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Stability of Regular Precessions of a Body with a Fixed Point Bounded by the Ellipsoid of Revolution in a Flow of Particles

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Abstract—Stability of regular precessions of a rigid body bounded by the ellipsoid of revolution in a free molecular flow of particles is studied. Conditions of stability of regular precessions of the body are obtained and the Poincare–Chetaev bifurcation diagrams are constructed.

Keywords: rigid body with a fixed point, free molecular flow of particles, regular precessions, stability of steady motions

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1. INTRODUCTION

The problem of motion of a rigid body with a fixed point under the action of external forces and moments is a generalization of the classical problem of motion of a heavy rigid body with a fixed point. One of the cases of integrability of this problem was discovered by Lagrange [1]. Subsequently many papers devoted to study the dynamics of the Lagrange top were published. Regular precessions of the Lagrange top were firstly considered in [2, 3], and their stability were studied in [2, 4, 5].

The mathematical model of interaction of a free molecular flow of particles with a rigid body was proposed by Karymov [6, 7] and Beletsky [8, 9]. In their papers the dynamics of satellites moving in the upper layers of the atmosphere or moving under the action of solar radiation pressure is considered. For the first time the corresponding model of interaction of a free molecular flow with a rigid body was applied to the problem of motion of a rigid body with a fixed point by Burov and Karapetyan [10]. In the paper by Burov and Karapetyan [10] the equations of motion of the body were obtained, and the integrable case, similar to the Lagrange case in the classical problem of motion of a heavy rigid body with a fixed point, was found. The problem of stability of steady motions of a satellite under the action of solar radiation pressure was considered, for example, by Sidorenko [11].

In this paper we study the stability of regular precessions of a dynamically symmetric body, bounded by the ellipsoid of revolution, in a flow of particles. Using the Routh theory of analyzing the stability of steady motions of mechanical systems with known first integrals, we obtain stability conditions for regular precessions of a rigid body with a fixed point located in a flow of particles.

2. PROBLEM FORMULATION

Let us consider the problem of motion of a rigid body with a fixed point O in a free molecular flow of particles of constant density ρ (see Fig. 1). The particles in the flow move with a constant velocity

$$-\boldsymbol{v} = v_0\boldsymbol{\gamma},$$

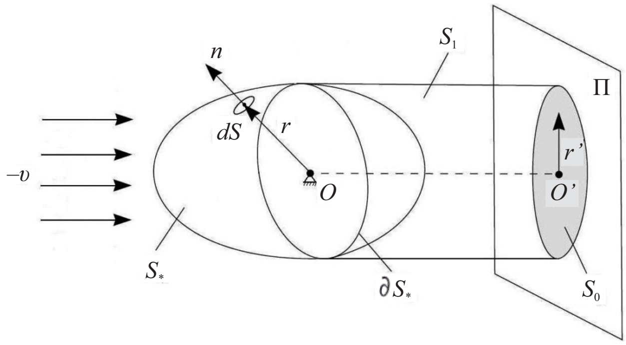

Fig. 1. Rigid body with a fixed point in a flow of particles.

where $\boldsymbol{\gamma}$ is a unit vector, fixed in inertial space and directed along the oncoming flow. Thus, particles move in a fixed direction, their thermal motion is neglected. The following model of particle—body interaction is chosen: a particle makes an inelastic collision with the body, transferring all its energy to the body and not being reflected. We will also consider sufficiently slow rotations of the body, i.e., we will assume that the flow velocity v_0 considerably exceeds the product of characteristic value of the angular velocity of the body and the characteristic length from any point of the body to a fixed point. Using the method, proposed by Beletsky [8, 9], we can write equations of motion of the body in the following form (see [10, 19–21]):

$$\begin{gathered} \mathbb{J}_0\dot{\boldsymbol{\omega}} + [\boldsymbol{\omega} \times \mathbb{J}_0\boldsymbol{\omega}] = -\rho v_0^2 S(\boldsymbol{\gamma})[\boldsymbol{\gamma} \times \boldsymbol{c}(\boldsymbol{\gamma})], \\ \dot{\boldsymbol{\gamma}} + [\boldsymbol{\omega} \times \boldsymbol{\gamma}] = 0, \end{gathered} \tag{1}$$

where $\mathbb{J}_0 = \operatorname{diag}(A_1, A_2, A_3)$ is the inertia matrix of the body relative to a fixed point O, written in the coordinate system $Oxyz$, which is rigidly connected with the body. The origin of this coordinate system is located at the fixed point O and its axes are directed along the principal axes of inertia of the body at O. We denote the unit vectors of this coordinate system by $\boldsymbol{e}_x$, $\boldsymbol{e}_y$, $\boldsymbol{e}_z$. Let $\boldsymbol{\omega} = \omega_1\boldsymbol{e}_x + \omega_2\boldsymbol{e}_y + \omega_3\boldsymbol{e}_z$ be the absolute angular velocity of the body in the same coordinate system. The expression $S(\boldsymbol{\gamma})$ is the area of the figure S_0—the orthogonal projection of the surface of the body onto the plane Π, perpendicular to the direction of the oncoming particle flow $\boldsymbol{\gamma}$. We can say that S_0 is the shadow of the body on the plane Π, i.e., the projection of the body on the Π plane along the $\boldsymbol{\gamma}$ direction. Vector $\boldsymbol{c}(\boldsymbol{\gamma})$ is the vector drawn from the projection O' of the fixed point O to the centroid of the shadow—coinciding with the center of mass of a homogeneous plate occupying the region S_0 (see Fig. 1).

3. REGULAR PRECESSIONS OF THE BODY

Let us consider a dynamically symmetric rigid body ($A_1 = A_2$) with a fixed point lying on the axis of dynamical symmetry. Assume that the body is bounded by the ellipsoid of revolution with semiaxes $a_1 = a_2 = a$, $a_3 = b$, the axis of symmetry of which coincides with the axis of dynamical symmetry of the body. In this case, the equations of motion of the body (1) admit the quadratic and two linear in generalized velocities first integrals [19–22]:

$$U_0 = \frac{A_1}{2}\left(\omega_1^2 + \omega_2^2\right) + \frac{A_3}{2}\omega_3^2 - f\int_0^{\gamma_3} S(\gamma_3)\, c_3(\gamma_3)\, d\gamma_3 = k_0 = \text{const}, \tag{2}$$

$$U_1 = A_1(\omega_1\gamma_1 + \omega_2\gamma_2) + A_3\omega_3\gamma_3 = k_1 = \text{const}, \tag{3}$$

$$U_2 = \omega_3 = k_2 = \text{const}, \tag{4}$$

where $f = \rho v_0^2$. In this case, to study the steady motions of the system, one can use the Routh theory for holonomic systems with explicitly known first integrals [12–15, 18]. The effective potential in the case, when the body is bounded by the elongated ellipsoid of revolution is explicitly written as follows:

$$W(\theta) = \frac{(k_1 - A_3 k_2 \cos\theta)^2}{2A_1 \sin^2\theta} - \frac{f\pi a^2 bl}{2} \cos\theta \sqrt{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}} - \frac{f\pi bl}{2\sqrt{\dfrac{1}{a^2} - \dfrac{1}{b^2}}} \arctan\left(\frac{\sqrt{\dfrac{1}{a^2} - \dfrac{1}{b^2}}\cos\theta}{\sqrt{\dfrac{\sin^2\theta}{a^2} + \dfrac{\cos^2\theta}{b^2}}}\right). \tag{5}$$

Here l is the distance from the fixed point O to the center of the ellipsoid of revolution, bounding the rigid body and θ is the angle between the axis of dynamical symmetry of the body and the direction of the flow of particles $\boldsymbol{\gamma}$, measured so that at $\theta = 0$ the body is oriented along the flow of particles. For any values of contants of the first integrals, the system of equations (1) admits a two-parametric family of particular solutions of the following form:

$$\begin{gathered} \omega_1 = \omega\gamma_1, \quad \omega_2 = \omega\gamma_2, \quad \omega_3 = \omega\cos\theta + \Omega, \\ \gamma_1^2 + \gamma_2^2 = 1 - \gamma_3^2 = \sin^2\theta, \quad \gamma_3 = \cos\theta, \end{gathered} \tag{6}$$

where ω and Ω are constants connected with the constants k_1 and k_2 of the first integrals (3) and (4) by the equations

$$\omega = \frac{k_1 - A_3 k_2 \cos\theta}{A_1 \sin^2\theta}, \quad \Omega = \frac{(A_1 \sin^2\theta + A_3 \cos^2\theta)\, k_2 - k_1 \cos\theta}{A_1 \sin^2\theta},$$

and the constant angle θ is determined from the condition of existence of regular precessions

$$\frac{dW(\theta)}{d\theta} = 0,$$

which is explicitly written as follows:

$$\frac{(k_1 - A_3 k_2 \cos\theta)(A_3 k_2 - k_1 \cos\theta)}{2A_1 \sin^3\theta} + f\pi a^2 bl \sin\theta \sqrt{\frac{\sin^2\theta}{a^2} + \frac{\cos^2\theta}{b^2}} = 0. \tag{7}$$

Let us write the equation (7) in dimensionless form. For this purpose, we introduce dimensionless constants of the first integrals

$$p_1 = \frac{k_1}{\sqrt{A_3 f\pi a^2 l}}, \quad p_2 = k_2 \sqrt{\frac{A_3}{f\pi a^2 l}} \tag{8}$$

and dimensionless parameters

$$y = \frac{A_1}{A_3} \in \left[\frac{1}{2}, +\infty\right), \quad z = \frac{b^2}{a^2}.$$

Then the condition for the existence of regular precessions (7) is dimensionless form is rewritten as follows:

$$\frac{(p_1 - p_2\cos\theta)(p_2 - p_1\cos\theta)}{y\sin^3\theta} + \sin\theta\sqrt{z\sin^2\theta + \cos^2\theta} = 0,$$

or, multiplying by $-y\sin^3\theta$,

$$a_{11}p_1^2 + 2a_{12}p_1p_2 + a_{22}p_2^2 + a_1 = 0, \tag{9}$$
$$a_{11} = \cos\theta, \quad a_{12} = -\frac{1+\cos^2\theta}{2}, \quad a_{22} = \cos\theta,$$
$$a_1 = -y\sin^4\theta\sqrt{z\sin^2\theta + \cos^2\theta}.$$

It is easy to see that for each fixed value of θ the equation (9) defines the second order curve. Let us transform it to its canonical form. To do this, we introduce new variables x_1 and y_1 according to the formulae:

$$x_1 = p_1 - p_2, \qquad y_1 = p_1 + p_2. \tag{10}$$

In order to understand the physical meaning of the variables x_1 and y_1, let us recall the definition of the constants k_1 and k_2 (see (3), (4)). For a dynamically symmetric body we can write

$$k_1 = A_1(\omega_1\gamma_1 + \omega_2\gamma_2) + A_3\omega_3\gamma_3 = (\mathbb{J}_O\boldsymbol{\omega},\, \boldsymbol{\gamma}), \quad k_2 = \omega_3 = \frac{(\mathbb{J}_O\boldsymbol{\omega},\, \boldsymbol{e}_z)}{A_3}.$$

Substituting these expressions into (8), we obtain the following equations:

$$p_1 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, \boldsymbol{\gamma}), \quad p_2 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, \boldsymbol{e}_z).$$

According to the formula (10) we can conclude that x_1 is, up to a constant factor, the projection of the angular momentum of the body onto the vector $\boldsymbol{\gamma} - \boldsymbol{e}_z$:

$$x_1 = p_1 - p_2 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, (\boldsymbol{\gamma} - \boldsymbol{e}_z)).$$

Similarly, y_1 is the projection of the angular momentum of the body onto the vector $\boldsymbol{\gamma} + \boldsymbol{e}_z$

$$y_1 = p_1 + p_2 = \frac{1}{\sqrt{A_3 f\pi a^2 l}}(\mathbb{J}_0\boldsymbol{\omega},\, (\boldsymbol{\gamma} + \boldsymbol{e}_z)).$$

Vectors $\boldsymbol{\gamma} - \boldsymbol{e}_z$ and $\boldsymbol{\gamma} + \boldsymbol{e}_z$ are orthogonal

$$((\boldsymbol{\gamma} - \boldsymbol{e}_z),\, (\boldsymbol{\gamma} + \boldsymbol{e}_z)) = \boldsymbol{\gamma}^2 - \boldsymbol{e}_z^2 = 0.$$

The third orthogonal vector is perpendicular to $\boldsymbol{\gamma}$ and $\boldsymbol{e}_z$ and therefore it is directed along the nodal line.

$$[(\boldsymbol{\gamma} - \boldsymbol{e}_z) \times (\boldsymbol{\gamma} + \boldsymbol{e}_z)] = 2\,[\boldsymbol{\gamma} \times \boldsymbol{e}_z].$$

Thus, $\boldsymbol{\gamma} - \boldsymbol{e}_z$ and $\boldsymbol{\gamma} + \boldsymbol{e}_z$ are orthogonal vectors in the plane, perpendicular to the nodal line, and x_1 and y_1, up to a constant factor, are projections of the angular momentum of the body onto these vectors.

Let us write the equation (9) in the variables x_1, y_1

$$\frac{1}{4}(1+\cos\theta)^2 x_1^2 - \frac{1}{4}(1-\cos\theta)^2 y_1^2 + a_1 = 0. \tag{11}$$

The given equation, up to a constant factor, is the canonical equation of a second-order curve. Let us determine the type of this curve. For $\theta \neq \pi n$ we have $a_1 < 0$, and the coefficients in the quadratic part are nonzero. This means (see, for example, [16, 17]), that in each section by planes

$\theta \neq \pi n$ of the surface, defined by the equation (11), we have a hyperbola. For $\theta = \pi n$ we have $a_1 = 0$, as well as one of the coefficients in the quadratic part will be equal to zero. Thus, in the section by planes $\theta = \pi n$ of the surface, defined by the equation (11) we have a straight line of the form

$$p_1 = (-1)^n p_2.$$

These straight lines correspond to a single parametric family of solutions of equations of motion of the body in a flow of particles, which correspond to permanent rotations of the body about its axis of dynamical symmetry, coinciding with the direction of the flow of particles. The stability of these permanent rotations has been studied in [10, 22].

The condition of stability of steady motions (6) in the case when a rigid body with a fixed point, located in the flow of particles, is bounded by the elongated ellipsoid of revolution, has the form

$$\frac{d^2W(\theta)}{d\theta^2} \geqslant 0$$

of, if we write it explicitly,

$$\frac{(1+2\cos^2\theta)}{y\sin^4\theta}p_1^2 - \frac{(5+\cos^2\theta)\cos\theta}{y\sin^4\theta}p_1p_2 + \frac{(1+2\cos^2\theta)}{y\sin^4\theta}p_2^2 + \frac{(2z\sin^2\theta + 2\cos^2\theta - 1)\cos\theta}{\sqrt{z\sin^2\theta+\cos^2\theta}} \geqslant 0.$$

After multiplying by the positive factor $y\sin^4\theta$, this inequality takes the form

$$b_{11}p_1^2 + 2b_{12}p_1p_2 + b_{22}p_2^2 + b_1 \geqslant 0, \tag{12}$$

$$b_{11} = 1 + 2\cos^2\theta, \quad b_{12} = -\frac{(5+\cos^2\theta)\cos\theta}{2}, \quad b_{22} = 1+2\cos^2\theta,$$

$$b_1 = \frac{y\,(2z\sin^2\theta + 2\cos^2\theta - 1)\sin^4\theta\cos\theta}{\sqrt{z\sin^2\theta+\cos^2\theta}}.$$

Let us write the inequality (12) using the variables x_1 and y_1:

$$\frac{1}{4}(1+\cos\theta)^2(2+\cos\theta)\,x_1^2 + \frac{1}{4}(1-\cos\theta)^2(2-\cos\theta)\,y_1^2 + b_1 \geqslant 0. \tag{13}$$

For each fixed θ, the boundary of the stability region (i.e., the curve, corresponding to the equal sign in the inequality (13)) will also be the second-order curve. The type of this curve depends on the sign of b_1. For $b_1 < 0$ the boundary of the stability region is an ellipse with the center at $x_1 = 0$, $y_1 = 0$. On the plane of dimensionless variables x_1, y_1 for each fixed θ, outside the corresponding ellipse there will be a stability region of regular precessions (6), and inside it—an instability region. For $b_1 > 0$ the boundary of the stability region degenerates into an imaginary ellipse, and for $b_1 = 0$ the boundary is a straight line. We can note here, that for $b_1 \geqslant 0$ the inequality (13) is satisfied for any values of x_1, y_1, and therefore the regular precessions of the body will be stable.

Considering simultaneously the condition of existence of regular precessions (11) and the condition of stability of regular precessions (13) of a body with a fixed point in the flow of particles, we can draw a conclusion about stability of such motions. Namely, if for some fixed θ the hyperbola (11) and the ellipse (13) do not intersect, then the steady motion (6), corresponding to this value of θ, is stable (Figs. 2–4).

Let us study the problem of stability of regular precessions in more details. From the condition of existence of regular precessions (11) we express y_1^2:

$$y_1^2 = \frac{(1+\cos\theta)^2}{(1-\cos\theta)^2}x_1^2 + \frac{4a_1}{(1-\cos\theta)^2}. \tag{14}$$

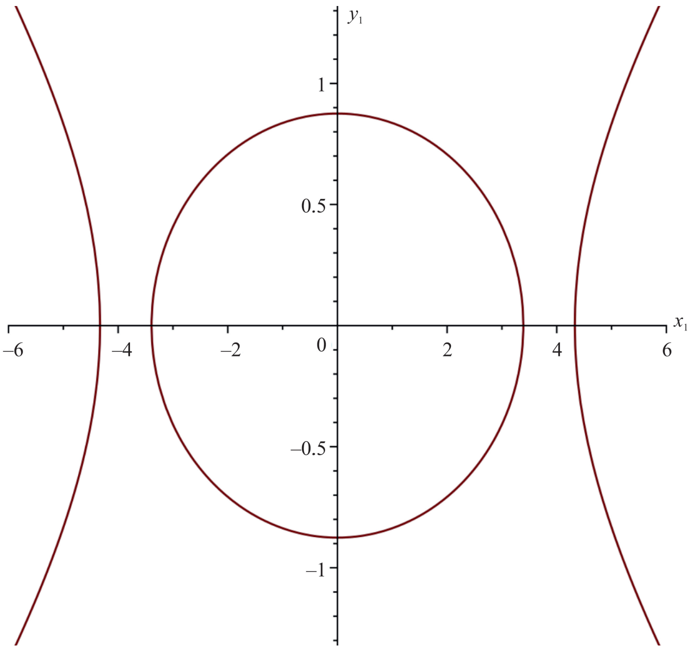

Fig. 2. Relative position of the hyperbola and the ellipse for $y = \frac{5}{6}$, $z = 8$, $\theta = \frac{2\pi}{3}$.

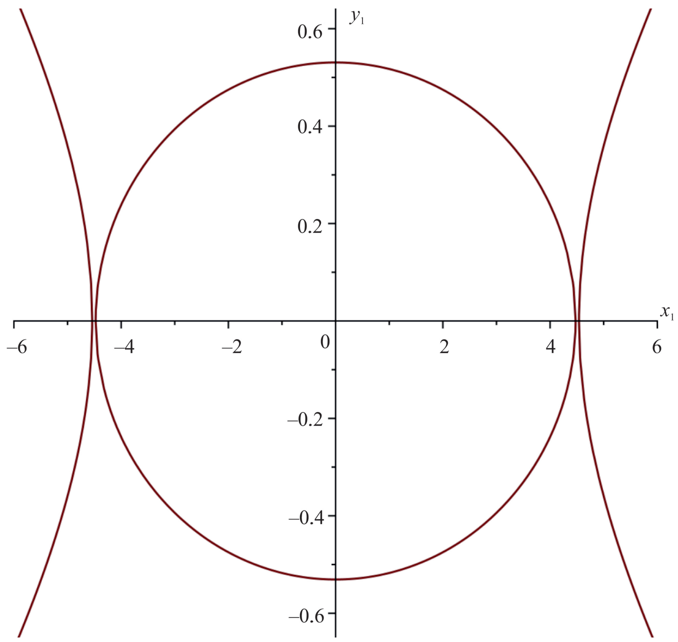

Fig. 3. Relative position of the hyperbola and the ellipse for $y = \frac{5}{6}$, $z = 8$, $\theta = \frac{3\pi}{4}$.

Since $y_1^2 \geqslant 0$, then from (14) we obtain that x_1 should satisfy the condition:

$$x_1^2 \geqslant \frac{4y\sin^4\theta}{(1+\cos\theta)^2}\sqrt{z\sin^2\theta + \cos^2\theta} = (x_1)_*^2. \tag{15}$$

Let us substitute the obtained expression (14) for y_1^2 into the stability condition (13). We obtain the following inequality

$$(1+\cos\theta)^2 x_1^2 + (2-\cos\theta)\,a_1 + b_1 \geqslant 0. \tag{16}$$

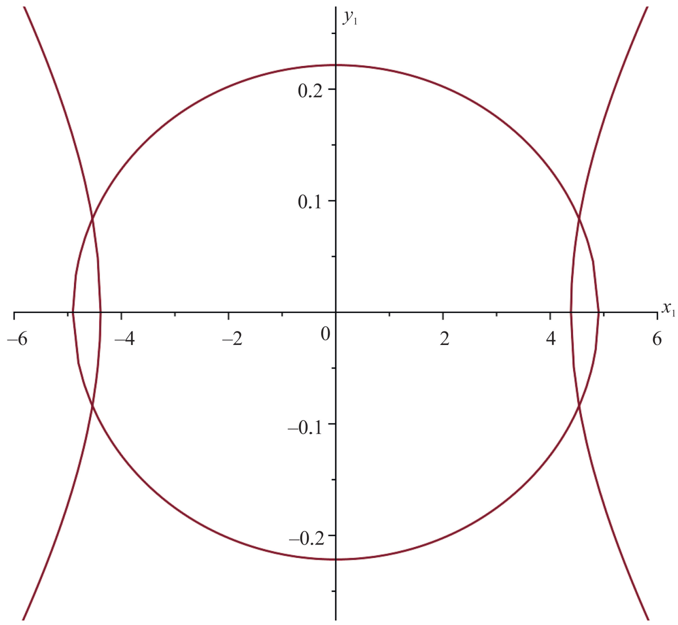

Fig. 4. Relative position of the hyperbola and the ellipse for $y = \frac{5}{6}$, $z = 8$, $\theta = \frac{5\pi}{6}$.

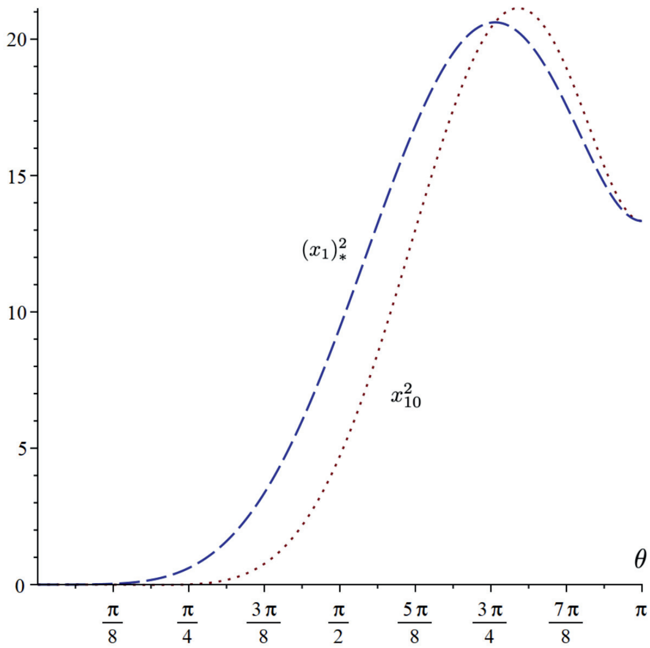

Fig. 5. Relative position of x^2_{10} and $(x_1)^2_*$ for $y = \frac{5}{6}$, $z = 8$.

The left hand side of defines a parabola whose branches are directed upwards. Let x_{10} be the abscissa of the intersection point of the parabola with the positive semiaxis Ox_1. Let us find x_{10} explicitly

$$x^2_{10} = -\frac{(2-\cos\theta)\,a_1 + b_1}{(1+\cos\theta)^2}.$$

After substitution the values of a_1 and b_1, we write the expression for x^2_{10} in explicit form:

$$x^2_{10} = \frac{y\sin^4\theta\sqrt{z\sin^2\theta+\cos^2\theta}}{(1+\cos\theta)^2}\left(2 - 3\cos\theta + \frac{\cos\theta}{z\sin^2\theta+\cos^2\theta}\right). \qquad (17)$$

For $x_1^2 \geqslant x_{10}^2$ the stability condition (3) will be satisfied. Thus, regular precessions exist when the condition (15) is satisfied, and they will be stable if the condition (3) is satisfied. From this fact we can draw the following conclusions (Fig. 5):

1. For $x_{10}^2 < (x_1)_*^2$, regular precessions of the body are stable for any admissible values of x_1, y_1 (and, accordingly, for any values of constants p_1 and p_2 of the first integrals).

2. For $x_{10}^2 > (x_1)_*^2$, regular precessions are stable only if the condition $x_1^2 > x_{10}^2$ holds.

3.1. The Case $z > 1$

First of all let us consider the case of elongated ellipsoid of revolution, $z > 1$. We will find the conditions under which $x_{10}^2 > (x_1)_*^2$. We obtain

$$3\,(1-z)\cos^3\theta + 2\,(1-z)\cos^2\theta + (3z-1)\cos\theta + 2z < 0.$$

By factoring the left hand side, we obtain the following inequality:

$$3\,(z-1)\,(\cos\theta+1)\left(\cos\theta - \frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right)\left(\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right) > 0. \tag{18}$$

The first three factors in the left hand side of inequality (18) are positive. It can be shown that for $z > 1$ the following condition is valid:

$$\frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}} > 1.$$

Thus, the factor

$$\cos\theta - \frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}$$

in the left hand side of (18) is always negative. Therefore the condition (18) is equivalent to the condition

$$\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}} < 0. \tag{19}$$

Let us consider the latter condition. The function

$$\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}$$

increases monotonically for $z > 1$. It takes the value -1 for $z = 2$ and then it asymptotically tends to $-\frac{2}{3}$. Thus for $1 < z \leqslant 2$ we obtain

$$\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{25z-1}{z-1}} \geqslant 0,$$

and therefore the condition (19) is not satisfied. From this fact we conclude that for $1 < z \leqslant 2$ regular precessions of the rigid body are stable for any value of θ and for any admissible values of the parameters x_1 and y_1 (and accordingly, for any admissible values of the constants p_1 and p_2 of the first integrals).

Figure 6 presents the Poincare–Chetaev bifurcation diagram for $y = 2$, $z = \frac{12}{11}$ at the level $y_1 = 0$. On this figure the straight line $\theta = \pi$ corresponds to permanent rotations of the body about its axis

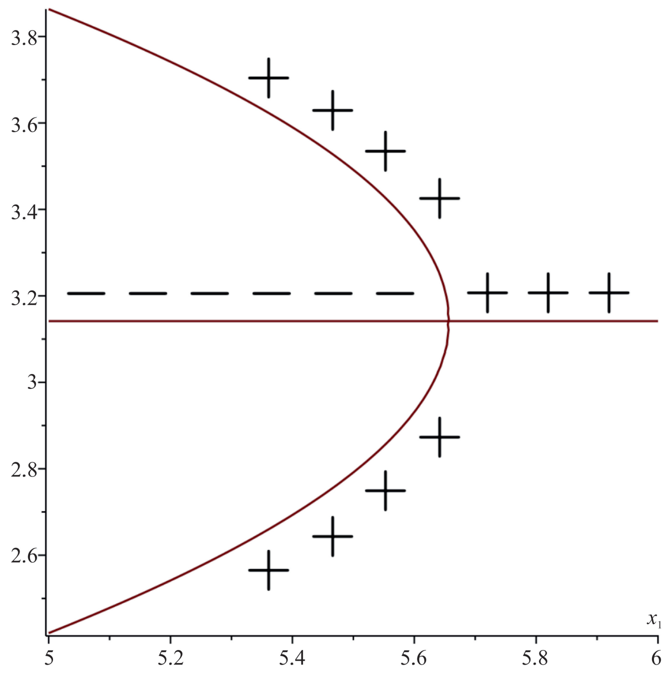

Fig. 6. Section of the surface (11) by the plane $y_1 = 0$ for $y = 2$, $z = \frac{12}{11}$.

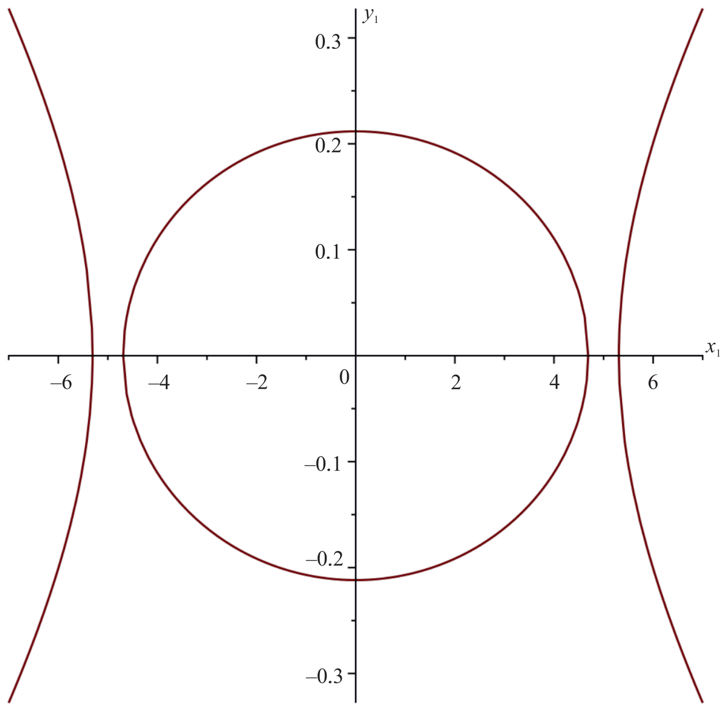

Fig. 7. Relative position of the hyperbola and the ellipse for $y = 2$, $z = \frac{12}{11}$, $\theta = \frac{5\pi}{6}$.

of dynamical symmetry. It is easy to see the stable regular precessions of the body which branch off from this straight line. In addition, on the plane of dimensionless variables x_1 and y_1 for the fixed values of θ the corresponding hyperbola (11) and ellipse (13) were constructed (see Fig. 7). It is easy to see, that they do not intersect in the considered case.

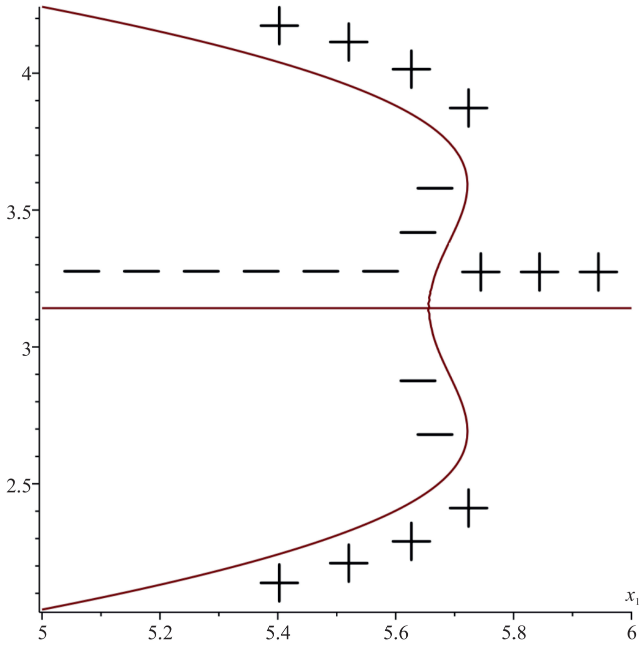

Fig. 8. Section of the surface (11) by the plane $y_1 = 0$ for $y = 2$, $z = \frac{5}{2}$.

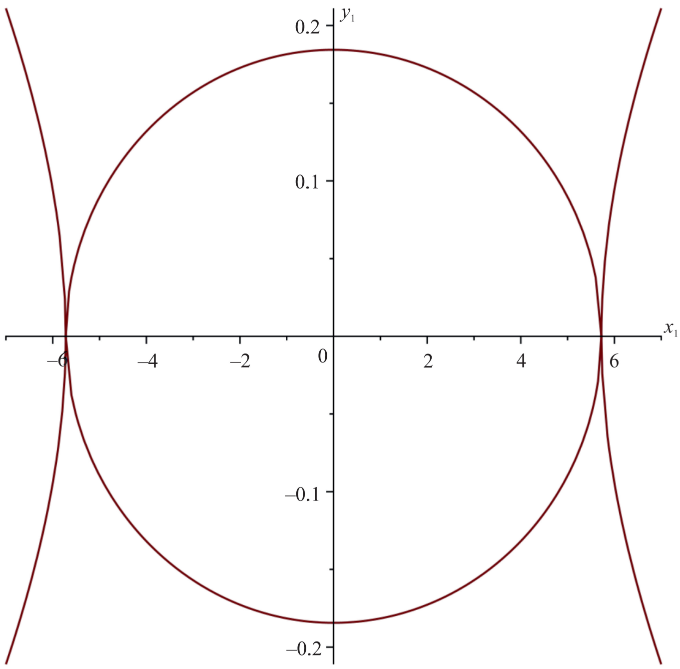

Fig. 9. Relative position of the hyperbola and the ellipse for $y = 2$, $z = \frac{5}{2}$, $\theta = \theta_2$.

Let us introduce the new value $\theta = \theta_2$:

$$\theta_2 = \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right). \tag{20}$$

We can conclude that for $z > 2$ the inequality (19) holds for $\cos\theta < \cos\theta_2$ and does not hold for $\cos\theta \geqslant \cos\theta_2$.

Figure 8 presents the Poincare–Chetaev bifurcation diagram for $y = 2$, $z = \frac{5}{2}$ at the level $y_1 = 0$. It is easy to see, that the regular precessions of the body, branching off the permanent rotations, are unstable near the value $\theta = \pi$, and then, passing through the inflection point, become stable. The inflection point corresponds to $\theta = \theta_2$, where θ_2 is determined by (20).

In addition Fig. 9 shows the hyperbola (11) and the ellipse (13) on the plane of dimensionless variables x_1 and y_1 for $\theta = \theta_2$, where θ_2 is determined by (20). It is clear that for $\theta = \theta_2$ the hyperbola and the ellipse touch but do not intersect.

Taking into account all the obtained results we can conclude that the condition (18) is valid for $z > 2$ and $\cos\theta < \cos\theta_2$. And it is not satisfied for any other values of the parameters. Thus, regular precessions of the rigid body located in the flow of particles are stable for any values of x_1, y_1 (and, accordingly, for any values of dimensionless constants p_1 and p_2 of the first integrals), if $1 < z \leqslant 2$ or if $z > 2$ and $\cos\theta > \cos\theta_2$. If $z > 2$ and $\cos\theta < \cos\theta_2$, then the regular precessions of the body are stable under condition $x_1^2 > x_{10}^2$.

3.2. The Case $z < 1$

Now we study the stability of regular precessions of the body bounded by the oblate ellipsoid of revolution, $z < 1$. All previously find expressions for x_{10}^2 and $(x_1)_*^2$ will have the same form (15), (17) for the oblate ellipsoid. The conditions, under which $x_{10}^2 > (x_1)_*^2$, will be the same (18). Simplifying them, we obtain the following condition:

$$\left(\cos\theta - \frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\left(\cos\theta - \frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right) < 0. \tag{21}$$

It is easy to see, that for $z \in [\frac{1}{25}, 1)$ the condition (21) does not hold. Thus, we can conclude, that for $z \in [\frac{1}{25}, 1)$ we have $x_{10}^2 \leqslant (x_1)_*^2$ and therefore regular precessions of the body in the flow of particles will be stable for any values of x_1, y_1.

3.3. The Case $z < \frac{1}{25}$

Finally, let us consider the case $z < \frac{1}{25}$. The condition (21) will be satisfied for the following values of θ:

$$\theta \in \left(\arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right), \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\right). \tag{22}$$

Let us introduce the value $\theta = \theta_1$ such that

$$\theta_1 = \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right).$$

It is easy to obtain that for $\theta \in (0, \theta_1)\cup(\theta_2, \pi)$ the condition (21) will not satisfied, and therefore, regular precessions of the body will be stable for any values of x_1, y_1. If $\theta \in (\theta_1, \theta_2)$, then regular precessions will be stable only for $x_1^2 > x_{10}^2$.

Figure 10 presents the Poincare–Chetaev bifurcation diagram for $y = \frac{101}{200}$, $z = \frac{1}{100}$ at the level $y_1 = 0$. It is easy to see, that the regular precessions of the body, branching off from the permanent rotations, are stable near $\theta = \pi$, and its stability is preserved as the angle θ decreases to θ_2. Further, for $\theta \in (\theta_1, \theta_2)$, the regular precessions of the body become instable, and for $\theta < \theta_1$ they become stable again.

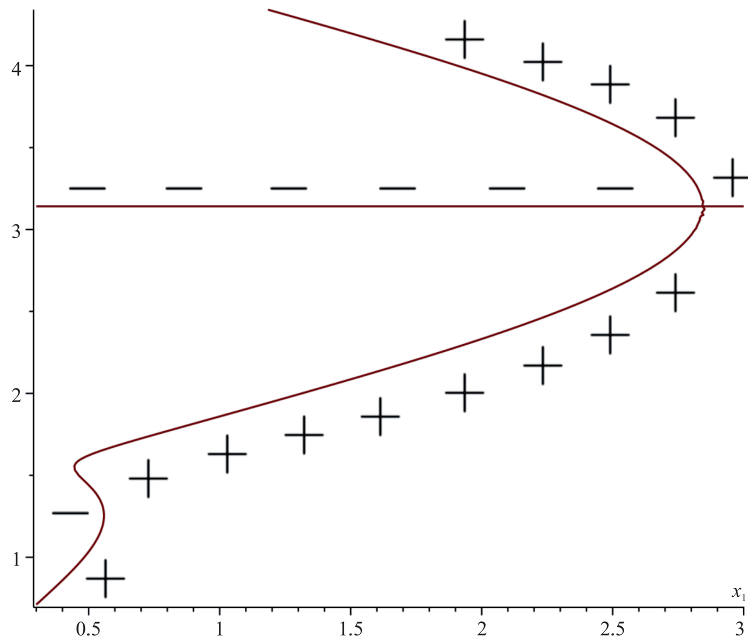

Fig. 10. Section of the surface (11) by the plane $y_1 = 0$ for $y = \frac{101}{200}, z = \frac{1}{100}$.

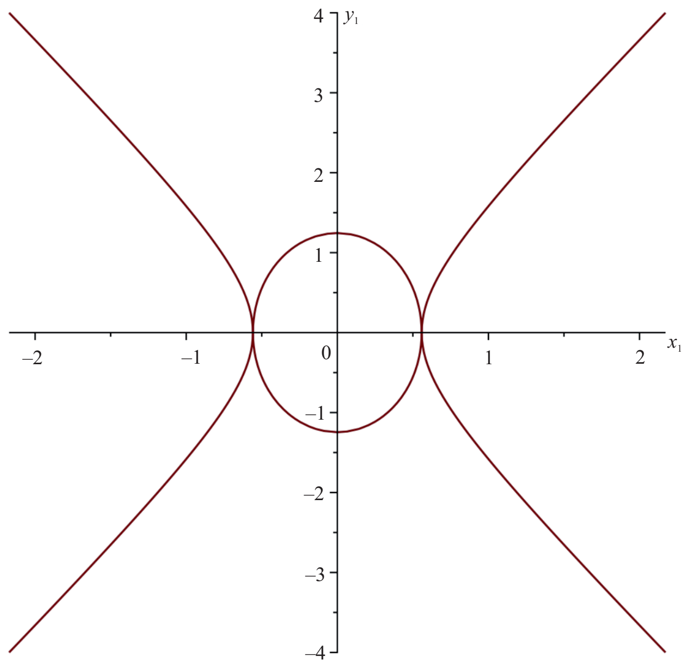

Fig. 11. Relative position of the hyperbola and the ellipse for $y = \frac{101}{200}$, $z = \frac{1}{100}$, $\theta = \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)$.

Figures 11 and 12 shows the hyperbola (11) and the ellipse (13) on the plane of dimensionless variables x_1 and y_1 for

$$z = \frac{1}{100}, \quad \theta = \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right) \text{ and } \quad \theta = \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right).$$

It is clear that for these values of θ the hyperbola (11) and the ellipse (13) touch, but do not intersect.

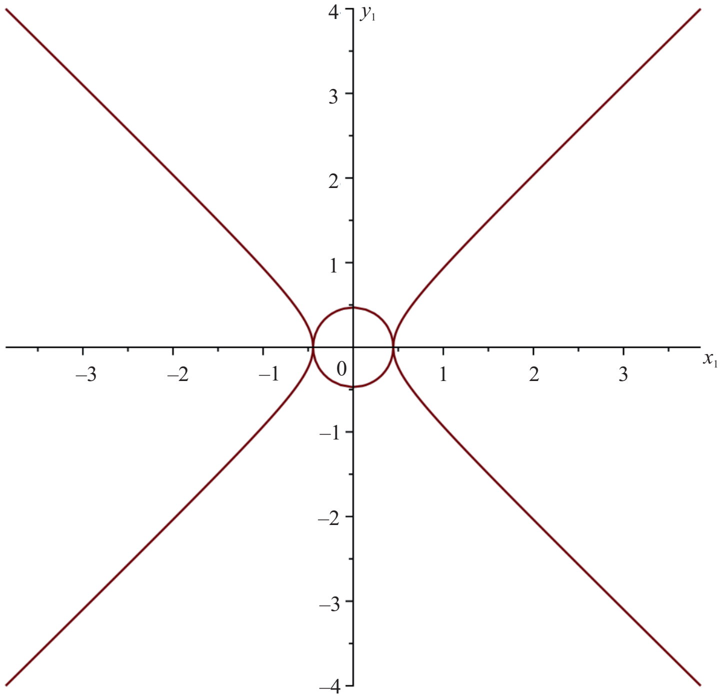

Fig. 12. Relative position of the hyperbola and the ellipse for $y = \frac{101}{200}$, $z = \frac{1}{100}$, $\theta = \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)$.

Thus we can finally conclude that all the regular precessions (6) of a dynamically symmetric rigid body with a fixed point, bounded by the ellipsoid of revolution and located in a flow of particles, will be stable for any values of x_1, y_1 (and, accordingly, for any values of the dimensionless constants p_1 and p_2 of the first integrals), if the ratio of the squares of the semiaxes of the ellipsoid is satisfied to the condition

$$\frac{1}{25} \leqslant z \leqslant 2.$$

For $z > 2$ the regular precessions of the body will be stable for any values of x_1 and y_1 if θ lies in the interval

$$\theta \in \left[0,\ \arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right)\right).$$

For values of θ, belonging to the interval

$$\theta \in \left[\arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{25z-1}{z-1}}\right), \pi\right)$$

the regular precessions are stable if the following condition is valid

$$x_1^2 > \frac{y\sin^4\theta\sqrt{z\sin^2\theta + \cos^2\theta}}{(1+\cos\theta)^2}\left(2 - 3\cos\theta + \frac{\cos\theta}{z\sin^2\theta + \cos^2\theta}\right).$$

For

$$0 < z < \frac{1}{25}$$

the regular precessions of the body will be stable for any values of x_1 and y_1, if θ belongs to the set

$$\theta \in \left(0,\ \arccos\left(\frac{1}{6} + \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\right) \bigcup \left(\arccos\left(\frac{1}{6} - \frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right),\ \pi\right).$$

For any θ, belonging to the interval

$$\theta \in \left[\arccos\left(\frac{1}{6}+\frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right), \arccos\left(\frac{1}{6}-\frac{1}{6}\sqrt{\frac{1-25z}{1-z}}\right)\right]$$

the regular precessions will be stable if the following condition is valid

$$x_1^2 > \frac{y\sin^4\theta\sqrt{z\sin^2\theta+\cos^2\theta}}{(1+\cos\theta)^2}\left(2-3\cos\theta+\frac{\cos\theta}{z\sin^2\theta+\cos^2\theta}\right).$$

These are the main results of study the stability of regular precessions of a dynamically symmetric body with a fixed point, bounded by the ellipsoid of revolution, in a flow of particles.

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