ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2025, Vol. 86, No. 10, pp. 953–968.

STOCHASTIC SYSTEMS

Application of a Linear Pseudomeasurement Filter to Tracking and Positioning Based on Observations with Random Delays

A. V. Bosov

Federal Research Center "Computer Science and Control," Russian Academy of Sciences, Moscow, Russia
e-mail: ABosov@frccsc.ru

Received April 2, 2025
Revised July 11, 2025
Accepted July 17, 2025

Abstract—The possibility of adaptation and effectiveness of a linear pseudomeasurement filter in a stochastic observation system model with random time delays between arriving observations and the factual state of a moving object are investigated. The method of pseudomeasurements is modified to combine the results of observations performed by several measuring devices located at different distances from the object and having different time delays. The filter is realized in a model that considers measurements of direction angles and range. Experimental computations are carried out for a model example describing the motion of an autonomous underwater vehicle that uses two stationary acoustic beacons for positioning.

Keywords: stochastic system with random observation delays, linear pseudomeasurements, extended Kalman filter (EKF), positioning, target tracking, sonars

DOI: 10.31857/S0005117925100055

1. INTRODUCTION

State filtering methods in stochastic dynamic systems find application in various fields, including the control of autonomous underwater vehicles (AUVs) [1]. Along with unmanned aerial vehicles [2] and autonomous cars [3], this field is currently a topical source of research problems. The aquatic environment itself has some features unnecessary to be considered in surface motion problems. For instance, these are such factors as variable water temperature, salinity, and pressure [4]; flows [5] are quite interesting as well. In addition to affecting the moving object, water creates significant challenges for measuring devices. Let us consider only external observers, without discussing the onboard accelerometers and gyroscopes of AUVs. Then all available measuring devices are based on general physical laws and use acoustic signals, i.e., belong to acoustic sensors or sonars [6]. A fundamental feature of such devices is the significant effect of random delays in arriving data about the observed AUV state on the measurement accuracy. This effect also occurs in measuring devices using electromagnetic radiation. For instance, a radar observing an object at a distance of 1 km will receive its coordinates with a delay of about 10^{-7} s. Such values can be neglected. For a sonar with the sound velocity in water being 1500 m/s, the delay in determining the object's coordinates at a distance of 1 km will reach about 0.7 s; at a distance of 10 km, 7 (!) s. Even if the object is not moving fast, such values cannot be neglected. This effect must be taken into account in models oriented to high-velocity AUVs.

A stochastic dynamic observation system model incorporating the delay factor of the acoustic signal was proposed in [7, 8] and extended to the identification of unknown motion model parameters in [9, 10]. The relations for optimal Bayesian filtering [11] were derived for state and parameter estimation.

As in most applications, it is impractical to use universal filtering methods, such as the extended Kalman filter (EKF) [12], particle filters [13], and various types of sigma-point filters [14], or conditionally optimal and minimax Pugachev–Pankov filters [15, 16] of standard structure, or (even more so) optimal Bayesian filters: either the realization turns out to be very costly in computational sense or suboptimal algorithms exhibit a tendency to diverge. An exception could be the method of linear pseudomeasurements, which occupies an intermediate position between universal methods applicable to any model and special ones (i.e., those intended exclusively for a particular model). Although the idea of this method is quite universal, it should be applied to particular measurements, linearizing them. In underwater navigation problems, various sensors with indirect information about the object's position are used. Among them, note direction angle and range sensors [17].

The idea of pseudomeasurements itself has been known for a long time and seems to be a logical supplement or development of the EKF as the most popular suboptimal filtering method [12]. The EKF reproduces the structure of the linear Kalman filter [18], which is optimal for state filtering in a linear Gaussian observation system and also possesses a series of outstanding properties in various problems of robust and adaptive estimation and control. Formal adherence to the linear filter structure implies linearization. In the case of the EKF, this is linearization around the state prediction to obtain heuristic estimates of the state prediction covariance and the filtering estimate. The linearization of observations improved through some functional transformations, making combinations of observations more linear, was apparently first demonstrated for direction angle measurements in [19]. A modern practical setting was presented in [20]. By updating the model, a new quality was attributed to the pseudomeasurement filter in a series of research works initiated in [21].

This paper aims to adapt the EKF based on linear pseudomeasurements for a model with time delays. To this end, Section 2 proposes a more universal pseudomeasurement model, developing the classical method [19, 20]. In Section 3, this model is used to derive the filtering equations based on the EKF for the stochastic observation system model with time delays. Section 4 is devoted to a computational experiment of tracking the motion of an AUV, observed by two stationary acoustic beacons, towards a given target. In the Conclusions, we summarize the results, including possible shortcomings of the EKF-based method of linear pseudomeasurements and some ways to eliminate them.

2. FILTERING BY THE METHOD OF LINEAR PSEUDOMEASUREMENTS

2.1. System Model and Definition of Pseudomeasurements

In this paper, the following notation is adopted: $\mathsf{E}\{X\}$ means the mathematical expectation of a random vector X; $\operatorname{cov}(X, Y)$ is the covariance of X and Y; X' stands for the transpose of X.

By assumption, the motion of an autonomous AUV (denoted by $\mathcal{A}$) is described in a reference frame $Oxyz$ where the plane Oxy coincides with the sea surface and the axis Oz is directed downward and corresponds to depth (Fig. 1).

Let the coordinates of $\mathcal{A}$ at some fixed (e.g., initial) time instant form the vector $(X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}})'$ and, when considered as time-varying functions, the vector $(X(t), Y(t), Z(t))'$.

First, suppose that the motion is observed by one measuring complex ($\mathcal{M}$) with coordinates $(X_{\mathcal{M}}, Y_{\mathcal{M}}, Z_{\mathcal{M}})$. This could be a passive acoustic device for estimating the direction of movement [22], allowing measurement aboard the AUV, or an active hydroacoustic beacon [23], forming measurements for an external observer.

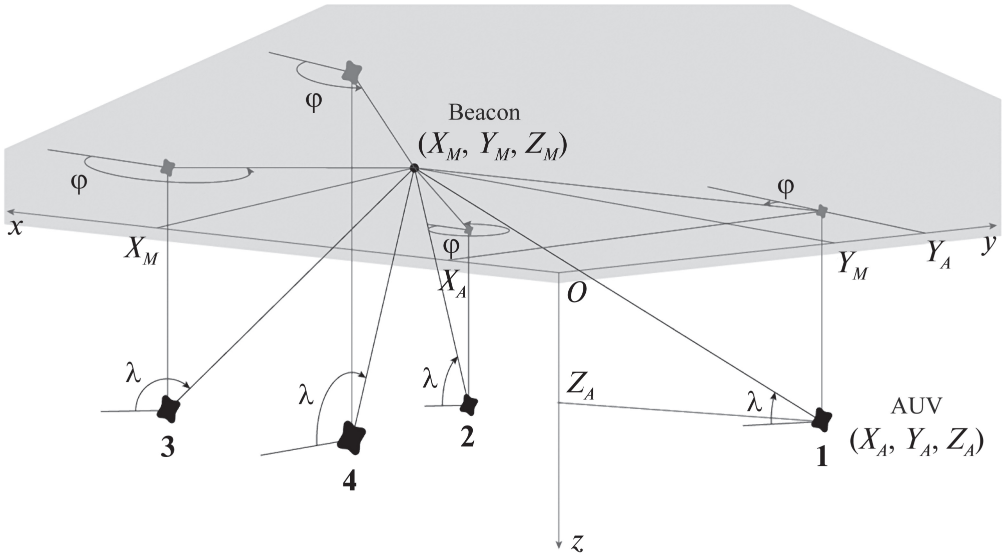

Fig. 1. Possible relative positions of AUVs and the observer.

The type of measuring device depends on the navigation task being solved. If the AUV interacts with the measuring device (the cooperative scenario), then positioning is performed aboard the AUV. In the case of opposing interests, an external device tracks a target.

Regardless of the task, it is necessary to measure the direction angle φ (azimuth or bearing) in the plane Oxy, the elevation angle λ (the inclination of the acoustic ray relative to the straight line Oz), and the range r (the distance between $\mathcal{A}$ and $\mathcal{M}$). Figure 1 illustrates possible relative positions of AUVs and the observer and the angle measurement rules.

The proposed approach to form linear pseudomeasurements from the measurements of φ, λ, and r combines the idea of a classical filter [19] and the model with tangent observations [20].

First, consider the measurement $y_\varphi = \varphi + v_\varphi$ of the bearing φ, where the error v_φ has a distribution with zero mean ($\mathsf{E}\{v_\varphi\} = 0$) and a standard deviation σ_φ ($\operatorname{cov}(v_\varphi, v_\varphi) = \sigma_\varphi^2$). Assuming that sonar errors in angle measurements are about 1–2°, we represent measurements in radians; then the value σ_φ can be set to $\sigma_\varphi = \frac{\pi}{180} \approx 0.0175$ and, consequently, $v_\varphi \ll 1$.

For the measurement y_φ, we write the sine and cosine and approximate them with the corresponding linear parts of the Taylor expansion for small v_φ :

$$y_\varphi^{\sin} = \sin(y_\varphi) = \sin(\varphi + v_\varphi) \approx \sin(\varphi) + \cos(\varphi)v_\varphi,$$
$$y_\varphi^{\cos} = \cos(y_\varphi) = \cos(\varphi + v_\varphi) \approx \cos(\varphi) - \sin(\varphi)v_\varphi.$$

Assume further that for the distribution of v_φ, only the moments $\mathsf{E}\{v_\varphi\} = 0$ and $\mathsf{E}\{v_\varphi^2\} = \sigma_\varphi^2$ are given, and the variables φ and v_φ are independent. In this case, we have $\mathsf{E}\{(\cos(\varphi)v_\varphi)^2\} \leqslant \sigma_\varphi^2$ and $\mathsf{E}\{(\sin(\varphi)v_\varphi)^2\} \leqslant \sigma_\varphi^2$. Moreover, $\mathsf{E}\{\cos(\varphi)v_\varphi \sin(\varphi)v_\varphi\} = \frac{1}{2}\sigma_\varphi^2 \mathsf{E}\{\sin(2\varphi)\} = 0$ if φ is distributed symmetrically about zero. In view of the physical meaning of φ, the latter assumption seems quite realistic. Recall the well-known minimax property of the Gaussian distribution, which maximizes the variance of a random variable in the class of distributions with known mean and bounded variance [24]. Therefore, we arrive at an approximation of the form

$$y_\varphi^{\sin} \approx \sin(\varphi) + v_1, \quad y_\varphi^{\cos} \approx \cos(\varphi) + v_2,$$

where v_1 and v_2 are independent Gaussian random variables with $\mathsf{E}\{v_1\} = \mathsf{E}\{v_2\} = 0$ and $\mathsf{E}\{v_1^2\} = \mathsf{E}\{v_2^2\} = \sigma_\varphi^2$. Here, the measurement error $(v_1, v_2)'$ is interpreted as the worst-case one.

Such an interpretation is expected to be excessively rough, which is characteristic of minimax estimates. Thus, it makes sense to focus on another approximation of $\mathsf{E}\{(\cos(\varphi)v_\varphi)^2\}$ and $\mathsf{E}\{(\sin(\varphi)v_\varphi)^2\}$. These moments cannot be computed without knowing the distribution of φ. But it can be supposed that φ takes any value with equal probability, i.e., has a “nearly” uniform distribution. This reflects the assumption that the target can appear anywhere; then

$$\mathsf{E}\{(\cos(\varphi)v_\varphi)^2\} = \mathsf{E}\{(\sin(\varphi)v_\varphi)^2\} \approx \frac{1}{4}\sigma_\varphi^2.$$

Which approximation is better, $\mathsf{E}\{v_{1,2}^2\} = \sigma_\varphi^2$ or $\mathsf{E}\{v_{1,2}^2\} = \frac{1}{4}\sigma_\varphi^2$? It is possible to test them experimentally.

Continuing the considerations, we obtain

$$\sin(\varphi) \approx y_\varphi^{\sin} - v_1, \quad \cos(\varphi) \approx y_\varphi^{\cos} - v_2,$$
$$\tan(\varphi) = \frac{Y_\mathcal{A} - Y_\mathcal{M}}{X_\mathcal{A} - X_\mathcal{M}} \approx \frac{y_\varphi^{\sin} - v_1}{y_\varphi^{\cos} - v_2},$$
$$(Y_\mathcal{A} - Y_\mathcal{M})y_\varphi^{\cos} - (X_\mathcal{A} - X_\mathcal{M})y_\varphi^{\sin} \approx (Y_\mathcal{A} - Y_\mathcal{M})v_2 - (X_\mathcal{A} - X_\mathcal{M})v_1.$$

By replacing the exact coordinates $(X_\mathcal{A}, Y_\mathcal{A})$ in the last expression with their estimates, one derives the pseudomeasurement residual figuring in the filtering equations. The pseudomeasurements themselves can be written as

$$-Y_\mathcal{M}y_\varphi^{\cos} + X_\mathcal{M}y_\varphi^{\sin} \approx (y_\varphi^{\sin}, -y_\varphi^{\cos}) \begin{pmatrix} X_\mathcal{A} \\ Y_\mathcal{A} \end{pmatrix} + (Y_\mathcal{A} - Y_\mathcal{M})v_2 - (X_\mathcal{A} - X_\mathcal{M})v_1,$$

which explains the meaning of the above transformations: the pseudomeasurements $-Y_\mathcal{M}y_\varphi^{\cos} + X_\mathcal{M}y_\varphi^{\sin}$ approximate the measurements of a linear combination of the estimated coordinates $(X_\mathcal{A}, Y_\mathcal{A})$ under an additive noise with known covariance.

Thus, for the measurement $y_\varphi = \varphi + v_\varphi$ of the bearing φ, the pseudomeasurement Y_φ is formed as follows:

$$Y_\varphi = -Y_\mathcal{M}y_\varphi^{\cos} + X_\mathcal{M}y_\varphi^{\sin}, \quad y_\varphi^{\sin} = \sin(y_\varphi), \quad y_\varphi^{\cos} = \cos(y_\varphi); \tag{1}$$

the filtering algorithm uses the observation model

$$Y_\varphi = (y_\varphi^{\sin}, -y_\varphi^{\cos}) \begin{pmatrix} X_\mathcal{A} \\ Y_\mathcal{A} \end{pmatrix} + (X_\mathcal{M} - X_\mathcal{A}, Y_\mathcal{A} - Y_\mathcal{M}) \begin{pmatrix} v_1 \\ v_2 \end{pmatrix}. \tag{2}$$

Next, consider the measurement $y_\lambda = \lambda + v_\lambda$ of the elevation angle λ. Similarly to the bearing, we approximate the sine and cosine based on the same assumptions about the measurement error v_λ:

$$y_\lambda^{\sin} = \sin(y_\lambda) \approx \sin(\lambda) + \cos(\lambda)v_\lambda \approx \sin(\lambda) + v_3,$$
$$y_\lambda^{\cos} = \cos(y_\lambda) \approx \cos(\lambda) - \sin(\lambda)v_\lambda \approx \cos(\lambda) + v_4,$$

where v_3 and v_4 are independent Gaussian random variables with $\mathsf{E}\{v_3\} = \mathsf{E}\{v_4\} = 0$ and $\mathsf{E}\{v_3^2\} = \mathsf{E}\{v_4^2\} = \sigma_\lambda^2$ (in the alternative approximation, $\mathsf{E}\{v_{3,4}^2\} = \frac{1}{4}\sigma_\lambda^2$). Hence,

$$\sin(\lambda) \approx y_\lambda^{\sin} - v_3, \quad \cos(\lambda) \approx y_\lambda^{\cos} - v_4,$$
$$\tan(\lambda) = \frac{Z_\mathcal{A} - Z_\mathcal{M}}{|X_\mathcal{A} - X_\mathcal{M}|}\cos(\varphi) \approx \frac{y_\lambda^{\sin} - v_3}{y_\lambda^{\cos} - v_4}.$$

To simplify manipulations with the measurement of λ, let the reference frame be chosen so that, for the relative position of $\mathcal{A}$ and $\mathcal{M}$, $X_{\mathcal{A}} > X_{\mathcal{M}}$ and $X(t) > X_{\mathcal{M}}$ during the further motion. Using the available bearing approximation, we replace $\cos(\varphi)$ with $y_\varphi^{\cos} - v_2$ to get

$$y_\varphi^{\cos} y_\lambda^{\cos}(Z_{\mathcal{A}} - Z_{\mathcal{M}}) - y_\lambda^{\sin}(X_{\mathcal{A}} - X_{\mathcal{M}})$$
$$\approx (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} v_4 - (Z_{\mathcal{A}} - Z_{\mathcal{M}}) v_2 v_4.$$

By the above assumption, all v_i are independent and centered, so the variance of the right-hand side of this expression (the pseudomeasurement error) has the form

$$\mathsf{E}\Big\{\big((Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} v_4 - (Z_{\mathcal{A}} - Z_{\mathcal{M}}) v_2 v_4\big)^2\Big\}$$
$$= \mathsf{E}\left\{(Z_{\mathcal{A}} - Z_{\mathcal{M}})^2 (y_\lambda^{\cos})^2\right\} \sigma_\varphi^2 + \mathsf{E}\left\{(X_{\mathcal{A}} - X_{\mathcal{M}})^2\right\} \sigma_\lambda^2$$
$$+ \mathsf{E}\left\{(Z_{\mathcal{A}} - Z_{\mathcal{M}})^2 (y_\varphi^{\cos})^2\right\} \sigma_\lambda^2 + \mathsf{E}\left\{(Z_{\mathcal{A}} - Z_{\mathcal{M}})^2\right\} \sigma_\varphi^2 \sigma_\lambda^2.$$

Appealing to the same arguments about the small values of v_i, σ_φ, and σ_λ, we neglect the last term and represent the pseudomeasurement residual as

$$y_\varphi^{\cos} y_\lambda^{\cos}(Z_{\mathcal{A}} - Z_{\mathcal{M}}) - y_\lambda^{\sin}(X_{\mathcal{A}} - X_{\mathcal{M}})$$
$$\approx (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_A - Z_M) y_\varphi^{\cos} v_4.$$

(Here, the estimates are substituted for the exact coordinates $(X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}})$.) The pseudomeasurements themselves become

$$-y_\varphi^{\cos} y_\lambda^{\cos} Z_{\mathcal{M}} + y_\lambda^{\sin} X_{\mathcal{M}}$$
$$\approx -y_\varphi^{\cos} y_\lambda^{\cos} Z_{\mathcal{A}} + y_\lambda^{\sin} X_{\mathcal{A}} + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} v_2 - (X_{\mathcal{A}} - X_{\mathcal{M}}) v_3 + (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} v_4.$$

Thus, for the measurement $y_\lambda = \lambda + v_\lambda$ of the elevation angle λ, the pseudomeasurement Y_λ is formed as follows:

$$\begin{gathered} Y_\lambda = -y_\varphi^{\cos} y_\lambda^{\cos} Z_{\mathcal{M}} + y_\lambda^{\sin} X_{\mathcal{M}}, \\ y_\lambda^{\sin} = \sin(y_\lambda), \quad y_\lambda^{\cos} = \cos(y_\lambda), \quad y_\varphi^{\cos} = \cos(y_\varphi); \end{gathered} \tag{3}$$

the filtering algorithm uses the observation model

$$Y_\lambda = \begin{pmatrix} y_\lambda^{\sin}, & -y_\varphi^{\cos} y_\lambda^{\cos} \end{pmatrix} \begin{pmatrix} X_{\mathcal{A}} \\ Z_{\mathcal{A}} \end{pmatrix} + \begin{pmatrix} (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos}, & X_{\mathcal{M}} - X_{\mathcal{A}}, & (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} \end{pmatrix} \begin{pmatrix} v_2 \\ v_3 \\ v_4 \end{pmatrix}. \tag{4}$$

Finally, consider the measurement $y_r = r + v_5$ of the range r with an error v_5 independent of the previous ones v_i: $\mathsf{E}\{v_5\} = 0$ and $\mathsf{E}\{v_5^2\} = \sigma_r^2$. Using the measurement of the elevation angle λ and the approximation $\sin(\lambda) \approx y_\lambda^{\sin} - v_3$, we write

$$r = \frac{Z_{\mathcal{A}} - Z_{\mathcal{M}}}{\sin(\lambda)} \quad \Rightarrow \quad y_r - v_5 \approx \frac{Z_{\mathcal{A}} - Z_{\mathcal{M}}}{y_\lambda^{\sin} - v_3}.$$

Similarly to the angle transformations, it follows that

$$Z_{\mathcal{A}} - Z_{\mathcal{M}} - y_r y_\lambda^{\sin} \approx -y_r v_3 - y_\lambda^{\sin} v_5 + v_3 v_5.$$

The centered error on the right has the variance

$$\mathsf{E}\{(-y_r v_3 - y_\lambda^{\sin} v_5 + v_3 v_5)^2\} = \mathsf{E}\left\{y_r^2\right\}\sigma_\lambda^2 + \mathsf{E}\left\{(y_\lambda^{\sin})^2\right\}\sigma_r^2 + \sigma_\lambda^2\sigma_r^2.$$

Here, the third term can be neglected compared to the first two, and the pseudomeasurement residual becomes

$$Z_{\mathcal{A}} - Z_{\mathcal{M}} - y_r y_\lambda^{\sin} \approx -y_r v_3 - y_\lambda^{\sin} v_5.$$

(Here, the estimate is substituted for the exact coordinate $Z_{\mathcal{A}}$.)

We write the pseudomeasurements themselves as

$$Z_{\mathcal{M}} + y_r y_\lambda^{\sin} \approx Z_{\mathcal{A}} + y_r v_3 + y_\lambda^{\sin} v_5.$$

Thus, for the measurement $y_r = d + v_5$ of the range, the pseudomeasurement Y_r is formed as follows:

$$Y_r = Z_{\mathcal{M}} + y_r y_\lambda^{\sin}, \quad y_\lambda^{\sin} = \sin(y_\lambda); \tag{5}$$

the filtering algorithm uses the observation model

$$Y_r = Z_{\mathcal{A}} + \begin{pmatrix} y_r, & y_\lambda^{\sin} \end{pmatrix} \begin{pmatrix} v_3 \\ v_5 \end{pmatrix}. \tag{6}$$

Now we combine all the three models (2), (4), and (6) into the single observation vector $Y = (Y_\varphi, Y_\lambda, Y_r)'$:

$$\begin{aligned} Y = &\begin{pmatrix} y_\varphi^{\sin} & -y_\varphi^{\cos} & 0 \\ y_\lambda^{\sin} & 0 & -y_\varphi^{\cos} y_\lambda^{\cos} \\ 0 & 0 & 1 \end{pmatrix} X \\ &+ \begin{pmatrix} X_{\mathcal{M}} - X_{\mathcal{A}} & Y_{\mathcal{A}} - Y_{\mathcal{M}} & 0 \\ 0 & (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\lambda^{\cos} & X_{\mathcal{M}} - X_{\mathcal{A}} \\ 0 & 0 & y_r \end{pmatrix} \begin{pmatrix} 0 \\ (Z_{\mathcal{A}} - Z_{\mathcal{M}}) y_\varphi^{\cos} \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ 0 \\ y_\lambda^{\sin} \end{pmatrix} V, \end{aligned} \tag{7}$$

where

$$X = (X_{\mathcal{A}}, Y_{\mathcal{A}}, Z_{\mathcal{A}})' \quad \text{and} \quad V = (v_1, v_2, v_3, v_4, v_5)'.$$

2.2. Application of the Extended Kalman Filter

Let the observation vector $y_t \in \mathbb{R}^{q_y}$ of the AUV be formed from the measurements y_φ, y_λ, and y_r at a time instant t. (For one observer, the dimension is $q_y = 3$.) The AUV has the state vector $X_t \in \mathbb{R}^{p_X}$; without loss of generality, assume that the state X_t is determined by the AUV coordinates in the $Oxyz$ system, denoted by $X_t = (X(t), Y(t), Z(t))'$, $p_X = 3$. Below, the motion model will be supplemented with other variables, but the objective is still to estimate the AUV position.

The estimation of X_t begins at the time instant $t = 0$ and is performed at discrete time instants $1, 2, \ldots, t, \ldots,$ corresponding to the partition of the observation interval with a step δ s: $\delta, 2\delta, \ldots, t\delta, \ldots$ The AUV initial position is given by the vector $X_0 = \eta = (\eta_X, \eta_Y, \eta_Z)' = (X(0), Y(0), Z(0))'$.

The vectors X_t and y_t are described by a discrete stochastic dynamic system of general form:

$$X_t = \Phi_t^{(1)}(X_{t-1}) + \Phi_t^{(2)}(X_{t-1})W_t, \quad t = 1, 2, \ldots, \quad X_0 = \eta,$$
$$y_t = \psi_t^{(1)}(X_t) + \psi_t^{(2)}(X_t)v_t. \tag{8}$$

By assumption, the random sequences of X_t and y_t have finite covariances, and the disturbances $W_t \in \mathbb{R}^{p_W}$ and the measurement errors $v_t \in \mathbb{R}^{q_v}$ are independent discrete white noises of the second order; the initial condition vector $\eta \in \mathbb{R}^{p_X}$ is independent of W_t and v_t and has finite covariance. The corresponding central moments are denoted, e.g., for W_t, by $m_W(t)$ and $D_W(t)$.

We supplement system (8) with an equation for the pseudomeasurements $Y_t \in \mathbb{R}^{q_Y}$:

$$Y_t = \boldsymbol{\Psi}_t^{(1)}(X_t, y_t) + \Psi_t^{(2)}(X_t, y_t)V_t. \tag{9}$$

(In all the available examples, exactly one pseudomeasurement is formed for one measurement and, accordingly, $q_Y = q_y = 3$; in the general case, the dimensions may differ.) Here, the matrix functions $\boldsymbol{\Psi}_t^{(1)}(X, y)$ and $\Psi_t^{(2)}(X, y)$ are given by (7), and due to its linearity, we have $\boldsymbol{\Psi}_t^{(1)}(X, y) = \Psi_t^{(1)}(y)X$.

Filtering by the method of linear pseudomeasurements [19] consists in applying the EKF [12] to system (8), with the observations y_t replaced by the pseudomeasurements (9). In the current notation, such a filter has the form

$$\begin{aligned}
\widetilde{X}_t &= \Phi_t^{(1)}(\widehat{X}_{t-1}) + \Phi_t^{(2)}(\widehat{X}_{t-1})m_W(t),\\
\widetilde{K}_t &= \widetilde{\Phi}_t^{(1)}\widehat{K}_{t-1}\left(\widetilde{\Phi}_t^{(1)}\right)' + \widetilde{\Phi}_t^{(2)}D_W(t)\left(\widetilde{\Phi}_t^{(2)}\right)',\\
\widetilde{\Phi}_t^{(1)} &= \left.\frac{\partial \Phi_t^{(1)}(X)}{\partial X}\right|_{X=\widetilde{X}_t}, \quad \widetilde{\Phi}_t^{(2)} = \Phi_t^{(2)}(\widetilde{X}_t),\\
\widehat{X}_t &= \widetilde{X}_t + K_t\left(Y_t - \boldsymbol{\Psi}_t^{(1)}(\widetilde{X}_t, y_t) - \Psi_t^{(2)}(\widetilde{X}_t, y_t)m_V(t)\right),\\
K_t &= \widetilde{K}_t\left(\widetilde{\Psi}_t^{(1)}\right)'\left(\widetilde{\Psi}_t^{(1)}\widetilde{K}_t\left(\widetilde{\Psi}_t^{(1)}\right)' + \widetilde{\Psi}_t^{(2)}D_V(t)\left(\widetilde{\Psi}_t^{(2)}\right)'\right)^{-1},\\
\widetilde{\Psi}_t^{(1)} &= \left.\frac{\partial \boldsymbol{\Psi}_t^{(1)}(X, y_t)}{\partial X}\right|_{X=\widetilde{X}_t} = \Psi_t^{(1)}(y_t), \quad \widetilde{\Psi}_t^{(2)} = \Psi_t^{(2)}(\widetilde{X}_t, y_t),\\
\widehat{K}_t &= \widetilde{K}_t - K_t\widetilde{\Psi}_t^{(1)}\widetilde{K}_t.
\end{aligned} \tag{10}$$

The difference from the classical EKF here is the mandatory linearity of the function $\boldsymbol{\Psi}_t^{(1)}$ in the estimated state, so that $\frac{\partial \boldsymbol{\Psi}_t^{(1)}(X, y_t)}{\partial X} = \Psi_t^{(1)}(y_t)$, which is determined by the pseudomeasurement equation (7) itself. The other elements are the same as in the standard EKF: the prediction $\widetilde{X}_t$ is constructed along the system trajectories; the heuristic prediction error covariance $\widetilde{K}_t$ is the result of linearizing the state equation in the neighborhood of the prediction; the correction is the observation residual with the Kalman gain K_t; the heuristic estimation error covariance $\widehat{K}_t$ is the result of linearizing the observation equation. Also, a feature of the EKF by the method of linear pseudomeasurements is the dependence of $\widetilde{\Psi}_t^{(1)}$ and $\widetilde{\Psi}_t^{(2)}$ on the "real" observations y_t. Indeed, according to (7), the values of y_t are used to compute not only the pseudomeasurements (1), (3), and (5) but also the approximate model matrices $\Psi_t^{(1)}$ and $\Psi_t^{(2)}$.

2.3. Pseudomeasurements with Time Delay

The dependence of observations y_t and pseudomeasurements Y_t (in (8) and (9), respectively) on the state X_t varies fundamentally if the information exchange time between the observed object and the observer cannot be neglected. This is the situation with sonars, see the discussion above. Accordingly, the current measurements turn out to match the position of A for some previous time instant $s < t$. This instant is determined as follows.

Let $v_s = \text{const}$ be the sound velocity in water. Regardless of the sonar type and the source of measurements (aboard the AUV or an external measuring complex), there is a difference between the time when the observer $\mathcal{M}$ receives the measurement and the time when $\mathcal{A}$ had the "measured" position. This difference is the time taken by the acoustic signal to travel the distance between $\mathcal{A}$ and $\mathcal{M}$, i.e., $\tau = t - s = r/(\delta v_s)$. Following the idea of pseudomeasurements, this random value can be approximated by $\widetilde{\tau} = y_r/(\delta v_s)$. Considering that the typical value is $v_s = 5400$ km/h (1500 m/s), the introduced error can be neglected, and the pseudomeasurement model (7) can be supplemented with the relation

$$X = (X(t-\widetilde{\tau}), Y(t-\widetilde{\tau}), Z(t-\widetilde{\tau}))', \quad \widetilde{\tau} = y_r/\delta v_s. \tag{11}$$

The general form of the observation–pseudomeasurement system is

$$\begin{gathered} X_t = \Phi_t^{(1)}(X_{t-1}) + \Phi_t^{(2)}(X_{t-1})W_t, \quad t = -T, -T+1, \ldots, 1, 2, \ldots, \quad X_{-T-1} = \eta, \\ y_t = \psi_t^{(1)}(X_{t-\tau_t}) + \psi_t^{(2)}(X_{t-\tau_t})v_t, \quad \tau_t = \tau_t(X_t), \\ Y_t = \Psi_t^{(1)}(y_t)X_{t-\widetilde{\tau}_t} + \Psi_t^{(2)}(X_{t-\widetilde{\tau}_t}, y_t)V_t, \quad \widetilde{\tau}_t = \widetilde{\tau}_t(y_t). \end{gathered} \tag{12}$$

This model rests on the following assumptions. First, the maximum possible time delay of observations, i.e., the value $T\delta > 0$, is known. (Essentially, this is the maximum detection range of the moving object.) Second, the motion of $\mathcal{A}$ starts at the time instant $-T\delta$, i.e., $t = -T$, so that the observer $\mathcal{M}$ will surely perform a measurement at the time instant $t = 0$. The initial position of $\mathcal{A}$ is given by the vector $\eta = (\eta_X, \eta_Y, \eta_Z)' = (X(-T-1), Y(-T-1), Z(-T-1))'$. The time delay τ_t is a function of the state X_t, i.e., the time required for the sound wave to travel the distance between $\mathcal{A}$ and $\mathcal{M}$. (Precisely from this consideration, the estimate $\widetilde{\tau}$ is included in the pseudomeasurements (11).)

The functions $\tau_t(X)$ and $\widetilde{\tau}_t(y)$ in (12) must take integer values from the set $\{0, 1, \ldots, T\}$. For "real" states and observations, they have the form

$$\begin{gathered} \tau_t = \min\left\{T, \left[\frac{\sqrt{(X(t) - X_{\mathcal{M}})^2 + (Y(t) - Y_{\mathcal{M}})^2 + (Z(t) - Z_{\mathcal{M}})^2}}{\delta v_s}\right]\right\}, \\ \widetilde{\tau}_t = \min\left\{T, \left[\frac{y_r}{\delta v_s}\right]\right\}, \end{gathered} \tag{13}$$

where $[\cdot]$ denotes the floor function.

2.4. Filtering in the Model with Several Observers

Now, let the observation vector y_t in (12) combine measurements of angles and range coming from q observers, i.e., $y_t = (y_{\varphi_t}^{(1)}, y_{\lambda_t}^{(1)}, y_{r_t}^{(1)}, \ldots, y_{\varphi_t}^{(q)}, y_{\lambda_t}^{(q)}, y_{r_t}^{(q)})'$. Since $y_t \in \mathbb{R}^{q_y}$, $q_y = 3q$. For each ith observer, we define a time delay $\tau_t^{(i)}$, $i = 1, \ldots, q$, with values in the set $\{0, 1, \ldots, T\}$.

The delays $\tau_t^{(i)}$ are combined into the vector $\tau_t = (\tau_t^{(1)}, \dots, \tau_t^{(q)})' \in \mathbb{R}^q$, which is a function of X_t just like τ_t in (12). Thus, the measurements $y_{\varphi_t}^{(i)}, y_{\lambda_t}^{(i)}$, and $y_{r_t}^{(i)}$ in each group can be represented as functions of the position $X_{t-\tau_t^{(i)}}$. The observation system takes the form

$$
\begin{gathered}
X_t = \Phi_t^{(1)}(X_{t-1}) + \Phi_t^{(2)}(X_{t-1}) W_t, \\
t = -T, -T+1, \dots, 0, 1, \dots, \quad X_{-T-1} = \eta, \\
y_t^{(i)} = \psi_t^{(i,1)}\left(X_{t-\tau_t^{(i)}}\right) + \psi_t^{(i,2)}\left(X_{t-\tau_t^{(i)}}\right) v_t^{(i)}, \quad i = 1, \dots, q, \\
Y_t^{(i)} = \Psi_t^{(i,1)}\left(y_t^{(i)}\right) X_{t-\widetilde{\tau}_t^{(i)}} + \Psi_t^{(i,2)}\left(X_{t-\widetilde{\tau}_t^{(i)}}, y_t^{(i)}\right) V_t^{(i)}.
\end{gathered}
\tag{14}
$$

For (14) to correctly reflect the above assumptions and turn into (8), (9) under $T = 0$ (no time delays), we introduce the following designations:

$y_t = \left((y_t^{(1)})', \dots, (y_t^{(q)})'\right)'$ is the observation vector composed of q groups of measurements $y_t^{(i)} = \left(y_{\varphi_t}^{(i)}, y_{\lambda_t}^{(i)}, y_{r_t}^{(i)}\right)'$;

$v_t^{(i)}$ is the vector of measurement errors in this group;

$Y_t = \left((Y_t^{(1)})', \dots, (Y_t^{(q)})'\right)'$ is the vector of q groups of pseudomeasurements;

$Y_t^{(i)} = \left(Y_t^{(i)}, Y_t^{(i)}, Y_t^{(i)}\right)'$, which is associated with the corresponding group $y_t^{(i)}$, and $V_t^{(i)}$ is the vector of measurement errors in this group.

The vector functions $\psi_t^{(i,1)}$, as well as the matrices $\psi_t^{(i,2)}$, $\Psi_t^{(i,1)}$, and $\Psi_t^{(i,2)}$, $i = 1, \dots, q$, are defined for each group of observations and pseudomeasurements. Their presence implies the independence of each observer forming the measurements of the group, i.e.,

$$
\psi_t^{(1)} = \left((\psi_t^{(1,1)})', \dots, (\psi_t^{(q,1)})'\right)', \quad \psi_t^{(2)} = \operatorname{diag}\left(\psi_t^{(1,2)}, \dots, \psi_t^{(q,2)}\right),
$$

$$
\Psi_t^{(1)} = \begin{pmatrix} \Psi_t^{(1,1)} \\ \vdots \\ \Psi_t^{(q,1)} \end{pmatrix}, \quad \Psi_t^{(2)} = \operatorname{diag}\left(\Psi_t^{(1,2)}, \dots, \Psi_t^{(q,2)}\right).
$$

Then the basic EKF equations (10) can be refined for the model with time delays (14) as follows:

$$
\begin{gathered}
\widetilde{X}_t = \Phi_t^{(1)}(\widehat{X}_{t-1}) + \Phi_t^{(2)}(\widehat{X}_{t-1}) m_W(t), \\
\widetilde{K}_t = \widetilde{\Phi}_t^{(1)} \widehat{K}_{t-1} (\widetilde{\Phi}_t^{(1)})' + \widetilde{\Phi}_t^{(2)} D_W(t) (\widetilde{\Phi}_t^{(2)})', \\
\widetilde{\Phi}_t^{(1)} = \left.\frac{\partial \Phi_t^{(1)}(X)}{\partial X}\right|_{X = \widetilde{X}_t}, \quad \widetilde{\Phi}_t^{(2)} = \Phi_t^{(2)}(\widetilde{X}_t), \\
\widehat{X}_t = \widetilde{X}_t + K_t \Delta \widetilde{Y}_t, \\
\Delta \widetilde{Y}_t = \left(Y_t^{(1)} - \Psi_t^{(1,1)}(y_t^{(1)}) \widetilde{X}_{t-\widetilde{\tau}_t^{(1)}}, \dots, Y_t^{(q)} - \Psi_t^{(q,1)}(y_t^{(q)}) \widetilde{X}_{t-\widetilde{\tau}_t^{(q)}}\right), \\
K_t = \widetilde{K}_t (\Psi_t^{(1)})' \left(\Psi_t^{(1)} \widetilde{K}_t (\Psi_t^{(1)})' + \widetilde{\Psi}_t^{(2)} D_V(t) (\widetilde{\Psi}_t^{(2)})'\right)^{-1}, \\
\Psi_t^{(1)} = \Psi_t^{(1)}(y_t), \widetilde{\Psi}_t^{(2)} = \operatorname{diag}\left\{\Psi_t^{(1,2)}(\widetilde{X}_{t-\widetilde{\tau}_t^{(1)}}, y_t), \dots, \Psi_t^{(q,2)}(\widetilde{X}_{t-\widetilde{\tau}_t^{(q)}}, y_t)\right\}, \\
\widehat{K}_t = \widetilde{K}_t - K_t \Psi_t^{(1)} \widetilde{K}_t.
\end{gathered}
\tag{15}
$$

Essentially, compared to the filter (10), the filter (15) simply incorporates the estimates $\widetilde{\tau}_t^{(i)}$, $i = 1, \ldots, q$, of the time delays. The observation residual $\Delta\widetilde{Y}_t$ and the matrix of measurement error deviations $\Psi_t^{(2)}$ are composed of the values of the position predictions $\widetilde{X}_{t-\widetilde{\tau}_t^{(i)}}$ corresponding to the time instants for which the current observations y_t were performed and the pseudomeasurements Y_t were composed. For this purpose, we use the position predictions shifted relative to the current time instant by the value of the estimate $\widetilde{\tau}_t^{(i)}$ of the time delay $\tau_t^{(i)}$ for the corresponding ith observer.

3. TRACKING OF AUV'S APPROACH USING ACOUSTIC BEACON MEASUREMENTS

3.1. Observation System Model

To apply the filtering algorithm (15), we adopt the same model as in [9, 10], with slight meaningful modifications for the tracking problem of an approaching unknown object. (In the papers cited, this model was studied for parameter identification.) Following Fig. 1, let the origin O of the reference frame $Oxyz$ define a stationary object ($\mathcal{O}$) located on the sea surface, to which an AUV is approaching. $\mathcal{A}$ is detected in the initial position $\eta = (\eta_X, \eta_Y, \eta_Z)'$, whose random elements are independent and have a uniform distribution: $\eta_X \sim R[10, 20]$, $\eta_Y \sim R[10, 20]$, and $\eta_Z \sim R[0.5, 1.5]$. Thus, the initial position of $\mathcal{A}$ is characterized by the mean $\mathsf{E}\{\eta\} = (15, 15, 1)'$ and covariance $\operatorname{cov}(\eta, \eta) \approx \operatorname{diag}\{2.9^2; 2.9^2; 0.29^2\}$. All distances are given in kilometers (km). By assumption, the detected AUV moves towards $\mathcal{O}$ with chaotic maneuvering but an average constant velocity of about 21 km/h.

There are two complexes ($\mathcal{F}$, first) and ($\mathcal{S}$, second) for observing the AUV. In the reference frame $Oxyz$, the z axis is directed vertically downward to the water surface (as in Fig. 1, corresponding to the AUV depth) whereas the y and x axes are directed from the object to the first and second observers, respectively ($\mathcal{O} \to \mathcal{F}$ and $\mathcal{O} \to \mathcal{S}$). The observers are considered to be stationary on the water surface, i.e., at zero depth. Thus, their coordinates are $\mathcal{F}(X_{\mathcal{F}}, Y_{\mathcal{F}}, Z_{\mathcal{F}}) = (0, Y_{\mathcal{F}}, 0)$ and $\mathcal{S}(X_{\mathcal{S}}, Y_{\mathcal{S}}, Z_{\mathcal{S}}) = (X_{\mathcal{S}}, 0, 0)$, where $X_{\mathcal{S}} = -2$ km and $Y_{\mathcal{F}} = -1$ km. Furthermore, assume that throughout the observation time, the coordinates $\mathcal{A}(X(t), Y(t), Z(t))$ are such that the AUV remains at depth without surfacing (i.e., $Z(t) > 0$), and the conditions $X(t) > X_{\mathcal{M}}$ used for the pseudomeasurement (3) are valid for both observers, i.e., $X(t) > 0$. The intersection of the motion trajectory with the Ox axis does not affect the pseudomeasurements, and a possible intersection

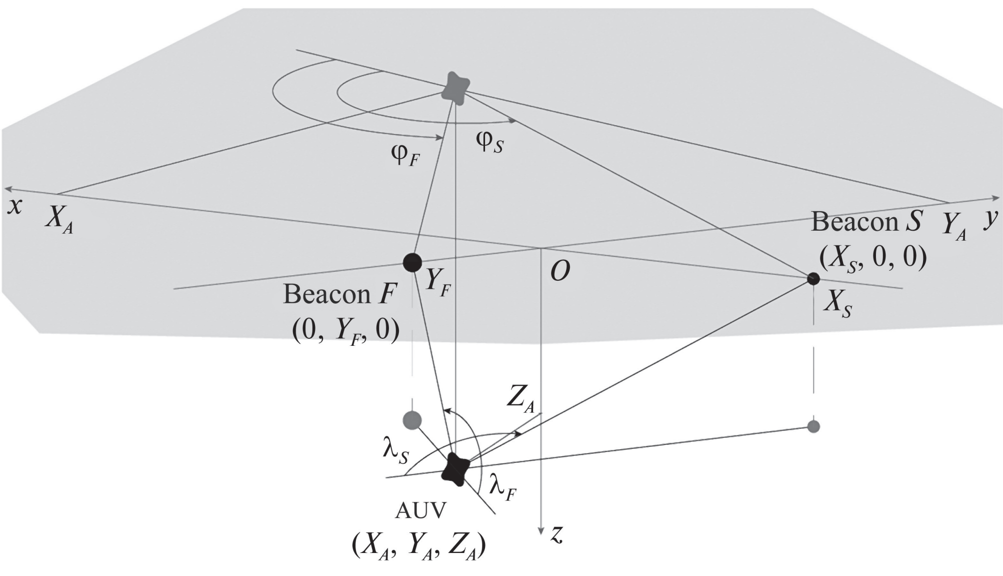

Fig. 2. The relative position of observers in the experiment.

with the Oy axis can be easily considered by using the cotangent instead of the tangent for the pseudomeasurements (3). The experiment is schematically illustrated in Fig. 2.

The vector $(X(t), Y(t), Z(t))'$ describes the position of $\mathcal{A}$ at discrete time instants $t = 0, \ldots, 1000$, which correspond to the partition of the observation time interval with a discretization step of $\delta = 0.0001$ h. Taking the maximum time delay T into account, the navigation task is thus solved in 0.1 h = 6 min. Measurements are performed at the same time instants, i.e., about three measurements per second by each complex. With an absolute constant average velocity of 21 km/h, during this time the AUV travels on average a distance of about 2.1 km, approaching $\mathcal{O}$. The maximum distance from $\mathcal{A}$ to $\mathcal{O}$ and to $\mathcal{F}$ or $\mathcal{S}$ is about 28 and 30 km, respectively; the minimum distances are 14 and 16 km, respectively. Hence, the maximum possible time delay at the AUV detection instant is $T = 56$ (i.e., 0.0056 h or about 20 s).

By assumption, the AUV moves with a constant average velocity $(s_x, s_y, s_z)'$, and the deviations from this velocity are described by the vector of additive disturbances $(w_x(t), w_y(t), w_z(t))'$:

$$\begin{aligned} X(t) &= X(t-1) + \delta S_x(t), \quad S_x(t) = s_x + \sigma_{s_x} w_x(t), \\ Y(t) &= Y(t-1) + \delta S_y(t), \quad S_y(t) = s_y + \sigma_{s_y} w_y(t), \\ Z(t) &= Z(t-1) + \delta S_z(t), \quad S_z(t) = s_z + \sigma_{s_z} w_z(t). \end{aligned} \tag{16}$$

On each trajectory, the average velocity $(s_x, s_y, s_z)'$ is specified by a random vector of independent uniformly distributed variables: $s_x \sim R[-20, -10]$, $s_y \sim R[-20, -10]$, and $s_z \sim R[-2, 0]$. Thus, the average velocity of $\mathcal{A}$ is characterized by the mean $\mathsf{E}\{S(t)\} = (-15, -15, -1)'$ (hence, the average velocity has an absolute value of about 21 km/h and the direction of motion is towards $\mathcal{O}(0, 0, 0)$) and the covariance $\mathrm{diag}\{D_{s_x}; D_{s_y}; D_{s_z}\} \approx \mathrm{diag}\{2.9^2; 2.9^2; 0.4^2\}$. The standard deviations of the additive velocity disturbance vector $W_t = (w_x(t), w_y(t), w_z(t))'$ are $\sigma_{s_x} = 15$, $\sigma_{s_y} = 15$, and $\sigma_{s_z} = 1$. As a result, the velocity covariance is $\mathrm{cov}(S(t), S(t)) \approx \mathrm{diag}\{15.3^2; 15.3^2; 1.1^2\}$.

In addition to (16), we model abrupt changes (jumps) in the average velocity. Consider a standard Poisson process $P(u)$ independent of the position of $\mathcal{A}$ and a known intensity λ_u of changes in the constant average velocity of the AUV (i.e., the average time between velocity jumps). The discrete time t is related to the continuous time u via the discretization step: $u = t\delta$. The state vector $X_t \in \mathbb{R}^{p_X}$ can be augmented by the $(p_X + 1)$th element, so that $X_{(p_X+1)_t} = P(\lambda_{t\delta} t\delta)$.

The constant, or rather piecewise constant, average velocity is described by a sequence $(s^p_x(t), s^p_y(t), s^p_z(t))'$, whose cross-section at $t = 0$ has the same distribution as $(s_x, s_y, s_z)'$. Thus, the motion model takes the form

$$\begin{aligned} X(t) &= X(t-1) + \delta S_x(t), \quad S_x(t) = s^p_x(t) + \sigma_{s_x} w_x(t), \\ Y(t) &= Y(t-1) + \delta S_y(t), \quad S_y(t) = s^p_y(t) + \sigma_{s_y} w_y(t), \\ Z(t) &= Z(t-1) + \delta S_z(t), \quad S_z(t) = s^p_z(t) + \sigma_{s_z} w_z(t). \end{aligned} \tag{17}$$

To determine the sequence $s^p(t) = (s^p_x(t), s^p_y(t), s^p_z(t))'$ for $t > 0$, we define $p(t) = X_{(p_X+1)_t} - X_{(p_X+1)_{t-1}}$ as the indicator of jumps of the process $P(\lambda_{t\delta} t\delta)$ on the current discretization interval. Assume that $s^p(t) = s^p(t-1)$ if $p(t) = 0$, i.e., the constant average velocity remains invariable without jumps. For $p(t) = 1$, $s^p(t)$ becomes a new random variable. To find its distribution, we use the same idea as for the distribution of the initial velocity $(s_x, s_y, s_z)'$, which means motion on average towards the object $\mathcal{O}$ (i.e., the origin) while preserving the variance. To this end, the new value of the average velocity $s^p(t)$ is modeled by the uniform distribution with mean $-X_t$ and the same covariance as in the previous model. Specifically, if we denote $s_x \sim R[a_x, b_x]$, then $s^p_x(t) \sim R[a_x, b_x] - (a_x + b_x)/2 - X(t-1)$, i.e., the conditional distribution of $s^p_x(t)$ given $X(t-1)$ has the mean $-X(t-1)$ (preserves on average the direction of $\mathcal{A}$'s motion towards the object $\mathcal{O}$) and the variance $\mathsf{D}[s^p_x(t) \,|\, X(t-1)] = D_{s_x}$. Similar expressions describe $s^p_y(t)$ and $s^p_z(t)$.

The process $P(u)$ used in the experiment has an intensity $\lambda_u = \frac{3}{6 \text{ min}}$, i.e., during the observation time, three changes in the constant average velocity $(s_x, s_y, s_z)'$ occur on average. (In other words, the average time between jumps is 2 min.) In other respects, model (17) retains the same parameters.

It remains to specify the parameters of the observers. According to the aforesaid, there are two observers with chosen coordinates. Hence, we have to set the parameters of the measurement accuracy of y_t. The used values can be represented as

$$\begin{aligned} &\operatorname{cov}(v_t^F, v_t^F) = \operatorname{cov}(v_t^S, v_t^S) = \operatorname{diag}\{\sigma_\varphi, \sigma_\lambda, \sigma_r\}, \\ &\sigma_\varphi = \sigma_\lambda = \frac{\pi}{180} \text{ rad } (1^\circ), \quad \sigma_r = 0.1 \text{ km } (100 \text{ m}). \end{aligned} \tag{18}$$

The distributions of the errors v_t^F and v_t^S are Gaussian.

3.2. Numerical Experiments

Using computer simulations of $N = 10,000$ motion trajectories of the form (16) and (17) and observations $y_t^{(\mathcal{F})} = (y_{\varphi_t}^{(\mathcal{F})}, y_{\lambda_t}^{(\mathcal{F})}, y_{r_t}^{(\mathcal{F})})'$ and $y_t^{(\mathcal{S})} = (y_{\varphi_t}^{(\mathcal{S})}, y_{\lambda_t}^{(\mathcal{S})}, y_{r_t}^{(\mathcal{S})})'$, we computed the position estimates $\widehat{X}_t = (\widehat{X}(t), \widehat{Y}(t), \widehat{Z}(t))'$ by formulas (15) for the observation modes with time delays ($T = 56$) and without them ($T = 0$) and the approximations of angular pseudomeasurements with the parameters $\mathsf{E}\{v_i^2\} = \sigma_{\varphi,\lambda}^2$ and $\mathsf{E}\{v_i^2\} = 1/4\,\sigma_{\varphi,\lambda}^2$. The estimation accuracy was determined by the root-mean-square deviations $\sigma_{\widehat{X}}(t), \sigma_{\widehat{Y}}(t)$, and $\sigma_{\widehat{Z}}(t)$ (indicated in meters in the figures below), computed by averaging the estimation errors over the simulated pencil.

Figure 3 illustrates the experiment with a typical example of the AUV trajectory: the coordinates $X(t)$ and $Y(t)$ with their estimates $\widehat{X}(t)$ and $\widehat{Y}(t)$ (Fig. 3a) and the velocities $S_x(t)$ and $S_y(t)$ (Fig. 3b). This example corresponds to model (17) and the approximation of angular pseudomeasurements with $\mathsf{E}\{v_i^2\} = 1/4\,\sigma_{\varphi,\lambda}^2$. The motion trajectories (16) differ by a more rectilinear form, as there are no changes in direction and velocity magnitude; the dynamics in depth $Z(t)$ are an order of magnitude smoother. Note that despite the quite chaotic velocity values, the general direction of $\mathcal{A}$ towards $\mathcal{O}$ is maintained both along the trajectory and when the velocity direction changes. For the presented trajectory, the time delays varied from 35 to 32; among all the simulated trajectories, from 55 to 25.

Note that in Fig. 3a, the beginning of the motion is accompanied by a group of inaccurate estimates. It corresponds to the first 56 steps (the initial period) without EKF estimation by the algorithm (15). At these steps, direct measurements were estimated: assuming the error-free nature of the two available measurements $y_t^{(\mathcal{F})}$ and $y_t^{(\mathcal{S})}$, the coordinates were computed from each set of angles and range, and the final position was estimated as their average value. Further, the root-mean-square deviations of this estimate are denoted by $\Sigma_{\widehat{X}}(t), \Sigma_{\widehat{Y}}(t)$, and $\Sigma_{\widehat{Z}}(t)$. The estimation accuracy is illustrated in Fig. 4. We pay the reader's attention to the initial period where $\sigma_{\widehat{X}}(t) = \Sigma_{\widehat{X}}(t), \sigma_{\widehat{Y}}(t) = \Sigma_{\widehat{Y}}(t)$, and $\sigma_{\widehat{Z}}(t) = \Sigma_{\widehat{Z}}(t)$. The EKF estimate was computed starting from $t = 57$.

Other variants of the computations (the motion model (16), no time delays ($T = 0$), and the approximation of angular pseudomeasurements with $\mathsf{E}\{v_i^2\} = \sigma_{\varphi,\lambda}^2$) have some differences. For instance, model (16) gives a more rectilinear trajectory, observations with $T = 0$ lead to no transition period with the direct measurement filter, and the parameters of the pseudomeasurement noises change the accuracy of the resulting estimates. These figures illustrate a qualitative picture of the effectiveness of the linear pseudomeasurement filter in the most complex model. A formal comparison in all models is given in the table below. To characterize the accuracy, the root-mean-square

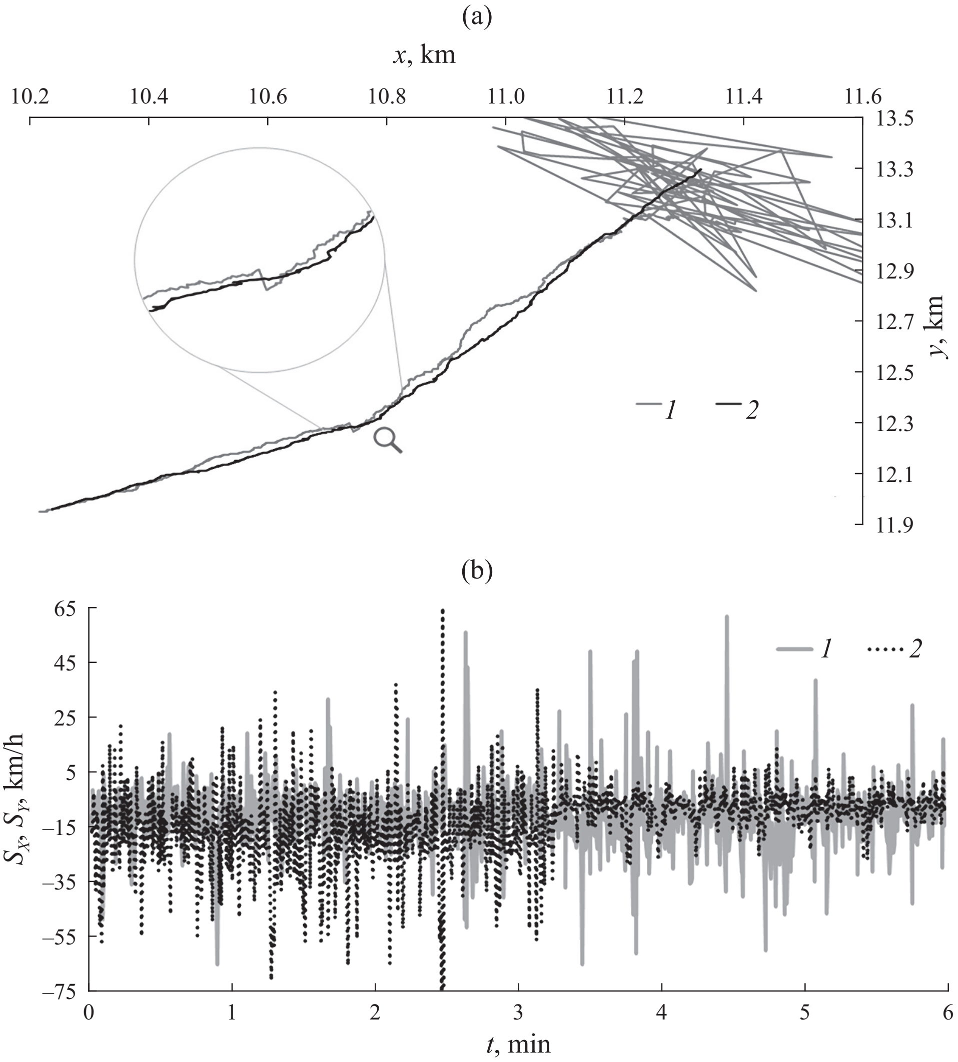

Fig. 3. A typical example of the AUV trajectory: (a) (*1*) the coordinates $X(t)$ and $Y(t)$ and (*2*) their estimates $\widehat{X}(t)$ and $\widehat{Y}(t)$; (b) the velocities (*1*) $S_x(t)$ and (*2*) $S_y(t)$.

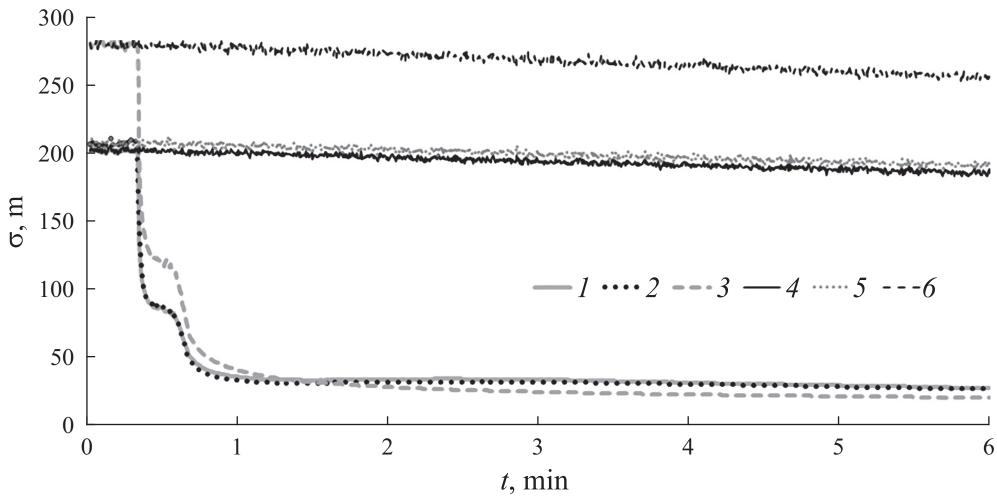

Fig. 4. Root-mean-square deviations: (*1*) $\sigma_{\widehat{X}}(t)$, (*2*) $\sigma_{\widehat{Y}}(t)$, (*3*) $\sigma_{\widehat{Z}}(t)$, (*4*) $\Sigma_{\widehat{X}}(t)$, (*5*) $\Sigma_{\widehat{Y}}(t)$, and (*6*) $\Sigma_{\widehat{Z}}(t)$.

deviations of the estimation errors were averaged over the trajectories: for $\widehat{X}(t)$ as an example, the values $\widehat{\sigma}_{\widehat{X}} = \frac{1}{1000}\sum_{t=1}^{1000}\sigma_{\widehat{X}}(t)$ and $\widehat{\Sigma}_{\widehat{X}} = \frac{1}{1000}\sum_{t=1}^{1000}\Sigma_{\widehat{X}}(t)$ were computed, etc. All deviations are presented in meters.

Comparison of estimation quality

Model	$\widehat{\sigma}_{\widehat{X}}$	$\widehat{\sigma}_{\widehat{Y}}$	$\widehat{\sigma}_{\widehat{Z}}$	$\widehat{\Sigma}_{\widehat{X}}$	$\widehat{\Sigma}_{\widehat{Y}}$	$\widehat{\Sigma}_{\widehat{Z}}$
(16), $T = 0$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	24.01	22.33	27.04	192.54	198.35	266.86
(16), $T = 0$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	**21.96**	**22.07**	**22.69**			
(16), $T = 56$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	37.82	37.09	44.76	193.42	199.23	267.89
(16), $T = 56$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	36.47	37.32	41.31			
(17), $T = 0$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	24.78	23.34	26.55	193.04	198.56	267.44
(17), $T = 0$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	22.73	22.72	24.55			
(17), $T = 56$, $\mathsf{E}\{v_i^2\} = \sigma^2_{\varphi,\lambda}$	50.46	47.37	49.23	193.95	199.48	268.49
(17), $T = 56$, $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$	**44.46**	**43.37**	**45.63**			

4. CONCLUSIONS

The experiment has confirmed the ability of the linear pseudomeasurements filter to estimate the system state in the model with time delays. Compared to direct estimation, the results have demonstrated the effectiveness of this filter and a twofold deterioration in estimation quality in the case of time delays. Regarding the absolute error values of tens of meters, we emphasize extreme estimation conditions: a large distance to the object and external disturbances of the same magnitude as the object's velocity. Moreover, from the standpoint of tracking, tens of meters is a quite satisfactory order of errors for an object located farther than 10 km. More accurate results are needed when solving the positioning task aboard the AUV. But in this case, onboard measurements (e.g., velocity) can be utilized besides external observers. This significantly increases the accuracy [25].

Furthermore, note the superiority of the filter with $\mathsf{E}\{v_i^2\} = \frac{1}{4}\sigma^2_{\varphi,\lambda}$ (i.e., the model with softer assumptions regarding the error in pseudomeasurements). This parameter gives the greatest advantage in the last (most complex) model; in the others, the difference is small. Here, we should mention the results not included in the table, namely, the experiments with other values of $\mathsf{E}\{v_i^2\}$. According to the results in the table, the filter seems to be insensitive to this value, since the filtering quality estimates change little for different $\mathsf{E}\{v_i^2\}$. However, this is true only for the values of $\mathsf{E}\{v_i^2\}$ in the range $[\frac{1}{4}, 1]\sigma^2_{\varphi,\lambda}$. Additional calculations (not included in the table) have shown that the filtering estimate deteriorates significantly for $\mathsf{E}\{v_i^2\}$ beyond this range (on the left or right).

Also, we underline that the results are in good agreement with computations performed for other similar models. For example, in [7–10], the same motion model was used together with observations of direction angle tangent, and the conditionally minimax nonlinear filtering method [15, 16] was applied for estimation.

Finally, it is crucial that in the experiments presented here, only the constant average velocity has been assumed to be known among the motion model parameters (including the model with abrupt velocity changes). Due to this feature, in particular, the accuracy of the direct measurement estimate does not deteriorate too much when passing from the model with $T = 0$ to the one with $T = 56$. According to the previous results, this parameter can be identified, which is the foundation of the approach.

While listing the positive aspects, some negative ones cannot be ignored. Despite the demonstrated effectiveness of the method of linear pseudomeasurements, the EKF has retained its worst features, primarily the tendency to diverge. Such an effect has manifested itself for the nonlinear motion model (due to the unknown parameter), when direct measurements are not used to set the initial condition for the EKF estimate, when the velocity increases and trajectories could approach coordinate planes. Therefore, although the method of linear pseudomeasurements is very good, it should be tried not only in the EKF but also in other, more reliable and stable, filtering schemes.

FUNDING

This work was supported by the Ministry of Science and Higher Education of the Russian Federation, project no. 075-15-2024-544. This work was performed using the infrastructure of the Shared Research Facilities "High Performance Computing and Big Data" (CKP "Informatics") of the Federal Research Center "Computer Science and Control," Russian Academy of Sciences (Moscow).

REFERENCES

1. *Autonomous Underwater Vehicles: Design and Practice (Radar, Sonar & Navigation)*, Ehlers, F., Ed., London: SciTech Publishing, 2020.
2. Mohsan, S.A.H., Khan, M.A., Noor, F., Ullah, I., and Alsharif, M.H., Towards the Unmanned Aerial Vehicles (UAVs): A Comprehensive Review, *Drones*, 2022, vol. 6, no. 6 (147).
3. Burns, L.D. and Shulgan, C., *Autonomy: The Quest to Build the Driverless Car – And How It Will Reshape Our World*, HarperCollins, 2018.
4. Christ, R.D. and Wernli, R.L., *The ROV Manual: A User Guide for Remotely Operated Vehicles*, 2nd ed., Oxford: Butterworth-Heinemann, 2013.
5. Zhu, Z., Hu, S.-L.J., and Li, H., Effect on Kalman Based Underwater Tracking Due to Ocean Current Uncertainty, *Proc. 2016 IEEE/OES Autonomous Underwater Vehicles*, Tokyo, Japan, November 6–9, 2016, pp. 131–137.
6. Kebkal, K.G. and Mashoshin, A.I., AUV Acoustic Positioning Methods, *Gyroscopy Navig.*, 2017, vol. 8, pp. 80–89.
7. Bosov, A.V., Observation-Based Filtering of State of a Nonlinear Dynamical System with Random Delays, *Autom. Remote Control*, 2023, vol. 84, no. 6, pp. 671–684.
8. Bosov, A., Tracking a Maneuvering Object by Indirect Observations with Random Delays, *Drones*, 2023, vol. 7, no. 7, art. no. 468.
9. Bosov, A.V., AUV Positioning and Motion Parameter Identification Based on Observations with Random Delays, *Autom. Remote Control*, 2024, vol. 85, no. 12, pp. 1149–1167.
10. Bosov, A., Maneuvering Object Tracking and Movement Parameters Identification by Indirect Observations with Random Delays, *Axioms*, 2024, no. 13, art. no. 668.
11. Bosov, A.V., Nonlinear Dynamic System State Optimal Filtering by Observations with Random Delays, *Informatics and Applications*, 2023, vol. 17, no. 3, pp. 8–17.
12. Bernstein, I. and Friedland, B., Estimation of the State of a Nonlinear Process in the Presence of Nongaussian Noise and Disturbances, *J. Franklin Instit.*, 1966, vol. 281, no. 6, pp. 455–480.
13. Arulampalam, S., Maskell, S., Gordon, N.J., and Clapp, T., A Tutorial on Particle Filters for Online Non-linear/Non-Gaussian Bayesian Tracking, *IEEE Trans. Signal Processing*, 2002, vol. 50, no. 2, pp. 174–188.
14. Julier, S.J., Uhlmann, J.K., and Durrant-Whyte, H.F., A New Approach for Filtering Nonlinear Systems, *Proc. IEEE Amer. Control Conf. (ACC'95)*, 1995, pp. 1628–1632.

15. Pugachev, V.S., Estimation of Variables and Parameters in Discrete-Time Nonlinear Systems, *Autom. Remote Control*, 1979, vol. 40, no. 4, pp. 39–50.

16. Pankov, A.R. and Bosov, A.V., Conditionally Minimax Algorithm for Nonlinear System State Estimation, *IEEE Trans. Autom. Control*, 1994, vol. 39, no. 8, pp. 1617–1620.

17. Su, X., Ullah, I., Liu, X., and Choi, D., A Review of Underwater Localization Techniques, Algorithms, and Challenges, *J. Sens.*, 2020, no. 1, art. no. 6403161.

18. Kalman, R.E., A New Approach to Linear Filtering and Prediction Problems, *J. Basic Eng. – T. ASME*, 1960, vol. 82, no. 1, pp. 35–45.

19. Lingren, A. and Gong, K., Position and Velocity Estimation Via Bearing Observations, *IEEE Trans. Aerosp. Electron. Syst.*, 1978, vol. AES-14, pp. 564–577.

20. Lin, X., Kirubarajan, T., Bar-Shalom, Y., and Maskell, S., Comparison of EKF, Pseudomeasurement, and Particle Filters for a Bearing-Only Target Tracking Problem, in *SPIE Proceedings. Signal and Data Processing of Small Targets 2002 (AEROSENSE 2002)*, Drummond, O.E., Ed., Orlando, April 1–5, 2002, SPIE: Bellingham, 2002, vol. 4728, pp. 240–250.

21. Konovalenko, I., Kuznetsova, E., Miller, A., et.al., New Approaches to the Integration of Navigation Systems for Autonomous Unmanned Vehicles (UAV), *Sensors*, 2018, no. 18, art. no. 3010.

22. Hodges, R., *Underwater Acoustics: Analysis, Design and Performance of Sonar*, New York: Wiley, 2011.

23. Holler, R.A., The Evolution of the Sonobuoy from World War II to the Cold War, *US Navy J. Underwater Acoust.*, 2014, vol. 25, no. 1, pp. 322–346.

24. Morris, J., The Kalman Filter: A Robust Estimator for Some Classes of Linear Quadratic Problems, *IEEE Trans. Inf. Theory*, 1976, vol. 22, no. 5, pp. 526–534.

25. Bosov, A.V., Doppler Measurements Application Analysis to Identify Motion Parameters from Observations with Random Delays, *Informatics and Applications*, 2025, vol. 19, no. 1, pp. 34–44.

This paper was recommended for publication by B.M. Miller, a member of the Editorial Board