

The Green's Function Method in the Problem of Fuzzy Signal Transformation by a Linear Dynamic System

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Abstract—In this paper, the problem of transforming a fuzzy signal by a linear dynamic system is reduced to studying the problem of bounded solutions for a high-order linear differential equation with constant coefficients and a fuzzy-valued inhomogeneity on the right-hand side. To solve the latter, a modification of the Green's function method for fuzzy problems is developed. A class of equations with positive coefficients and a nonnegative Green's function is identified, and some results on the existence and smoothness of a fuzzy-valued bounded solution on the entire axis are established for this class of equations. As shown, in the case of a triangular right-hand side, the solution will also be triangular. Applications to radio circuits with fuzzy input signals are considered. A relationship between the modal values of fuzzy input and output signals for a linear dynamic system is derived.

Keywords: fuzzy numbers, dynamic systems with constant coefficients and fuzzy input signals, Green's function method

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1. INTRODUCTION

Dynamic models of many applied processes are characterized by uncertainty in input data. When the probabilistic characteristics of an input signal can be estimated during modeling, the theory of stochastic processes is used [1].

On the other hand, in recent decades, the interval approach has been widely applied: expert assessments serve to indicate the bounds of intervals for input variables [2].

Along with the interval approach, researchers actively employ methods of fuzzy mathematics [3, 4]. If a formal probability distribution is a priori unknown, but it is still possible to specify some possibilistic estimates, then fuzzy methods are often used. They are characterized by a rather simple apparatus producing an intuitively understandable result. Furthermore, approaches based on fuzzy methods allow transforming the possibilistic estimate of the initial data into that of the results.

Within the fuzzy approach, the membership function of a fuzzy number is used, which characterizes the possibility that the fuzzy number will take a given real value. Thus, the fuzzy approach contains more information about the existing uncertainty than the interval counterpart. Accordingly, fuzzy modeling yields more meaningful results compared to interval modeling. In practice, membership functions are constructed using expert assessments.

This paper considers fuzzy dynamic systems described by linear differential equations of order n with constant coefficients and fuzzy-valued right-hand sides. Such systems are encountered in automatic control theory, signal processing (radio engineering), and other applications.

The foundations of the theory of fuzzy differential equations were laid in [5–7] and further developed in [8–11]. Various applications were reflected in [12, 13]. Among recent works, we mention [14–17].

The literature considers various definitions of differentiability for fuzzy-valued functions. In this paper, we use the classical definition of a Hukuhara derivative [5] and the related one of a Seikkala derivative [7].

As a rule [18], when dealing with linear fuzzy differential equations, the system of equations for the corresponding α -levels is written and solved. Then it is necessary to check whether the derivatives of the resulting α -levels define the derivative of the fuzzy-valued function. According to illustrative examples [18, 19], this is not always the case. Note that the so-called operator method was developed in [19]: differential equations for α -levels are reduced to integral equations, which are then solved.

In recent years, the method of the fuzzy Laplace transform has become widespread for solving linear fuzzy differential equations of high order [20–22]. However, this method does not determine in advance whether the resulting functions will be smooth (and, accordingly, the desired solutions).

In contrast to conventional approaches, the one proposed below rests on a development of the Green's function method, widely used in the theory of ordinary differential equations [23, Chs. 1 and 2; 24], to the case of fuzzy differential equations.

The Green's function method is fruitful as it gives formulas for the α -levels of fuzzy solutions. Hence, conditions can be provided under which the Seikkala derivatives of the α -levels will define the Seikkala derivative of the fuzzy-valued solution. In this paper, such conditions are the positivity of the coefficients of the dynamic system, the nonnegativity of the corresponding Green's function, and the Seikkala differentiability of the fuzzy input signal. These conditions are natural for several applications. In particular, the positivity of the coefficients is a necessary condition for the stability of the characteristic polynomial corresponding to the linear dynamic system. Nonnegativity conditions for Green's functions are well-known; for example, see [23]. They have been studied in connection with various applications.

Let us clarify a significant aspect: this paper involves the concept of an ultra-weak fuzzy solution as a fuzzy-valued function whose α -levels satisfy the equations for the α -levels derived from a given fuzzy differential equation. Note the following fact established below: the j -times differentiability of a fuzzy-valued inhomogeneity implies the j -times differentiability of the ultra-weak fuzzy-valued solution (in the Seikkala sense).

Furthermore, an important issue concerns the form of the output fuzzy signal of a linear dynamic system receiving a fuzzy input signal of a given type (e.g., triangular). As shown below, under definite conditions (the positivity of the system coefficients and the nonnegativity of the corresponding Green's function), the output is also a triangular fuzzy signal.

Let us add that fuzzy differential equations with the generalized (Bede–Gal) derivatives have been recently investigated by several authors [10, 18–22]. The approach developed in this paper is applicable to this case as well.

As applications, this paper considers models of radio circuits with fuzzy input signals. The relationship between the modal values of the triangular fuzzy input and output signals of a linear dynamic system is obtained. The concept of a possibilistic confidence interval is introduced and used.

2. FUZZY NUMBERS AND FUZZY-VALUED FUNCTIONS

Let $\mathbb{R}$ be the set of all real numbers. A fuzzy number $\tilde{u}$ is a subset of $\mathbb{R}$ defined by its membership (possibility) function $\mu_{\tilde{u}} : \mathbb{R} \rightarrow [0, 1]$ (e.g., see [3, Ch. 5; 4, Chs. 2 and 3]), which assigns to each number $x \in \mathbb{R}$ a number $\mu_{\tilde{u}}(x)$ from the interval $[0, 1]$ characterizing the grade of membership of the element x in the set $\tilde{u}$. Here, 0 and 1 represent the lowest and highest grades of membership of an element in a given set, respectively. Thus, a fuzzy number $\tilde{u}$ can be treated as a pair $\{x, \mu_{\tilde{u}}(x) : x \in \mathbb{R}\}$.

Note that the concept of a membership function is introduced due to the insufficiency of the probabilistic approach for describing problems with uncertainty. In particular, in many applications, it is difficult to determine an a priori probability distribution. Here, the membership function is a certain analog of the probability distribution of a random variable in probability theory. Let us clarify that the number $\mu_{\tilde{u}}(x)$ is interpreted as the possibility of taking the value x for $\tilde{u}$. We emphasize that, as a rule, expert assessments are used to construct membership functions.

Also note that the membership function of a fuzzy number conceptually generalizes the characteristic function of a set, which can take only two values: 0 and 1 (0 when the element does not belong to the set and 1 otherwise).

The number x_M for which $\mu(x_M) = \max_{x \in \mathbb{R}} \mu(x)$ is called the modal value (or mode) of the fuzzy number. It is interpreted as the most possible value.

By a common assumption, the support of a fuzzy number $\tilde{u}$ (i.e., the set $\{x : \mu_{\tilde{u}}(x) > 0\}$) is bounded, and its membership function is convex, upper semicontinuous, and normal (i.e., $\sup_x \mu_{\tilde{u}}(x) = 1$). Let J denote the set of such fuzzy numbers.

Below, the interval representation of fuzzy numbers will be considered.

As is well known [3, Ch. 5], the α -level intervals (α -levels) of a fuzzy number $\tilde{u} \in J$ with a membership function $\mu_{\tilde{u}}(x)$ are defined by the relations

$$u_{\alpha} = \{x | \mu_{\tilde{u}}(x) \geq \alpha\}, \quad (\alpha \in (0, 1]), \quad z_0 = cl\{x | \mu_{\tilde{u}}(x) > 0\},$$

where cl indicates the closure of an appropriate set. According to the accepted assumptions, all α -levels of a fuzzy number are closed and bounded intervals of the real axis.

We denote by u_{α}^{-} and u_{α}^{+} the left and right bounds of an α -interval, respectively: $u_{\alpha} = [u_{\alpha}^{-}, u_{\alpha}^{+}]$. The expressions u_{α}^{-} and u_{α}^{+} are called the left and right α -indices (or, simply, indices) of the fuzzy number, respectively.

The α -indices of a fuzzy number $\tilde{u} \in J$ have the following properties:

1. $u_{\alpha}^{-} \leq u_{\alpha}^{+} \forall \alpha \in [0, 1]$.
2. The function u_{α}^{-} is bounded, nondecreasing, left continuous on the interval $(0, 1]$, and right continuous at the point 0.
3. The function u_{α}^{+} is bounded, nonincreasing, left continuous on the interval $(0, 1]$, and right continuous at the point 0.

Conversely, a pair of functions on the interval $[0, 1]$ satisfying conditions 1–3 defines a fuzzy number whose α -interval has the form $[u_{\alpha}^{-}, u_{\alpha}^{+}]$.

The sum of fuzzy numbers with indices $u_{\alpha}^{-}, u_{\alpha}^{+}$ and $v_{\alpha}^{-}, v_{\alpha}^{+}$ is understood as a fuzzy number with the α -level intervals $[u_{\alpha}^{-} + v_{\alpha}^{-}, u_{\alpha}^{+} + v_{\alpha}^{+}]$. Multiplication by a positive real number c is characterized by the α -level intervals $[cu_{\alpha}^{-}, cu_{\alpha}^{+}]$ whereas multiplication by a negative real number c by the α -level intervals $[cu_{\alpha}^{+}, cu_{\alpha}^{-}]$. Equality for fuzzy numbers is understood as equality for all the corresponding α -indices $\forall \alpha \in [0, 1]$.

Triangular numbers, for which membership functions have a triangular shape, are widely used in applications.

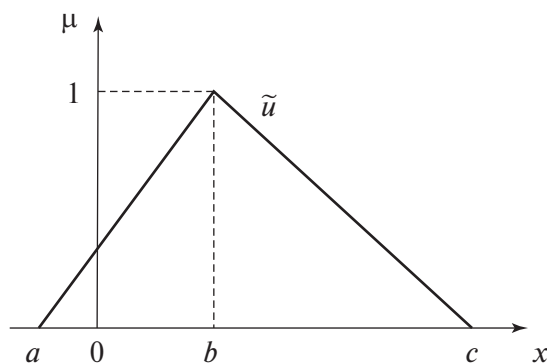


Fig. 1. A triangular fuzzy number.

Example 1. A triangular fuzzy number $\tilde{u}$ characterized by a triple of real numbers (a, b, c) with $a < b < c$ is defined by the membership function

$$\mu_{\tilde{u}}(x) = \begin{cases} \frac{x-a}{b-a} & \text{if } x \in [a, b] \\ \frac{x-c}{b-c} & \text{if } x \in [b, c] \\ 0 & \text{otherwise.} \end{cases}$$

Its graph generates the triangle shown in Fig. 1.

Note that the number b is the modal value (or mode) of the considered fuzzy number. For this number, we have $\mu_{\tilde{u}}(b) = 1$. The triangular fuzzy number in Fig. 1 is interpreted as the one near b .

As is well known, in the case of a triangular number, the lower and upper bounds of the α -interval have the form

$$u_{\alpha}^{-} = (b-a)\alpha + a, \quad u_{\alpha}^{+} = -(c-b)\alpha + c.$$

Distances between fuzzy numbers can be defined in different ways. Within the interval approach, the Hausdorff distance between the α -level sets of fuzzy numbers is often used: for fuzzy numbers $\tilde{u}$ and $\tilde{v}$ with α -levels u_{α} and v_{α} , this metric [25] is given by

$$\rho(\tilde{u}, \tilde{v}) = \sup_{0 \leq \alpha \leq 1} \max\{|u_{\alpha}^{-} - v_{\alpha}^{-}|, |u_{\alpha}^{+} - v_{\alpha}^{+}|\}. \quad (1)$$

Here, $[u_{\alpha}^{-}, u_{\alpha}^{+}]$ and $[v_{\alpha}^{-}, v_{\alpha}^{+}]$ are the α -level intervals of the fuzzy numbers $\tilde{u}$ and $\tilde{v}$, respectively.

Note that by (1), the condition $\rho(\tilde{u}, \tilde{v}) = 0$ is equivalent to the equality of fuzzy numbers $\tilde{u}$ and $\tilde{v}$ (see the definition above).

We fix an interval T of the real axis. A mapping $\tilde{z} : T \rightarrow J$ will be called a fuzzy-valued function.

Let a fuzzy-valued function $\tilde{z}(t) \forall t \in T$ be characterized by a membership function $\mu_{\tilde{z}(t)}(x)$. For a fixed number $\alpha \in (0, 1]$, we consider the α -interval $z_{\alpha}(t) = \{x \in \mathbb{R} : \mu_{\tilde{z}(t)}(x) \geq \alpha\}$ and $z_0(\alpha) = cl\{x \in \mathbb{R} : \mu_{\tilde{z}(t)}(x) > 0\}$. The left and right bounds of the α -interval will be denoted by $z_{\alpha}^{-}(t)$ and $z_{\alpha}^{+}(t)$, respectively: $z_{\alpha}(t) = [z_{\alpha}^{-}(t), z_{\alpha}^{+}(t)]$.

The continuity of a function $\tilde{z}(t)$ in t will be understood in terms of the metric (1) whereas its boundedness in the following sense: there exists a constant $C > 0$ such that, for all $t \in T$,

$$\rho(\tilde{z}(t), \tilde{0}) = \sup_{0 \leq \alpha \leq 1} \max\{|z_{\alpha}^{-}(t)|, |z_{\alpha}^{+}(t)|\} \leq C.$$

Here, $\tilde{0}$ is the fuzzy number with the α -indices $0_{\alpha}^{\pm} = 0 \forall \alpha \in [0, 1]$.

Remark 1. The indices $z_{\alpha}^{-}(t)$ and $z_{\alpha}^{+}(t)$ of a continuous fuzzy-valued function $\tilde{z}(t)$ are continuous in t for any $\alpha \in [0, 1]$. And if $\tilde{z}(t)$ is bounded for $t \in T$, then $z_{\alpha}^{\pm}(t)$ is bounded in $t \in T$ for any $\alpha \in [0, 1]$.

The integral of a continuous fuzzy-valued function $\tilde{z}(t)$ over an interval T is defined as the fuzzy number $\tilde{g}$ with the α -level intervals $g_{\alpha} = \int_T z_{\alpha}(t)dt$ for any $\alpha \in [0, 1]$ [6]. The integral is denoted by $\int_T \tilde{z}(t)dt$.

Essentially, this is the Aumann integral [26] of a multivalued mapping $z_{\alpha}(t)$. In fact, we have the interval representation [27]

$$\int_T \tilde{z}(t)dt = \left[\int_T z_{\alpha}^{-}(t)dt, \int_T z_{\alpha}^{+}(t)dt \right].$$

Let us proceed to the derivatives of fuzzy-valued functions. Various definitions are used in the literature. One of the most common rests on the definition of the Hukuhara difference [28]: a set C is called the Hukuhara difference of sets A and B if $A = B + C$, and is denoted by $A \ominus B$.

A function $\tilde{z} : T \rightarrow J$ is said to be Hukuhara differentiable (H -differentiable) at a point $t \in T$ [5] if, for all sufficiently small $h > 0$, there exist Hukuhara differences $\tilde{z}(t+h) \ominus \tilde{z}(t)$ and $\tilde{z}(t) \ominus \tilde{z}(t-h)$ and an element $\tilde{z}'(t) \in J$ such that

$$\lim_{h \rightarrow 0^+} \rho \left(\frac{\tilde{z}(t+h) \ominus \tilde{z}(t)}{h}, \tilde{z}'(t) \right) = \lim_{h \rightarrow 0^+} \rho \left(\frac{\tilde{z}(t) \ominus \tilde{z}(t-h)}{h}, \tilde{z}'(t) \right) = 0,$$

where the distance ρ is given by (1). In this case, the element $\tilde{z}'(t)$ is called the H -derivative at the point t .

A fuzzy-valued function $\tilde{z} : T \rightarrow J$ is said to be Seikkala differentiable (S -differentiable) at a point $t \in T$ [7] if its α -indices $z_{\alpha}^{-}(t)$ and $z_{\alpha}^{+}(t)$ are differentiable and their derivatives $(z_{\alpha}^{-})'(t)$ and $(z_{\alpha}^{+})'(t) \forall \alpha \in [0, 1]$ form a fuzzy number with the α -interval $[\tilde{z}'(t)]_{\alpha} = [(z_{\alpha}^{-})'(t), (z_{\alpha}^{+})'(t)]$.

Proposition 1 [8]. *Let a fuzzy-valued function $\tilde{z}(t)$ be H -differentiable at a point $t \in T$. Then it is S -differentiable at the point $t \in T$.*

For example, a fuzzy-valued function $\tilde{z}(t)$ of the form $\tilde{z}(t) = g(t)\tilde{r}$, where $g(t)$ is a real-valued differentiable function and $\tilde{r} \in J$ is a given fuzzy number, is H -differentiable (hence, S -differentiable) at a point t provided that $g(t) \cdot g'(t) \geq 0$ [9].

Remark 2. By definition, the fuzzy S -derivative is additive and positively homogeneous, i.e., for S -differentiable fuzzy-valued functions $\tilde{z}(t)$ and $\tilde{w}(t)$, we have $(\tilde{z}(t) + \tilde{w}(t))' = \tilde{z}'(t) + \tilde{w}'(t)$ and $(c\tilde{z}(t))' = c\tilde{z}'(t)$ for any real constant $c \geq 0$.

The second Seikkala derivative $\tilde{z}''(t)$ at a point $t \in T$ is defined as the S -derivative of the first derivative, i.e., as the fuzzy number $\tilde{z}''(t)$ with the left $(z_{\alpha}^{-})''(t)$ and right $(z_{\alpha}^{+})''(t)$ α -index $\forall \alpha \in [0, 1]$.

Higher-order S -derivatives are defined by analogy.

3. TRANSFORMATION OF A CONTINUOUS FUZZY SIGNAL BY A LINEAR DYNAMIC SYSTEM

A device is called a linear dynamic system if the relationship between its input and output is described by a differential equation of order n with constant coefficients. If fuzzy signals $\tilde{f}(t)$ and $\tilde{z}(t)$ ($t \in T$) are observed at the input and output, respectively, then the linear dynamic system is described by a fuzzy differential equation of the form

$$a_n \tilde{z}^{(n)}(t) + a_{n-1} \tilde{z}^{(n-1)}(t) + \dots + a_1 \tilde{z}'(t) + a_0 \tilde{z}(t) = \tilde{f}(t). \quad (2)$$

Here, the coefficients a_i ($i = 0, \dots, n$) are real numbers, $\tilde{f}(t)$ is an input fuzzy-valued function, and the derivatives of a fuzzy-valued function $\tilde{z}(t)$ are understood as S -derivatives.

Below, the interval T is taken as $T = (-\infty, \infty)$.

Consider the problem of bounded solutions for a real differential equation with constant coefficients of the form

$$a_n x^{(n)}(t) + a_{n-1} x^{(n-1)}(t) + \dots + a_1 x'(t) + a_0 x(t) = f(t), \quad t \in (-\infty, \infty). \quad (3)$$

A function $G(t)$ is called a Green's function in the problem of bounded solutions of equation (3) if it has the following properties (for details, e.g., see [23, Ch. 1, § 4; 24]):

1) $G(t)$ is continuously differentiable $(n-2)$ times for all t , the n th and $(n-1)$ th derivatives are continuously differentiable for all t except $t = 0$, and

$$G^{(n-1)}(+0) - G^{(n-1)}(-0) = \frac{1}{a_n}.$$

2) At all points except $t = 0$, the function $G(t)$ satisfies the homogeneous differential equation corresponding to (3) (with $f(t) \equiv 0$).

3) The Green's function and its derivatives are estimated as

$$|G^{(i)}(t)| \leq M e^{-\gamma|t|} \quad (i = 0, 1, \dots, n, \quad -\infty < t < +\infty),$$

where M and γ are some positive constants.

Proposition 2 [23, Ch. 1; 24]. *Let the roots of the characteristic equation $a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$ contain no points on the imaginary axis. Then for any continuous function $f(t)$ bounded on the entire real axis, equation (3) has a unique bounded solution on the entire real axis given by*

$$x(t) = \int_{-\infty}^{\infty} G(t-s) f(s) ds, \quad (4)$$

where $G(t)$ is the Green's function in the problem of bounded solutions of equation (3).

Moreover,

$$x^{(j)}(t) = \int_{-\infty}^{\infty} G_t^{(j)}(t-s) f(s) ds, \quad j = (0, 1, \dots, n-1),$$

$$x^{(n)}(t) = f(t) + \int_{-\infty}^{\infty} G_t^{(n)}(t-s) f(s) ds.$$

We emphasize that the convergence of the improper integral (4) and the corresponding integrals for the derivatives is ensured by the exponential estimates of the Green's function and its derivatives, as well as by the continuity and boundedness of the function $f(t)$ on the entire real axis.

Note that in the problem of bounded solutions of equation (3), the Green's function has a known general form; for example, see [23, Ch. 2, § 8].

Proposition 3 [23, Ch. 2]. *Under the hypotheses of Proposition 2, let all roots of the characteristic equation lie in the left half-plane ($\operatorname{Re} \lambda_i < 0$, $i = 1, \dots, n$). Then the bounded solution of equation (3) is asymptotically Lyapunov stable. Moreover, the solution (4) takes the form*

$$x(t) = \int_{-\infty}^t G(t-s) f(s) ds. \quad (5)$$

Here, the Green's function is $G(\sigma) = \begin{cases} K(\sigma) & \text{for } \sigma \geq 0 \\ 0 & \text{for } \sigma < 0, \end{cases}$ with $K(\sigma)$ being the Cauchy function that represents the solution of the homogeneous differential equation corresponding to equation (3) (with $f(t) \equiv 0$) and satisfies the initial conditions

$$K^{(j)}(0) = 0, \quad j = 0, 1, \dots, n-2, \quad K^{(n-1)}(0) = 1.$$

Now we address the case of fuzzy input and output signals. In some cases, the representation (4) can be used to write explicitly the α -indices of the fuzzy signal at the output of the dynamic system (2).

A strong solution of the fuzzy differential equation (2) is an n times continuously S -differentiable fuzzy-valued function satisfying (2) on the corresponding interval.

Lemma 1. *Let the coefficients of the fuzzy differential equation (2) be positive: $a_i > 0$, $i = 0, \dots, n$. If a fuzzy-valued function $\tilde{z}(t)$ is a strong solution of equation (2) on the interval T , then the corresponding α -indices $z_\alpha^\pm(t)$ for all $\alpha \in [0, 1]$ and $t \in T$ satisfy the ordinary differential equations*

$$a_n(z_\alpha^-)^{(n)}(t) + a_{n-1}(z_\alpha^-)^{(n-1)}(t) + \dots + a_1(z_\alpha^-)'(t) + a_0 z_\alpha^-(t) = f_\alpha^-(t), \quad (6)$$

$$a_n(z_\alpha^+)^{(n)}(t) + a_{n-1}(z_\alpha^+)^{(n-1)}(t) + \dots + a_1(z_\alpha^+)'(t) + a_0 z_\alpha^+(t) = f_\alpha^+(t). \quad (7)$$

Indeed, we substitute the strong solution $\tilde{z}(t)$ of the fuzzy differential equation into (2). Recall that the equality of fuzzy numbers means the equality of all the corresponding α -indices. Hence, by the rules of interval arithmetic, the positivity of the coefficients a_i , the definition of a fuzzy Seikkala derivative, and Remark 2, for all $\alpha \in [0, 1]$ and $t \in (-\infty, \infty)$ equation (2) implies equalities (6) and (7).

Lemma 2. *Let the coefficients of the fuzzy differential equation (2) be positive, ($a_i > 0$, $i = 0, \dots, n$), and let the roots of the characteristic equation $a_n \lambda^n + a_{n-1} \lambda^{n-1} + \dots + a_1 \lambda + a_0 = 0$ contain no points on the imaginary axis. In addition, let the fuzzy-valued function $\tilde{f}(t)$ on the right-hand side of equation (2) be continuous and bounded in the metric (1) for $t \in (-\infty, \infty)$. Then for all $\alpha \in [0, 1]$ there exists a unique bounded solution for each of the two equations (6), (7) on the entire real axis, and they can be represented as*

$$z_\alpha^-(t) = \int_{-\infty}^{\infty} G(t-s) f_\alpha^-(s) ds, \quad z_\alpha^+(t) = \int_{-\infty}^{\infty} G(t-s) f_\alpha^+(s) ds. \quad (8)$$

Indeed, due to the hypotheses and Remark 1, for all $\alpha \in [0, 1]$ the functions $f_\alpha^\pm(t)$ are continuous and bounded on the entire real axis. Then, by Proposition 2, for all $\alpha \in [0, 1]$ the solutions of equations (6), (7) exist, are unique, and equalities (8) hold.

Lemma 3. *Under the hypotheses of Lemma 2, let the Green's function G in problem (3) be nonnegative. Then for all $t \in (-\infty, \infty)$ the expressions (8) satisfy conditions 1–3 for the indices of fuzzy numbers (see Section 2).*

Proof. We fix $t \in (-\infty, \infty)$. By the hypothesis, $f_\alpha^-(s) \leq f_\alpha^+(s) \forall s \in (-\infty, \infty)$. Then, due to the nonnegativity of the Green's function,

$$G(t-s) f_\alpha^-(s) \leq G(t-s) f_\alpha^+(s).$$

Therefore, based on (8) and the monotonicity of the integral, we have $z_\alpha^-(t) \leq z_\alpha^+(t)$. That is, for all $t \in (-\infty, \infty)$ the expressions (8) satisfy condition 1 for the α -indices of fuzzy numbers (see Section 2).

Now we fix an arbitrary $t \in (-\infty, \infty)$ and show that the function $z_\alpha^-(t)$ is nondecreasing in α . Let $\alpha_1, \alpha_2 \in [0, 1]$ and $\alpha_1 < \alpha_2$. By the hypothesis, the condition $f_{\alpha_1}^-(s) \leq f_{\alpha_2}^-(s)$ holds for all $s \in (-\infty, \infty)$. Then, due to the nonnegativity of the Green's function, $G(t-s)f_{\alpha_1}^-(s) \leq G(t-s)f_{\alpha_2}^-(s)$. Consequently, in view of the monotonic property of the integral, we obtain $z_{\alpha_1}^-(t) \leq z_{\alpha_2}^-(t)$, i.e., the function $z_\alpha^-(t)$ is monotonically nondecreasing in α . The monotonic non-increase of the function $z_\alpha^+(t)$ in α can be established by analogy.

By the hypothesis, the function $\tilde{f}(t)$ is bounded in t for $t \in (-\infty, \infty)$ in the metric (1). This means the existence of a constant $C > 0$ such that $\rho(\tilde{f}(t), \tilde{0}) \leq C$ for any $t \in (-\infty, \infty)$; consequently, $\sup_{0 \leq \alpha \leq 1} |f_\alpha^\pm(t)| \leq C$. Then according to (8), for fixed t and all $\alpha \in [0, 1]$, we have

$$|z_\alpha^\pm(t)| \leq \int_{-\infty}^{\infty} |G(t-s)| |f_\alpha^\pm(s)| ds \leq C \int_{-\infty}^{\infty} |G(t-s)| ds,$$

which ensures the boundedness of the expressions (8) in $\alpha \in [0, 1]$ for each $t \in (-\infty, \infty)$.

For fixed t , the left continuity of the functions $z_\alpha^\pm(t)$ in $\alpha \in (0, 1]$ is immediate from the following consideration. Let us fix $\alpha_0 \in [0, 1)$ and consider the equality

$$\lim_{\alpha \rightarrow \alpha_0 - 0} z_\alpha^\pm(t) = \int_{-\infty}^{\infty} G(t-s) \lim_{\alpha \rightarrow \alpha_0 - 0} f_\alpha^\pm(s) ds = \int_{-\infty}^{\infty} G(t-s) f_{\alpha_0}^\pm(s) ds = z_{\alpha_0}^\pm(t).$$

Here, we take advantage of the representation (8), the possibility of passing to the limit under the sign of an absolutely convergent improper integral, and the left continuity of the α -indices $f_\alpha^\pm(s)$ of the fuzzy-valued function $\tilde{f}(s)$ in $\alpha \in (0, 1]$ for arbitrary $s \in (-\infty, \infty)$. The right continuity of the functions $z_\alpha^\pm(t)$ in α at $\alpha = 0$ is verified similarly. In other words, $z_\alpha^\pm(t)$ satisfy conditions 2 and 3 of Section 2 as well.

Thus, the expressions $z_\alpha^\pm(t)$ (8) satisfy all the conditions for the α -levels of fuzzy numbers (see Section 2).

Let us emphasize the significance of Lemma 3. According to illustrative examples [18, 19], the solutions of systems for the α -indices of linear fuzzy differential equations are not always the α -indices of some fuzzy-valued function.

Theorem 1. *Under the hypotheses of Lemma 3, the fuzzy-valued function generated by the α -indices (8) $\forall t \in (-\infty, \infty)$ is characterized by the representation*

$$\tilde{z}(t) = \int_{-\infty}^{\infty} G(t-s) \tilde{f}(s) ds. \quad (9)$$

Indeed, by Lemmas 1–3 and the definition of the integral of a fuzzy-valued function, for all $\alpha \in [0, 1]$ the α -indices satisfy the relations

$$\begin{aligned} \left(\int_{-\infty}^{\infty} G(t-s) \tilde{f}(s) ds \right)_\alpha^- &= \int_{-\infty}^{\infty} G(t-s) f_\alpha^-(s) ds, \\ \left(\int_{-\infty}^{\infty} G(t-s) \tilde{f}(s) ds \right)_\alpha^+ &= \int_{-\infty}^{\infty} G(t-s) f_\alpha^+(s) ds. \end{aligned}$$

In view of (8), these relations imply the representation (9).

Theorem 2. *Under the hypotheses of Theorem 1, the fuzzy-valued function $\tilde{z}(t)$ (9) is continuous and bounded in t on the entire real axis.*

Proof. Let us fix real numbers t_1 and t_2 . Due to (8),

$$z_{\alpha}^{\pm}(t_1) - z_{\alpha}^{\pm}(t_2) = \int_{-\infty}^{\infty} (G(t_1 - s) - G(t_2 - s)) f_{\alpha}^{\pm}(s) ds$$

and, consequently,

$$|z_{\alpha}^{\pm}(t_1) - z_{\alpha}^{\pm}(t_2)| \leq \int_{-\infty}^{\infty} |G(t_1 - s) - G(t_2 - s)| |f_{\alpha}^{\pm}(s)| ds.$$

Note that $G(t_1 - s) - G(t_2 - s) = G'_t(\tau - s)(t_1 - t_2)$, where $\tau \in ((1 - \theta)t_1 + \theta t_2)$ and $\theta \in (0, 1)$. Therefore, by the previous considerations,

$$|z_{\alpha}^{\pm}(t_1) - z_{\alpha}^{\pm}(t_2)| \leq \left(\int_{-\infty}^{\infty} |G'_t(\tau - s)| |f_{\alpha}^{\pm}(s)| ds \right) |t_1 - t_2| \leq C |t_1 - t_2| \int_{-\infty}^{\infty} |G'_t(\tau - s)| ds,$$

where the constant $C > 0$ characterizes the boundedness condition: $\rho(\tilde{f}(t), \tilde{0}) \leq C \forall t \in (-\infty, \infty)$.

Next, we utilize the exponential estimate of the derivative of the Green's function $|G'_t(t)| \leq M e^{-\gamma|t|}$ (see property 3 of Green's functions in Section 3). As a result,

$$|z_{\alpha}^{\pm}(t_1) - z_{\alpha}^{\pm}(t_2)| \leq MC |t_1 - t_2| \int_{-\infty}^{\infty} e^{-\gamma|\tau - s|} ds = 2 \frac{MC}{\gamma} |t_1 - t_2|.$$

Therefore, by the definition of the metric (1), we obtain $\rho(z_{\alpha}^{\pm}(t_1), z_{\alpha}^{\pm}(t_2)) \leq 2 \frac{MC}{\gamma} |t_1 - t_2|$, which implies the continuity of the fuzzy-valued function $\tilde{z}(t)$ (9).

Let us show the boundedness of the fuzzy-valued function $\tilde{z}(t)$ (9) for $t \in (-\infty, \infty)$. By definition,

$$\rho(\tilde{z}(t), \tilde{0}) = \sup_{0 \leq \alpha \leq 1} |z_{\alpha}^{\pm}(t)|.$$

Moreover, according to (8),

$$|z_{\alpha}^{\pm}(t)| \leq \int_{-\infty}^{\infty} |G(t - s)| |f_{\alpha}^{\pm}(s)| ds \leq \left(\int_{-\infty}^{\infty} |G(t - s)| ds \right) \rho(\tilde{f}(t), \tilde{0}) \leq 2 \frac{M \rho(\tilde{f}(t), \tilde{0})}{\gamma}.$$

Here, the exponential estimate of the Green's function is used again. Thus, $\rho(\tilde{z}(t), \tilde{0}) \leq 2 \frac{M \rho(\tilde{f}(t), \tilde{0})}{\gamma} \leq 2 \frac{MC}{\gamma} \forall t \in (-\infty, \infty)$.

We call a continuous fuzzy-valued function $\tilde{z}(t)$ an ultra-weak solution of the fuzzy differential equation (2) if its α -indices $z_{\alpha}^{\pm}(t) \forall \alpha \in [0, 1]$ are n times continuously differentiable with respect to t and satisfy equations (6) and (7) on the corresponding time interval. A weak solution of the fuzzy differential equation (2) is a fuzzy-valued function satisfying the integro-differential or integral fuzzy equation corresponding to (2) (for example, see [19]). Such solutions will not be considered in this paper. The following result is true.

Theorem 3. *Under the hypotheses of Theorem 1, there exists a unique ultra-weak solution of the fuzzy differential equation (2) that is bounded on the entire real axis.*

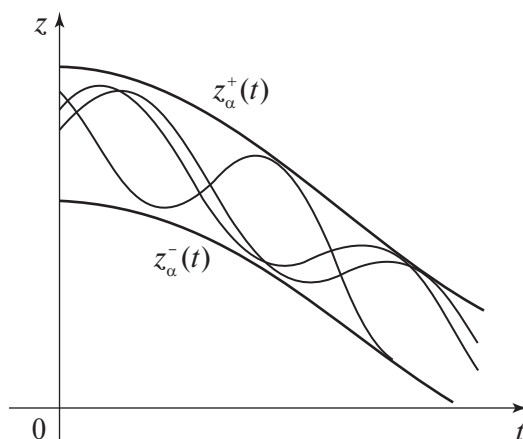


Fig. 2. The graphical representation of an ultra-weak fuzzy-valued solution in interval form.

Existence is ensured by Theorems 1 and 2. Let us establish uniqueness. Assume on the contrary that $\tilde{z}(t)$ and $\tilde{w}(t)$ are two different ultra-weak solutions of equation (2) that have boundedness on the entire real axis. Then the indices $z_\alpha^-(t)$ and $w_\alpha^-(t)$ are bounded solutions of equation (6). Therefore, by Proposition 2, $z_\alpha^-(t) = w_\alpha^-(t)$ for all $\alpha \in [0, 1]$ and $t \in (-\infty, \infty)$. Similarly, $z_\alpha^+(t) = w_\alpha^+(t)$ for all $\alpha \in [0, 1]$ and $t \in (-\infty, \infty)$. But in this case, according to the equality criterion of fuzzy numbers, we have $\tilde{z}(t) = \tilde{w}(t)$ for all $t \in (-\infty, \infty)$. This obvious contradiction concludes the proof of the theorem.

Theorem 4. *Under the hypotheses of Theorem 1, let all roots of the characteristic equation lie in the left half-plane. Then there exists a unique ultra-weak solution of the fuzzy differential equation (2) that is bounded on the entire real axis, and this solution is given by*

$$\tilde{z}(t) = \int_{-\infty}^t G(t-s)\tilde{f}(s)ds. \quad (10)$$

This fact is immediate from Theorem 1 and Proposition 3.

Figure 2 provides the geometric illustration of a fuzzy-valued solution.

We say that a fuzzy-valued function $\tilde{f}: \mathbb{R} \rightarrow J$ is triangular, or has the triangular type $(a(t), b(t), c(t))$, if there exist continuous real-valued functions $a(t)$, $b(t)$, and $c(t)$ on the entire real axis such that $a(t) < b(t) < c(t) \forall t \in \mathbb{R}$ and the membership functions $\mu_{\tilde{f}(t)}(x) \forall t \in \mathbb{R}$ have the triangular form

$$\mu_{\tilde{f}(t)}(x) = \begin{cases} \frac{x - a(t)}{b(t) - a(t)} & \text{if } x \in [a(t), b(t)] \\ \frac{x - c(t)}{b(t) - c(t)} & \text{if } x \in [b(t), c(t)] \\ 0 & \text{otherwise.} \end{cases}$$

(Also, see Example 1.)

In particular, a fuzzy-valued function of the form $\tilde{f}(t) = g(t)\tilde{r}$, where $g(t)$ is a continuous and nonnegative real-valued function and $\tilde{r} \in J$ is a given triangular fuzzy number, has triangular type as well.

Consider a special case when a triangular fuzzy signal is supplied to the input of a dynamic system described by equation (2). The following result is valid.

Theorem 5. *Under the hypotheses of Theorem 1, let the right-hand side of equation (2), i.e., the fuzzy-valued function $\tilde{f}(t) \forall t \in (-\infty, \infty)$, have the triangular type $(a(t), b(t), c(t))$. Then the bounded ultra-weak solution (9) of equation (2) on the entire real axis has the triangular type $(\int_{-\infty}^{\infty} G(t-s)a(s)ds, \int_{-\infty}^{\infty} G(t-s)b(s)ds, \int_{-\infty}^{\infty} G(t-s)c(s)ds) \forall t \in (-\infty, \infty)$.*

Indeed, by the hypothesis, $\tilde{f}(t)$ is generated by a triple of continuous and bounded functions $a(t)$, $b(t)$, $c(t)$ on the entire axis, with $a(t) < b(t) < c(t) \forall t \in (-\infty, \infty)$. Then, according to Example 1, for all $\alpha \in [0, 1]$ and $t \in (-\infty, \infty)$ we have

$$f_{\alpha}^{-}(t) = (b(t) - a(t))\alpha + a(t), \quad f_{\alpha}^{+}(t) = (b(t) - c(t))\alpha + c(t).$$

Therefore, due to (8), it follows that

$$z_{\alpha}^{-}(t) = (B(t) - A(t))\alpha + A(t), \quad z_{\alpha}^{+}(t) = (B(t) - C(t))\alpha + C(t),$$

where

$$A(t) = \int_{-\infty}^{\infty} G(t-s)a(s)ds, \quad B(t) = \int_{-\infty}^{\infty} G(t-s)b(s)ds, \quad C(t) = \int_{-\infty}^{\infty} G(t-s)c(s)ds.$$

Thus, $\tilde{z}(t) \forall t \in (-\infty, \infty)$ is a fuzzy number of the triangular type $(A(t), B(t), C(t))$.

A similar result holds for trapezoidal fuzzy-valued functions $\tilde{f}(t)$ [3, Ch. 5]: the solution $\tilde{z}(t)$ will also have the trapezoidal type.

Let us find conditions under which the bounded ultra-weak solution (10) of the fuzzy differential equation (2) will be S -differentiable with respect to t .

Under the hypotheses of Proposition 3, the following representation will be used below for the bounded solution of the scalar differential equation (3):

$$x(t) = \int_0^{\infty} G(\sigma)f(t-\sigma)d\sigma. \quad (11)$$

It is obtained from (5) by substituting $t-s=\sigma$.

Theorem 6. *Under the hypotheses of Theorem 4, let the fuzzy-valued function $\tilde{f}(t)$ be Seikkala differentiable for all $t \in (-\infty, \infty)$, and let its S -derivative $\tilde{f}'(t)$ be continuous and bounded for $t \in (-\infty, \infty)$. Then the bounded ultra-weak solution (10) of the fuzzy differential equation (2) is S -differentiable for $t \in (-\infty, \infty)$ and*

$$\tilde{z}'(t) = \int_0^{\infty} G(\sigma)\tilde{f}'_t(t-\sigma)d\sigma. \quad (12)$$

Indeed, due to (6), (7), and (11), for the α -indices of the ultra-weak solution $\tilde{z}(t)$ of the fuzzy differential equation (2) we write

$$z_{\alpha}^{\pm}(t) = \int_0^{\infty} G(\sigma)f_{\alpha}^{\pm}(t-\sigma)d\sigma. \quad (13)$$

Similar to Lemma 3, they generate the fuzzy-valued function

$$\tilde{z}(t) = \int_0^{\infty} G(\sigma)\tilde{f}(t-\sigma)d\sigma. \quad (14)$$

By the hypothesis, for all $\alpha \in [0, 1]$ the α -indices $f_\alpha^\pm(t)$ are differentiable with respect to $t \in (-\infty, \infty)$, and the derivatives $(f_\alpha^\pm)'_t(t)$ are continuous and bounded scalar functions on the entire real axis. Then, in view of (13), for the derivatives of the α -indices $z_\alpha^\pm(t)$ with respect to t we have

$$(z_\alpha^\pm)'(t) = \int_0^\infty G(\sigma)(f_\alpha^\pm)'_t(t - \sigma)d\sigma. \quad (15)$$

(Here, differentiation is performed with respect to the parameter under the sign of an absolutely convergent integral.)

By the nonnegativity of the Green's function (similar to the proof of Lemma 3), for each t the expressions (15) are the α -indices of the S -derivative $\tilde{z}'(t)$. Therefore, used jointly with Theorem 1, formulas (14) and (15) imply (12).

Then, by Theorem 3 on the uniqueness of the bounded ultra-weak solution of equation (2) on the entire real axis, based on (10) and (14) we obtain

$$\left(\int_{-\infty}^t G(t-s)\tilde{f}(s)ds \right)'_t = \left(\int_0^\infty G(\sigma)\tilde{f}(t-\sigma)d\sigma \right)'_t = \int_0^\infty G(\sigma)\tilde{f}'_t(t-\sigma)d\sigma.$$

And the desired conclusion follows.

Corollary 1. *Under the hypotheses of Theorem 6, the fuzzy derivative (12) is continuous and bounded for $t \in (-\infty, \infty)$.*

This fact is established by analogy with Theorem 2.

Proposition 4. *Under the hypotheses of Theorem 4, let the fuzzy-valued function $\tilde{f}(t)$ be twice S -differentiable, and let the first $\tilde{f}'_t(t)$ and second $\tilde{f}''_t(t)$ S -derivatives be continuous and bounded for $t \in (-\infty, \infty)$. Then the bounded ultra-weak solution $\tilde{z}(t)$ (14) on the entire real axis is twice S -differentiable for $t \in (-\infty, \infty)$.*

Indeed, due to equality (15), for all $\alpha \in [0, 1]$ we write the following relation for the α -indices $z_\alpha^\pm(t)$:

$$(z_\alpha^\pm)''(t) = \left(\int_0^\infty G(\sigma)(f_\alpha^\pm)'_t(t - \sigma)d\sigma \right)'_t = \int_0^\infty G(\sigma)(f_\alpha^\pm)''_t(t - \sigma)d\sigma.$$

Therefore, based on the nonnegativity of the Green's function, for all $t \in (-\infty, \infty)$ the integrals on the right-hand side generate a fuzzy number $\tilde{z}''(t)$, i.e., the second S -derivative of $\tilde{z}(t)$.

Corollary 2. *Under the hypotheses of Proposition 4, the derivatives $\tilde{z}'(t)$ and $\tilde{z}''(t)$ are continuous and bounded for $t \in (-\infty, \infty)$.*

Corollary 3. *Under the hypotheses of Theorem 4, let the fuzzy-valued function $\tilde{f}(t)$ be continuously S -differentiable $j > 2$ times, and let all its derivatives up to order j be bounded on the entire real axis. Then the bounded ultra-weak solution (14) of the fuzzy differential equation (2) is a continuously j times S -differentiable fuzzy-valued function.*

Remark 3. According to Proposition 2, the derivatives of the α -indices $z_\alpha^\pm(t)$ (8) are represented as

$$(z_\alpha^\pm)^{(j)}_t(t) = \int_{-\infty}^\infty G_t^{(j)}(t-s)f_\alpha^\pm(s)ds \quad (j = 1, \dots, n-1).$$

However, the derivatives $G_t^{(j)}(t-s)$ of the Green's function generally do not preserve sign (see Examples 2 and 3 below). So, it is more convenient to prove the S -differentiability of the fuzzy-valued solution using the representation (13).

In many applications, the fuzzy-valued solution $\tilde{z}(t)$ must be nonnegative, which means $z_\alpha^-(t) \geq 0$ $\forall \alpha \in [0, 1]$, $\forall t \geq 0$. The following result is true.

Corollary 4. *Under the hypotheses of Theorem 1, let the fuzzy-valued inhomogeneity $\tilde{f}(t)$ of equation (2) be nonnegative, i.e., $f_\alpha^-(t) \geq 0$ $\forall \alpha \in [0, 1]$, $\forall t \in (-\infty, \infty)$. Then the bounded ultra-weak solution of equation (2) is nonnegative on the entire real axis.*

This fact is immediate from the representation (9) due to the assumed nonnegativity of the Green's function $G(t-s)$ and $f_\alpha^-(s)$.

In particular, for a triangular fuzzy number (a, b, c) , nonnegativity means that $a \geq 0$.

4. EXAMPLES OF RADIO CIRCUITS WITH FUZZY INPUT SIGNALS

Here, we consider some applications of the results of Section 3 to elementary radio circuits (e.g., see [29]) with fuzzy input signals.

Example 2. Consider an RC filter, i.e., a radio circuit shown in Fig. 3, where R and C are resistance and capacitance, respectively.

This filter is a dynamic system described by the first-order differential equation with constant coefficients

$$\tilde{z}'(t) + \beta \tilde{z}(t) = \tilde{y}(t), \quad \beta = \frac{1}{RC} > 0. \quad (16)$$

Let a continuous fuzzy signal $\tilde{y}(t)$ bounded on the entire real axis be supplied to the system input.

We determine the characteristics of the bounded fuzzy output signal $\tilde{z}(t)$ of the RC filter. Note that in the problem of bounded solutions of the scalar equation $x' + \beta x = y(t)$ with $\beta > 0$, the Green's function can be represented as

$$G_1(t) = \begin{cases} e^{-\beta t} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Thus, $G_1(t) \geq 0$ for all $t \in (-\infty, \infty)$.

Proposition 5. *Let the coefficient β of the fuzzy differential equation (16) be positive, and let the right-hand side $\tilde{y}(t)$ be a continuous fuzzy-valued function bounded for $t \in (-\infty, \infty)$. Then the continuous fuzzy ultra-weak signal at the output of the dynamic system (16) is bounded on the entire*

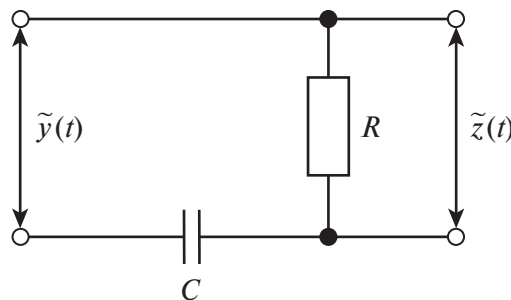


Fig. 3. An RC filter.

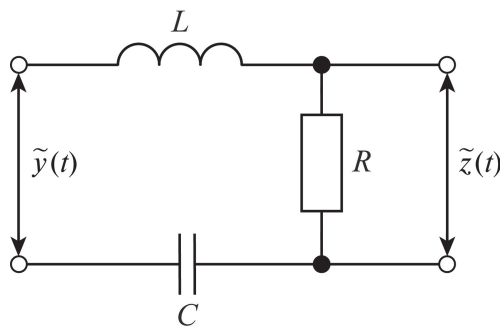


Fig. 4. A series oscillatory circuit.

real axis and has the form

$$\tilde{z}(t) = \int_{-\infty}^t e^{-\beta(t-s)} \tilde{y}(s) ds. \quad (17)$$

This fact follows from Theorem 4.

Proposition 6. *Under the hypotheses of Proposition 5, let the fuzzy-valued function $\tilde{y}(t)$ on the right-hand side of equation (16) have triangular type for $t \geq 0$. Then the solution (17) is also triangular for all $t \geq 0$.*

This fact is immediate from the representation (17) and Theorem 5.

The following case was considered in [9]: the right-hand side of equation (16) has the form $\tilde{y}(t) = \tilde{r}f(t)$, where $\tilde{r} \in J$ is a fuzzy number, and the function $f: \mathbb{R} \rightarrow \mathbb{R}$ is almost periodic and $f(t) \geq 0 \forall t \in \mathbb{R}$. As established in this case, the condition $f(t) > \int_{-\infty}^t e^{-\beta(t-s)} f(s) ds$ ensures the H -differentiability of the solution (17) of the fuzzy differential equation (16).

This condition imposes an additional constraint on either the range of t or the relationship between the parameters β and $f(t)$ of equation (16).

The next proposition, following from Theorem 6, ensures the S -differentiability of the fuzzy-valued solution of equation (16).

Proposition 7. *Under the hypotheses of Proposition 5, let the fuzzy-valued right-hand side $\tilde{y}(t)$ of equation (16) be continuously S -differentiable for all $t \in (-\infty, \infty)$, and let the S -derivative $\tilde{y}'(t)$ be bounded on the entire real axis. Then the bounded ultra-weak solution (17) is continuously S -differentiable for all $t \in (-\infty, \infty)$ and satisfies the fuzzy differential equation (16) for all $t \in (-\infty, \infty)$.*

Thus, under the hypotheses of Proposition 7, formula (17) gives a bounded strong solution of the fuzzy equation (16).

Example 3. Consider a series oscillatory circuit, i.e., a radio circuit in Fig. 4, where R , C , and L are resistance, capacitance, and inductance, respectively.

This oscillatory circuit is a dynamic system described by the second-order differential equation with constant coefficients

$$a_2 \tilde{z}''(t) + a_1 \tilde{z}'(t) + a_0 \tilde{z}(t) = \tilde{y}(t), \quad a_2 = L \geq 0, \quad a_1 = R > 0, \quad a_0 = \frac{1}{RC} > 0. \quad (18)$$

Let a continuous fuzzy signal $\tilde{y}(t)$ bounded on the entire real axis be supplied to the system input (see Fig. 4).

We determine the characteristics of the bounded fuzzy output signal $\tilde{z}(t)$ of this circuit.

Theorem 7. *Let the coefficients of the fuzzy differential equation (18) satisfy the conditions $a_i > 0$ ($i = 0, 1, 2$) and $a_1^2 - 4a_0a_2 > 0$. In addition, let the input signal $\tilde{y}(t)$ be a continuous fuzzy-valued function bounded for $t \in (-\infty, \infty)$. Then the fuzzy ultra-weak signal $\tilde{z}(t)$ at the output of the dynamic system (18) is represented as*

$$\tilde{z}(t) = \int_{-\infty}^t G_2(t-s)\tilde{y}(s)ds, \quad (19)$$

where $G_2(t)$ is the Green's function in the problem of bounded solutions of the real differential equation $a_2x'' + a_1x' + a_0x = f(t)$ with the continuous real-valued function $f(t)$ bounded on the entire real axis, which has the form

$$G_2(t) = \begin{cases} (e^{\lambda_2 t} - e^{\lambda_1 t})(\lambda_2 - \lambda_1)^{-1} & \text{for } t \geq 0 \\ 0 & \text{for } t < 0. \end{cases}$$

In this formula, λ_1 and λ_2 are distinct negative real-valued roots of the characteristic equation $a_2\lambda^2 + a_1\lambda + a_0 = 0$ ($\lambda_1 < \lambda_2 < 0$).

Indeed, note that $G_2(t) \geq 0 \forall t \in (-\infty, \infty)$. Then, by Theorem 4, equality (19) holds for the fuzzy-valued signal $\tilde{z}(t)$ at the system output.

Moreover, the expression (19) is an ultra-weak solution of the fuzzy differential equation (18).

Proposition 8. *Under the hypotheses of Theorem 7, let the right-hand side $\tilde{y}(t)$ of (18) be a triangular fuzzy number for all $t \in (-\infty, \infty)$. Then the bounded ultra-weak solution (19) of the fuzzy differential equation (18) on the entire real axis is also a triangular fuzzy number for $t \in (-\infty, \infty)$.*

This fact follows from the representation (19) and Theorem 5.

Let us find conditions under which the bounded ultra-weak solution (19) of the fuzzy differential equation (18) will be an S -differentiable fuzzy-valued function.

Theorem 8. *Under the hypotheses of Theorem 7, let the fuzzy-valued function $\tilde{y}(t)$ be continuously S -differentiable for all $t \in (-\infty, \infty)$, and let the S -derivative $\tilde{y}'(t)$ be bounded for $t \in (-\infty, \infty)$. Then the bounded ultra-weak solution $\tilde{z}(t)$ (19) of the fuzzy differential equation (18) is S -differentiable for all $t \in (-\infty, \infty)$.*

This fact is immediate from Theorem 6.

Moreover, another result is valid.

Theorem 9. *Under the hypotheses of Theorem 7, let the right-hand side $\tilde{y}(s)$ of equation (18) be twice continuously S -differentiable, and let the S -derivatives $\tilde{y}'(t)$ and $\tilde{y}''(t)$ be bounded for $t \in (-\infty, \infty)$. Then the bounded ultra-weak solution $\tilde{z}(t)$ (19) is also twice continuously S -differentiable and satisfies the fuzzy differential equation (18).*

This fact follows from Proposition 4.

Thus, under the hypotheses of Theorem 9, formula (19) provides a bounded strong solution of the fuzzy equation (18).

5. ON THE RELATIONSHIP BETWEEN MODAL VALUES OF FUZZY INPUT AND OUTPUT SIGNALS

The peculiarity of considering fuzzy input signals in radio engineering (when the a priori probability distribution of the input signal is unknown) consists in identifying the most possible input signal. After that, the system parameters are used to characterize the most possible fuzzy output signal.

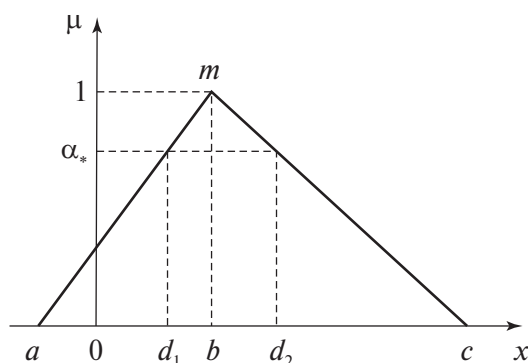


Fig. 5. A possibilistic confidence interval for the modal value of a fuzzy input signal.

In probability theory, for the corresponding problem, the term “possible” would have to be replaced by the term “probable.” This is especially clear for triangular fuzzy input and output signals (see Example 1). For instance, let a triangular fuzzy input signal $(a(t), b(t), c(t))$ be supplied to the input of a radio circuit described by the fuzzy differential equation (2). According to Example 1, for each t , $b(t)$ is its modal value. Based on Theorem 5, we arrive at the following result.

Proposition 9. *Under the hypotheses of Theorem 5, the modal value of the triangular fuzzy signal at the output of the linear dynamic system (2) is determined from the modal value $b(t)$ of its triangular fuzzy input signal $(a(t), b(t), c(t))$ by the formula $B(t) = \int_{-\infty}^{\infty} G(t-s)b(s)ds$, with the designations of Theorem 5.*

An important branch of mathematical statistics is the theory of confidence intervals; for example, see [30, Ch. 2]. By analogy, let us discuss the problem of possibilistic confidence intervals for the values of fuzzy input and output signals.

Suppose that for an arbitrary $t \geq 0$, experts have modeled a fuzzy input signal with a possibilistic confidence level $\alpha_* \in [0.7, 1)$ as triangular with a membership function $\mu_t(x)$ having support $(a(t), c(t))$. Moreover, for a given $t \geq 0$, let the modal value lie in a possibilistic confidence interval $[d_1(t), d_2(t)]$ with the possibilistic confidence level α_* , where $[d_1(t), d_2(t)] \subset (a(t), c(t))$ (Fig. 5).

Remark 4. Under the above assumptions, the modal value of the triangular fuzzy signal with the possibilistic confidence level α_* has the form

$$b(t) = \frac{1}{\alpha_*}(d_1(t) - (1 - \alpha_*)a(t)).$$

Indeed, according to Fig. 5, the point m with coordinates $(b, 1)$ lies on the straight line passing through the points with coordinates $(a, 0)$ and (d_1, α_*) ; hence, this line is described by the equation $\frac{\mu}{\alpha_*} = \frac{x-a}{d_1-a}$.

Consider the issue regarding the possibilistic confidence interval of the modal value of the fuzzy output signal with the same possibilistic confidence level α_* .

Proposition 10. *Under the hypotheses of Theorem 5, let a triangular fuzzy signal with support $(a(t), c(t))$ be supplied, with a possibilistic confidence level $\alpha_* \in [0.7, 1)$, to the input of the dynamic system described by differential equation (2), and let its modal value for an arbitrary $t \geq 0$ and this possibilistic confidence level lie in a possibilistic confidence interval $[d_1(t), d_2(t)]$. Then the possibilistic confidence interval for the modal value of the fuzzy output signal has the form $[D_1(t), D_2(t)]$, where $D_1(t) = (1 - \alpha_*)A(t) + \alpha_*B(t)$, $D_2(t) = \alpha_*B(t) + (1 - \alpha_*)C(t)$, and $A(t)$, $B(t)$, and $C(t)$ are given by Theorem 5. Moreover, the modal value $B(t)$ is determined using the value $b(t)$ from Remark 4.*

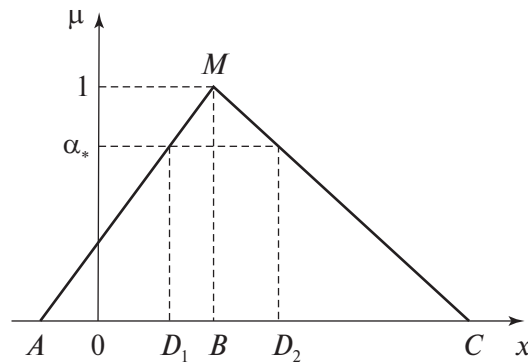


Fig. 6. A possibilistic confidence interval for the modal value of a fuzzy output signal.

Indeed, by Theorem 5 and Remark 4, the fuzzy output signal has the triangular type $(A(t), B(t), C(t))$. Then, according to Fig. 6, we obtain the possibilistic confidence interval $[D_1(t), D_2(t)]$ for the modal value of the fuzzy output signal for fixed $t > 0$ with a possibilistic confidence level α_* .

According to Fig. 6, $D_1 = (1 - \alpha_*)A + \alpha_*B$ is the abscissa of the intersection point of the line $[A, M]$, described by the equation $\mu = (x - A)(B - A)^{-1}$ (see Example 1), and the line $\mu = \alpha_*$. In turn, the point $D_2 = \alpha_*B + (1 - \alpha_*)C$ is the abscissa of the intersection point of the line $[M, C]$, described by the equation $\mu = (x - C)(B - C)^{-1}$, and the line $\mu = \alpha_*$.

Note that for practical use of the results of Sections 4 and 5, it is convenient to model the fuzzy input signal in the form $\tilde{y}(t) = \tilde{r}g(t)$, where $\tilde{r} \in J$ and $g(t)$ are a triangular fuzzy number and a real-valued function, respectively. Alternatively, one should choose a fuzzy signal model with $\tilde{y}(t)$ for $t \geq 0$ and $\tilde{y}(t) = \tilde{y}(0)$ for $t \leq 0$.

6. CONCLUSIONS

The main results of this paper concern fuzzy dynamic systems described by linear differential equations of order n with constant coefficients under the assumption of the continuity and boundedness of a fuzzy input signal (Section 3). They are based on a development of the Green's function method for the case of fuzzy differential equations. An important part of the research has been devoted to the smoothness of solutions. Also note the above result on the triangular type of the fuzzy output signal of a dynamic system receiving a fuzzy signal at its input.

The applications in Section 4 have illustrated the use of the general theoretical constructs in radio circuits with fuzzy input signals. They further refine some results of Section 3 for the case of dynamic systems described by first- and second-order differential equations. Section 5 plays a significant role as well: by analogy with confidence intervals of mathematical statistics, the concept of a possibilistic confidence interval has been introduced and used therein.

The approach presented in this paper is an alternative to the conventional analysis of linear dynamic systems with constant coefficients, which involves the frequency response, the direct and inverse Fourier transform, and the Laplace transform.

Note that the developed approach can be extended to the case of periodic and almost periodic signals and can be fruitful in the study of boundary value problems for fuzzy differential equations.

Many results can be modified to the case of generalized fuzzy derivatives.

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