ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 7, pp. 701–710.

Synthesis of Itô Equations for a Shaping Filter with a Given Spectrum

M. M. Khrustalev*† and D. S. Rumyantsev*,[a]

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: [a]n3030@mail.ru

Received August 6, 2023
Revised January 15, 2024
Accepted January 20, 2024

Abstract—The analytical method for the synthesis of a generator of a random process with a given spectrum in the form of a linear system of Ito's equations is proposed. The stationarity of a random process is assumed, the spectral and corresponding transfer functions of which are defined in the form of rational fractions. The coefficients of the system of Ito's equations of the generator are found from recurrent algebraic relations. The method is focused on working with mathematical models of nature random processes, such as the Dryden's wind model. The transformation of the spectra of the wind gust model in three directions is presented in detail and the corresponding stochastic equations are given.

Keywords: Ito's stochastic differential equation, spectral density, transfer function, shaping filter, random disturbance generator, Dryden wind turbulence model

DOI: 10.31857/S0005117924070059

1. INTRODUCTION

The shaping filter allows you to generate a random signal with a given spectral density from a white noise signal [1, Section 6.6; 2, Section 10.1; 3, Section 5.1.5]. The shaping filter and the analyzed system form some extended system, the input of which is affected by white noise (Fig. 1). This shows a way to move from representing a system in terms of transfer functions (shown in the diagram) to stochastic differential equations. The results of the article will be useful to researchers for adding random factors to a dynamic model and simulating natural phenomena (movement of air masses, water flow, etc.).

There are many known models of wind gusts [4], but in the article only the Dryden [5] turbulence model is considered in detail, which at the output gives a stochastic process determined by velocity spectra. The spectral density of the signal is an even fractional-rational function of frequency and can be represented in the form of two complex conjugate factors, from which the transfer function of the shaping filter is found [1, Section 6.6; 3, Section 5.1.5].

Trying to directly write a high-order differential equation whose output has a given spectrum usually results in high-order white noise derivatives. The representation of these derivatives in the form of generalized functions [6] and the generalization of the Itô equations in the form of Leontief-type equations [6] are known, but such equations are complicated and little studied. In [3, Section 3.3.3] the transition from a linear stochastic differential equation of higher order to a linear system of stochastic differential equations of first order is considered, but to find the coefficients of the system it is necessary to differentiate the coefficients of the original equation (if it is not stationary). The proposed method allows us to describe a natural random process using the well-studied Ito

† Deceased.

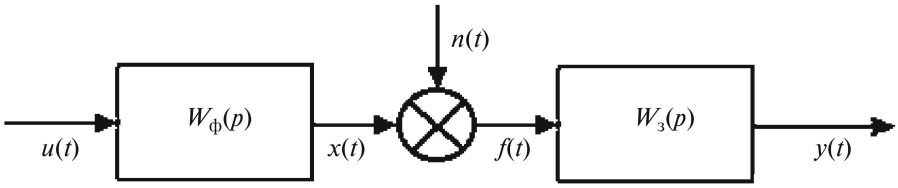

Fig. 1. Connection diagram for the shaping filter.

equations [3]. The resulting equations, for example, can be used in conjunction with the equations of the mathematical model of the aircraft [7].

A method is proposed for obtaining relatively simple stochastic differential equations for synthesizing the output signal using a known transfer function. Next, we will consider transfer functions under the assumption that the corresponding spectra are known.

Two ways of transforming any fractional-rational transfer function, leading to the same result, are presented below. The function is decomposed into a sum of fractions, the numerators of which are real numbers, and the denominators of which are polynomials. In this case, all operations are arithmetic. And the process flow diagram can be depicted as a sum of integrating links. Based on the new notation, it is possible to construct a system of linear stochastic differential Ito equations.

There is a slightly more complex way of similar transformation of the transfer function [8, Section 2.3]. If the transfer function $W(p)$ is given, then for the corresponding system of linear equations of the form

$$\begin{gathered} \dot{x} = Ax + bu \\ y = cx \end{gathered} \tag{1}$$

it is necessary to find the matrix A and the vector b. The output vector c is given. The first Frobenius form of the state equation matrix A is selected so that its characteristic polynomial coincides with the denominator of the transfer function. The elements of the vector b are found from solving the system of equations

$$W(p) = c(Ep - A)^{-1}b$$

by the method of indefinite coefficients by equating factors with equal powers of the variable p of polynomials of numerators on the left and right [8, Example 2.7].

In the proposed approach, the output vector c is in the process of being solved and is not known in advance. As a result, matrix inversion is not required and only the coefficients of the Ito equation are calculated using arithmetic operations.

The transformation to obtain the equation (1) is not unique [8]. Therefore, it is not always possible to achieve "minimal implementation" (1), i.e., obtain the minimum possible number of variables in the Ito equation.

2. MATHEMATICAL PROBLEM STATEMENT

Let the spectral density of the disturbance under study be defined as $\Phi(\omega) = |W(i\omega)|^2$, here

$$W(p) = \frac{P_m(p)}{Q_n(p)} = \frac{a_0p^m + a_1p^{m-1} + \ldots + a_{m-1}p + a_m}{b_0p^n + b_1p^{n-1} + \ldots + b_{n-1}p + b_n}, \tag{2}$$

and a_i $(i = \overline{0,m})$, b_j $(j = \overline{0,n})$ are constant real coefficients. The poles and zeros of the function $W(p)$ are located in the left half-plane. $W(p)$ is the transfer function of the linear differential

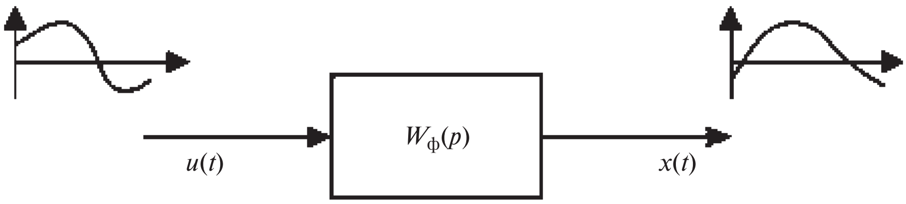

Fig. 2. Block diagram of a shaping filter.

equation

$$\begin{aligned} &b_0x^{(n)} + b_1x^{(n-1)} + \ldots + b_{n-1}x' + b_nx \\ &= a_0u^{(m)} + a_1u^{(m-1)} + \ldots + a_mu. \end{aligned} \tag{3}$$

The block diagram of the shaping filter is shown in Fig. 2. If standard white noise (the derivative of the standard Wiener process) $u(t)$ is supplied to the input, then the equation (3) becomes stochastic, but in the general case contains higher derivatives of white noise.

The goal is to replace the equation with a system of linear differential equations that satisfies two conditions: a) the system does not contain derivatives of the input signal, b) its output, linearly dependent on the state, coincides with the output of the equation. Such a system of equations, as shown below, can easily be converted into the Ito system of equations.

It turns out that for the transformation it is enough to represent the transfer function (2) as a sum of rational fractions, the numerators of which are real coefficients (zero-order polynomials). In the case when all the zeros of the denominator of a rational fraction are real, such a transformation is known [9], but requires finding the zeros of the denominator, which in the general case is only possible numerically. The proposed transformation does not require finding zeros and for any proper rational fraction with both real and complex zeros of the denominator, it can be performed analytically. The coefficients of the modified transfer function are found sequentially from a chain of linear equations.

A method for transforming a n order linear stochastic equation (3) to an equivalent first order linear system of equations not containing white noise derivatives is shown in [3, Section 3.3.3]. In this case, the equation is not stationary and the coefficients a_i $(i = \overline{0, m})$, b_j $(j = \overline{0, n})$ depend on time t, and to find the coefficients of an equivalent system, it is necessary to differentiate the functions a_i, b_j. For a stationary system, new coefficients are found from recurrent arithmetic relations.

3. TRANSFER FUNCTION CONVERSION

The main idea is to represent the original function $W(p)$ as the sum of several functions $W_i(p)$, $i = \overline{1, m+n}$, see Fig. 3. The input and output signals will not change as a result of this conversion.

The proposed representation of the transfer function (2) has the form

$$\begin{aligned} W(p) = \frac{P_m(p)}{Q_n(p)} &= \frac{\alpha_1}{p^{n-m}} + \frac{\alpha_2}{p^{n-m+1}} + \ldots + \frac{\alpha_m}{p^{n-1}} \\ &+ \frac{1}{Q_n(p)}\left(\frac{\beta_{n-1}}{p^{n-1}} + \frac{\beta_{n-2}}{p^{n-2}} + \ldots + \frac{\beta_1}{p} + \beta_0\right). \end{aligned} \tag{4}$$

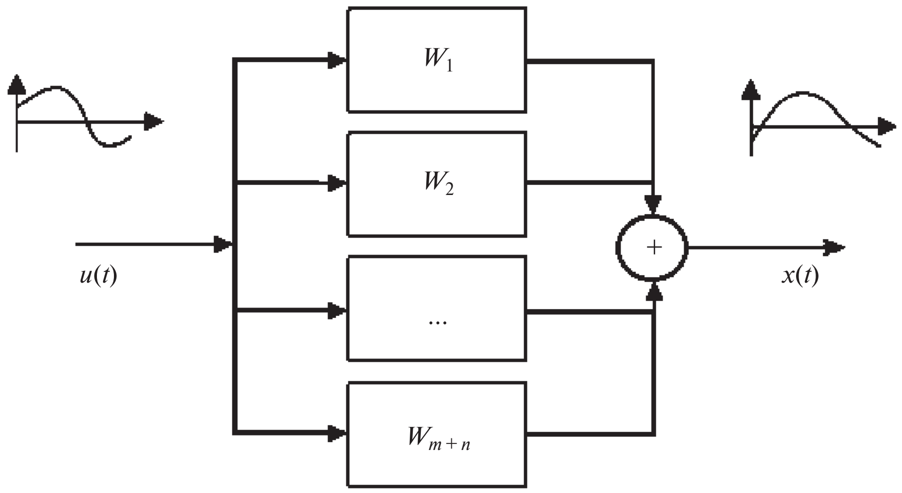

Fig. 3. Block diagram of the sum of several shaping filters.

The number of coefficients α_i $(i = \overline{1, m})$, β_j $(j = \overline{0, n-1})$ is $m + n$. They can be obtained by equating the left and right sides of the equations (2) and (4).

$$
\begin{array}{llr}
\beta_{n-1} & + \alpha_m b_n & = 0, \\
\beta_{n-2} & + \alpha_m b_{n-1} + \alpha_{m-1} b_n & = 0, \\
\beta_{n-3} & + \alpha_m b_{n-2} + \alpha_{m-1} b_{n-1} + \alpha_{m-2} b_n & = 0, \\
\ldots & & \\
\beta_{n-m+1} & + \alpha_m b_{n-m+2} + \ldots + \alpha_2 b_n & = 0, \\
\beta_{n-m} & + \alpha_m b_{n-m+1} + \ldots + \alpha_2 b_{n-1} + \alpha_1 b_n & = 0, \\
\beta_{n-m-1} & + \alpha_m b_{n-m} + \ldots + \alpha_2 b_{n-2} + \alpha_1 b_{n-1} & = 0, \\
\ldots & & \\
\beta_2 & + \alpha_m b_3 + \alpha_{m-1} b_4 + \ldots + \alpha_2 b_{m+1} + \alpha_1 b_{m+2} & = 0, \\
\beta_1 & + \alpha_m b_2 + \alpha_{m-1} b_3 + \ldots + \alpha_1 b_{m+1} & = 0, \\
\beta_0 & + \alpha_m b_1 + \alpha_{m-1} b_2 + \ldots + \alpha_1 b_m & = a_m, \\
 & \alpha_m b_0 + \alpha_{m-1} b_1 + \ldots + \alpha_1 b_{m-1} & = a_{m-1}, \\
 & \alpha_{m-1} b_0 + \ldots + \alpha_1 b_{m-2} & = a_{m-2}, \\
\ldots & & \\
 & \alpha_2 b_0 + \alpha_1 b_1 & = a_1, \\
 & \alpha_1 b_0 & = a_0.
\end{array}
$$

Let us write a short form, which is a system of recurrent equations, with the help of which the coefficients can be calculated sequentially:

$$
\alpha_1 = \frac{a_0}{b_0}, \quad \alpha_k = \frac{1}{b_0}\left[a_{k-1} - \sum_{s=1}^{k-1} \alpha_s b_{k-s}\right], \quad k = \overline{2, m},
$$

$$
\beta_0 = a_m - \sum_{s=1}^{m} \alpha_s b_{m-s+1},
$$

$$
\beta_k = -\sum_{s=1}^{m} \alpha_s b_{m+k-s+1}, \quad k = \overline{1, n-m},
$$

$$
\beta_k = -\sum_{s=1}^{n-k} \alpha_{-n+m+k+s} b_{n-s+1}, \quad k = \overline{n-m+1, n-1}.
$$

There is another way to converse the transfer function. Let the transfer function have the form

$$W^{(r)}(p) = \frac{a_0^{(r)} p^{n-r} + a_1^{(r)} p^{n-r-1} + \ldots + a_{n-r-1}^{(r)} p + a_{n-r}^{(r)}}{b_0 p^n + b_1 p^{n-1} + \ldots + b_{n-1} p + b_n}, \tag{5}$$

$1 \leqslant r \leqslant n$. The superscript (r) indicates the function number and the degree of the numerator polynomial. The degree of the numerator polynomial $W^{(r+1)}(p)$ is less than the degree of the numerator polynomial $W^{(r)}(p)$, since $n-(r+1) < n-r$. Let us denote $B(p) = b_0 p^n + b_1 p^{n-1} + \ldots + b_{n-1} p + b_n$ and perform a series of conversions of the function $W^{(r)}(p)$, consisting of successively reducing the degree of the numerator polynomial to zero.

$$\begin{aligned} W^{(r)}(p) &= \frac{a_0^{(r)} p^n + a_1^{(r)} p^{n-1} + \ldots + a_{n-r}^{(r)} p^r}{p^r B(p)} \\ &= \frac{1}{p^r B(p)} \left[\frac{a_0^{(r)}}{b_0} B(p) - \frac{a_0^{(r)}}{b_0} \left(b_1 p^{n-1} + \ldots + b_{n-1} p + b_n \right) \right. \\ &\qquad \left. + \left(a_1^{(r)} p^{n-1} + a_2^{(r)} p^{n-2} + \ldots + a_{n-r}^{(r)} p^r \right) \right] \\ &= \frac{a_0^{(r)}}{b_0} \frac{1}{p^r} + \frac{1}{B(p)} \left[\left(a_1^{(r)} - \frac{a_0^{(r)}}{b_0} b_1 \right) p^{n-r-1} + \ldots \right. \\ &\qquad \left. + \left(a_{n-r-1}^{(r)} - \frac{a_0^{(r)}}{b_0} b_{n-r-1} \right) p + \left(a_{n-r}^{(r)} - \frac{a_0^{(r)}}{b_0} b_{n-r} \right) \right] \\ &\qquad - \frac{a_0^{(r)}}{b_0} \frac{1}{B(p)} \left[\frac{b_{n-r+1}}{p} + \frac{b_{n-r+2}}{p^2} + \ldots + \frac{b_n}{p^r} \right]. \end{aligned}$$

Let's determine the coefficients $a_\alpha^{(r+1)} = a_{1+\alpha}^{(r)} - (a_0^{(r)}/b_0) b_{1+\alpha}$, $\alpha = \overline{0, n-r-1}$, for the new function $W^{(r+1)}$. Then

$$W^{(r)}(p) = \frac{a_0^{(r)}}{b_0} \frac{1}{p^r} + W^{(r+1)}(p) - \frac{a_0^{(r)}}{b_0} \frac{1}{B(p)} \sum_{k=1}^{r} \frac{b_{n-r+k}}{p^k}, \tag{6}$$

$$\text{and} \quad W^{(r+1)}(p) = W^{(r)}(p) - \frac{a_0^{(r)}}{b_0} \frac{1}{p^r} + \frac{a_0^{(r)}}{b_0} \frac{1}{B(p)} \sum_{k=1}^{r} \frac{b_{n-r+k}}{p^k}.$$

The maximum number of steps is $n-r$. The function $W^{(r)}(p)$ is defined at the rth step, it is necessary to find $W^{(r+1)}(p)$, $W^{(r+2)}(p)$, ... The numerator of the last function $W^{(r+s)}(p)$ is a zeroth order polynomial, and then the calculation will be completed. Each next found function $W^{(r+k+1)}(p)$ must be substituted into the current function $W^{(r+k)}(p)$.

4. CREATING A RANDOM DISTURBANCE GENERATOR

Let's consider the transfer function (4), which is the sum of integrating links with their own gain factors [10]. For link $1/p^{n-m}$ the corresponding equation will be $x_1 = u/p^{n-m}$, or $x_1^{(n-m)} = u$. The link $1/p^{n-m+1}$ will give the equation $x_2 = u/p^{n-m+1} = u/(p^{n-m}p) = u/p^{n-m} \times 1/p = x_1 \times 1/p$, or $x_2' = x_1$. This is how differential equations for the first m outputs are successively found. In the same way, using the denominator $Q_n(p)$, we obtain the output equation x_{m+1}, which then needs to be integrated another $n-1$ times using the terms in brackets from (4). The last step will be the summation of all outputs with the corresponding coefficients α, β. Let us write down the system of differential equations and the output equation corresponding to the transfer function (4) and (3):

$$
\begin{gathered}
x_1^{(n-m)} = u, \quad x_2' = x_1, \quad x_3' = x_2, \quad \ldots \quad x_m' = x_{m-1}, \\
b_0 x_{m+1}^{(n)} + b_1 x_{m+1}^{(n-1)} + \ldots + b_{n-1} x_{m+1}' + b_n x_{m+1} = u, \\
x_{m+2}' = x_{m+1}, \quad \ldots \quad x_{m+n}' = x_{m+n-1}, \\
x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots + \alpha_m x_m + \beta_0 x_{m+1} + \beta_1 x_{m+2} + \ldots + \beta_{n-1} x_{m+n}.
\end{gathered}
\tag{7}
$$

The order of the resulting system is $N \leqslant 3n - 2$.

The system (3) can be easily written as a system of first-order equations (7) and, assuming that $u(t)$ is standard white noise, transformed into a system of Ito equations Nth order.

The transformation does not change the transfer function $W(p)$ for the output $x = \alpha_1 x_1 + \alpha_2 x_2 + \ldots$ (see (7)). Nevertheless, the transfer functions of the outputs x_1, x_2, $\ldots$ have a certain number of zero poles. Thus, $W_k(p)$ for x_k, $k = \overline{1, m}$ has the form $W_k(p) = 1/p^{n-m+k-1}$. In practice, this will lead to instability in process modeling (due to calculation errors). However, this situation can be corrected by making the replacement $p = (q - \Delta)/\lambda$ $(q = \lambda p + \Delta)$, $\Delta/\lambda > 0$ in the original transfer function $W(p)$. Then

$$
W^*(q) = \frac{P_m^*(q)}{Q_n^*(q)} = \frac{P_m\left(\frac{q-\Delta}{\lambda}\right)}{Q_n\left(\frac{q-\Delta}{\lambda}\right)} = \frac{\alpha_1^*}{q^{n-m}} + \frac{\alpha_2^*}{q^{n-m+1}} + \ldots
$$

$$
+ \frac{\alpha_m^*}{q^{n-1}} + \frac{1}{Q_n^*(q)} \left(\frac{\beta_{n-1}^*}{q^{n-1}} + \frac{\beta_{n-2}^*}{q^{n-2}} + \ldots + \frac{\beta_1^*}{q} + \beta_0^* \right).
$$

By inverse transformation $q = \lambda p + \Delta$ we get

$$
W(p) = \frac{P_m(p)}{Q_n(p)} = \frac{\alpha_1^*}{(\lambda p + \Delta)^{n-m}} + \frac{\alpha_2^*}{(\lambda p + \Delta)^{n-m+1}} + \ldots.
$$

The transfer functions of the output components x_1, x_2, $\ldots$ of such an expansion will have poles in the left half-plane.

5. EXAMPLE

Let the transfer function be given

$$
W(p) = \frac{p^2 + 2p + 1}{p^3 + 3p^2 + 2p + 2}.
$$

It is required to write it in the form of a sum of fractions with zero-order polynomials in the numerators.

As a result of the transformation we get

$$
W^*(q) = W(q-1) = \frac{(q-1)^2 + 2(q-1) + 1}{(q-1)^3 + 3(q-1)^2 + 2(q-1) + 2} = \frac{q^2}{q^3 - q + 2}.
$$

Here $n = 3$, $m = 2$, $a_0^* = 1$, $a_1^* = 0$, $a_2^* = 0$, $b_0^* = 1$, $b_1^* = 0$, $b_2^* = -1$, $b_3^* = 2$, $\alpha_1^* = \frac{a_0^*}{b_0^*} = 1$, $\alpha_2^* = \frac{1}{b_0^*}[a_1^* - \alpha_1^* b_1^*] = 0$, $\beta_0^* = a_2^* - [\alpha_1^* b_2^* + \alpha_2^* b_1^*] = 1$, $\beta_1^* = -[\alpha_1^* b_3^* + \alpha_2^* b_2^*] = -2$, $\beta_2^* = -[\alpha_2^* b_3^*] = 0$.

$$
W^*(q) = \frac{1}{q} + \left(1 - \frac{2}{q}\right) \frac{1}{q^3 - q + 2},
$$

$$
W(p) = \frac{1}{p+1} + \left(1 - \frac{2}{p+1}\right) \frac{1}{p^3 + 3p^2 + 2p + 2}.
$$

Let's write the solution in the second way

$$W^{(1)}(q) = \frac{q^2}{q^3 - q + 2} = \frac{q^2}{B(q)}, \quad B(q) = q^3 - q + 2,$$
$$W^{(2)}(q) = \frac{q^2}{B(q)} - \frac{1}{q} + \frac{1}{B(q)} \times \frac{2}{q} = \frac{q^3 - B(q) + 2}{qB(q)} = \frac{1}{B(q)}.$$

We use (6)

$$W^{(1)}(q) = \frac{1}{q} + \frac{1}{B(q)} - \frac{1}{B(q)}\frac{2}{q} = \frac{1}{q} + \left(1 - \frac{2}{q}\right)\frac{1}{B(q)} = W^*(q).$$

The results are the same.

Let us write down the derivation of the Ito equations for $W(p)$ in accordance with (7). The first term gives the equation $u = (p+1)x_1$. The second equation will be $u = (p^3 + 3p^2 + 2p + 2)x_2 = ((p+1)^3 - (p+1) + 2)x_2$. Let us denote $(p+1)x_2 = x_4$, $(p+1)x_4 = x_5$, then $u = (p+1)x_5 - x_4 + 2x_2$. The third equation would be $u = (p+1)(p^3 + 3p^2 + 2p + 2)x_3$, or $(p+1)x_3 = x_2$.

The required system of Ito equations and the output equation have the form

$$dx_1 + x_1 dt = dw, \quad dx_2 + (x_2 - x_4)dt = 0,$$
$$dx_3 + (x_3 - x_2)dt = 0, \quad dx_4 + (x_4 - x_5)dt = 0,$$
$$dx_5 + (x_5 - x_4 + 2x_2)dt = dw,$$
$$x = x_1 + x_2 - 2x_3.$$

Based on the transfer function, a linear system of Ito differential equations was obtained that does not contain derivatives of the input signal. Of course, the choice of a new variable was made so that it would be easy to isolate the cube of the sum in the denominator of the transfer function, and then obtain first-order linear equations.

6. DRYDEN WIND TURBULENCE MODEL

The US Department of Defense uses the Dryden gust model in some aircraft design and simulation applications. This mathematical model considers the speed components of continuous gusts of wind as random processes [5, 11]. The MATLAB documentation provides an implementation of the transfer function for wind gusts [12]. Twelve transfer functions are defined for gust models in the longitudinal, horizontal and vertical directions. However, only three types of different functions can be distinguished, differing from the model functions only by constant coefficients A, B, C, D (see [12]):

$$G_1(p) = A\frac{1}{1 + Cp}, \quad G_2(p) = A\frac{1 + Bp}{(1 + Cp)^2}, \quad G_3(p) = \frac{Ap}{1 + Cp} \times \frac{1 + Bp}{(1 + Dp)^2}.$$

The first type of function $G_1(p)$ is a simple integrator and does not require any transformation. The required system of Ito equations for $G_1(p)$ has the form

$$dx + \frac{1}{C}x dt = \frac{A}{C}dw.$$

Let's look at the second one. It is required to obtain a system of Ito equations for the transfer function $G_2(p)$. Then

$$G_2^*(q) = G_2\left(\frac{q-1}{C}\right) = \frac{A}{C}\frac{Bq + C - B}{q^2} = \frac{A}{C}\left(\frac{B}{q} + \frac{C - B}{q^2}\right).$$
$$G_2(p) = \frac{A}{C}\left(\frac{B}{1 + Cp} + \frac{C - B}{(1 + Cp)^2}\right).$$

The desired system of Ito equations and the output equation for $G_2(p)$ have the form

$$dx_1 + \frac{1}{C}x_1 dt = \frac{1}{C}dw, \quad dx_2 + \frac{1}{C}(x_2 - x_1)dt = 0,$$
$$x = \frac{A}{C}\Big(Bx_1 + (C-B)x_2\Big).$$

If we make another change of variables $q = p + 1$, we will get a rather cumbersome system of 5th order equations. We invite readers to see this for themselves.

Let's consider the third function $G_3(p)$ and replace the variable: $p = (q-1)/D$. Then

$$G_3^*(q) = G_3\left(\frac{q-1}{D}\right) = A\frac{Bq^2 + (D-2B)q + B - D}{DCq^3 + (D^2 - DC)q^2}.$$

Let's represent the last expression using (4):

$$G_3^*(q) = A\left[\frac{\alpha_1}{q} + \frac{\alpha_2}{q^2} + \frac{1}{b_0 q^3 + b_1 q^2}\left(\frac{\beta_2}{q^2} + \frac{\beta_1}{q} + \beta_0\right)\right],$$

then $b_0 = DC$, $b_1 = D^2 - DC$, $\alpha_1 = B/(DC)$, $\alpha_2 = \left[D - 2B - B(D^2 - DC)/(DC)\right]/(DC)$, $\beta_0 = B - D - (D^2 - DC)\left[D - 2B - B(D^2 - DC)/(DC)\right]/(DC)$, $\beta_1 = 0$, $\beta_2 = 0$.

Let's do the reverse change of variables and get the function

$$G_3(p) = A\left[\frac{\alpha_1}{1 + Dp} + \frac{\alpha_2}{(1+Dp)^2} + \frac{\beta_0}{b_0(1+Dp)^3 + b_1(1+Dp)^2}\right].$$

Let us write down the derivation of the Ito equations for $G_3(p)$ in more detail. The first and second terms give $x_1 = \frac{u}{1+Dp}$, $x_2 = \frac{u}{(1+Dp)^2} = \frac{u}{1+Dp} \times \frac{1}{1+Dp} = \frac{x_1}{1+Dp}$. Let's consider the third term: $x_3 = \frac{u}{b_0(1+Dp)^3 + b_1(1+Dp)^2} = \frac{u}{(1+Dp)^2} \times \frac{1}{b_0(1+Dp)+b_1} = \frac{x_2}{b_0(1+Dp)+b_1}$. Then the desired system of Ito equations and the output equation for $G_3(p)$ have the form

$$dx_1 + \frac{1}{D}x_1 dt = \frac{1}{D}dw, \quad dx_2 + \frac{1}{D}(x_2 - x_1)dt = 0,$$
$$dx_3 - \frac{1}{Db_0}x_2 dt + \frac{b_0 + b_1}{Db_0}x_3 dt = 0,$$
$$x = A\big[\alpha_1 x_1 + \alpha_2 x_2 + \beta_0 x_3\big].$$

Dryden's wind turbulence model is not the only one. For example, the von Karman model [13] has other transfer functions such as

$$G(p) = A\frac{1 + Bp}{1 + Cp + Dp^2}.$$

The corresponding system of Ito equations for this function will contain 6 variables. We do not present the transformed function here because it turned out to be too cumbersome. Perhaps a not very successful variable replacement was chosen. Therefore, the researchers themselves, depending on the coefficients C and D of the denominator polynomial, must choose a manner for replacing the variable.

The discussion about the choice of turbulence model continues [14]. It can be seen that the number of variables in the Ito equation for the Dryden model is no more than three, and in the von Karman model no less than six. Accordingly, the computational complexity of the wind gust modeling algorithm increases.

7. CONCLUSION

The proposed method for conversing the transfer function allows us to bring it to such a form that the system of differential equations equivalent to the differential equation (3) does not contain derivatives of the input signal $u(t)$. Assuming that $u(t)$ is white noise, the system can easily be transformed into a system of Ito equations.

The results can be used not only for stochastic differential equations, but also for ordinary differential equations with constant coefficients of the form (3) with scalar input and output signals [3, Section 1.3.4].

In the presented method it is impossible to influence the number of variables, but in the method [8] it is possible to influence the number of output variables for the output signal $y = cx$ (see (1)) and, accordingly, the type shaping filter shown in Fig. 3. Therefore, in the approach discussed above, the structure of the output signal becomes known only as a result of solving the problem. And in the method described in [8], the type of the output signal is known in advance. But the proposed solution, unlike [8], does not require matrix inversion, but uses only recursive arithmetic operations to find the coefficients of polynomials.

The results of calculations and numerical modeling of dynamic processes for the systems considered in the article are not specifically presented here. In the opinion of the authors, a fairly complete study with various modeling results was carried out in [14].

FUNDING

The work was carried out in part with the financial support of the Russian Foundation for Basic Research (project no. 20-08-00400).

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This paper was recommended for publication by A.V. Nazin, a member of the Editorial Board