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Design of Hybrid Nonlinear Control Systems Based on a Quasilinear Approach

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Abstract—A method for designing hybrid nonlinear control systems for plants with differentiable nonlinearities and a measurable state vector is developed based on continuous quasilinear models and quasilinear discretization. The hybrid system is designed with an increased control discretization period and zero static error for a reference signal. A solution of the control design problem exists if the nonlinear plant satisfies state and output controllability criteria and some additional conditions. The stability of the hybrid system is proven using the Aizerman–Pyatnitsky "technical" approach and the Lyapunov function method. The effectiveness of the design method proposed for hybrid control systems is illustrated by a numerical example. This method can be applied to create hybrid control systems for different-purpose nonlinear plants.

Keywords: differentiable nonlinearity, quasilinear model, state controllability criterion, output controllability criterion, quasilinear discretization, hybrid system, stability, static error

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1. INTRODUCTION

Recently, significant attention has been paid to the development of methods for designing hybrid nonlinear control systems, which are characterized by continuous and discrete nonlinear dynamics, requiring the use of differential and difference equations [1–3]. In reality, such systems represent a combination of continuous (hardware) and digital (programmable) elements [4]. In known publications, a wide variety of systems are referred to as hybrid.

Hybrid systems of the first class include either one nonlinear plant operating in switched modes or several plants that must be switched on in a certain sequence. In this case, difference equations describe the digital part, which ensures the switching of the continuous part elements [1, 5–8]. To create hybrid nonlinear optimal control systems, the Hybrid Necessary Principle was used in [1]. This principle allows considering the constraints due to the switching strategy. In [5], the same goal was achieved by applying the Hamilton–Jacobi–Bellman equation and the spectral Galerkin method.

The hybrid nature of the control system considered in [6] is due to the switching of several continuous subsystems, while that of the one created in [7] is due to the switching of its operating modes. A quadcopter with in-flight switchable morphology was considered in [8]. Its hybrid control system includes a nonlinear PID controller and a discrete controller that stabilize the control system for all possible quadcopter configurations.

The second class of hybrid systems seems to be more defined [2, 3]. Here, a controlled continuous plant is equipped with a discrete (digital) controller. The main problem is ensuring the stable operation of the hybrid system with a relatively large period of the digital controller's operation.

This necessity arises when creating control systems for inertial plants, such as baking ovens, incubators, greenhouses, etc., as well as plants operating under harsh temperature conditions where controller cooling is difficult. Due to the large period, the conditions of the Kotelnikov theorem fail, violating the stability of the system, which holds in classical cases with a sufficiently small discretization period. Therefore, various methods are employed to create hybrid control systems of this class [2, 3, 9–12]. For example, predictive control based on quadratic and integer programming was used in [2]. The effectiveness of the approach was illustrated by an example of designing a hybrid control system for a plant with three spherical tanks.

A hybrid terminal control method with an identified model of the controlled plant was developed in [3]. A two-layer artificial neural network was proposed for identifying nonlinear plants. For nonlinear plants with delay and parametric uncertainty, hybrid control systems were designed in [9–11] using the hyperstability criterion, the L-dissipativity condition, and a filter-corrector. The discrete control was obtained by discretizing a continuous one. The problem of tracking a given trajectory by a quadcopter under uncertainty was considered in [12]. A hybrid autopilot implementing predictive and fuzzy control was applied in the tracking system.

This paper develops a new method for designing hybrid nonlinear control systems for single-input single-output (SISO) plants with differentiable nonlinearities and a measurable state vector. Quasilinear models of nonlinear plants and the quasilinear discretization method [13–17] are used. By assumption, the plant satisfies state and output controllability criteria and some additional conditions. Hybrid systems of the second class with an increased discretization period are designed, which significantly reduces the performance requirements for computing resources. The main limitation of the developed method is the differentiability of the plant's nonlinearities. If the state vector is unmeasurable, a state observer can be used.

2. PROBLEM STATEMENT

Consider control-affine nonlinear SISO plants described by the following equations in deviations:

$$\dot{x} = \varphi(x, u, f), \quad y = \xi(x, u), \tag{1}$$

where $x = [\underset{\sim}{x}_1, \ldots, \underset{\sim}{x}_n]^{\mathrm{T}} \in \mathrm{R}^n$ denotes the vector of state variables; $u, f, y \in \mathrm{R}$ are scalar control signal, disturbance, and controlled output, respectively; $\varphi(x, u, f)$ is a nonlinear n-dimensional vector function, and $\xi(x, u)$ is a scalar nonlinear function. These functions are bounded and differentiable in all arguments; moreover, $\varphi(\mathbf{0}, 0, 0) = \mathbf{0}$ and $\xi(\mathbf{0}, 0) = 0$. The state vector x and the output y or the deviation $\varepsilon = g - y$ are measured. Here, $\mathbf{0} \in \mathrm{R}^n$ stands for the zero vector, $g = g(t) \in \mathrm{R}$ and $f = f(t) \in \mathrm{R}$ are a reference signal and disturbance, representing arbitrary time-varying functions bounded by absolute value, and $f(t)$ is not measured.

Since the nonlinearities $\varphi(x, u, f)$ and $\xi(x, u)$ in (1) are differentiable, the method described in [15, 16] yields a quasilinear model (QLM) of the form

$$\dot{x} = A(x)x + b(x)u + h(x)f, \quad y = c^{\mathrm{T}}(x)x + d(x)u, \tag{2}$$

where $A(x) \in \mathrm{R}^{n\times n}$ and $b(x), h(x), c(x) \in \mathrm{R}^n$ are functional matrix and vectors, respectively, whose all elements, as well as $d(x) \in \mathrm{R}$, are known bounded differentiable nonlinear functions or numbers. Let us emphasize that QLMs describe the corresponding plants with differentiable nonlinearities with the same accuracy as equations (1). In other words, the properties of equations (2) fully match those of (1). Various methods for building QLMs have long been known. For example, the equation $\dot{x} = D(x)x$ was used by N.N. Krasovskii et al. to construct Lyapunov functions for nonlinear systems as early as the middle of the previous century [18].

By assumption, the QLM (2) satisfies the state controllability criterion

$$\det U_s(x) = \det[b(x) \;\; A(x)b(x) \;\; \dots \;\; A^{n-1}(x)b(x)] \neq 0, \;\; \forall\, x \in \Omega_{Us}, \tag{3}$$

as well as the output controllability criterion

$$\gamma_{\mathrm{pl}}(x) \neq 0, \;\; \forall\, x \in \Omega_{Uo}, \tag{4}$$

where $\gamma_{\mathrm{pl}}(x)$ is the output controllability index of the plant (1), defined by the expression

$$\gamma_{\mathrm{pl}}(x) = d(x) \det A(x) - c^{\mathrm{T}}(x) \mathrm{adj} A(x)\, b(x). \tag{5}$$

In (3)–(5), $\Omega_{Us} = \{x \in \mathrm{R}^n : \; \det U_s(x) \neq 0\}$; $\Omega_{Uo} = \{x \in \mathrm{R}^n : \; \gamma_{\mathrm{pl}}(x) \neq 0\}$; $\mathrm{adj}A(x)$ is the adjoint matrix for $A(x)$ [19]; $\Omega_{Uu} = \Omega_{Uo} \cap \Omega_{Us}$ is the set of vectors $x \in \mathrm{R}^n$ for which conditions (3) and (4) hold; moreover, both Ω_{Us} and Ω_{Uo} include the vector $x = \mathbf{0}$.

The objective of this work is to develop a method for designing second-class hybrid control systems for nonlinear plants of the form (1). The discretization periods of these systems must be significantly larger than those of discrete control systems created by conventional methods. To solve this problem, we apply a piecewise-constant control obtained by the quasilinear discretization method of nonlinear plants [17].

3. THE QUASILINEAR DISCRETIZATION METHOD

In this method, not the equations of nonlinear plants but their quasilinear models are discretized using the trapezoidal method. It is possible due to the boundedness of the right-hand sides of the QLM equations (2) for bounded x, u, g, and f.

Let T be a certain discretization period for the solutions $x = x(t) \in \Omega_{Uu}$ of the differential equation (2). With each time instant $t = kT$, $k = 0, 1, 2, \dots$, a discrete value $x_k = x(kT)$ of this solution is associated. The exact value $x_{k+1} = x(kT + T)$ is given by the expression

$$x_{k+1} = x_k + \int\limits_{kT}^{kT+T} F(t)dt, \tag{6}$$

where $F(t) = A(x)x + b(x)u + h(x)f(t\,)|_{x=x(t)}$ is the right-hand side of the first equation in (2). Assume that the input $u = u_k$ is a bounded and piecewise constant function. Nowadays, exact methods for computing the integrals (6) are unknown, so based on the modified trapezoidal method, the integrand in (6) is replaced by

$$\bar{F} = 0.5[A_k x_k + A_k x_{k+1} + 2b_k u_k + 2h_k f_k] + \Delta_k,$$

where $\Delta_k = 0.5[(A_{k+1} - A_k)x_{k+1} + b_{k+1}u_{k+1} - b_k u_k + h_{k+1}f_{k+1} - h_k f_k]$. (For brevity, $A_k = A(x_k)$, $b_k = b(x_k)$, and $h_k = h(x_k)$.) Replacing $F(t)$ in (6) with $\bar{F}$ for $\Delta_k = \mathbf{0}$ and integrating, we obtain the difference equation

$$[E - 0.5TA(x_k)]x_{k+1} = [E + 0.5TA(x_k)]x_k + Tb(x_k)u_k + Th(x_k)f_k, \quad x_k \in \Omega_{Uu}. \tag{7}$$

Note that the modification of the trapezoidal method consists in adding and subtracting the sum $A_k x_{k+1} + b_k u_k + h_k f_k$ when deriving the expression for $\bar{F}(kT)$ from $F(t)$.

Equation (7) can be solved for x_{k+1} if the matrix $[E - 0.5TA(x_k)]$ has an inverse, i.e., under the following condition imposed on the choice of the period T :

$$\det[E - 0.5TA(x)] \neq 0, \quad x \in \Omega_{Uu}. \tag{8}$$

To find T, we determine the roots η_i of the auxiliary equation $\det[E - 0.5\eta A(x)] = 0$. Let this equation for $x \in \Omega_{Uu}$ have $0 < m(x) \le n$ [20] positive real roots, of which $m_1(x)$ are independent of x and $m_2(x) = m(x) - m_1(x)$ depend on x. Then

$$0 < T < \min\{\eta_{\min,1},\ \eta_{\min,2}\}, \tag{9}$$

where $\eta_{\min,1} = \min\{\eta_i, i = \overline{1, m_1(x)}, x \in \Omega_{Uu}\}$, $\eta_{\min,2} = \inf\{\eta > 0 : \eta_i = \eta_i(x), i = \overline{1, m_2(x)}, x \in \Omega_{Uu}\}$.

If $m(x) \equiv 0$ (i.e., the equation $\det[E - 0.5\eta A(x)] = 0$ has no positive real roots η_i), the condition on the matrix $[E - 0.5TA(x_k)]$ will not imply any constraints on T.

In this case, the value of T in (8) is taken arbitrarily, based on constructive constraints; and the value of T can be refined later.

If the period T is chosen according to (8), then from (7) and the second equation in (2) it follows that

$$\begin{aligned} x_{k+1} &= A_d(x_k)x_k + b_d(x_k)u_k + h_d(x_k)f_k, \\ y_k &= c^{\mathrm{T}}(x_k)x_k + d(x_k)u_k, \quad x_k \in \Omega_{Uu}, \end{aligned} \tag{10}$$

where

$$A_d(x_k) = [E - 0.5TA(x_k)]^{-1}[E + 0.5TA(x_k)], \tag{11}$$

$$\begin{aligned} b_d(x_k) &= [E - 0.5TA(x_k)]^{-1}Tb(x_k), \\ h_d(x_k) &= [E - 0.5TA(x_k)]^{-1}Th(x_k). \end{aligned} \tag{12}$$

The relations (6)–(12) represent the quasilinear discretization method, and the expressions (10)–(12) are the discrete quasilinear model (DQLM) of the plant (1) [17]. In contrast to the exact QLM (2), this model is approximate. However, as shown below under certain conditions, some control signal u_k stabilizing the equilibrium of the DQLM (10) also ensures the stability of the equilibrium of the control system for the plant (1). In this sense, quasilinear discretization is analogous to classical linearization in the continuous case, where the control law based on first-approximation equations stabilizes the equilibrium of the nonlinear system in the small.

The application of the DQLM (10)–(12) allows hybrid systems to have a significantly larger discretization period compared to conventional approaches, thereby substantially reducing the performance requirements for system controllers.

4. STABILIZING CONTROL

This control is constructed by the algebraic polynomial-matrix (APM) method [21, 22]. Let the period T be chosen so that condition (8) holds, and let the corresponding DQLM (10)–(12) satisfy the state controllability criterion[1] for nonlinear discrete plants:

$$\det U_d(x_k) = \det[b_d(x_k)\ \ A_d(x_k)b_d(x_k)\ \ \ldots\ \ A_d^{n-1}(x_k)b(x_k)] \neq 0, \quad x_k \in \Omega_{Ud}, \tag{13}$$

where $\Omega_{Ud} = \{x_k \in \Omega_{Uu} : \det U_d(x_k) \neq 0\}$. In other words, $\Omega_{Ud} \subset \Omega_{Uu}$ is the domain where conditions (3), (4), (8), and (13) hold, and it contains the point $x = \mathbf{0}$.

The discrete control law stabilizing system (10)–(12) has the form

$$u_k(x_k) = -l^{\mathrm{T}}(x_k)x_k = -[l_1(x_k)\ l_2(x_k)\ \ldots\ l_n(x_k)]x_k. \tag{14}$$

The gains $l_i(x_k)$ are determined by the algorithm with the following steps [21].

[1] The question of whether condition (13) holds under conditions (3) and (8) is open.

1) Using (11) and (12), it is necessary to find the functional polynomials

$$A_d(z, x_k) = \det[zE - A_d(x_k)] = z^n + \alpha_{n-1}(x_k)z^{n-1} + \ldots + \alpha_1(x_k)z + \alpha_0(x_k), \quad (15)$$

$$V_{d,i}(z, x_k) = e_i^{\mathrm{T}} \operatorname{adj}[zE - A_d(x_k)]b_d(x_k) = v_{i,n-1}(x_k)z^{n-1} + v_{i,n-2}(x_k)z^{n-2} + \ldots + v_{i,0}(x_k), \quad (16)$$

where e_i is the ith column of the identity matrix E of dimensions $n \times n$; $\alpha_j(x_k)$ and $v_{i,j}(x_k)$ are functional or numerical coefficients, $i = 1, \ldots, n$, and $j = 0, 1, \ldots, n-1$.

2) This step is to form the polynomial

$$D^*(z) = \prod_{i=1}^{n}(z - \sigma_i^*) = z^n + \delta_{n-1}^* z^{n-1} + \ldots + \delta_1^* z + \delta_0^*, \quad (17)$$

where σ_i^* are real numbers for which there exist $0 < \varsigma_1 < 1$ and $0 < \varsigma_2$, independent of i and κ, such that

$$|\sigma_i^*| \le 1 - \varsigma_1, \quad \varsigma_2 < |\sigma_i^* - \sigma_\kappa^*|, \quad i \ne \kappa, \quad i, \kappa = 1, \ldots, n. \quad (18)$$

3) One determines the coefficients of the difference

$$D^*(z) - A_d(z, x_k) = \rho_{n-1}(x_k)z^{n-1} + \ldots + \rho_1(x_k)z + \rho_0(x_k), \quad (19)$$

where $\rho_j(x_k) = \delta_j^* - \alpha_j(x_k)$, $j = 0, 1, \ldots, n-1$. Next, it is necessary to equate the coefficients of the sum of the products of the polynomials $V_{d,i}(z, x_k)$ (16) by the coefficients $l_i(x_k)$ (14), $i = 1, \ldots, n$, to those of the polynomial (19) at the same powers of z. The resulting equations, written in the vector-matrix form, constitute the system of linear algebraic equations (SLAE)

$$\begin{bmatrix} v_{10}(x_k) & v_{20}(x_k) & \cdots & v_{n0}(x_k) \\ v_{11}(x_k) & v_{21}(x_k) & \cdots & v_{n1}(x_k) \\ \vdots & \vdots & \ddots & \vdots \\ v_{1,n-1}(x_k) & v_{2,n-1}(x_k) & \cdots & v_{n,n-1}(x_k) \end{bmatrix} \begin{bmatrix} l_1(x_k) \\ l_2(x_k) \\ \vdots \\ l_n(x_k) \end{bmatrix} = \begin{bmatrix} \rho_0(x_k) \\ \rho_1(x_k) \\ \vdots \\ \rho_{n-1}(x_k) \end{bmatrix}. \quad (20)$$

The SLAE (20) has a unique solution due to condition (13). Its solution—the vector $l(x_k)$—is substituted into (14), and the resulting control law u_k is then substituted into the DQLM equation (10). Thus, one arrives at the following equation of the virtual discrete system:

$$x_{k+1} = D_d(x_k)x_k + h_d(x_k)f_k, \quad x_k \in \Omega_{Ud}, \quad = 0, 1, 2, \ldots, \quad (21)$$

where

$$D_d(x_k) = A_d(x_k) - b_d(x_k)l^{\mathrm{T}}(x_k). \quad (22)$$

The lemma below establishes the effectiveness of the APM method.

Lemma 1. *Under condition* (13)*, the SLAE* (20) *has a unique solution* $l(x_k)$*. Moreover, for any* $x_k \in \Omega_{Ud}$*, the eigenvalues of the matrix* $D_d(x_k)$ (22) *coincide with the roots of the polynomial* $D^*(z)$ (17)*, i.e., they do not depend on* x_k*, are real, distinct, and less than one by absolute value.*

The proof of Lemma 1 is postponed to the Appendix. We emphasize that the relations (8)–(22) can be used for designing discrete nonlinear control systems [17, 22].

5. HYBRID SYSTEM DESIGN

Proceeding to the solution of this problem, we introduce the matrix

$$\tilde{D}(x) = [E - 0.5Tb(x)l^{\mathrm{T}}(x)]H_g(x), \tag{23}$$

with $H_g(x) = [D_d(x) - E][D_d(x) + E]^{-1}$, the matrix $D_d(x)$ (22), and the vector $l(x) = l(x_k)$ for $x_k = x$.

Let $\lambda_i^{\tilde{D}(x)}$ be the eigenvalues of the matrix $\tilde{D}(x) \in \mathrm{R}^{n\times n}$. Assume that the period T in (10)–(20) and (23) satisfies conditions (8) and (13) and the inequalities

$$\mathrm{Re}\lambda_i^{\tilde{D}(x)} < 0, \quad i = 1, \ldots, n, \quad x \in \Omega_{sys}, \tag{24}$$

where $\Omega_{sys} = \{x \in \Omega_{Ud} : \mathrm{Re}\lambda_i^{\tilde{D}(x)} < 0,\ i = 1, \ldots, n\}$. In other words, the eigenvalues of the matrix $\tilde{D}(x)$ can be either real or complex conjugate but with negative real parts (i.e., the matrix $\tilde{D}(x)$ is Hurwitz in the domain Ω_{sys}). Note that the choice of T can be iterative: if condition (24) fails for some value of the period T, then this value in (8)–(20) and (23) is decreased.

Under conditions (3), (4), (8), (13), and (24), the control law of the hybrid system is a discrete, piecewise-constant function of the form

$$u = u_{\mathrm{hyb}}(x_k, g_k) = l_g(x_k)g_k - l^{\mathrm{T}}(x_k)x_k, \quad k = 0, 1, 2, \ldots, \tag{25}$$

where $x_k \in \Omega_{sys}$; $g_k = g(t\ )|_{t=kT}$ are the values of the reference signal $g(t)$; the gain $l_g(x)$ and the matrix $D_{\mathrm{hyb}}(x) \in \mathrm{R}^{n\times n}$ are given by

$$l_g(x) = \det D_{\mathrm{hyb}}(x)/\gamma_{\mathrm{pl}}(x) \tag{26}$$

and

$$D_{\mathrm{hyb}}(x) = A(x) - b(x)l^{\mathrm{T}}(x), \tag{27}$$

respectively. From now on, the vector $x = x(t) \in \mathrm{R}^n$ is the solution of system (1), (25) or (2), (25).

From the expressions (2) and (25) we derive the following QLM equations of the hybrid system:

$$\dot{x} = A(x)x - b(x)l^{\mathrm{T}}(x_k)x_k + b(x)l_g(x_k)g_k + h(x)f, \quad kT \le t < (k+1)T, \tag{28}$$

$$y = c^{\mathrm{T}}(x)x - d(x)l^{\mathrm{T}}(x_k)x_k + d(x)l_g(x_k)g_k, \quad kT \le t < (k+1)T, \quad k = 0, 1, 2, \ldots. \tag{29}$$

According to the definition (25), on the surfaces $\varphi(t, x) = t - kT = 0$, $k = 1, 2, 3, \ldots$, the control signal $u = u(t)$ undergoes discontinuities of the first kind, i.e., instantaneously changes its value [23]. Such instantaneously changing controls were used as admissible in [24–26]. In the case under consideration, for each $k \ge 1$, the above discontinuity surfaces form two continuity sectors for both the control signal u (25) and the right-hand side of equation (28) [23]. By formula (25), in the left continuity sectors $kT - T < t < kT$, the control law is given by the expression $u^-(t) = l_g(x_{k-1})g_{k-1} - l^T(x_{k-1})x_{k-1}$; in the right continuity sectors $kT < t < kT + T$, by the expression $u^+(t) = l_g(x_k)g_k - l^T(x_k)x_k$. Here, x_k are the values of the solution of the differential system (28) on the discontinuity surfaces, i.e., at $t = kT$, $k = 1, 2, 3, \ldots$. Following [24] or [25], we assume the existence of finite right and left limits in each continuity sector.

However, the values x_k are not determined by the differential system (28) in the classical sense due to the discontinuities of its right-hand side. There are several approaches to overcome this problem [26]. The so-called “technical” [23] (or “physical” [26]) one was proposed by M.A. Aizerman and E.S. Pyatnitsky: the idea is to consider the physical meaning of the problem, using "additional information about the 'original system' to narrow the domain of possible solutions" on the discontinuity surfaces [23, p. 39]. Recall that x_0 is given, and the subsequent values of x_k, $k = 1, 2, 3, \dots$, are measured in the hybrid system of the above type. Having this in mind, we assume that the solution of equation (28) in the left continuity sectors is given by

$$x^-(t) = x_{k-1} + \int\limits_{kT-T}^{t} [A(x(\tau))x(\tau) - b(x(\tau))l^T(x_{k-1})x_{k-1} + \upsilon^-(\tau)]d\tau, \qquad kT - T \le t < kT, \quad k = 1, 2, 3, \dots, \tag{30}$$

where $x_0 = x(0)$, $\upsilon^-(\tau) = b(x(\tau))l_g(x_{k-1})g_{k-1} + h(x(\tau))f(\tau)$, and the integral is a Lebesgue integral [23]. Following [24], it seems convenient to define the measured values as $x_k = \lim_{t\to kT} x^-(t)$. Replacing the subscript k with $k+1$ in equality (30), we derive an explicit expression for $x(t)$ in the right continuity sectors:

$$x^+(t) = x_k + \int\limits_{kT}^{t} [A(x(\tau))x(\tau) - b(x(\tau))l^T(x_k)x_k + \upsilon^+(\tau)]d\tau, \qquad kT \le t < kT + T, \quad k = 1, 2, 3, \dots, \tag{31}$$

where $\upsilon^+(\tau) = b(x(\tau))l_g(x_k)g_k + h(x(\tau))f(\tau)$.

Due to the assumed existence of right and left limits in each continuity sector, both formulas (30) and (31) yield the same value: $x_k = \lim_{t\to kT} x^-(t) = \lim_{t\to kT} x^+(t)$. Thus, the Aizerman–Pyatnitsky approach allows obtaining the values of the continuous solution $x(t)$ of the differential system (28) for all t under the piecewise-constant control law (25) by utilizing the additional information about the properties of the hybrid system.

Let us formulate a theorem on the properties of system (1), (25).

Theorem. *Assume that conditions* (3), (4), (8), (13), (18), *and* (24) *hold, and the vector* $l(x_k)$ *in* (25) *is given by the solution of the SLAE* (20). *Then for* $g(t) = f(t) \equiv 0$ *and all* $t \ge 0$, *there exists a set of solutions* $x(t, x_0)$ *of equation* (28) *such that*

$$\lim_{t\to\infty} x(t, x_0) = \mathbf{0}, \quad x \in \Omega_{sys}. \tag{32}$$

If the gain $l_g(x_k)$ *in* (25) *is given by* (26), *then the static error of system* (1), (25) *with respect to the reference signal* $g(t)$ *is zero:*

$$\lim_{t\to\infty} \varepsilon_g(t) = \lim_{t\to\infty} [g(t) - y_g(t)] = 0 \tag{33}$$

for $g(t) = g_0 1(t)$ *and* $f(t) \equiv 0$.

Here, $1(t)$ indicates the unit step function (the Heaviside function); $y_g(t)$ is the response of system (28), (29), i.e., (1), (25) with $f(t) \equiv 0$, to the reference signal $g(t) = g_0 1(t)$ for some $x_0 = x(0)$ and g_0 such that $x(t, x_0, g_0) \in \Omega_{sys}$, $t \ge 0$.

The proof of this theorem is provided in the Appendix. The following lemma establishes that the controllability of the plant implies the "controllability" of the closed-loop system, i.e., the possibility of ensuring the necessary change in the system output by an appropriate reference signal.

Lemma 2. *If the matrices* $U_s(x)$ *and* $D_{\text{hyb}}(x)$ *are given by* (3) *and* (27), *then*

$$\det Q_{\text{hyb}}(x) = \det[b(x) \;\; D_{\text{hyb}}(x)b(x) \;\; \dots \;\; D_{\text{hyb}}^{n-1}(x)b(x)] = \det U_s(x), \quad x \in \Omega_{sys}. \tag{34}$$

The proof of Lemma 2 can be found in the Appendix. The relations (2), (5), (9)–(20), (22), (23), and (25)–(29) constitute the mathematical foundation of the proposed method for designing hybrid nonlinear control systems; inequalities (3), (4), (8), (13), (18) and (24) express the solvability conditions of the design problem by this method. The effectiveness of the developed method is illustrated by a numerical example below.

6. A NUMERICAL EXAMPLE

It is required to design a hybrid pitch control system (HPCS) for an autonomous underwater vehicle (AUV). Pitch control is carried out using bow and stern tanks of variable volume [27] and is described by the system of equations

$$\ddot{\psi} = \alpha_1 U_\psi \cos\psi - \alpha_2 U_a \sin\psi - \beta\left|\dot{\psi}\right|\dot{\psi}, \quad \dot{U}_\psi = -k_v U_\psi + k_u u, \quad y = \psi, \tag{35}$$

with the following notation: ψ and $\dot{\psi}$ are the pitch angle and its rate of change, respectively; U_ψ stands for the difference in the volumes of the bow and stern tanks; U_a is the AUV displacement; α_1 and α_2 mean hydrodynamic coefficients; β is the pitch change resistance coefficient; k_v and k_u are parameters of the device changing U_ψ; u is the control signal of this device; y is the controlled output of the HPCS; finally, ψ, $\dot{\psi}$, and U_ψ are measured variables. The HPCS must have zero static error in pitch and transients of a duration not exceeding 5 s under zero initial conditions and the desired pitch $g(t) = \psi^*(t) = -0.5236 \times 1(t)$ rad.

Solution. Setting $\utilde{x}_1 = \psi$, $\utilde{x}_2 = \dot{\psi}$ and $\utilde{x}_3 = U_\psi$, we write equations (35) in the Cauchy form:

$$\begin{aligned} &\dot{\utilde{x}}_1 = \utilde{x}_2, \\ &\dot{\utilde{x}}_2 = \alpha_1 \utilde{x}_3 \cos\utilde{x}_1 - \alpha_2 U_a \sin\utilde{x}_1 - \beta\left|\utilde{x}_2\right|\utilde{x}_2, \\ &\dot{\utilde{x}}_3 = -k_v \utilde{x}_3 + k_u u, y = \utilde{x}_1. \end{aligned} \tag{36}$$

Since $\sin\utilde{x}_1 = \omega(\utilde{x}_1)\utilde{x}_1$, where $\omega(\utilde{x}_1) = (\sin\utilde{x}_1)/\utilde{x}_1$ is the QLM of the function $\sin\utilde{x}_1$ [16], the QLM of the nonlinear system of equations (36) has the form (2) with

$$\begin{aligned} A(x) &= \begin{bmatrix} 0 & 1 & 0 \\ -a_{21}(x) & -a_{22}(x) & a_{23}(x) \\ 0 & 0 & -a_{33}(x) \end{bmatrix}, \\ b(x) &= \begin{bmatrix} 0 \\ 0 \\ k_u \end{bmatrix}, \quad c(x) = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \quad h(x) = \mathbf{0}, \quad d(x) = 0. \end{aligned} \tag{37}$$

Here, $x = [\utilde{x}_1 \quad \utilde{x}_2 \quad \utilde{x}_3]^{\mathrm{T}}$, $a_{21}(x) = \alpha_2 U_a \omega(\utilde{x}_1)$, $a_{22}(x) = \beta\left|\utilde{x}_2\right|$, $a_{23}(x) = \alpha_1 \cos\utilde{x}_1$, and $a_{33}(x) = k_v$.

Consider the solution of the HPCS design problem for $g(t) = \psi^*(t)$ and the following model values of the coefficients in (37): $a_{21}(x) = 7.044\omega(\utilde{x}_1)$, $a_{22}(x) = 1.192\left|\utilde{x}_2\right|$, $a_{23}(x) = 6.48\cos\utilde{x}_1$, $a_{33}(x) = 1.326$, and $k_v = k_u = 0.12$. In this case, in view of (5) and (37), conditions (3) and (4) become $\det U_s(x) = 0.0933(\cos\utilde{x}_1)^2 \neq 0$ and $\gamma_{\mathrm{pl}}(x) = 0.7776\cos\utilde{x}_1 \neq 0$. In other words, the domain Ω_{Uu} is given by $|\utilde{x}_1| < \pi/2$, $|\utilde{x}_2| \leq \utilde{x}_{2,\max}$, and $|\utilde{x}_3| \leq \utilde{x}_{3,\max}$, where $\utilde{x}_{2,\max}$ and $\utilde{x}_{3,\max}$ are some design bounds. Let $\utilde{x}_{2,\max} = 3.5$ rad/s and $L(\eta, x) = [E - 0.5\eta A(x)]$; then, taking (37) into account, we arrive at the equation

$$\det L(\eta, x) = (1 + 0.663\eta)\left[1 + 0.596\left|\utilde{x}_2\right|\eta + 3.522\omega(\utilde{x}_1)\eta^2\right] = 0. \tag{38}$$

Equation (38) has no positive real roots η_i in the domain Ω_{Uu}. That is, condition (8) on the matrix $[E - 0.5TA(x_k)]$ does not yield any constraints on the period T. Therefore, we take $T_1 = 0.6$ s and

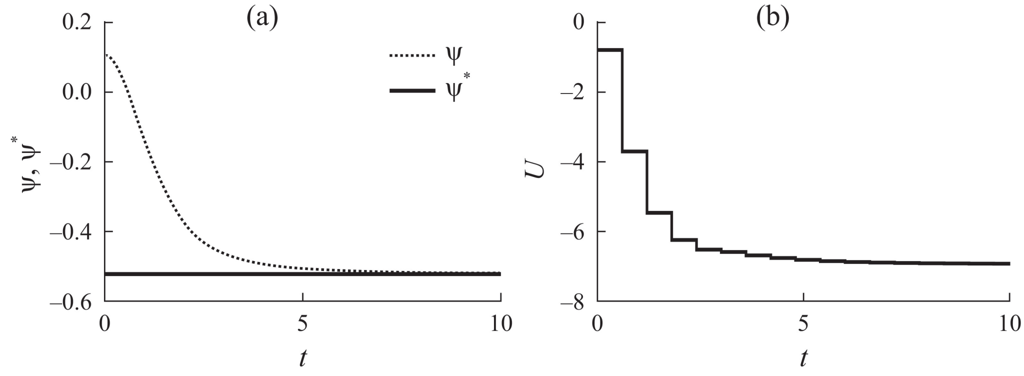

Fig. 1. The plots of the variables for $T = 0.6$ s: (a) pitch angles and (b) control input.

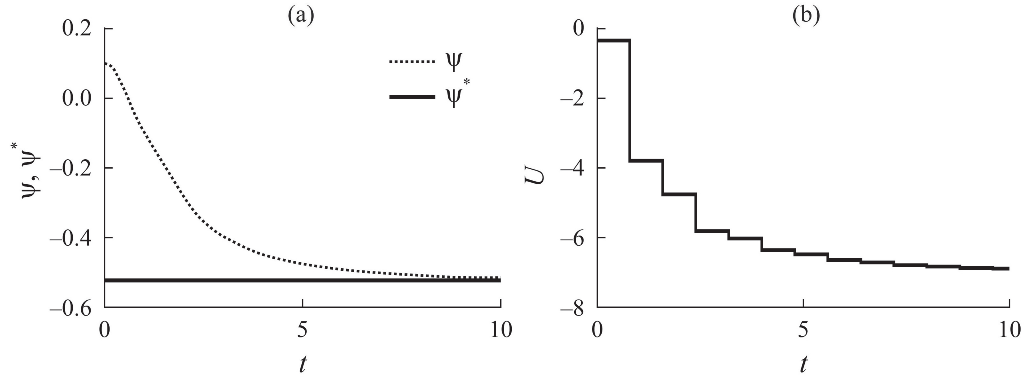

Fig. 2. The plots of the variables for $T = 0.8$ s: (a) pitch angles and (b) control input.

$T_2 = 0.8$ s based on constructive constraints. Let us determine the controller. The matrix $A_d(x_k)$ and the vector $b_d(x_k)$ are found from (11), (12), and (37); the fulfillment of condition (13) in the domain $\Omega_{Ud} = \Omega_{Uu}$ is verified numerically; the polynomials $A_d(z, x_k)$ and $V_{d,i}(z, x_k)$, $i = 1, 2, 3$, are constructed by (15) and (16).

The polynomial $D^*(z)$, found from (17) and (18), allows calculating the coefficients $\rho_j(x_k)$, $j = 0, 1, 2$, from (19) and compiling the SLAE (20). Its solution determines the three-dimensional vector $l(x_k)$. Next, the matrix $D_d(x_k)$ is obtained from (22) and the matrix $\tilde{D}(x)$ from (23). It is verified numerically that condition (24) holds both for $T = 0.6$ s and for $T = 0.8$ s in the domain $\Omega_{sys} = \Omega_{Uu}$. Finally, $D_{\rm hyb}(x)$, $l_g(x)$, and $u_{\rm hyb}(x_k, g_k)$ are determined by formulas (27), (26), and (25), respectively.

We emphasize that during the operation of the HPCS, almost all computations (first, the matrix $A(x)$ and the vector $b(x)$ (37), then $A_d(x_k)$ and $b_d(x_k)$, and, finally, the control input $u_{\rm hyb}(x_k, g_k)$) are performed by a digital control device for all $k = 0, 1, 2, \ldots$ with period T. (The only exception is the formation of the polynomial $D^*(z)$.) This is due to the nonlinear nature of the plant (35).

Analysis of the designed HPCS. For this purpose, we used MATLAB to compute the values of the discrete control signal $u_{\rm hyb}(x_k, g_k)$ for each $t = kT$ (see the description above) and integrate (via the ode45 function) the system of equations (36) with $u = u_{\rm hyb}(x_k, g_k)$ and the initial conditions $x_{0,k} = x(kT)$ and $\psi^*(t) = \psi_0 1(t)$, $k = 0, 1, 2, \ldots$, on each time interval $kT \le t < (k+1)T$. Figures 1 and 2 show the transients of the designed HPCS for $D^*(z) = z^3 - 0.8z^2 + 0.2032z - 0.01613$, $x_{0,0} = [0.1 \;\; 0.01 \;\; 0]^{\rm T}$, and $\psi_0 = -0.5236$ rad.

Clearly, the variables of the controlled plant are continuous functions, although the control signal changes with a significant period, which is characteristic of hybrid systems. The transients are similar under other conditions as well. A small increase in the discretization period just slightly extends the transients.

Table contains the eigenvalues $\lambda_i(x)$, $i = 1, 2, 3$, of the matrix $D_{\mathrm{hyb}}(x_k)$ of the HPCS for two values of the period T and several values of k.

Table

	$T = 0.6$		$T = 0.8$	
k	λ_1	$\lambda_{2,3}$	λ_1	$\lambda_{2,3}$
0	-1.3463	$-0.7002 \pm 1.5983i$	-1.3250	$-0.2280 \pm 1.3446i$
1	-1.3484	$-0.8130 \pm 1.6057i$	-1.3249	$-0.3112 \pm 1.3637i$
5	-1.3468	$-0.7261 \pm 1.5810i$	-1.3249	$-0.2556 \pm 1.3322i$
10	-1.3466	$-0.7138 \pm 1.5753i$	-1.3249	$-0.2487 \pm 1.3249i$

According to this table, the eigenvalues of the matrix $D_{\mathrm{hyb}}(x)$ have negative real parts, and increasing the period T reduces these parts by absolute value; for a large T, system stability is lost. Note that the real parts of the eigenvalues of the above matrix change insignificantly during the transient process of the hybrid system.

7. CONCLUSIONS

This paper has proposed a method for designing hybrid nonlinear control systems for continuous plants with differentiable nonlinearities and a measurable state vector. The problem has been solved using continuous and discrete quasilinear models, the algebraic polynomial-matrix method for designing nonlinear systems, and the Aizerman–Pyatnitsky solution approach to differential equations with a discontinuous right-hand side. The method proposed is applicable if the continuous and discrete quasilinear models of the nonlinear plant satisfy state and output controllability criteria and some additional conditions. The effectiveness of this method has been illustrated by a numerical example of designing a hybrid nonlinear pitch control system for an autonomous underwater vehicle. The method can be used to create hybrid control systems for nonlinear plants of industrial, social, and special purpose using moderate-performance computing means.

FUNDING

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APPENDIX

Proof of Lemma 1. Due to the expressions (6.3) and (6.55) [28, p. 145 and p. 169], the controllability conditions for the systems in the continuous and discrete cases coincide in form. Therefore, we use a theorem on the properties of controllable continuous systems to prove Lemma 1, formulated for the discrete system. In view of this remark, Theorems 1.1 and 1.2 [29, pp. 29, 31, 32] lead to the following assertion: under inequality (13), there is a unique gain vector $l^{\mathrm{T}}(x_k)$ for the control law $u_k = -l^{\mathrm{T}}(x_k)x_k$ (14) under which the eigenvalues of the matrix of the closed-loop discrete system (21) have a specified location on the complex plane z.

The matrix of the indicated closed-loop discrete system (21) is $D_d(x_k) = A_d(x_k) - b_d(x_k)l^{\mathrm{T}}(x_k)$ (22) with the characteristic polynomial

$$D_d(z, x_k) = \det[zE - A_d(x_k) + b_d(x_k)l^{\mathrm{T}}(x_k)], \quad x_k \in \Omega_{Ud}. \tag{A.1}$$

Thus, for a given polynomial $D_d(z, x_k) = D^*(z)$, the expression (A.1) is an equation with respect to the vector $l^{\mathrm{T}}(x_k)$; according to Theorems 1.1 and 1.2 from [29], this equation has a unique solution under condition (13). By applying to (A.1) equality (П.25) from [30, p. 233] for $\mu = 1$, we obtain

$$D_d(z, x_k) = \det[zE - A_d(x_k)] + l^{\mathrm{T}}(x_k)\mathrm{adj}\,[zE - A_d(x_k)]\,b_d(x_k).$$

(In particular, (П.25) is immediate from formulas (I) and (II) [19].) Hence, considering the notation (15), (16) and the vector $l^{\mathrm{T}}(x_k) = [l_1(x_k)\ l_2(x_k)\ \ldots\ l_n(x_k)]$, following [14, 15], we derive an equivalent representation of the same polynomial (A.1):

$$D_d(z, x_k) = A_d(z, x_k) + \sum_{i=1}^{n} l_i(x_k)V_{d,i}(z, x_k). \tag{A.2}$$

Moreover, by construction, system (20) is equivalent to the polynomial equation

$$\sum_{i=1}^{n} l_i(x_k)V_{d,i}(z, x_k) = \rho_{n-1}(x_k)z^{n-1} + \ldots + \rho_1(x_k)z + \rho_0(x_k).$$

In view of (19), it can be written as

$$\sum_{i=1}^{n} l_i(x_k)V_{d,i}(z, x_k) = D^*(z) - A_d(z, x_k), \quad x_k \in \Omega_{Ud}. \tag{A.3}$$

Based on (17), the roots of the polynomial $D^*(z)$ are the numbers σ_i^*, i.e., $D^*(\sigma_i^*) = 0$. Then according to (A.3), $A_d(\sigma_i^*, x_k) + \sum_{i=1}^{n} l_i(x_k)V_{d,i}(\sigma_i^*, x_k) = 0$. By (A.2), it follows that $D_d(\sigma_i^*, x_k) = 0$, $i = 1, 2, \ldots, n$, and the proof of Lemma 1 is complete.

Proof of Lemma 2. For this purpose, let us utilize a well-known property of determinants: if one column of the determinant's matrix, multiplied by a *number*, is added or subtracted from another column, the determinant value will not change [31, p. 143]. For brevity, we omit the arguments of the matrices $A(x)$, $D_{\mathrm{hyb}}(x)$, and $Q_{\mathrm{hyb}}(x)$ and vectors $b(x)$ and $l(x)$ from (3), (27), (34) and emphasize that $D_{\mathrm{hyb}} = A - bl^{\mathrm{T}}$ and $\det Q_{\mathrm{hyb}} = \det[b\ \ D_{\mathrm{hyb}}b\ \ \ldots\ \ D_{\mathrm{hyb}}^{n-1}b]$ for all n. Lemma 2 will be proved by induction. First, we show its validity for $n = 1$ and $n = 2$.

For $n = 1$, we have $A = a_1$, $b = b_1$, $l = l_1$, and $\det U_s = b_1$ by (3). Here, $D_{\mathrm{hyb}} = a_1 - b_1 l_1$, and $\det Q_{\mathrm{hyb}} = b_1$; obviously, $\det Q_{\mathrm{hyb}} = \det U_s$. Let $n = 2$; in this case, by (3), $\det U_s = \det[b\ \ Ab]$, and by (34), $\det Q_{\mathrm{hyb}} = \det[b\ D_{\mathrm{hyb}}b] = \det[b\ \ Ab - \breve{\beta}b]$, where $\breve{\beta} = l^{\mathrm{T}}b$ is a scalar number, since for each particular value of x the vectors $l(x)$ and $b(x)$ are numerical. Hence, due to the above property of determinants, $\det Q_{\mathrm{hyb}} = \det\begin{bmatrix} b & Ab \end{bmatrix}$, i.e., $\det Q_{\mathrm{hyb}} = \det U_s$. Thus, Lemma 2 is valid for $n = 1$ and $n = 2$.

Now, under the inductive hypotheses $\det U_s = \det[b\ \ Ab\ \ldots\ A^{\mu-1}b]$ and $\det Q_{\mathrm{hyb}} = \det[b\ \ D_{\mathrm{hyb}}b\ \ \ldots\ \ D_{\mathrm{hyb}}^{\mu-1}b] = \det U_s$ (Lemma 2 for $n = \mu$), the validity of this lemma has to be shown for $n = \mu + 1$. To this end, we expand the left-hand side of the expression (34) with $n = \mu + 1$ as follows:

$$\det Q_{\mathrm{hyb}} = \det[b\ \ D_{\mathrm{hyb}}b\ \ D_{\mathrm{hyb}}^2 b\ \ \ldots\ \ D_{\mathrm{hyb}}^{\mu-3}b\ \ D_{\mathrm{hyb}}^{\mu-2}b\ \ D_{\mathrm{hyb}}^{\mu-1}b\ \ D_{\mathrm{hyb}}^{\mu}b]. \tag{A.4}$$

Further, we transform the columns of the matrix of the determinant (A.4) step by step, starting from $D^{\mu}_{\rm hyb}b$, considering the above property of determinants and $D_{\rm hyb} = A - bl^{\rm T}$.

Step 1.1. $D^{\mu}_{\rm hyb}b = D^{\mu-1}_{\rm hyb}(A - bl^{\rm T})b = D^{\mu-1}_{\rm hyb}Ab + \beta_0 D^{\mu-1}_{\rm hyb}b \succ D^{\mu-1}_{\rm hyb}Ab$ since $\beta_0 = -l^{\rm T}b$ is a scalar number and the column $\beta_0 D^{\mu-1}_{\rm hyb}b$ equals the column $D^{\mu-1}_{\rm hyb}b$ of the matrix of the determinant (A.4) multiplied by β_0. From this point onwards, $\succ$ is the correspondence sign, indicating that the value of the determinant (A.4) will not change when replacing the column $D^{\mu}_{\rm hyb}b$ in (A.4) with the column $D^{\mu-1}_{\rm hyb}Ab$.

Step 1.2. $D^{\mu}_{\rm hyb}b \succ D^{\mu-1}_{\rm hyb}Ab = D^{\mu-2}_{\rm hyb}(A - bl^{\rm T})Ab = D^{\mu-2}_{\rm hyb}A^2b + \beta_1 D^{\mu-2}_{\rm hyb}b \succ D^{\mu-2}_{\rm hyb}A^2b$ due to the above property of determinants since $\beta_1 = -l^{\rm T}Ab$ is a scalar number and the column $\beta_1 D^{\mu-2}_{\rm hyb}b$ equals to the column $D^{\mu-2}_{\rm hyb}b$ of the matrix of the determinant (A.4) multiplied by β_1. (In other words, the column $\beta_1 D^{\mu-2}_{\rm hyb}b$ is proportional to the column $D^{\mu-2}_{\rm hyb}b$.)

Step 1.3. $D^{\mu}_{\rm hyb}b \succ D^{\mu-2}_{\rm hyb}A^2b = D^{\mu-3}_{\rm hyb}(A - bl^{\rm T})A^2b = D^{\mu-3}_{\rm hyb}A^3b + \beta_2 D^{\mu-3}_{\rm hyb}b \succ D^{\mu-3}_{\rm hyb}A^3b$ due to the above property of determinants since $\beta_2 = -l^{\rm T}A^2b$ is a scalar number and the column $\beta_2 D^{\mu-3}_{\rm hyb}b$ is proportional to the column $D^{\mu-3}_{\rm hyb}b$ of the matrix of the determinant (A.4). Continuing this process at Step 1.μ, we arrive at $D^{\mu}_{\rm hyb}b \succ D^{\mu-\mu}_{\rm hyb}A^{\mu}b = A^{\mu}b$.

Let us proceed to transforming the column $D^{\mu-1}_{\rm hyb}b$ of the matrix of the determinant (A.4).

Step 2.1. $D^{\mu-1}_{\rm hyb}b = D^{\mu-2}_{\rm hyb}(A - bl^{\rm T})b = D^{\mu-2}_{\rm hyb}Ab + \beta_0 D^{\mu-2}_{\rm hyb}b \succ D^{\mu-2}_{\rm hyb}Ab$.

Step 2.2. $D^{\mu-1}_{\rm hyb}b \succ D^{\mu-2}_{\rm hyb}Ab = D^{\mu-3}_{\rm hyb}(A - bl^{\rm T})Ab = D^{\mu-3}_{\rm hyb}A^2b + \beta_1 D^{\mu-3}_{\rm hyb}b \succ D^{\mu-3}_{\rm hyb}A^2b$.

Continuing the transformation, at Step 2.$(\mu - 1)$, we obtain $D^{\mu-1}_{\rm hyb}b \succ A^{\mu-1}b$. Obviously, applying this transformation to each column $D^{j}_{\rm hyb}b$ of the matrix of the determinant (A.4) yields the column A^jb, $j = 1, \ldots, \mu$. Based on the above property of determinants, this transformation does not change the value of (A.4); therefore, $\det Q_{\rm hyb} = \det U_s$ for $n = \mu + 1$ as well.

So, Lemma 2 is valid for $n = 1, 2$, and its validity for $n = \mu$ implies the same for $n = \mu + 1$. By induction, Lemma 2 is valid for any positive integer n, and the proof is complete.

Proof of Theorem. As shown above, the continuous solution of equation (28) is defined for all $t \geq 0$ and $x \in \Omega_{sys}$. Moreover, its right-hand side depends on the time t, in addition to the vector $x(t)$, which is reflected in the additional expressions: $kT \leq t < kT + T$ and $k = 0, 1, 2, \ldots$. To make the dependence on t more explicit and eliminate k, we replace $x_k = x(kT)$ with $x(T\lfloor t/T \rfloor)$, where $\lfloor t/T \rfloor$ is the floor function of the ratio t/T. As a result, the state equation (28) of the hybrid system (1), (25) or, which is the same, (2) and (25), takes the form

$$\dot{x}(t) = D_{\rm hyb}(x)x + \Upsilon_1(t, x) + b(x)l_g(x(T\lfloor t/T \rfloor))g(T\lfloor t/T \rfloor) + h(x)f(t), \tag{A.5}$$

$$\Upsilon_1(t, x) = b(x)[l^T(x)x - l^T(x(T\lfloor t/T \rfloor))x(T\lfloor t/T \rfloor)], \tag{A.6}$$

where $D_{\rm hyb}(x)$ is the matrix given by (27), and still $x = x(t)$.

To prove the theorem, we first demonstrate that the eigenvalues of the matrix $D_{\rm hyb}(x)$ have negative real parts. For this purpose, in view of (11) and (12), with $x_k = x$ for brevity, equality (22) can be written as follows:

$$\begin{gathered} {}^{-1}[E + 0.5TA(x)] \\ - [E - 0.5TA(x)]^{-1}Tb(x)l^{\rm T}(x) = D_d(x). \end{gathered} \tag{A.7}$$

Multiplying both sides of equality (A.7) by the matrix $[E - 0.5TA(x)]$ on the left, we expand the square brackets and factor the terms with the matrix $A(x)$ to the left-hand side. As a result,

$$0.5TA(x)[D_d(x) + E] = D_d(x) - E + Tb(x)l^{\rm T}(x). \tag{A.8}$$

By Lemma 1, all eigenvalues σ_i^* of the matrix $D_d(x)$ are such that $\sigma_i^* \neq \sigma_\kappa^*$, $i \neq \kappa$, and $|\sigma_i^*| < 1$. Therefore, the matrix $[D_d(x) + E]^{-1}$ exists, and (A.8) implies the equality

$$A(x) = 2T^{-1}[D_d(x) - E + Tb(x)l^{\mathrm{T}}(x)][D_d(x) + E]^{-1}. \tag{A.9}$$

Adding the term $-b(x)l^{\mathrm{T}}(x)$ to both sides of (A.9) and again factoring the matrix $[D_d(x) + E]^{-1}$ to the right, we obtain the expression

$$D_{\mathrm{hyb}}(x) = \Big\{2T^{-1}\Big[D_d(x) - E + Tb(x)l^{\mathrm{T}}(x)\Big] - b(x)l^{\mathrm{T}}(x)[D_d(x) + E]\Big\}[D_d(x) + E]^{-1}.$$

(Here, formula (27) is taken into account.) Expanding both bracketed expressions in the curly braces and collecting terms, we factor the matrix $[D_d(x) - E]$ out of the curly braces to the right and the term $2T^{-1}$ to the left. These manipulations yield

$$D_{\mathrm{hyb}}(x) = 2T^{-1}\Big[E - 0.5Tb(x)l^{\mathrm{T}}(x)\Big]H_g(x), \quad x \in \Omega_{sys}, \tag{A.10}$$

where $H_g(x) = [D_d(x) - E][D_d(x) + E]^{-1}$. Under the conditions of this theorem, the matrix $H_g(x)$ is Hurwitz and has distinct eigenvalues. This is easy to verify since the matrix $D_d(x)$ has distinct eigenvalues, i.e., it is similar to a diagonal matrix [20].

Comparing (A.10) with (23), we conclude that $D_{\mathrm{hyb}}(x) = 2T^{-1}\tilde{D}(x)$. Therefore, by formula (2.15.8) from [20] and condition (24), under the conditions of this theorem, the eigenvalues of the matrix $D_{\mathrm{hyb}}(x)$ (A.5) have negative real parts for $x \in \Omega_{sys}$.

Consider first the free motion of the hybrid system (28), (29) by letting $g(t) = f(t) \equiv 0$. Moreover, bearing in mind Lyapunov's theorem [32, p. 257] and (A.6), we represent equation (A.5) for $t \geq 0$ as follows:

$$\dot{x} = D_{\mathrm{hyb},0}x + \Upsilon(t, x), \tag{A.11}$$

where $D_{\mathrm{hyb},0} = D_{\mathrm{hyb}}(0)$, $\Upsilon(t, x) = \Upsilon_1(t, x) + \Upsilon_2(x)$, $\Upsilon_2(x) = [D_{\mathrm{hyb}}(x) - D_{\mathrm{hyb},0}]x$, and the vector $\Upsilon_1(t, x)$ is given by (A.6).

As established above, the matrix $D_{\mathrm{hyb}}(x)$ $\forall\, x \in \Omega_{sys}$ is Hurwitz; consequently, the constant matrix $D_{\mathrm{hyb},0}$ in (A.11) is also Hurwitz. Let us show that under the conditions of this theorem, the vector function $\Upsilon(t, x) = o(\|x\|)$ uniformly in t [32, p. 257]. For this purpose, we find the limits of the ratios $\Upsilon_1(x)/\|x\|^2$ and $\Upsilon_2(x)/\|x\|^2$ as $x(t) \to 0$. Obviously, for all $t \geq 0$,

$$\lim_{x \to \mathbf{0}} \Big(\Upsilon_2(x)/\|x\|^2\Big) = \lim_{x \to \mathbf{0}} \Big(x^{\mathrm{T}}P[D_{\mathrm{hyb}}(x) - D_{\mathrm{hyb},0}]x/\|x\|^2\Big) = 0. \tag{A.12}$$

Considering the limit of the ratio $\Upsilon_1(t, x)/\|x\|^2$, we observe that for all t, according to (31) and (32), $x \to 0$ implies $x(T\lfloor t/T \rfloor) \to 0$. Therefore, taking (A.6) into account,

$$\lim_{x \to \mathbf{0}} \Big(\Upsilon_1(t, x)/\|x\|^2\Big) = \lim_{x \to \mathbf{0}} \Big(b(x)\Big[l^T(x)x - l^T(x(T\lfloor t/T \rfloor))x(T\lfloor t/T \rfloor)\Big]/\|x(t)\|^2\Big) = 0 \tag{A.13}$$

since for all t, both vectors $l^T(x)$ and $l^T(x(T\lfloor t/T \rfloor))$ in the above expression are multiplied by those tending to zero. Thus, from (A.12) and (A.13) it follows that the vector function $\Upsilon(t, x) = o(\|x\|)$ uniformly in t, i.e.,

$$\frac{\Upsilon(t, x)}{\|x\|} \underset{t}{\rightrightarrows} 0 \ \text{ as } \ x \to 0. \tag{A.14}$$

The matrix $D_{\text{hyb},0}$ is Hurwitz; hence, due to (A.14), the differential system (A.11) satisfies the conditions of Lyapunov's theorem [32, p. 257], stating that the solution $x = \mathbf{0}$ of this system is asymptotically stable. In other words, condition (32) holds in the domain Ω_{sys}. Moreover, according to [32, pp. 258–260], there exists a Lyapunov function $V(x) = x^{\mathrm{T}} S_L x > 0$ with $\dot{V}(x) < 0$ along the trajectories of this system. Here, S_L is a real symmetric matrix.

On the other hand, equation (A.11) corresponds to equation (70.1) whereas equation (A.5) to equation (70.3) from the monograph [33]. Moreover, according to I.G. Malkin, equation (A.11) describes the perturbed motion of the Hurwitz system (1), (25) and (28), (29), and the term $b(x)l_g(x(T\lfloor t/T \rfloor))g(T\lfloor t/T \rfloor) + h(x)f(t)$ in (A.5) characterizes the constantly acting perturbations of this system. In addition, there exists a positive definite function $V(x) = x^{\mathrm{T}} S_L x$ for the differential system (A.11) whose total time derivative along the trajectories of this system is negative definite. In the domain $t \geq 0$, $x \in \Omega_{sys}$, the partial derivatives $(\partial V(x)/\partial x_i) = 2S_{Li}x$, where S_{Li} is the ith row of the matrix S_L, $i = 1, \ldots, n$, are obviously bounded. Therefore, by Malkin's theorem [33], the unperturbed motion of the hybrid system described by equations (A.5) and (28) is stable under the constantly acting perturbations. In other words, under sufficiently small initial conditions and external perturbations (the reference signal $g(t)$ and the disturbance $f(t)$) such that $x(t) \in \Omega_{sys}$, a steady-state regime arises in system (A.5) or, which is the same, in system (1), (25), whose QLM has the form (28), (29).

Consider this regime for $f(t) \equiv 0$, $g(t) = g_0 1(t)$, and sufficiently small $\|x_0\|$ and $|g_0|$. In this regime, as $t \to \infty$, we have $\dot{x}(t) \to \mathbf{0}$, $x(t) \to x^\circ$, $x(T\lfloor t/T \rfloor) \to x^\circ$, and $y_g(t) \to y_g^\circ$, where x° and y_g° are the steady-state values of the variables $x(t)$, $x(T\lfloor t/T \rfloor)$ and $y_g(t)$, respectively, due to $g_0 1(t)$ (see Lemma 2). Then equations (28) and (29) take the form

$$\begin{aligned} \mathbf{0} &= D_{\text{hyb}}(x^\circ)x^\circ + b(x^\circ)l_g(x^\circ)g_0, \\ y_g^\circ &= [c^{\mathrm{T}}(x^\circ) - d(x^\circ)l^{\mathrm{T}}(x^\circ)]x^\circ + d(x^\circ)l_g(x^\circ)g_0, \end{aligned} \tag{A.15}$$

where the matrix $D_{\text{hyb}}(x^\circ)$ is given by (27) for $x = x^\circ$.

Since the matrix $D_{\text{hyb}}(x)$ is Hurwitz for $x \in \Omega_{sys}$, the matrix $D_{\text{hyb}}^{-1}(x^\circ)$ exists, so (A.15) implies the equalities $x^\circ = -D_{\text{hyb}}^{-1}(x^\circ)b(x^\circ)l_g(x^\circ)g_0$ and

$$y_g^\circ = \Big\{[d(x^\circ)l^{\mathrm{T}}(x^\circ) - c^{\mathrm{T}}(x^\circ)]D_{\text{hyb}}^{-1}(x^\circ)b(x^\circ) + d(x^\circ)\Big\}\, l_g(x^\circ)g_0. \tag{A.16}$$

Equality (33) will obviously be satisfied if $y_g^\circ = g_0$. Therefore, from (A.16) we obtain the following necessary and sufficient condition for this: $\{[d(x^\circ)l^{\mathrm{T}}(x^\circ) - c^{\mathrm{T}}(x^\circ)]D_{\text{hyb}}^{-1}(x^\circ)b(x^\circ) + d(x^\circ)\}l_g(x^\circ) = 1$. However, the value of x° is unknown in advance, so this condition is replaced, in view of the formula $D_{\text{hyb}}^{-1}(x) = \mathrm{adj}D_{\text{hyb}}(x)/\det D_{\text{hyb}}(x)$, by the equality

$$\Big\{[d(x)l^{\mathrm{T}}(x) - c^{\mathrm{T}}(x)]\mathrm{adj}D_{\text{hyb}}(x)b(x) + d(x)\det D_{\text{hyb}}(x)\Big\}\, l_g(x) = \det D_{\text{hyb}}(x). \tag{A.17}$$

The equality $y_g^\circ = g_0$ is immediate from (A.17) and (A.16) by Malkin's theorem (see above). Based on the definition (27) and formulas (П.25) and (П.26) from [30, p. 233], we have the equalities

$$\begin{aligned} &\mathrm{adj}D_{\text{hyb}}(x)b(x) = \mathrm{adj}\left[A(x) - b(x)l^{\mathrm{T}}(x)\right] b(x) = \mathrm{adj}A(x)b(x) \quad \text{and} \\ &\det D_{\text{hyb}}(x) = \det\left[A(x) - b(x)l^{\mathrm{T}}(x)\right] = \det A(x) - l^{\mathrm{T}}(x)\mathrm{adj}A(x)b(x). \end{aligned}$$

Substituting them into (A.17) yields, after trivial simplifications, the relation

$$\Big\{d(x)\det A(x) - c^{\mathrm{T}}(x)\mathrm{adj}A(x)b(x)\Big\}\, l_g(x) = \det D_{\text{hyb}}(x).$$

Taking (5) into account, this result finally leads to equality (33) under condition (4) and the gain $l_g(x)$ (26). The proof of the theorem is complete.

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