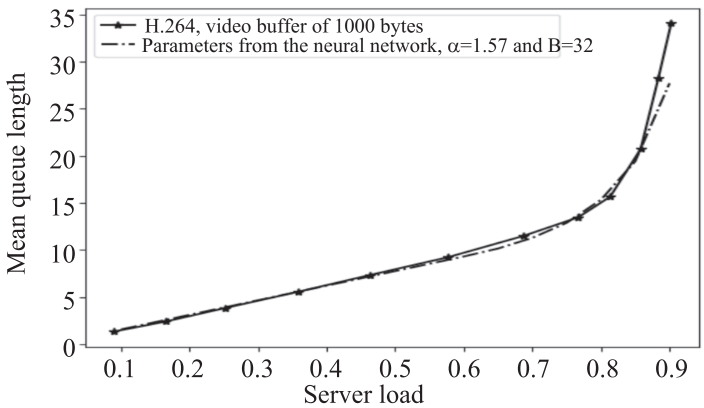

Fig. 9. Approximation of the dependence of the mean queue length on the server load for H.264 video traffic when determining the parameters using a neural network. Video buffer size is 1000 bytes.

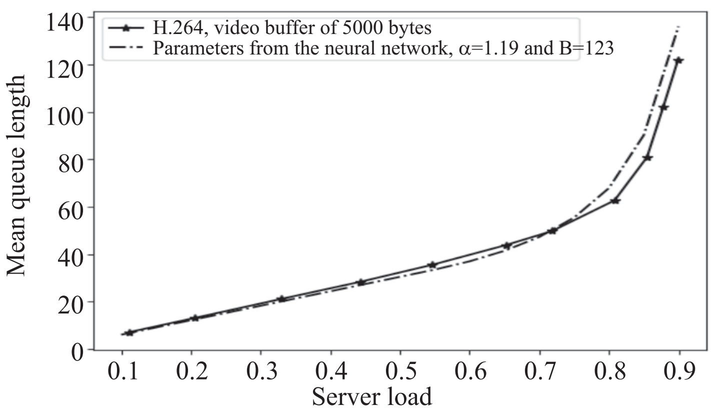

Fig. 10. Approximation of the dependence of the mean queue length on the server load for H.264 video traffic when determining the parameters using a neural network. Video buffer size is 5000 bytes.

The input data for the neural network are a set of mean queue lengths for given server loads. In this work, nine values corresponding to loads from 0.1 to 0.9 with a step of 0.1 were used. There are two output values, namely, the values of the parameters α and B. After supervised training of the neural network, the set of mean queue lengths obtained during the passage of the real traffic through the queuing system under the same loads is supplied as the input parameters for recognition. The output of the neural network is an estimate of the parameters of the quasi-Poisson flow for approximating the real traffic.

Figure 8 shows the architecture of the neural network used. Two neurons, each representing an output layer to estimate one parameter of the quasi-Poisson flow, have no nonlinear element; the others have ReLU. All layers are fully connected.

The training set consisted of 2000 examples, but with a particular sample for each real trace; training examples were generated with parameters from a small range containing the desired estimate. (The graph of the mean queue length of the real traffic lay between the graphs for the boundary values of the parameters of the examples from the training set.)

Figures 9 and 10 demonstrate the results of the experiment: the parameters predicted by the neural network were used for simulation, and the graphs show the mean queue length for both the real and simulated traffic (the solid line and the dashed line with markers, respectively). Clearly, the estimate is also quite good.

6. CONCLUSIONS

This paper has presented a quasi-Poisson flow with identical pauses as a model for approximating the mean queue length of real traffic using the H.264 video traffic as an example.

The analytical formula for estimating the mean queue length, derived in the paper, provides a good approximation to the results of simulation modeling.

The parameters of the flow for approximating the mean queue length of real traffic, obtained using this formula, provide an approximation that is significantly more accurate than the batch Poisson flow approximation.

The feasibility of determining the flow parameters for solving the approximation problem using a neural network has been demonstrated, also giving good results.

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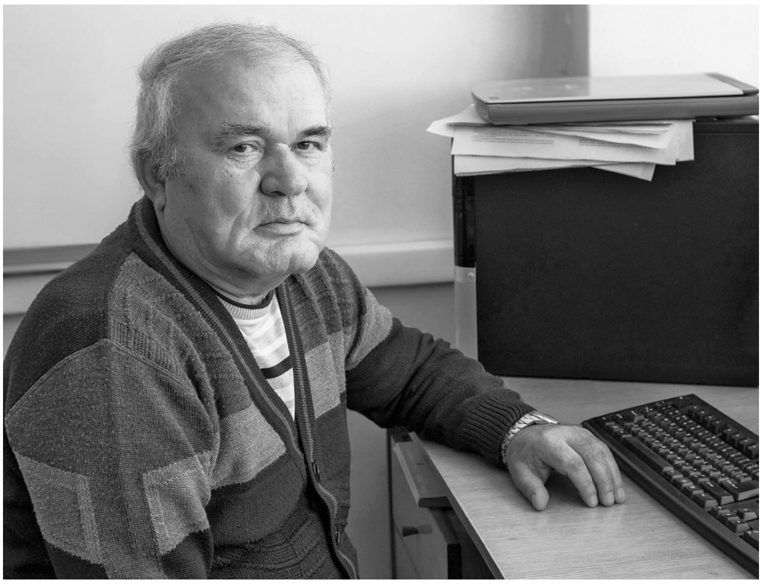
NOTES, CHRONICLES, INFORMATION



Dykhta Vladimir Aleksandrovich
(1949–2025)

On April 2, 2025, most regrettably, Vladimir Alexandrovich Dykhta, a remarkable Russian and Soviet mathematician, an Honored Scientist of the Russian Federation (1997), a remarkable person, Doctor of Physical and Mathematical Sciences, Professor, passed away. He worked until the end of his life in the positions of Chief Researcher at the Matrosov Institute for System Dynamics and Control Theory, Siberian Branch of Russian Academy of Sciences (ISDCT SB RAS)[1] and Professor of the Department of Computational Mathematics and Optimization at the Institute of Mathematics and Information Technology, Irkutsk State University (IMIT ISU)[2]. On the websites of ISDCT SB RAS, ISU, and IMIT ISU, the corresponding obituaries are posted[3,4,5]. Vladimir Alexandrovich has made an outstanding contribution to the theory of optimal control and its applications. The given below list of the publications [1–92] does not pretend to be complete. In these publications, V.A. Dykhta is the single author or a co-author. The articles were published in the journals “Automation and Remote Control” (“Avtomatika i Telemekhanika”), “Doklady Mathematics” (“Doklady Akademii Nauk”), “Engineering Cybernetics” (“Izvestija Akademii Nauk SSSR. Tehnicheskaja Kibernetika”), “Journal of Computer and System Sciences International” (“Izvestija RAN. Teorija i Sistemy Upravlenija”), “Mathematical Notes of the Academy of Sciences of the USSR” (“Matematicheskie Zametki”), “Differentsial’nye Uravneniya”, “Proceedings of the Steklov Institute of Mathematics” (“Trudy Matematicheskogo Instituta imeni V.A. Steklova”), “Compu-

[1] http://idstu.irk.ru/en
[2] https://math.isu.ru/
[3] http://idstu.irk.ru/en/inews/ushel-iz-zhizni-zasluzhennyy-deyatel-nauki-rf-vladimir-dyhta (the photo is from this web page).
[4] https://isu.ru/ru/news/2025/details/news-id2025necrologDYCHTA
[5] https://math.isu.ru/export/sites/math/ru/media/announces/2025/.galleries/docs/nekrolog.pdf

tational Mathematics and Mathematical Physics" ("Zhurnal Vychislitel'noj Matematiki i Matematicheskoj Fiziki"), "Soviet Mathematics (Izvestiya VUZ. Matematika)" and "Russian Mathematics" ("Izvestija Vysshih Uchebnyh Zavedenij. Matematika"), "Journal of Mathematical Sciences" ("Itogi Nauki i Tehniki. Sovremennaja Matematika i ee Prilozhenija. Tematicheskie Obzory"), "Proceedings of Krasovskii Institute of Mathematics and Mechanics UB RAS" ("Trudy Instituta Matematiki i Mekhaniki UrO RAN"), "Siberian Mathematical Journal" ("Sibirskij Matematicheskij Zhurnal"), "Bulletin of Irkutsk State University. Series Mathematics" ("Izvestija Irkutskogo Gosudarstvennogo Universiteta. Serija Matematika"), "Buryat State University Bulletin. Mathematics, Informatics" ("Vestnik Burjatskogo Gosudarstvennogo Universiteta. Matematika, Informatika"), "Journal of Optimization Theory and Applications", "European Journal of Control", etc. V.A. Dykhta is a co-author of the monographs' series [5–15]. In particular, [14] is the second edition of the monograph "Optimal Impulse Control with Applications" by V.A. Dykhta, O.N. Samsonyuk.

V.A. Dykhta was born on October 1, 1949 in Irkutsk. He graduated from the Faculty of Mathematics, Irkutsk University in 1972 and worked at his Alma Mater for decades in various positions: assistant, senior lecturer, associate professor, professor, and the head of a department. Under the supervision of Vladimir Iosifovich Gurman[6], a well-known Russian and Soviet scientist and specialist in the theory of optimal control and its applications, V.A. Dykhta prepared the candidate's dissertation "Dostatochnye uslovija optimal'nosti osobyh rezhimov" ("Sufficient Conditions for Optimality of Singular Regimes") in Irkutsk and defended this dissertation in 1979 in Sverdlovsk [1, 2]. In 1992, the doctoral dissertation "Rasshirenie zadach optimal'nogo upravlenija i variacionnyj princip maksimuma" ("Extension of Optimal Control Problems and the Variational Maximum Principle")[7] prepared by Vladimir Aleksandrovich during his doctoral studies at the Irkutsk Computing Center, Siberian Branch of the USSR Academy of Sciences was defended by Vladimir Aleksandrovich at the Institute of Mathematics, Siberian Branch of the Russian Academy of Sciences (Novosibirsk) [3, 4]. In 1992–2007, Vladimir Alexandrovich was the head of the Department of Higher Mathematics at the Irkutsk State Academy of Economics (now Baikal State University). Since 2008, the main place of the work for Vladimir Alexandrovich was ISDCT SB RAS, where he headed the Laboratory of Optimal Control, a department and was a Chief Researcher. V.A. Dykhta was a member of some dissertation councils, the organizing committees' head of several international school-seminars "Nonlinear Analysis and Extremal Problems" (NLA), including, for example, the first school-seminar (2008)[8]. We also note the work of V.A. Dykhta as a supervisor and a lecturer, who has prepared 11 candidates of sciences (for example, N.V. Derenko (1994), O.N. Samsonyuk (1999), N.V. Antipina (2003), and S.P. Sorokin (2012)) and 2 doctors of sciences, has published a number of textbooks, including [16–21], has given a large number of university courses, mainly in Irkutsk, and also as a visiting professor in Ulan-Ude (in the Buryat State University and East Siberian State Technological University).

An important component of the scientific heritage of V.A. Dykhta is a development of the fundamental results of V.I. Gurman and V.F. Krotov[9]. One of the key characteristics of the scientific work of Vladimir Aleksandrovich is his holistic vision of the subject of the various necessary and sufficient conditions for optimality, taking into account subtle theoretical aspects. V.A. Dykhta's

[6] About V.I. Gurman (1934–2016), there are the articles [80, 81], [90th Anniversary of the Birth Vladimir Iosifovich Gurman // Autom. Remote Control, 2024, pp. 1128–1130.]

[7] The scientific advisor was Doctor of Physical and Mathematical Sciences, Professor Aleksandr Aleksandrovich Tolstonogov. He is an Honored Scientist of the Russian Federation since 2002 and a Corresponding Member of the Russian Academy of Sciences since 2006 (http://idstu.irk.ru/en/aatol). At ISDCT SB RAS, he is an Advisor to the Director, head of a department, and a Chief Researcher.

[8] http://idstu.irk.ru/en/node/93/95

[9] About V.F. Krotov (1932–2015), a well-known Russian and Soviet scientist, specialist in the theory of optimal control theory and its applications, there are the articles [Khrustalev M.M., *AiT*, 2022, No. 5, pp. 164–168 (in Russian)], ["Krotov Vadim Fedorovich", https://www.ipu.ru/en/node/71453].

scientific heritage is impressive. The works of V.A. Dykhta, including the works with the co-authors, cover:

- various classes of optimal control problems (with concentrated parameters, just continuous, — including degenerate (in many publications, beginning with [22, 23]), non-smooth [41, 43, 86], — discrete [85, 90], discrete-continuous [66], as well as with distributed parameters [12, 28, 30, 54]);
- problems' transformations (including the nonlinear Goh transformation developed by Vladimir Aleksandrovich [14, 25]);
- various necessary and sufficient conditions for optimality (including the developed by Vladimir Aleksandrovich variational maximum principle [3, 33, 35, 49], positional minimum principle [82, 83, 88–90, 92] which are important, in particular, from the point of view of the Pontryagin maximum principle's strengthening in regular problems, studies of degenerate problems);
- methods for solving optimal control problems (for example, [39, 44] are for a constructing the methods for solving optimal impulse control problems based on the variational maximum principle);
- numerous analytical examples, including the investigation examples (in [14, 42, 46], etc.) for the optimal control problems on modeling a robotic manipulator, quantum systems, (ecological-)economic processes, etc., with the application and discussion of the conditions for optimality, problems' transformations, taking into account the issues of the constructions' mathematical correctness;
- researches in numerical experiments (in this regard, for example, [10, § 4.1, 4.2]);
- modeling of ecological-economic systems — this certain point notes the co-authorship of V.A. Dykhta in the collective monographs [5–9].

V.A. Dykhta did an active scientific creativity until his recent passing. We note the work [91], which was presented in December 2024 at the 40th conference 'Lyapunov Readings' and addresses positional strengthenings of the V.F. Krotov's method, and the article [92] developing the positional minimum principle. The collective computer program for solving optimal control problems of a certain type using a gradient method has been registered [93].

For the numerical solution of various optimal control problems — including degenerate problems and those with terminal constraints — the creation of new algorithms and computer programs based on the fundamental results of V.A. Dykhta and from the co-authored works, including, for example, the positional minimum principle, seems promising. For example, it is of interest to take into account the results on impulse control which were obtained for some quantum model optimization problems and presented in [14, § 3.7, 6.8], [42, 46] (V.A. Dykhta, et al.).

One of the goals of this article is to draw its readers' attention to both the scientific heritage of V.A. Dykhta and the scientific heritage of V.I. Gurman, V.F. Krotov, etc., and more broadly, to the development of the theory of optimal control. Young readers unfamiliar with the works of V.A. Dykhta are recommended to start with the video recordings of the talks of Vladimir Alexandrovich [94, 95].

This article was written taking into account the discussions with colleagues about its preliminary versions.

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