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STOCHASTIC SYSTEMS

The Asymptotic Behavior of Anisotropic DOF Controller at Infinitesimal Values of Upper Bound of Input's Mean Anisotropy

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Abstract—An asymptotic formula is obtained for the optimal anisotropic controller for linear discrete time-invariant system driven by random disturbance with infinitesimal mean anisotropy. The result is accompanied with the asymptotic formula for the anisotropic norm of the closed-loop system. An upper bound is computed for the mean anisotropy at which the optimal anisotropic controller can be approximated by the $\mathcal{H}_2$-optimal controller with loss in performance gain less than a given threshold.

Keywords: anisotropy-based theory, linear discrete time invariant systems, optimal controller design, asymptotic behavior

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1. INTRODUCTION

The optimal control design problem remains to be one of the most relevant problems in control theory. To design the optimal controller, one needs to determine a plant to control, a family of control laws with parameters to be adjusted, and a certain function that quantitatively specifies the quality of the closed-loop system performance. Such a criterion is chosen based on the control objectives and the operating conditions of the system. The dynamic output feedback (DOF) control, in both of its versions – strictly proper (causal) and non-strictly proper, is frequently used to solve the linear quadratic control problems. In practice, the measurements used to define a control actions contain random noise, in most cases with somewhat uncertain statistical parameters. When the disturbances driven the linear system are Gaussian white noise, and the quadratic loss function is defined as the performance gain, the corresponding control problem is referred to as the linear-quadratic Gaussian (LQG) problem. A significant amount of publications exists on this topic [1–4]. Nevertheless, usually, external disturbance is rarely happen to be white noise, for which case the LQG controller loses its efficiency.

In the period from the late 60-s to the early 80-s, the cornerstone of the $\mathcal{H}_\infty$-theory has been developed [5–8]. This theory addresses the optimal control design under assumption that the disturbances are the square-integrable signals, and the L_2 operator norm is used as performance gain of the system. However, the optimal $\mathcal{H}_\infty$-controller is too conservative in the sense that it only performs in the best manner when the inputs are of the worst case corresponding to the maximum value of the closed-loop system performance gain.

In the 1990-s, the so-called anisotropy-based control theory has been developed by I.G. Vladimirov as an attempt to generalize $\mathcal{H}_2$- and $\mathcal{H}_\infty$-control approaches to optimal control design [9]. The fundamental concepts of the anisotropy of a random vector, the mean anisotropy of a sequence of random vectors, and the anisotropic norm of a system were introduced [10, 11]. The anisotropy of random vector is defined as a measure of the divergence (in informational sense) of the distribution of this vector with respect to a uniform distribution on the unit sphere. Later, it was re-defined as the divergence of the vector's distribution from isotropic Gaussian distributions [12]. Subsequently, the anisotropy-based theory apparatus has been also developed to solve the analysis problems, optimal control and filtering problems [13–16].

In [17], the problem of the asymptotic representation of the anisotropic norm of a linear discrete time invariant (LDTI) system has been solved when mean anisotropy upper bound was infinitesimal (the so-called left asymptotic) or infinitely large (the right asymptotic). Based on the aforementioned results, the asymptotic formula of the optimal anisotropy-based filter in terms of its deviation from the $\mathcal{H}_2$-optimal filter for infinitesimal values of mean anisotropy was presented in [18]. The formula of the mean anisotropy maximum upper bound threshold at which the $\mathcal{H}_2$-optimal filter approximates the anisotropy-based filter with a given accuracy is also obtained. In the subsequent study [19], a solution for the special case of the similar anisotropy-based control problem is presented. All the results obtained for the left asymptotic of anisotropic filters and controllers form the basis for the present study.

This paper presents a solution for the general case of the left asymptotic representation for the optimal anisotropic controller for an LDTI system. The first section provides a brief overview of the object of study, anisotropy-based control theory, and addresses the methods for the optimal $\mathcal{H}_2$ and anisotropic controllers design. In the second section, a solution of the general optimal anisotropic control problem is given. The third section of the article addresses the asymptotic representation of the DOF anisotropic controller in a general form.

2. BACKGROUND

2.1. Fundamental Notations

The following notations are used in the paper: $\mathbb{R}^n$ – a set of n-dimensional real vectors; $\mathbb{R}^{n\times m}$ – a set of $(n\times m)$-dimensional real matrices; $\mathbb{C}$ – a set of complex numbers; $\mathbb{L}_2^n$ – a set of n-dimensional real valued square integrated random vectors; $\mathcal{H}_\infty^{p\times m}$ – the Hardy space of $(n \times m)$-dimensional complex-valued matrix functions, which are analytical in a unit circle $\mathbb{C}_\odot = \{z \in \mathbb{C} : \ |z| < 1\}$ and have limited $\mathcal{H}_\infty$–norm $\|F\|_\infty = \sup_{|z|<1} \overline{\sigma}(F(z))$; $\overline{\sigma}(X) = \max \sqrt{\lambda(X^*X)}$ – maximum singular number of the matrix X; $\lambda(X)$ – eigenvalue of matrix X; $X^* = \overline{X}^T$ – Hermitian conjugate matrix to X; $\mathcal{H}_2^{p\times m}$ – the Hardy space of analytical in a unit circle $\mathbb{C}_\odot$ complex-valued matrix functions $F(z) = \sum_{k=0}^{+\infty} f_k z^k$ with limited $\mathcal{H}_2$-norm $\|F\|_2 = \left(\sum_{k=0}^{+\infty} \mathrm{tr}(f_k f_k^T)\right)^{1/2}$, where $f_k \in \mathbb{R}^{p\times m}$.

2.2. Research Object

The research object of the paper is linear discrete time invariant system F with state space realization

$$x_{k+1} = Ax_k + B_w w_k + B_u u_k, \quad k = 0, 1, \dots, \tag{1}$$

where $x_k \in \mathbb{L}_2^{n_x}$ – state vector, $x_0 = 0$; $w_k \in \mathbb{L}_2^{n_w}$ is the input disturbance vector; $u_k \in \mathbb{L}_2^{n_u}$ is the input control vector. Controlled output of system (1) denoted as a vector $z_k \in \mathbb{L}_2^{n_z}$ is determined by

$$z_k = C_z x_k + D_z u_k. \tag{2}$$

Sensor measurements are used to determine the control input u_k of the system F. This data is represented as a sequence of vectors $y_k \in \mathbb{L}_2^{n_y}$ described as follows:

$$y_k = C_y x_k + D_y w_k. \tag{3}$$

Matrices A, B_w, B_u, C_z, D_z, C_y, D_y are known real matrices of corresponding dimensions. The system of equations (1)–(3) is associated with transfer function $T_{yw}(z) = D_y + C_y(zI_{n_x} - A)^{-1}B_w$, which is described by four matrices

$$T_{yw} \sim (A, B_w, C_y, D_y), \tag{4}$$

and transfer function $T_{zu}(z) = D_z + C_z(zI_{n_x} - A)^{-1}B_u$ with quadruple of matrices

$$T_{zu} \sim (A, B_u, C_z, D_z). \tag{5}$$

The general formulation of the DOF control problem is to find a controller K of the form

$$K \sim \begin{cases} h_{k+1} = \widehat{A}h_k + \widehat{B}y_k, \\ u_k = \widehat{C}h_k + \widehat{D}y_k \end{cases} \tag{6}$$

with state vector $h_k \in \mathbb{L}_2^{n_x}$, input vector $y_k \in \mathbb{L}_2^{n_y}$ and output vector $u_k \in \mathbb{L}_2^{n_u}$, which provides the fulfillment of some quality criterion. In (6), the matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$ and $\widehat{D}$ are to be derived. The following section presents basic information regarding two controller types based on its quality criteria: the $\mathcal{H}_2$ controller which minimizes the trace of the state or controlled output covariance matrix of the closed-loop system and the anisotropy-based controller, which minimizes the anisotropic norm of the linear operator mapping external disturbances to the controlled output of the closed-loop system.

2.3. $\mathcal{H}_2$-Optimal Control

To facilitate further expositions, we introduce the following matrices :

$$U_L = (D_z^{\mathrm{T}}D_z + B_u^{\mathrm{T}}\widehat{P}_\star B_u)^{-1}, \quad U_R = -(D_z^{\mathrm{T}}C_z + B_u^{\mathrm{T}}\widehat{P}_\star A), \quad U_\star = U_L U_R, \tag{7}$$

$$V_L = -(A\widehat{Q}_\star C_y^{\mathrm{T}} + B_w D_y^{\mathrm{T}}), \quad V_R = (D_y D_y^{\mathrm{T}} + C_y\widehat{Q}_\star C_y^{\mathrm{T}})^{-1}, \quad V_\star = V_L V_R. \tag{8}$$

The optimal $\mathcal{H}_2$-control problem is to find a controller that minimizes the $\mathcal{H}_2$ norm of the closed-loop system. Consider the linear discrete time invariant system (1), controlled output (2) and measured output (3) with external random disturbance w_k distributed normally with zero mean $\mathbf{E}[w_k] = 0$ and an identity covariance matrix $\mathbf{E}[w_k w_k^{\mathrm{T}}] = I_{n_w}$. Consider the problem of designing $\mathcal{H}_2$-optimal controller of the form (6). Thus, we have the following solution to the stated optimal $\mathcal{H}_2$-optimal control problem [2]:

$$\begin{aligned} \widehat{A}_\star &= A + B_u U_\star + V_\star C_y - B_u\widehat{D}_\star C_y, \\ \widehat{B}_\star &= B_u\widehat{D}_\star - V_\star, \\ \widehat{C}_\star &= U_\star - \widehat{D}_\star C_y, \\ \widehat{D}_\star &= -U_L(D_z^{\mathrm{T}}C_z\widehat{Q}_\star C_y^{\mathrm{T}} + B_u^{\mathrm{T}}\widehat{P}_\star A\widehat{Q}_\star C_y^{\mathrm{T}} + B_u^{\mathrm{T}}\widehat{P}_\star B_w D_y^{\mathrm{T}})V_R, \end{aligned}$$

where $\widehat{P}_\star$ and $\widehat{Q}_\star$ are the stabilizing solutions of algebraic Riccati equations (for control and filtering, respectively):

$$\begin{aligned} \widehat{P}_\star &= A^{\mathrm{T}}\widehat{P}_\star A + C_z^{\mathrm{T}}C_z - U_R^{\mathrm{T}}U_\star, \\ \widehat{Q}_\star &= A\widehat{Q}_\star A^{\mathrm{T}} + B_w B_w^{\mathrm{T}} - V_\star V_L^{\mathrm{T}}. \end{aligned}$$

The matrix V_* is related to the coefficient matrix of the Kalman filter (as part of the $\mathcal{H}_2$-controller) with respect to the update sequence, while U_* is responsible for forming the control action based on the filter's estimate of the current state of the plant (due to the separation principle inherent in Linear-Quadratic-Gaussian (LQG) control).

Next, we consider the fundamental concepts and principles of anisotropy-based theory upon which the solution to the problem presented in the article is based.

2.4. Anisotropic Norm

The synthesis of an $\mathcal{H}_2$-optimal controller typically assumes that the input of the system under research is a Gaussian white noise. In practice, external disturbances affecting systems are frequently correlated (and not necessarily Gaussian) noise, and its statistical characteristics are often imprecisely known.

Let us assume that the input of the system (1) is a random disturbance in the form of a stationary sequence of mutually independent random vectors $W = (w_k)_{0\leqslant k<+\infty}$, $w_k \in \mathbb{L}_2^{n_w}$, whose properties deviate from the standard normal distribution. To characterize the deviation of the random vector's distribution from the normal distribution, the concepts of anisotropy of the random vector and the mean anisotropy of a sequence of random vectors is used within the framework of anisotropy-based theory.

Definition 1 [12]. Anisotropy $\mathbf{A}(W)$ of n_w-dimensional random vector W is a nonnegative function defined by the following expression:

$$\mathbf{A}(w) = \min_{\lambda>0} \mathbf{D}(f \| p_{n_w,\lambda}),$$

where $\mathbf{D}(f\|p_{n_w,\lambda})$ is the relative entropy (Kulback-Leibler information divergence) of probability density function (pdf) f regarding to the Gaussian pdf $p_{n_w,\lambda}$ with zero mean and scalar covariance matrix λI_{n_w}, $\lambda > 0$, and $\mathbf{h}(W) = -\int_{\mathbb{R}^{n_w}} f(w)\ln f(w)dw$ is the differential entropy of W.

Characterizing a sequence of random vectors using the concept of anisotropy of random vector defined above is not feasible, as it tends towards infinity with an increasing number of sequence elements. Therefore, the concept of mean anisotropy for a sequence of random vectors was introduced.

Definition 2 [12]. The mean anisotropy of a (stationary ergodic) sequence $W = (w_k)_{0\leqslant k<+\infty}$ is defined as limit

$$\overline{\mathbf{A}}(W) = \lim_{N\to\infty} \frac{\mathbf{A}(W_{0:N-1})}{N},$$

where $W_{s:t} = (w_s^{\mathrm{T}},\, w_{s+1}^{\mathrm{T}},\, \ldots,\, w_t^{\mathrm{T}})^{\mathrm{T}}$ is the vector formed by the vectors of the sequence fragment $(w_k)_{s\leqslant k\leqslant t}$.

As is known [20], the vectors of a stationary Gaussian sequence of random disturbances $W = (w_k)_{0\leqslant k<+\infty}$ can be represented as

$$w_j = \sum_{k=0}^{+\infty} g_k v_{j-k},$$

where $V = (v_k)_{0\leqslant k<+\infty}$ is a sequence of independent n_w-dimensional random vectors with a standard normal distribution; g_k is the impulse response of the generating filter, and $G(z) \in \mathcal{H}_2^{n_w\times n_w}$ is the transfer function of the generating filter with the sequence of vectors V as input and the sequence W as output. Since the sequence of vectors W is generated by the filter G, the notation $\overline{\mathbf{A}}(G)$ can be used to denote the mean anisotropy $\overline{\mathbf{A}}(W)$ of the sequence. It has been shown (see [11, formula (4) and Lemma 1]) that the mean anisotropy $\overline{\mathbf{A}}(G)$ of a sequence of random vectors W generated by the shaping filter can be computed using the following formula:

$$\overline{\mathbf{A}}(G) = -\frac{1}{4\pi}\int_{-\pi}^{\pi} \ln\det\left(\frac{n_w}{\|G\|_2^2}\widehat{G}(w)(\widehat{G}(w))^{\star}\right)dw,$$

where $\widehat{G}(w) = \lim_{r\to 1-0} G(re^{iw})$, $w \in [-\pi,\pi)$, $i^2 = -1$.

One of the system response measures for system F of the form (4) in case of input disturbance represented as a sequence of vectors W with mean anisotropy $\overline{\mathbf{A}}(G) \leqslant a$ is the anisotropic norm of the system [11], defined as follows:

$$F|||F|||_a = \sup_{G \in \mathbf{G}_a} \frac{\|FG\|_2}{\|G\|_2}, \tag{9}$$

where $\mathbf{G}_a = \{G \in \mathcal{H}_2^{n_w \times n_w} : \overline{\mathbf{A}}(G) \leqslant a\}$ is a set of generating filters with a bounded mean anisotropy of the sequence W.

To compute the anisotropic norm, it is necessary to determine the parameters of the generating filter G that provides the supremum in the expression (9). This filter is called the worst-case shaping filter and has the representation [11, formulas (32), (33)]

$$G \sim \left[\begin{array}{c|c} A + BL & B\Sigma^{1/2} \\ \hline L & \Sigma^{1/2} \end{array}\right] \tag{10}$$

with state vector x_k, input vector v_k and output vector w_k. Next, the formulation of the lemma concerning the computation of the anisotropic norm for a linear discrete-time invariant system is presented.

Lemma 1 [11, Lemma 3]. *Given a stable linear discrete time-invariant system F of the form (4), defined by the matrix quadruple A, B, C, D. For any $a > 0$, there exists a unique pair (q, R), where $q \in (0, \|F\|_\infty^{-2})$ is a scalar parameter that satisfies the equation*

$$-\frac{1}{2} \ln \det \frac{n_w \Sigma}{\operatorname{tr}(LPL^{\mathrm{T}} + \Sigma)} = a, \tag{11}$$

and $R \in \mathbb{R}^{n_x \times n_x}$ is a matrix that is a stabilizing solution of the Riccati equation

$$\begin{aligned} R &= A^{\mathrm{T}} R A + q C^{\mathrm{T}} C + L^{\mathrm{T}} \Sigma^{-1} L, \\ \Sigma &= (I_{n_w} - q D^{\mathrm{T}} D - B^{\mathrm{T}} R B)^{-1}, \\ L &= \Sigma (B^{\mathrm{T}} R A + q D^{\mathrm{T}} C). \end{aligned}$$

Furthermore, the anisotropic norm of the system F is computed as

$$F|||F|||_a = \left(\frac{1}{q} \left(1 - \frac{n_w}{\operatorname{tr}(LPL^{\mathrm{T}} + \Sigma)} \right) \right)^{1/2}, \tag{12}$$

where matrix $P \in \mathbb{R}^{n_x \times n_x}$ satisfies Lyapunov equation

$$P = (A + BL) P (A + BL)^{\mathrm{T}} + B \Sigma B^{\mathrm{T}}. \tag{13}$$

The aforementioned concepts and principles of anisotropy-based control theory will be subsequently used in addressing the problem of determining the asymptotic representation of a general anisotropy-based controller and the maximum anisotropy threshold below which the anisotropy-based controller can be approximated by an $\mathcal{H}_2$-controller with a specified accuracy.

3. OPTIMAL ANISOTROPIC CONTROLLER

The optimal anisotropy-based control problem (6) for a linear discrete-time invariant system (5) with a measured output (3) is considered in this section. In [19], a solution is presented for the asymptotic representation problem with small values of the mean anisotropy a for a static state controller $u_k = K x_k$. In a similar way, asymptotic representation problem for a dynamic anisotropy-based output controller is solved.

Initially, the representation of the original system with a dynamic controller is expressed as the result of substituting the controller's expression (6) into the system (1)–(3):

$$\mathcal{L}(F, K) \sim \left[\begin{array}{c|c} \overline{A} & \overline{B} \\ \hline \overline{C} & \overline{D} \end{array}\right], \tag{14}$$

where matrices $\overline{A}$, $\overline{B}$, $\overline{C}$ and $\overline{D}$ have the form

$$\overline{A} = \begin{pmatrix} A + B_u \widehat{D} C_y & B_u \widehat{C} \\ \widehat{B} C_y & \widehat{A} \end{pmatrix}, \quad \overline{B} = \begin{pmatrix} B_w + B_u \widehat{D} D_y \\ \widehat{B} D_y \end{pmatrix},$$
$$\overline{C} = \begin{pmatrix} C_z + D_z \widehat{D} C_y & D_z \widehat{C} \end{pmatrix}, \quad \overline{D} = D_z \widehat{D} D_y.$$

It is assumed that the input disturbance vectors, denoted as w_k, of the system under consideration are the output of a worst-case generating filter of the form (10) and can be represented as

$$w_k = L_x x_k + L_h h_k + \Sigma^{1/2} v_k.$$

The Riccati equation from the lemma no.1 of calculating the anisotropic norm (11)–(13) for the system (14) has the form

$$R = \overline{A}^{\mathrm{T}} R \overline{A} + q \overline{C}^{\mathrm{T}} \overline{C} + L^{\mathrm{T}} \Sigma^{-1} L, \tag{15}$$
$$\Sigma = (I_{n_w} - q \overline{D}^{\mathrm{T}} \overline{D} - \overline{B}^{\mathrm{T}} R \overline{B})^{-1}, \tag{16}$$
$$L = (L_x \quad L_h) = \Sigma(\overline{B}^{\mathrm{T}} R \overline{A} + q \overline{D}^{\mathrm{T}} \overline{C}). \tag{17}$$

Thus, the control problem is decomposed into two subproblems: determining the worst-case generating filter for the closed-loop system (14), and synthesizing an optimal dynamic anisotropy-based controller in the form of an LQG controller that minimizes the trace of the covariance matrix of the regulated output of the closed-loop system (14) when affected by the worst-case noise. In [16], a solution to a similar control problem is presented for the case $\widehat{D} = 0$. After performing a similar analysis for the controller case (6), one has that the matrices $\widehat{A}$ and $\widehat{B}$ satisfy the following formulas:

$$\widehat{A} = A + B_w M + B_u \widehat{C} + (B_u \widehat{D} - \Lambda)(C_y + D_y M), \quad \widehat{B} = \Lambda, \tag{18}$$

where

$$M = L_x + L_h, \tag{19}$$
$$\begin{aligned} S = {} & (A + B_w L_x + B_u \widehat{D} D_y L_x) S (A + B_w L_x + B_u \widehat{D} D_y L_x)^{\mathrm{T}} \\ & + (B_w + B_u \widehat{D} D_y) \Sigma (B_w + B_u \widehat{D} D_y)^{\mathrm{T}} - \Lambda \Theta \Lambda^{\mathrm{T}}, \end{aligned} \tag{20}$$
$$\Theta = (C_y + D_y L_x) S (C_y + D_y L_x)^{\mathrm{T}} + D_y \Sigma D_y^{\mathrm{T}}, \tag{21}$$
$$\begin{aligned} \Lambda = {} & ((A + B_w L_x + B_u \widehat{D} C_y + B_u \widehat{D} D_y L_x) S (C_y + D_y L_x) \\ & + (B_w + B_u \widehat{D} D_y) \Sigma D_y^{\mathrm{T}}) \Theta^{-1}. \end{aligned} \tag{22}$$

To determine the unknown matrices $\widehat{C}$ and $\widehat{D}$ of the controller, the methodology for solving synthesis problems of dynamic $\mathcal{H}_2$-optimal output controllers presented in [2] should be employed.

Therefore, one expresses the system (1)–(3) with the dynamic controller (6) and the worst-case generating filter (10) in the form

$$\begin{cases} \widetilde{x}_{k+1} = \widetilde{A}\widetilde{x}_k + \widetilde{B}_w v_k + \widetilde{B}_u u_k, \\ \widetilde{z}_k = \widetilde{C}_z \widetilde{x}_k + \widetilde{D}_z u_k, \\ \widetilde{y}_k = \widetilde{C}_y \widetilde{x}_k + \widetilde{D}_y v_k, \end{cases} \tag{23}$$

where state vector $\widetilde{x}$ includes the state vector x_k of the initial system (1) and state vector h_k of controller (6), i.e. $\widetilde{x} = (x_k^{\mathrm{T}} \quad h_k^{\mathrm{T}})^{\mathrm{T}}$, $\widetilde{z}_k = z_k$, $\widetilde{y} = (y_k^{\mathrm{T}} \quad h_k^{\mathrm{T}})^{\mathrm{T}}$, and system matrices have the form

$$\widetilde{A} = \begin{pmatrix} A + B_w L_x & B_w L_h \\ \widehat{B} C_y + \widehat{B} D_y L_x & \widehat{A} + \widehat{B} D_y L_h \end{pmatrix}, \; \widetilde{B}_w = \begin{pmatrix} B_w \Sigma^{1/2} \\ \widehat{B} D_y \Sigma^{1/2} \end{pmatrix}, \; \widetilde{B}_u = \begin{pmatrix} B_u \\ 0 \end{pmatrix}, \tag{24}$$

$$\widetilde{C}_z = (C_z \quad 0), \quad \widetilde{D}_z = D_z, \tag{25}$$

$$\widetilde{C}_y = \begin{pmatrix} C_y + D_y L_x & D_y L_h \\ 0 & I_{n_x} \end{pmatrix}, \quad \widetilde{D}_y = \begin{pmatrix} D_y \Sigma^{1/2} \\ 0 \end{pmatrix}. \tag{26}$$

Consequently, the desired control u_k is determined by the following formula:

$$u_k = \widetilde{N} \widetilde{y}_k,$$

where $\widetilde{N} = (\widehat{D} \quad \widehat{C})$.

Applying the $\mathcal{H}_2$-optimal control method for the system (23) yields

$$\widetilde{N} = -\widetilde{U}_L(\widetilde{D}_z^{\mathrm{T}} \widetilde{C}_z Q_\star \widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}} P_\star \widetilde{A} Q_\star \widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}} P_\star \widetilde{B}_w \widetilde{D}_y^{\mathrm{T}}) \widetilde{V}_R, \tag{27}$$

where matrices $\widetilde{U}_L$ and $\widetilde{V}_R$ are introduced analogously to (7) and (8) by replacing the corresponding matrices with similar matrices marked with a tilde $\widehat{P}_\star$, $\widehat{Q}_\star$ to $P_\star$, $Q_\star$, and the matrices $P_\star$ and $Q_\star$ satisfy equations

$$P_\star = \widetilde{A}^{\mathrm{T}} P_\star \widetilde{A} + \widetilde{C}_z^{\mathrm{T}} \widetilde{C}_z - \widetilde{U}_R^{\mathrm{T}} \widetilde{U}_\star, \tag{28}$$

$$Q_\star = \widetilde{A} Q_\star \widetilde{A}^{\mathrm{T}} + \widetilde{B}_w \widetilde{B}_w^{\mathrm{T}} - \widetilde{V}_\star \widetilde{V}_L^{\mathrm{T}}. \tag{29}$$

From (27), it follows that desired controller matrices $\widehat{C}$ and $\widehat{D}$ are expressed as follows:

$$\widehat{C} = \widetilde{N} \begin{pmatrix} 0 \\ I_{n_x} \end{pmatrix}, \quad \widehat{D} = \widetilde{N} \begin{pmatrix} I_{n_y} \\ 0 \end{pmatrix}.$$

Thus, the matrices $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, and $\widehat{D}$ of the desired dynamical anisotropy-based output controller are uniquely determined by the system of equations (18), (27)–(29).

The subsequent section details a solution to the problem of determining an asymptotic representation for the derived optimal anisotropy-based controller as $a \to 0^+$.

4. ASYMPTOTIC REPRESENTATION OF CONTROLLER

The next step in solving the stated problem is to derive the formulas for the asymptotic representation of the obtained anisotropy-based dynamical controller. To achieve this goal, it is necessary to determine the components of the matrix decomposition for the controller, the system (23), and all related matrices. Let us express the matrices of the system (23) as the following series:

$$X(a) = \sum_{k=0}^{n} X_k a^{k/2} + o(a^{n/2}), \quad a \to 0 + 0, \tag{30}$$

where X denotes *any* variable, except for the matrices A, B_w, B_u, C_z, C_y, D_z, D_y of the initial system, which, by the problem statement, are independent of a (for example, the matrix Σ depends on a, so the representation (30) applies to it; i.e. $\Sigma(a) = \Sigma_0 + \Sigma_1\sqrt{a} + \Sigma_2 a + o(a)$, if we set $n = 2$). Note that $\widetilde{X}(\sqrt{a}) \doteq X(a)$ has to be a sufficiently smooth function of its argument $\sqrt{a}$. All matrices obtained from sums and products of individual matrices that can be represented in the form of (30) also have a similar form.

In similar way as in case of the static control problem, to determine the zero components of the expansions of matrix functions, it is necessary to determine the function values when $a = 0$. This case corresponds to the matrices of the $\mathcal{H}_2$-controller. For convenience, let us introduce the auxiliary matrix $\Upsilon = -\widetilde{U}_L^{-1}\widetilde{N}\widetilde{V}_R^{-1}$. All variables X_0 corresponding to the case $a = 0$ are not presented here, as they are trivially obtained by substituting the values $q = 0$, $L = 0$, and $\Sigma = I_{n_w}$ into all the necessary formulas.

Based on the results presented in [18, 19], the second-order terms in the expansions of the matrix functions $R(a)$, $\Sigma(a)$, $L(a)$, and $q(a)$ are expressed as follows:

$$\begin{aligned}
&q_1^2 = 4n_w/\left(2n_w \mathrm{tr}(\overline{B}_0^{\mathrm{T}}\mathcal{Q}\overline{A}_0\overline{P}_0\overline{A}_0^{\mathrm{T}}\mathcal{Q}\overline{B}_0 + n_w(\overline{B}_0^{\mathrm{T}}\mathcal{Q}\overline{B}_0)^2) - \mathrm{tr}^2(\overline{B}_0^{\mathrm{T}}\mathcal{Q}\overline{B}_0)\right), \\
&R_1 = q_1\mathcal{Q}, \quad \Sigma_1 = \overline{B}_0^{\mathrm{T}}R_1\overline{B}^0, \quad L_1 = \overline{B}_0^{\mathrm{T}}R_1\overline{A}_0,
\end{aligned} \tag{31}$$

where matrices $\mathcal{Q}$ $\overline{P}_0$ satisfy equations

$$\mathcal{Q} = \overline{A}_0^{\mathrm{T}}\mathcal{Q}\overline{A}_0 + \overline{C}_0^{\mathrm{T}}\overline{C}_0, \quad \overline{P}_0 = \overline{A}_0\overline{P}_0\overline{A}_0^{\mathrm{T}} + \overline{B}_0\overline{B}_0^{\mathrm{T}}.$$

With that, one obtains the following expressions for the first components of the non-zero matrices in the closed system (note the dependence of these matrices on various X_0 and X_1):

$$\widetilde{A}_1 = \begin{pmatrix} B_w L_{x,1} & B_w L_{h,1} \\ \widehat{B}_1 C_y + \widehat{B}_0 D_y L_{x,1} & \widehat{B}_0 D_y L_{h,1} \end{pmatrix}, \quad \widetilde{B}_{w,1} = \begin{pmatrix} B_w \Sigma_1^{1/2} \\ \widehat{B}_1 D_y + \widehat{B}_0 D_y \Sigma_1^{1/2} \end{pmatrix},$$

$$\widetilde{C}_{y,1} = \begin{pmatrix} D_y L_{x,1} & D_y L_{h,1} \\ 0 & 0 \end{pmatrix}, \quad \widetilde{D}_{y,1} = \begin{pmatrix} D_y \Sigma_1^{1/2} \\ 0 \end{pmatrix}.$$

Having derived derivations of the anisotropy-based controller matrices, one obtains equations for the first components of the anisotropy-based controller matrices:

$$\widehat{A}_1 = B_w M_1 + B_u \widehat{C}_1 + B_u \widehat{D}_1 C_y - \Lambda_1 C_y + (B_u \widehat{D}_0 - \Lambda_0) D_y M_1, \quad \widehat{B}_1 = \Lambda_1,$$

$$\widehat{C}_1 = \widetilde{N}_1 \begin{pmatrix} 0 \\ I_{n_x} \end{pmatrix}, \quad \widehat{D}_1 = \widetilde{N}_1 \begin{pmatrix} I_{n_y} \\ 0 \end{pmatrix}.$$

Although the expressions for the second terms in the expansion (30) of different matrix variables are quite complex, they are all obtained in a similar manner and share a similar structure. Therefore, to conserve space, we will present only the general principle of their derivation, using the matrix Υ as an illustrative example. According to the established notation,

$$\Upsilon = \widetilde{D}_z^{\mathrm{T}}\widetilde{C}_z Q_\star \widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}} P_\star \widetilde{A} Q_\star \widetilde{C}_y^{\mathrm{T}} + \widetilde{B}_u^{\mathrm{T}} P_\star \widetilde{B}_w \widetilde{D}_y^{\mathrm{T}}, \tag{32}$$

where all forming matrices depend on a. Therefore, for the first term in its decomposition according to formula (30), we have the following representation:

$$\Upsilon_0 = \widetilde{D}_{z,0}^{\mathrm{T}}\widetilde{C}_{z,0} Q_{\star,0} \widetilde{C}_{y,0}^{\mathrm{T}} + \widetilde{B}_{u,0}^{\mathrm{T}} P_{\star,0} \widetilde{A}_0 Q_{\star,0} \widetilde{C}_{y,0}^{\mathrm{T}} + \widetilde{B}_{u,0}^{\mathrm{T}} P_{\star,0} \widetilde{B}_{w,0} \widetilde{D}_{y,0}^{\mathrm{T}}, \tag{33}$$

and for the second term – as follows:

$$\Upsilon_1 = \sum_{\substack{i,j,k,l\geqslant 0\\ i+j+k+l=1}} \widetilde{D}_{z,i}^{\mathrm{T}}\widetilde{C}_{z,j}Q_{\star,k}\widetilde{C}_{y,l}^{\mathrm{T}} + \sum_{\substack{i,j,k,l,m\geqslant 0\\ i+j+k+l+m=1}} \widetilde{B}_{u,i}^{\mathrm{T}}P_{\star,j}\widetilde{A}_kQ_{\star,l}\widetilde{C}_{y,m}^{\mathrm{T}} + \sum_{\substack{i,j,k,l\geqslant 0\\ i+j+k+l=1}} \widetilde{B}_{u,i}^{\mathrm{T}}P_{\star,j}\widetilde{B}_{w,k}\widetilde{D}_{y,l}^{\mathrm{T}}. \tag{34}$$

It is easy to notice the general principle behind the formation of the matrix Υ_1: among all possible index combinations forming its matrices, only one index in each matrix product takes the value of 1. Similarly, to write out the third term Υ_2, it is necessary to consider all possible combinations of indices whose sum equals 2 (the total number of terms in this case will be 35). Therefore, one can assume that all the necessary matrices in the representation (30) have been written out; i.e. the asymptotic representation of the dynamic anisotropy-based controller has been determined with the specified accuracy as $a \to 0+0$. The obtained results are summarized in the following statement:

Theorem 1. *Consider a linear time-invariant system of the form* (1)–(3) *and a dynamical controller of the form* (6) *in an output feedback configuration. For small values of the mean anisotropy* $a \to 0+0$ *of the input disturbances, the following asymptotic expansions, given by equation* (30), *are valid for the matrices* $\widehat{A}$, $\widehat{B}$, $\widehat{C}$, *and* $\widehat{D}$ *of the controller. The terms of the series are determined analogously to equations* (32)–(34) *for the matrix* Υ, *and its dependence on* a *is given by equation* (11).

The following section presents a solution to the problem of the asymptotic representation of the anisotropic norm for a closed-loop system with an obtained controller.

5. ASYMPTOTIC REPRESENTATION OF ANISOTROPIC NORM

The next step in problem solving is to obtain an asymptotic representation of the anisotropic norm of the closed-loop system with the obtained controller and to determine the maximum mean anisotropy level $a_{\max}$ at which the corresponding optimal anisotropy-based controller can be approximated by an $\mathcal{H}_2$-optimal controller with a specified accuracy level ε. Therefore, it is necessary to determine the first components of the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$. By determining the partial derivatives of these matrix functions with respect to $\sqrt{a}$ and substituting $a = 0$, we readily obtain the required first components of the expansions of the matrices $\overline{A}$, $\overline{B}$, $\overline{C}$, and $\overline{D}$.

To obtain the asymptotic representation of the anisotropic norm, it is necessary to determine the second components of the matrix functions $R(a)$ and $\Sigma(a)$. Having determined the second partial derivatives of the matrices (15)–(17) with respect to $\sqrt{a}$, and substituting the zero value of the mean anisotropy a into them, one have

$$\begin{aligned} R_2 &= \overline{A}_0^{\mathrm{T}} R_2 \overline{A}_0 + Y_{R_2} + Y_{R_2}^{\mathrm{T}},\\ \Sigma_2 &= \overline{B}_0^{\mathrm{T}} R_2 \overline{B}_0 + Y_{\Sigma_2} + Y_{\Sigma_2}^{\mathrm{T}}, \end{aligned} \tag{35}$$

where $Y_{R_2} = q_1\left(\overline{A}_1^{\mathrm{T}}\mathcal{Q}\overline{A}_0 + \overline{C}_1^{\mathrm{T}}\overline{C}_0\right)$, $Y_{\Sigma_2} = q_1\left(\overline{B}_1^{\mathrm{T}}\mathcal{Q}\overline{B}_0 + \overline{D}_1^{\mathrm{T}}\overline{D}_0\right)$. Substituting the derived expansions into the series of matrix functions R, Σ, L, and P in formula (12) for the anisotropic norm, the asymptotic representation of the anisotropic norm for the system (14) as $a \to 0+$ is expressed as follows:

$$\mathcal{L}(F,K_\star)|\!|\!|\mathcal{L}(F,K_\star)|\!|\!|_a = \frac{\|\mathcal{L}(F,K_{\star,0})\|_2}{\sqrt{n_w}}\left(1+\left(\sqrt{\frac{\Xi}{n_w}}+\frac{\mathrm{tr}(\Sigma_2)}{2q_1\|\mathcal{L}(F,K_{\star,0})\|_2^2}\right)\sqrt{a}\right)+o(\sqrt{a}), \tag{36}$$

where $\mathcal{L}(F, K_{\star,0})$ represents a system of the form (14), closed by the optimal controller at the mean anisotropy level $a = 0$, and Ξ is of the form

$$\Xi = \frac{n_w \|\mathcal{L}(F, K_{\star,0})\|_4^4 - \|\mathcal{L}(F, K_{\star,0})\|_2^4}{\|\mathcal{L}(F, K_{\star,0})\|_2^4}. \tag{37}$$

The formulas for $\|\cdot\|_4^4$ and $\|\cdot\|_2^4$ are known and can be found in [17].

The final step is to determine the maximum level of mean anisotropy for a specified accuracy level $\varepsilon = \overline{o}(\|\mathcal{L}(F, K_{\star,0})\|_2)$, with which the $\mathcal{H}_2$-optimal controller approximates the anisotropy-based controller. This condition takes the form $a \leqslant a_{\max}$, where $a_{\max}$ satisfies the inequality:

$$\left| \mathcal{L}(F, K_\star) \| \mathcal{L}(F, K_\star) \|_{a_{\max}} - \frac{\|\mathcal{L}(F, K_{\star,0})\|_2}{\sqrt{n_w}} \right| < \varepsilon \frac{\|\mathcal{L}(F, K_{\star,0})\|_2}{\sqrt{n_w}}. \tag{38}$$

Substituting the asymptotic representation formula (36) for the anisotropic norm into inequality (38), one obtain

$$a \leqslant a_{\max} = \varepsilon^2 \left(\sqrt{\frac{\Xi}{n_w}} + \frac{\mathrm{tr}(\Sigma_2)}{2q_1 \|\mathcal{L}(F, K_{\star,0})\|_2^2} \right)^{-2}. \tag{39}$$

The described above results of solving the problem of the asymptotic representation of the anisotropic norm are presented as the following theorem.

Theorem 2. *Consider a linear time-invariant system of the form* (1)–(3) *and a dynamical controller of the form* (6) *in an output feedback configuration. For small values of the mean anisotropy* $a \to 0 + 0$ *of input disturbances, the anisotropic norm of the system closed by the controller* (6) *admits the asymptotic representation* (36)*, and the maximum level of mean anisotropy at which the relative deviation of the anisotropic norm* $\mathcal{L}(F, K_\star) \| \mathcal{L}(F, K_\star) \|_a$ *from the scaled* $\mathcal{H}_2$*-norm of the closed-loop system does not exceed a specified threshold* ε*, is determined by formula* (39)*, where* q_1*,* Σ_2*, and* Ξ *are defined according to formulas* (31)*,* (35)*, and* (37)*.*

Obviously, the maximum mean anisotropy level is determined by the matrices of the original system. Earlier papers devoted to asymptotic representation of the anisotropy-based filter [18] and the static anisotropy-based controller [19] have clearly shown that its $\mathcal{H}_2$-optimal analogues sufficiently effectively approximate the anisotropy-based filter and controller, respectively, when the mean anisotropy of the input disturbance is small.

6. CONCLUSION

The paper addresses the problems of synthesis of a dynamic optimal anisotropy-based controller for linear discrete stationary systems and the determination of the maximum mean anisotropy threshold below which the anisotropy-based controller can be approximated by an $\mathcal{H}_2$-optimal controller with a specified level of accuracy. As a result of solving these problems, asymptotic representations were derived for all matrices of the anisotropy-based controller, the matrices of the closed-loop system, and its anisotropic norm for small values of mean anisotropy. Future research may address a similar anisotropy-based control problem for the right asymptotics, deriving asymptotic representations for the anisotropy-based controller and the closed-loop system norm as the mean anisotropy tends to infinity.

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