Volume 85, Number 5,
May 2024

ISSN 0005-1179
CODEN: AURCAT

AUTOMATION AND REMOTE CONTROL

Automation and Remote Control

Vol. 85, No. 5, May 2024

http://ait.mtas.ru

Automation and Remote Control

ISSN 0005-1179

SCOPE

Automation and Remote Control is one of the first journals on control theory. The scope of the journal is control theory problems and applications. The journal publishes reviews, original articles, and short communications (deterministic, stochastic, adaptive, and robust formulations) and its applications (computer control, components and instruments, process control, social and economy control, etc.).

Automation and Remote Control is abstracted and/or indexed in *ACM Digital Library, BFI List, CLOCKSS, CNKI, CNPIEC Current Contents/ Engineering, Computing and Technology, DBLP, Dimensions, EBSCO Academic Search, EBSCO Advanced Placement Source, EBSCO Applied Science & Technology Source, EBSCO Computer Science Index, EBSCO Computers & Applied Sciences Complete, EBSCO Discovery Service, EBSCO Engineering Source, EBSCO STM Source, EI Compendex, Google Scholar, INSPEC, Japanese Science and Technology Agency (JST), Journal Citation Reports/Science Edition, Mathematical Reviews, Naver, OCLC WorldCat Discovery Service, Portico, ProQuest Advanced Technologies & Aerospace Database, ProQuest-ExLibris Primo, ProQuest-ExLibris Summon, SCImago, SCOPUS, Science Citation Index, Science Citation Index Expanded (Sci-Search), TD Net Discovery Service, UGC-CARE List (India), WTI Frankfurt eG, zbMATH.*

Journal website: http://ait.mtas.ru

0005-1179/24. *Automation and Remote Control* (ISSN: 0005-1179 print version, ISSN: 1608-3032 electronic version) is published monthly by Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, 65 Profsoyuznaya street, Moscow 117997, Russia.

Volume 85 (12 issues) is published in 2024.

Publisher: Trapeznikov Institute of Control Sciences, Russian Academy of Sciences.

65 Profsoyuznaya street, Moscow 117997, Russia; e-mail: redacsia@ipu.rssi.ru; http://ait.mtas.ru, http://ait-arc.ru

Contents

Automation and Remote Control
Vol. 85, No. 5, 2024

Topical Issue

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, p. 451.

TOPICAL ISSUE

DOI: 10.31857/S0005117924050019

On February 3, 2023, an outstanding mathematician and a wonderful person, Boris Teodorovich Polyak passed away. For a long time he headed Laboratory 7 of the Institute of Control Sciences, RAS, and was a generator of bright scientific ideas for decades, so we decided to arrange a special issue of the journal Automation and Remote Control, dedicated to his memory. The articles in this collection are mostly authored by the members of our lab, and the publication of this issue is not devoted to the sad anniversary of the death of Boris Teodorovich, but to his birthday on May 4, 1935.

The range of scientific interests of Prof. Polyak is striking in its diversity, as can be seen from the list of research papers which he authored and which we present in our issue. It is also worth mentioning that, in addition to the breadth of interests of Boris Teodorovich, this bibliography list testifies to a large number of his co-authors, — he has always generously shared his ideas with both students and younger followers and older colleagues.

Of course, within the framework of one issue, it is impossible to cover all areas of research interest of Boris Teodorovich, but we tried to collect articles on the topics that interested him most in recent years. These include the optimization theory, which he loved since his youth, and the methods of the classical theory of automatic control, robustness and rejection of exogenous disturbances, linear matrix inequalities, superstability, applied problems of energy systems research, and the peak effect. Due to the limitations on the volume of one issue, some of the submitted articles were transferred to the 6th issue of the journal.

Next year we plan to publish another special issue dedicated to the 90th anniversary of Prof. Polyak; the circle of authors is expected to be much wider.

On behalf of all members of lab. 7 of the Instutute of Control Science,
Editior of the issue,
P.S. Shcherbakov

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, pp. 452–478.

TOPICAL ISSUE

Boris Polyak: The List of Research Works[1]

DOI: 10.31857/S0005117924050023

MONOGRAPHS

1.1. Polyak, B.T., *Vvedenie v optimizatsiyu* (Introduction to Optimization), Moscow: Nauka, 1983; 2nd ed., revised and supplemented, Moscow: URSS, 2014, and Moscow: Lenand, 2019; 3rd ed., Moscow: URSS, 2023.

1.2. Polyak, B.T., *Introduction to Optimization*, New York: Optimization Software, 1987. (An extended and refined translation of the book [1.1], 1983.)

1.3. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.

1.4. Granichin, O.N. and Polyak, B.T., *Randomizirovannye algoritmy otsenivaniya i optimizatsii pri pochti proizvol'nykh pomekhakh* (Randomized Algorithms for Estimation and Optimization under Almost Arbitrary Disturbances), Moscow: Nauka, 2003.

1.5. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., *Upravlenie lineinymi sistemami pri vneshnikh vozmushcheniyakh: tekhnika lineinykh matrichnykh neravenstv* (Control of Linear Systems under Exogenous Disturbances: The Technique of Linear Matrix Inequalities), Moscow: URSS, 2014.

1.6. Polyak, B.T., Khlebnikov, M.V., and Rapoport, L.B., *Matematicheskaya teoriya avtomaticheskogo upravleniya* (Mathematical Theory of Automatic Control), Moscow: URSS, 2019.

DISSERTATIONS

2.1. Polyak, B.T., Gradient Methods for Minimizing Functionals and Solving Equations and Inequalities, *Cand. Sci. (Phys.–Math.) Dissertation*, Moscow: Moscow State University, 1963.

2.2. Polyak, B.T., Optimization Methods in the Presence of Disturbances, *Doctoral (Phys.–Math.) Dissertation*, Moscow: Institute of Control Sciences, 1978.

ARTICLES IN PEER-REVIEWED JOURNALS

3.1. Polyak, B.T., A Method for Determining the Mutual Independence of the Dimensions of Quantities, *Nauchn. Dokl. Vyssh. Shkol. Metallurg.*, 1958, no. 2, pp. 118–120.

3.2. Kagan, A.S., Polyak, B.T., and Shil'shtein, S.Sh., The Influence of Surface Topography on the Intensity of X-Ray Diffraction Maxima, *Zhurn. Tekh. Fiz.*, 1959, vol. 29, no. 2, pp. 1142–1145.

3.3. Ivantsov, G.P., Lyubov, B.Y., Polyak, B.T., and Roitburd, A.L., Calculation of the Crystallization Rate of a Metallic Ingot with Various Heat Flows through Its Surface, *Journal of Engineering Physics and Thermophysics*, 1960, vol. 3, no. 3, pp. 41–48.

3.4. Ivantsov, G.P. and Polyak, B.T., The Cooling Mode of a Continuously Casted Flat Ingot with Minimum Thermal Stresses in Its Crust, *Kristallizats. Metall.*, 1960, pp. 139–149.

3.5. Polyak, B., Gradient Methods for the Minimisation of Functionals, *USSR Computational Mathematics and Mathematical Physics*, 1963, vol. 3, no. 4, pp. 864–878.

[1] Compiled by P.S. Shcherbakov, A.A. Tremba, and Ya.I. Kvinto. The texts of almost all *journal articles* (and other publications) can be found at https://sites.google.com/site/lab7polyak/ © A.A. Tremba.

3.6. Polyak, B., Some Methods of Speeding Up the Convergence of Iteration Methods, *USSR Computational Mathematics and Mathematical Physics*, 1964, vol. 4, no. 5, pp. 1–17.

3.7. Polyak, B., Gradient Methods for Solving Equations and Inequalities, *USSR Computational Mathematics and Mathematical Physics*, 1964, vol. 4, no. 6, pp. 17–32.

3.8. Poljak, B.T., Existence Theorems and Convergence of Minimizing Sequences in Extremal Problems with Restrictions, *Soviet Math. Dokl.*, 1966, vol. 7, pp. 72–75.

3.9. Levitin, E.S. and Poljak, B.T., Convergence of Minimizing Sequences in Conditional Extremal Problems, *Soviet Math. Dokl.*, 1966, vol. 5, pp. 764–767.

3.10. Levitin, E.S. and Polyak, B.T., Constrained Minimization Methods, *USSR Computational Mathematics and Mathematical Physics*, 1966, vol. 6, no. 5, pp. 1–50.

3.11. Poljak, B.T., A General Method of Solving Extremum Problems, *Soviet Math. Dokl.*, 1967, vol. 8, no. 6, pp. 593–597.

3.12. Polyak, B.T., A Minimization Method for Functions of Several Variables, *Economics and Mathematical Methods*, 1967, vol. 3, no. 6, pp. 881–902.

3.13. Gurin, L.G., Polyak, B.T., and Raik, E.V., The Method of Projections for Finding the Common Point of Convex Sets, *USSR Computational Mathematics and Mathematical Physics*, 1967, vol. 7, no. 6, pp. 1–24.

3.14. Polyak, B.T., Toward a Theory of Nonlinear Optimal Control Problems, *Vest. Mosk. Univ. Ser. 1. Mat. Mekh.*, 1968, no. 2, pp. 30–40.

3.15. Poljak, B.T., Semicontinuity of Integral Functionals and Existence Theorems on Extremal Problems, *Mathematics of the USSR – Sbornik*, 1969, vol. 7, no. 1, pp. 59–77.

3.16. Poljak, B.T., Corrections to the Paper "Semicontinuity of Integral Functionals and Existence Theorems on Extremal Problems," Mathematics of the USSR – Sbornik, 1969, vol. 7, no. 1, pp. 59–77, *Mathematics of the USSR – Sbornik*, 1969, vol. 9, no. 4, p. 570.

3.17. Polyak, B.T., Minimization of Unsmooth Functionals, *USSR Computational Mathematics and Mathematical Physics*, 1969, vol. 9, no. 3, pp. 14–29.

3.18. Polyak, B.T., The Conjugate Gradient Method in Extremal Problems, *USSR Computational Mathematics and Mathematical Physics*, 1969, vol. 9, no. 4, pp. 94–112.

3.19. Polyak, B.T., Iterative Methods Using Lagrange Multipliers for Solving Extremal Problems with Constraints of the Equation Type, *USSR Computational Mathematics and Mathematical Physics*, 1970, vol. 10, no. 5, pp. 42–52.

3.20. Polyak, B.T., The Convergence Rate of the Penalty Function Method, *USSR Computational Mathematics and Mathematical Physics*, 1971, vol. 11, no. 1, pp. 1–12.

3.21. Polyak, B.T., Convergence of Methods of Feasible Directions in Extremal Problems, *USSR Computational Mathematics and Mathematical Physics*, 1971, vol. 11, no. 4, pp. 53–70.

3.22. Polyak, B.T. and Tretyakov, N.V., On a Linear Programming Iterative Method and Its Economic Interpretation, *Matekon*, 1974, vol. 10, no. 3, pp. 81–100.

3.23. Polyak, B.T. and Tret'yakov, N.V., The Method of Penalty Estimates for Conditional Extremum Problems, *USSR Computational Mathematics and Mathematical Physics*, 1973, vol. 13, no. 1, pp. 42–58.

3.24. Polyak, B.T. and Tsypkin, Ya.Z., Pseudogradient Adaptation and Training Algorithms, *Autom. Remote Control*, 1973, vol. 34, no. 3, pp. 377–397.

3.25. Tsypkin, J.Z. and Poliak, B.T., Attainable Accuracy of the Adaptation Algorithms, *Sov. Phys. Dokl.*, 1974, vol. 19, pp. 562–563.

3.26. Bakušinskiĭ, A.B. and Poljak, B.T., On the Solution of Variational Inequalities, *Sov. Math. Dokl.*, 1974, vol. 15, no. 6, pp. 1705–1710.

3.27. Polyak, B.T., Convergence and Convergence Rate of Iterative Stochastic Algorithms. I. General Case, *Autom. Remote Control*, 1976, vol. 37, no. 12, pp. 1858–1868.

3.28. Polyak, B.T., Convergence and Convergence Rate of Iterative Stochastic Algorithms. II. The Linear Case, *Autom. Remote Control*, 1977, vol. 38, no. 4, pp. 537–542.

3.29. Polyak, B.T., Comparison of the Convergence Rates for Single-Step and Multi-Step Optimization Algorithms in the Presence of Noise, *Eng. Cybern.*, 1977, vol. 15, no. 1, pp. 6–10.

3.30. Poliak, B.T., On the Comparison of Gradient Method and Random Search Method, *Autom. Control. Comput. Sci.*, 1977, vol. 11, no. 3, pp. 59–62.

3.31. Poljak, B.T., Nonlinear Programming Methods in the Presence of Noise, *Mathematical Programming. Series B*, 1978, vol. 14(1), pp. 87–97.

3.32. Belov, E.N., Polyak, B.T., and Skokov, V.A., A Set of Optimization Programs, *Economics and Mathematical Methods*, 1978, vol. 14, no. 4, pp. 792–796.

3.33. Polyak, B.T., Minimization of Complex Regression Functions, *Kibernetika*, 1978, no. 4, pp. 148–149.

3.34. Polyak, B.T. and Skokov, V.A., Solving Problems on the Minimum Sum of Squares, *Economics and Mathematical Methods*, 1978, vol. 14, no. 6, pp. 1173–1180.

3.35. Polyak, B.T., Methods for Solving Constrained Extremum Problems in the Presence of Random Noise, *USSR Computational Mathematics and Mathematical Physics*, 1979, vol. 19, no. 1, pp. 72–81.

3.36. Polyak, B.T. and Tsypkin, Ya.Z., Adaptive Estimation Algorithms (Convergence, Optimality, Stability), *Autom. Remote Control*, 1979, vol. 40, no. 3, pp. 378–389.

3.37. Poljak, B.T. and Tsypkin, J.Z., Robust Identification, *Automatica*, 1980, vol. 16(1), pp. 53–63.

3.38. Poliak, B.T. and Tsypkin, J.Z., Optimal Pseudogradient Algorithms for Stochastic Optimization, *Sov. Phys. Dokl*, 1980, vol. 25, no. 2, pp. 85–87.

3.39. Poliak, B.T. and Tsypkin, J.Z., Optimal Pseudogradient Adaptation Algorithms, *Autom. Remote Control*, 1981, vol. 41, no. 8, pp. 1101–1110.

3.40. Poliak, B.T. and Tsypkin, J.Z., Robust Pseudogradient Adaptation Algorithms, *Autom. Remote Control*, 1981, vol. 41, no. 10, pp. 1404–1409.

3.41. Poliak, B.T. and Tsypkin, J.Z., Optimal and Robust Estimation of Autoregression Coefficients, *Eng. Cybern.*, 1983, vol. 21, no. 1, pp. 100–109.

3.42. Polyak, B.T. and Tsypkin, Ya.Z., Optimal Algorithms of Criterial Optimization Under Conditions of Uncertainty, *Soviet Physics – Doklady*, 1983, vol. 28, no. 11, pp. 919–920.

3.43. Nemirovskii, A.S., Polyak, B.T., and Tsybakov, A.B., Estimators of Maximum Likelihood Type for Nonparametric Regression, *Soviet Math. Dokl.*, 1983, vol. 28, no. 3, pp. 788–792.

3.44. Polyak, B.T., Nemirovskii, A.S., and Tsybakov, A.B., The Maximum Likelihood Method in Nonparametric Regression, *Theor. Probab. Appl.*, 1984, vol. 28, no. 4, pp. 830–831.

3.45. Nemirovskiy, A.S. and Polyak, B.T., Iterative Methods for Solving Linear Ill-Posed Problems under Precise Information. I, *Engineering Cybernetics*, 1984, vol. 22, no. 3, pp. 1–11.

3.46. Nemirovskiy, A.S. and Polyak, B.T., Iterative Methods for Solving Linear Ill-Posed Problems under Precise Information. II, *Engineering Cybernetics*, 1984, vol. 22, no. 4, pp. 50–56.

3.47. Poliak, B.T. and Tsypkin, Ya.Z., Criterion Algorithms of Stochastic Optimization, *Autom. Remote Control*, 1984, vol. 45, no. 6, pp. 766–774.

3.48. Nemirovskii, A.S., Polyak, B.T., and Tsybakov, A.B., Signal Processing by the Nonparametric Maximum-Likelihood Method, *Problems of Information Transmission*, 1984, vol. 20(3), pp. 177–192.

3.49. Verulava, Yu.Sh., Gorgadze, Z.N., and Polyak, B.T., Algorithms for Estimation of Autoregression Coefficients, *Autom. Remote Control*, 1984, vol. 45, no. 11, pp. 1428–1434.

3.50. Nemirovskii, A.S., Polyak, B.T., and Tsybakov, A.B., Rate of Convergence of Nonparametric Estimates of Maximum-Likelihood Type, *Problems of Information Transmission*, 1985, vol. 21, no. 4, pp. 258–272.

3.51. Nemirovskii, A.S., Polyak, B.T., and Tsypkin, Ya.Z., Optimum Algorithms of Stochastic Optimization with Multiplicative Error, *Sov. Phys. Dokl.*, 1985, vol. 30, no. 9, pp. 744–745.

3.52. Verulava, Yu.Sh. and Polyak, B.T., Selection of the Regression Model Order, *Autom. Remote Control*, 1988, vol. 49, no. 11, pp. 1482–1494.

3.53. Polyak, B.T. and Tsybakov, A.B., Optimal Projection Estimates for a Regression Function of Unknown Smoothness, *Soviet Math. Dokl.*, 1989, vol. 39, no. 1, pp. 73–77.

3.54. Nazin, A.V., Polyak, B.T., and Tsybakov, A.B., Passive Stochastic Approximation, *Autom. Remote Control*, 1989, vol. 50, no. 11, pp. 1563–1569.

3.55. Polyak, B.T., and Tsypkin, Ya.Z., Gradient Methods of Stochastic Optimization, *Izmereniya, Kontrol', Avtomatizatsiya*, 1989, no. 3(71), pp. 50–54.

3.56. Polyak, B.T., and Tsybakov, A.B., Asymptotic Optimality of the C_p-Test for the Orthogonal Series Estimation of Regression, *Theory of Probability & Its Applications*, 1991, vol. 35, no. 2, pp. 293–306.

3.57. Polyak, B.T., New Method of Stochastic Approximation Type, *Autom. Remote Control*, 1990, vol. 51, no. 7, pp. 937–946.

3.58. Polyak, B.T. and Tsypkin, Ya.Z., Frequency Criteria of Robust Stability and Aperiodicity of Linear Systems, *Autom. Remote Control*, 1990, vol. 51, no. 9, pp. 1192–1201.

3.59. Polyak, B.T. and Tsybakov, A.B., Optimal Order of Accuracy for Search Algorithms in Stochastic Optimization, *Problems of Information Transmission*, 1990, vol. 26(2), pp. 126–133.

3.60. Tsypkin, Y.Z. and Polyak, B.T., Frequency Criteria for Robust Modality of Linear Discrete Systems, *Sov. J. Autom. Inform. Sci.*, 1990, vol. 23, no. 4, pp. 1–7.

3.61. Tsypkin, Ya.Z. and Polyak, B.T., Frequency Domain Criteria for l^p-Robust Stability of Continuous Linear Systems, *IEEE Trans. Autom. Control*, 1991, vol. 36, no. 12, pp. 1464–1469.

3.62. Polyak, B.T. and Tsypkin, Ya.Z., Robust Stability of Discrete Linear Systems, *Sov. Phys. Dokl.*, 1991, vol. 36, no. 2, pp. 111–113.

3.63. Kiselev, O.N. and Polyak, B.T., Ellipsoidal Estimation with Respect to a Generalized Criterion, *Autom. Remote Control*, 1991, vol. 52, no. 9, pp. 1281–1292.

3.64. Polyak, B.T. and Tsypkin, Ya.Z., Robust Stability under Complex Perturbations of the Parameters, *Autom. Remote Control*, 1991, vol. 52, no. 8, pp. 1069–1077.

3.65. Petrov, N.P. and Polyak, B.T., Robust *D*-Partition, *Autom. Remote Control*, 1991, vol. 52, no. 11, pp. 1513–1523.

3.66. Polyak, B.T., Frequency Methods in the Theory of Robust Stability, *Autom. Remote Control*, 1992, vol. 53, no. 1, pp. 140–142.

3.67. Polyak, B.T. and Juditsky, A.B., Acceleration of Stochastic Approximation by Averaging, *SIAM J. Control Optim.*, 1992, vol. 30(4), pp. 838–855.

3.68. Nazin, A.V., Polyak, B.T., and Tsybakov, A.B., Optimal and Robust Kernel Algorithms for Passive Stochastic Approximation, *IEEE Trans. Inform. Theory*, 1992, vol. 38, no. 5, pp. 1577–1583.

3.69. Polyak, B.T. and Tsypkin, Ya.Z., Robust Nyquist Test, *Autom. Remote Control*, 1992, vol. 53, no. 7, pp. 972–977.

3.70. Tsypkin, Ya.Z. and Polyak, B.T., Optimal Recurrent Algorithms for Identification of Nonstationary Plants, *Comput. Elect. Engin.*, 1992, vol. 18(5), pp. 365–371.

3.71. Polyak, B.T. and Tsybakov, A.B., A Family of Asymptotically Optimal Methods for Choosing the Order of a Projective Regression Estimate, *Theory of Probability & Its Applications*, 1993, vol. 37, no. 3, pp. 471–481.

3.72. Goldenshluger, A.V. and Polyak, B.T., Estimation of Regression Parameters with Arbitrary Noise, *Mathematical Methods of Statistics*, 1993, vol. 2(1), pp. 18–29.

3.73. Kiselyov, O.N. and Polyak, B.T., Robust Stability of a Chain of Simple Links, *Autom. Remote Control*, 1993, vol. 54, no. 12, pp. 1824–1834.

3.74. Tsypkin, Ya.Z. and Polyak, B.T., Robust Absolute Stability of Continuous Systems, *Int. J. Robust Nonlin. Control*, 1993, vol. 3, no. 3, pp. 231–239.

3.75. Polyak, B.T. and Tsypkin, Ya.Z., Robust Aperiodicity, *Phys. Dokl.*, 1994, vol. 39, no. 3, pp. 149–152.

3.76. Polyak, B.T., Scherbakov, P.S., and Shmulyian, S.B., Construction of Value Set for Robustness Analysis via Circular Arithmetic, *Int. J. Robust. Nonlin.*, 1994, vol. 4(3), pp. 371–385.

3.77. Polyak, B.T., Tsypkin, Ya.Z., Shi, Y.Q., Yen, K.K., and Chen, C.M., Comments on "Two Necessary Conditions for a Complex Polynomial to Be Strictly Hurwitz and Their Applications in Robust Stability Analysis" (with Reply), *IEEE Trans. Autom. Control*, 1994, vol. 39(5), pp. 1147–1148.

3.78. Tsypkin, Ya.Z., Polyak, B.T., Katbab, A., and Jury, E.I., On the Strict- and Wide-Sense Stability Robustness of Uncertain Systems: Application of a New Frequency Criterion, *Syst. Control Lett.*, 1994, vol. 22(5), pp. 377–383.

3.79. Polyak, B.T. and Shmulyian, S.B., Frequency Domain Criteria for Robust Stability of Bivariate Polynomials, *IEEE Trans. Circ. Syst. I: Fundamental Theory and Applications*, 1994, vol. 41(2), pp. 161–167.

3.80. Nemirovskii, A.S. and Polyak, B.T., Necessary Conditions for the Stability of Polynomials and Their Use, *Autom. Remote Control*, 1994, vol. 55, no. 11, pp. 1644–1649.

3.81. Kiselev, O.N. and Polyak, B.T., Robust Gain Margin for a Cascade of Uncertain Links, *Int. J. Syst. Sci.*, 1995, vol. 26(4), pp. 965–974.

3.82. Panchenko, O.B. and Polyak, B.T., Estimation of the Measure of Stable Polynomials in an Interval Family, *Autom. Remote Control*, 1995, vol. 56, no. 7, pp. 997–1003.

3.83. Tsypkin, Ya.Z. and Polyak, B.T., Robust Stability of a Class of Distributed-Parameter Systems, *Dokl. Math.*, 1995, vol. 51, no. 2, pp. 304–306.

3.84. Polyak, B.T. and Kogan, J., Necessary and Sufficient Conditions for Robust Stability of Linear Systems with Multiaffine Uncertainty Structure, *IEEE Trans. Autom. Control*, 1995, vol. 40, no. 7, pp. 1255–1260.

3.85. Kiselev, O.N. and Polyak, B.T., Gain Margin for a Cascade of Uncertain Components, *Autom. Remote Control*, 1995, vol. 56, no. 9, pp. 1278–1286.

3.86. Polyak, B.T. and Shcherbakov, P.S., Algorithms of Matrix Estimation, *Autom. Remote Control*, 1995, vol. 56, no. 11, pp. 1605–1619.

3.87. Tsypkin, Ya.Z. and Polyak, B.T., Frequency Domain Criteria for Robust Stability of a Family of Linear Difference Equations, *J. Diff. Equat. Appl.*, 1995, vol. 1(2), pp. 137–149.

3.88. Polyak, B.T. and Vishnyakov, A.N., Multiplying Disks: Robust Stability of a Cascade Connection, *Eur. J. Control*, 1996, vol. 2, no. 2, pp. 101–111.

3.89. Polyak, B.T. and Tsypkin, Ya.Z., Stability and Robust Stability of Uniform Systems, *Autom. Remote Control*, 1996, vol. 57, no. 11, pp. 1606–1617.

3.90. Polyak, B.T. and Shcherbakov, P.S., A Probabilistic Approach to Robust Stability of Time Delay Systems, *Autom. Remote Control*, 1996, vol. 57, no. 12, pp. 1770–1779.

3.91. Polyak, B.T. and Shcherbakov, P.S., Robust Stability of Systems with Uncertain Delays, *Mathematics Today*, 1996, vol. 32, no. 7–8, pp. 118–121.

3.92. Kiselev, O.N., Le Hung Lan, and Polyak, B.T., Frequency Responses under Parametric Uncertainty, *Autom. Remote Control*, 1997, vol. 58, no. 4, pp. 645–661.

3.93. Polyak, B.T. and Panchenko, O.B., Probabilistic Approach to Stability Problem for Interval Matrices, *Doklady Mathematics*, 1997, vol. 55(2), pp. 309–311.

3.94. Polyak, B.T. and Tsypkin, Ya.Z., Optimal and Robust Methods for Stochastic Optimization, *Nov. J. Math. Gam. Theor. Algebra*, 1997, vol. 6(2/3), pp. 163–176.

3.95. Tsypkin, Ya.Z. and Polyak, B.T., Stability of Linear Difference Equations with Unmodelled Higher Order Terms, *J. Diff. Equat. Appl.*, 1998, vol. 3(5–6), pp. 539–546. (Dedicated to Gerry Ladas on his sixtieth birthday.)

3.96. Polyak, B.T., Convexity of Quadratic Transformations and Its Use in Control and Optimization, *J. Optim. Theor. Appl.*, 1998, vol. 99(3), pp. 553–583.

3.97. Tsypkin, Ya.Z. and Polyak, B.T., High-Gain Robust Control, *Eur. J. Control*, 1999, vol. 5, no. 1, pp. 3–9.

3.98. Kiselev, O.N. and Polyak, B.T., Design of Low-Order Controllers by the H^∞ and Maximal-Robustness Performance Indices, *Autom. Remote Control*, 1999, vol. 60, no. 3, pp. 393–402.

3.99. Polyak, B.T. and Halpern, M.E., Robust Stability and Design of Linear Discrete-Time SISO Systems under l_1 Uncertainties, *IEEE Trans. Autom. Control*, 1999, vol. 44(11), pp. 2076–2080.

3.100. Polyak, B.T., Haddad, W.M., Chellaboina, V., and Kumar, R., Discussion on: "Multiobjective L_1/H_∞ Controller Design for Systems with Frequency and Time Domain Constraints," *Eur. J. Control*, 2000, vol. 6(2), pp. 184–185.

3.101. Polyak, B.T. and Shcherbakov, P.S., Random Spherical Uncertainty in Estimation and Robustness, *IEEE Trans. Autom. Control*, 2000, vol. 45(11), pp. 2145–2150.

3.102. Vishnyakov, A.N. and Polyak, B.T., Low-Order Controllers for Discrete Control Systems under Nonrandom Disturbances: A Synthesis Method, *Autom. Remote Control*, 2000, vol. 61, no. 9, pp. 1515–1521.

3.103. Polyak, B. and Halpern, M., Optimal Design for Discrete-Time Linear Systems via New Performance Index, *Int. J. Adaptive Control Signal Proc.*, 2001, vol. 15(2), pp. 129–152.

3.104. Polyak, B.T., Local Programming, *Computational Mathematics and Mathematical Physics*, 2001, vol. 41(9), pp. 1259–1266.

3.105. Polyak, B.T., Convexity of Nonlinear Image of a Small Ball with Applications to Optimization, *Set-Valued Analysis*, 2001, vol. 9(1–2), pp. 159–168.

3.106. Kiselev, O.N. and Polyak, B.T., Minimization of Overshoot in Linear Discrete-Time Systems via Low-Order Controllers, *Autom. Remote Control*, 2001, vol. 62, no. 4, pp. 597–606.

3.107. Calafiore, G. and Polyak, B.T., Stochastic Algorithms for Exact and Approximate Feasibility of Robust LMIs, *IEEE Trans. Automatic Control*, 2001, vol. 46(11), pp. 1755–1759.

3.108. Polyak, B.T. and Tempo, R., Probabilistic Robust Design with Linear Quadratic Regulators, *Syst. Control Lett.*, 2001, vol. 43(5), pp. 343–353.

3.109. Durieu, C., Walter, É., and Polyak, B., Multi-Input Multi-Output Ellipsoidal State Bounding, *J. Optim. Theor. Appl.*, 2001, vol. 111(2), pp. 273–303.

3.110. Polyak, B.T., History of Mathematical Programming in the USSR: Analyzing the Phenomenon, *Mathematical Programming. Series B*, 2002, vol. 91(3), pp. 401–416.

3.111. Halpern, M.E. and Polyak, B.T., Optimization-Based Design of Fixed-Order Controllers for Command Following, *Automatica*, 2002, vol. 38(9), pp. 1615–1619.

3.112. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. I. Analysis, *Autom. Remote Control*, 2002, vol. 63, no. 8, pp. 1239–1254.

3.113. Polyak, B.T. and Shcherbakov, P.S., Superstable Linear Control Systems. II. Design, *Autom. Remote Control*, 2002, vol. 63, no. 11, pp. 1745–1763.

3.114. Polyak, B.T., The Convexity Principle and Its Applications, *Bul. Brazil. Mat.Soc.*, 2003, vol. 34(1), pp. 59–75.

3.115. Dabbene, F., Gay, P., and Polyak, B.T., Recursive Algorithms for Inner Ellipsoidal Approximation of Convex Polytopes, *Automatica*, 2003, vol. 39(10), pp. 1773–1781.

3.116. Polyak, B.T., Convexity of the Reachable Set of Nonlinear Systems under L_2 Bounded Controls, *Dynamics of Continuous, Discrete and Impulsive Systems. Series A. Mathematical Analysis*, 2004, vol. 11, nos. 2–3, pp. 255–267.

3.117. Polyak, B.T., Nazin, S.A., Durieu, C., and Walter, E., Ellipsoidal Parameter or State Estimation under Model Uncertainty, *Automatica*, 2004, vol. 40(7), pp. 1171–1179.

3.118. Polyak, B.T. and Nazin, S.A., Interval Solutions for Interval Algebraic Equations, *Mat. Comput. Simulat.*, 2004, vol. 66, no. 2–3, pp. 207–217.

3.119. Polyak, B.T., Extended Superstability in Control Theory, *Autom. Remote Control*, 2004, vol. 65, no. 4, pp. 567–576.

3.120. Nazin, A.V., Nazin, S.A., and Polyak, B.T., On Convergence of External Ellipsoidal Approximations of the Reachability Domains of Discrete Dynamic Linear Systems, *Autom. Remote Control*, 2004, vol. 65, pp. 1210–1230.

3.121. Polyak, B.T., Newton–Kantorovich Method and Its Global Convergence, *J. Mat. Sci.*, 2006, vol. 133(4), pp. 1513–1523.

3.122. Nazin, S.A. and Polyak, B.T., Interval Parameter Estimation under Model Uncertainty, *Mathematical and Computer Modelling of Dynamical Systems: Methods, Tools and Applications in Engineering and Related Sciences*, 2005, vol. 11(2), pp. 225–237.

3.123. Polyak, B.T. and Shcherbakov, P.S., Hard Problems in Linear Control Theory: Possible Approaches to Solution, *Autom. Remote Control*, 2005, vol. 66, no. 5, pp. 681–718.

3.124. Polyak, B.T., Stabilizing Chaos with Predictive Control, *Autom. Remote Control*, 2005, vol. 66, no. 11, pp. 1791–1804.

3.125. Efremov, S.V. and Polyak, B.T., Using Predictive Control to Synchronize Chaotic Systems, *Autom. Remote Control*, 2005, vol. 66, no. 12, pp. 1905–1915.

3.126. Gryazina, E.N. and Polyak, B.T., Stability Regions in the Parameter Space: D-Decomposition Revisited, *Automatica*, 2006, vol. 42(1), pp. 13–26.

3.127. Polyak, B.T. and Nazin, S.A., Estimation of Parameters in Linear Multidimensional Systems under Interval Uncertainty, *J. Autom. Inform. Sci.*, 2006, vol. 38, no. 2, pp. 19–33.

3.128. Polyak, B.T., Newton–Kantorovich Method and Its Global Convergence, *J. Math. Sci.*, 2006, vol. 133(4), pp. 1513–1523.

3.129. Polyak, B.T., Controlling Chaos with Predictive Control, *Appl. Comput. Mat.*, 2006, vol. 5(1), pp. 66–78.

3.130. Polyak, B.T. and Shcherbakov, P.S., The D-Decomposition Technique for Linear Matrix Inequalities, *Autom. Remote Control*, 2006, vol. 67, no. 11, pp. 1847–1861.

3.131. Nesterov, Yu. and Polyak, B.T., Cubic Regularization of Newton Method and Its Global Performance, *Mathematical Programming. Series A*, 2006, vol. 108(1), pp. 177–205.

3.132. Gryazina, E.N., Polyak, B.T., and Tremba, A.A., Design of the Low-order Controllers by the H_∞ Criterion: A Parametric Approach, *Autom. Remote Control*, 2007, vol. 68, no. 3, pp. 456–466.

3.133. Dabbene, F., Polyak, B.T., and Tempo, R., On the Complete Instability of Interval Polynomials, *Syst. Control Lett.*, 2007, vol. 56(6), pp. 431–438.

3.134. Nazin, S.A., Polyak, B.T., and Topunov, M.V., Rejection of Bounded Exogenous Disturbances by the Method of Invariant Ellipsoids, *Autom. Remote Control*, 2007, vol. 68, no. 3, pp. 467–486.

3.135. Polyak, B.T. and Nazin, S.A., Invariant Ellipsoids Technique for Persistent Disturbance Rejection, *Int. J. Tomograph. Stat.*, 2007, vol. 5(W07), pp. 165–170.

3.136. Polyak, B.T., Newton's Method and Its Use in Optimization, *Eur. J. Oper. Res.*, 2007, vol. 181(3), pp. 1086–1096.

3.137. Nazin, S.A. and Polyak, B.T., Ellipsoid-based Parametric Estimation in the Linear Multidimensional Systems with Uncertain Model Description, *Autom. Remote Control*, 2007, vol. 68, no. 6, pp. 993–1005.

3.138. Gryazina, E.N. and Polyak, B.T., Multidimensional Stability Domain of Special Polynomial Families, *Autom. Remote Control*, 2007, vol. 86, no. 12, pp. 2128–2141.

3.139. Polyak, B.T. and Topunov, M.V., Filtering under Nonrandom Disturbances: The Method of Invariant Ellipsoids, *Doklady Mathematics*, 2008, vol. 77, no. 1, pp. 158–162.

3.140. Polyak, B.T. and Topunov, M.V., Suppression of Bounded Exogenous Disturbances: Output Feedback, *Autom. Remote Control*, 2008, vol. 69, no. 5, pp. 801–818.

3.141. Gryazina, E.N., Polyak, B.T., and Tremba, A.A., D-Decomposition Technique State-of-the-Art, *Autom. Remote Control*, 2008, vol. 69, no. 12, pp. 1991–2026.

3.142. Nazin, A.V. and Polyak, B.T., The Randomized Algorithm for Finding an Eigenvector of the Stochastic Matrix with Application to PageRank, *Doklady Mathematics*, 2009, vol. 79, no. 3, pp. 424–427.

3.143. Polyak, B.T., Research on Automatic Control Theory, *Problemy Upravleniya*, 2009, no. 3.1, pp. 13–18.

3.144. Kuntsevich, V.M. and Polyak, B.T., Invariant Sets of Nonlinear Discrete Systems with Bounded Disturbances and Control Problems, *J. Autom. Inform. Sci.*, 2009, vol. 41, no. 11, pp. 1–16.

3.145. Dabbene, F., Shcherbakov, P., and Polyak, B., A Randomized Cutting Plane Method with Probabilistic Geometric Convergence, *SIAM J. Optim.*, 2010, vol. 2, no. 6, pp. 3185–3207.

3.146. Polyak, B.T. and Gryazina, E.N., Randomized Methods Based on New Monte Carlo Schemes for Control and Optimization, *Ann. Oper. Res.*, 2011, vol. 189(1), pp. 343–356.

3.147. Nazin, A.V. and Polyak, B.T., Randomized Algorithm to Determine the Eigenvector of a Stochastic Matrix with Application to the PageRank Problem, *Autom. Remote Control*, 2011, vol. 72, no. 2, pp. 342–352.

3.148. Khlebnikov, M.V., Polyak, B.T., and Kuntsevich, V.M., Optimization of Linear Systems Subject to Bounded Exogenous Disturbances: The Invariant Ellipsoid Technique, *Autom. Remote Control*, 2011, vol. 72, no. 11, pp. 2227–2275.

3.149. Polyak, B.T. and Tremba, A.A., Regularization-based Solution of the PageRank Problem for Large Matrices, *Autom. Remote Control*, 2012, vol. 73, no. 11, pp. 1877–1894.

3.150. Juditsky, A., Karzan, F.K., Nemirovski, A., and Polyak, B., Accuracy Guarantees for l_1 Recovery of Block-sparse Signals, *Annal. Stat.*, 2012, vol. 40, no. 6, pp. 3077–3107.

3.151. Polyak, B.T., Scientometrics: Who Is the Patient for This Medicine?, *Upravlenie Bol'shimi Sistemami*, 2013, vol. 44, pp. 161–170.

3.152. Gryazina, E.N. and Polyak, B.T., Random Sampling: Billiard Walk Algorithm, *Eur. J. Oper. Res.*, 2014, vol. 238(2), pp. 497–504.

3.153. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Sparse Feedback in Linear Control Systems, *Autom. Remote Control*, 2014, vol. 75, no. 12, pp. 2099–2111.

3.154. Polyak, B.T., Tremba, A.A., Khlebnikov, M.V., Shcherbakov, P.S., and Smirnov, G.V., Large Deviations in Linear Control Systems with Nonzero Initial Conditions, *Autom. Remote Control*, 2015, vol. 76, no. 6, pp. 957–976.

3.155. Barabanov, N., Ortega, R., Griñó, R., and Polyak, B., On Existence and Stability of Equilibria of Linear Time-Invariant Systems with Constant Power Loads, *IEEE Trans. on Circuits and Systems I: Regular Papers*, 2016, vol. 63(1), pp. 114–121.

3.156. Polyak, B.T., Kuznetsov, O.N., and Chumachenko, V.V., Stability Study of a Power System with Unipolar Electromagnetic Brake, *Autom. Remote Control*, 2016, vol. 77, no. 9, pp. 1557–1566.

3.157. Polyak, B.T. and Smirnov, G., Large Deviations for Non-Zero Initial Conditions in Linear Systems, *Automatica*, 2016, vol. 74, pp. 297–307.

3.158. Polyak, B.T. and Khlebnikov, M.V., Principle Component Analysis: Robust Versions, *Autom. Remote Control*, 2017, vol. 78, no. 3, pp. 490–506.

3.159. Polyak, B. and Shcherbakov, P., Why Does Monte Carlo Fail to Work Properly in High-Dimensional Optimization Problems?, *J. Optim. Theor. Appl.*, 2017, vol. 173(2), pp. 612–627.

3.160. Polyak, B.T. and Kvinto, Ya.I., Stability and Synchronization of Oscillators: New Lyapunov Functions, *Autom. Remote Control*, 2017, vol. 78, no. 7, pp. 1234–1242.

3.161. Machado, J.E., Griñó, R., Barabanov, N., Ortega, R., and Polyak, B., On Existence of Equilibria of Multi-Port Linear AC Networks With Constant-Power Loads, *IEEE Trans. on Circuits and Systems I: Regular Papers*, 2017, vol. 64(10), pp. 2772–2782.

3.162. Polyak, B.T. and Shcherbakov, P.S., Optimization and Asymptotic Stability, *Int. J. Control*, 2018, vol. 91(11), pp. 2404–2410.

3.163. Polyak, B.T., Shcherbakov, P.S., and Smirnov, G., Peak Effects in Stable Linear Difference Equations, *J. Differ. Equat. Appl.*, 2018, vol. 24(9), pp. 1488–1502.

3.164. Polyak, B. and Tremba, A., Sparse Solutions of Optimal Control via Newton Method for Under-Determined Systems, *J. Global Optim.*, 2020, vol. 76(3), pp. 613–623.

3.165. Polyak, B.T. and Smirnov, G.V., Transient Response in Matrix Discrete-Time Linear Systems, *Autom. Remote Control*, 2019, vol. 80, no. 9, pp. 1645–1652.

3.166. Polyak, B.T. and Shalby, L.A., Minimum Fuel-Consumption Stabilization of a Spacecraft at the Lagrangian Points, *Autom. Remote Control*, 2019, vol. 80, no. 12, pp. 2217–2228.

3.167. Balashov, M.V., Polyak, B.T., and Tremba, A.A., Gradient Projection and Conditional Gradient Methods for Constrained Nonconvex Minimization, *Numer. Function. Anal. Optim.*, 2020, vol. 41(7), pp. 822–849.

3.168. Polyak, B. and Tremba, A., New Versions of Newton Method: Step-size Choice, Convergence Domain and Under-determined Equations, *Optim. Method. Softwar.*, 2020, vol. 35(6), pp. 1272–1303.

3.169. Polyak, B.T. and Fatkhullin, I.F., Use of Projective Coordinate Descent in the Fekete Problem, *Comput. Math. and Math. Phys.*, 2020, vol. 60, no. 5, pp. 795–807.

3.170. Fatkhullin, I. and Polyak, B., Optimizing Static Linear Feedback: Gradient Method, *SIAM J. on Control and Optimization*, 2021, vol. 59, no. 5, pp. 3887–3911.

3.171. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Linear Matrix Inequalities in Control Systems with Uncertainty, *Autom. Remote Control*, 2021, vol. 82, no. 1, pp. 1–40.

3.172. Polyak, B.T. and Khlebnikov, M.V., Static Controller Synthesis for Peak-to-Peak Gain Minimization as an Optimization Problem, *Autom. Remote Control*, 2021, vol. 82, no. 9, pp. 1530–1553.

3.173. Polyak, B.T. and Khlebnikov, M.V., Observer-Aided Output Feedback Synthesis as an Optimization Problem, *Autom. Remote Control*, 2022, vol. 83, no. 3, pp. 303–324.

3.174. Polyak, B.T. and Khlebnikov, M.V., New Criteria for Tuning PID Controllers, *Autom. Remote Control*, 2022, vol. 83, no. 11, pp. 1724–1741.

3.175. Polyak, B.T. and Khlebnikov, M.V., Erratum to: Observer-Aided Output Feedback Synthesis as an Optimization Problem. Autom. Remote Control, 2022, vol. 83, no. 3, pp. 303–324, *Autom. Remote Control*, 2022, vol. 83, no. 11, pp. 1818–1819.

3.176. Stonyakin, F.S., Kuruzov, I.A., and Polyak, B.T., Stopping Rules for Gradient Methods for Non-Convex Problems with Additive Noise in Gradient, *J. Optim. Theor. Appl.*, 2023, vol. 198, no. 2, pp. 531–551.

3.177. Beznosikov, A., Polyak, B., Gorbunov, E., Kovalev, D., and Gasnikov, A., Smooth Monotone Stochastic Variational Inequalities and Saddle Point Problems: A Survey, *Eur. Mat. Soc. Magazin.*, 2023, vol. 127, pp. 15–28.

ARTICLES IN COLLECTIONS, BOOK CHAPTERS, PREPRINTS, AND OTHERS

4.1. Ivantsov, G.P. and Polyak, B.T., Stability Issues for the Regular Shape of a Crystal, in *Sbornik trudov TsNIIChM* (Proceedings of the Central Research Institute of Ferrous Metallurgy), 1960, no. 21, pp. 464–479.

4.2. Polyak, B.T., Multi-Move Choice Problems, in *Voprosy teorii matematicheskikh mashin* (Theoretical Issues of Mathematical Machines), Bazilevskii, Yu.Ya., Ed., Moscow: Fizmatlit, 1962, vol. 2, pp. 156–173.

4.3. Poljak, B.T. and Schreider, Ju.A., Application of Walsh Polynomials in Approximate Calculation, in *AMS Nine Papers on Foundations, Measure Theory and Analysis*, 1967, vol. 57, pp. 171–189.

4.4. Borisova, T.N., Vlashek, Z., Karmanov, V.G., and Polyak, B.T., Some Methods for Solving Placement Problems, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1965, vol. 3, pp. 441–451.

4.5. Ber, A.M., Belov, E.N., and Polyak, B.T., On Some Network Optimization Problems, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1966, vol. 5, pp. 115–123.

4.6. Polyak, B.T., and Shostakovskii, B.I., One Problem on the Maximum of a Multivariable Function, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1966, vol. 5, pp. 107–114.

4.7. Karmanov, V.G., Rudneva, T.L., and Polyak, B.T., On Some Methods for Solving Extremal Problems of Economic and Technical-Economic Content, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1967, vol. 9, pp. 143–147.

4.8. Polyak, B.T., and Shostakovskii, B.I., One Method for Solving the Best Uniform Approximation Problem, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1967, vol. 9, pp. 160–166.

4.9. Polyak, B.T. and Skokov, V.A., Selecting the Parameters of Kinetic Equations Based on Experimental Data, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1967, vol. 9, pp. 167–178.

4.10. Orlov, V.S., Polyak, B.T., Rebrii, V.A., and Tret'yakov, N.V., Experience in Solving Optimal Control Problems, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1967, vol. 9, pp. 179–192.

4.11. Ivanova, T.P., Polyak, B.T., and Pukhova, G.V., Numerical Methods for Solving Some Extremal Problems with Partial Derivatives, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1967, vol. 9, pp. 193–202.

4.12. Polyak, B.T., One Method for Solving Linear and Quadratic Programming Problems of High Dimension, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1969, vol. 12, pp. 10–17.

4.13. Polyak, B.T., Iterative Methods for Solving Some Ill-Posed Variational Problems, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1969, vol. 12, pp. 38–52.

4.14. Polyak, B.T. and Pukhova, G.V., One Multidimensional Variational Problem with Constraints, in *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1970, vol. 14, pp. 186–197.

4.15. Polyak, B.T., Minimization Methods under Constraints, in *Itogi Nauki Tekh.* (Results of Science and Technology), Gamkrelidze, R.V., Ed., Moscow: VINITI, 1974, vol. 12, pp. 147–197.

Polyak, B.T., Minimization Methods with Constraints, *J. Sov. Mat.*, 1976, vol. 5, no. 1, pp. 97–128.

4.16. Polyak, B.T., Stable Methods for Estimating Parameters, in *Strukturnaya adaptatsiya slozhnykh sistem upravleniya* (Structural Adaptation of Complex Control Systems), Rastrigin, L.A., Ed., Voronezh: Voronezh Polytechnic Institute, 1977, pp. 66–71.

4.17. Tsypkin, Ya.Z., and Polyak, B.T., Stable Estimation under Incomplete Information, in *Voprosy kibernetiki. Adaptivnye sistemy upravleniya* (Issues of Cybernetics. Adaptive Control Systems), Moscow: Council on Cybernetics, the USSR Academy of Sciences, 1977, pp. 6–15.

4.18. Tsypkin, Ya.Z. and Polyak, B.T., A Coarse Maximum Likelihood Method, in *Dinamika sistem. Matematicheskie metody teorii kolebanii* (System Dynamics. Mathematical Methods of Oscillation Theory), Neimark, Yu.I., Butenin, N.V., et al., Eds., Gorky: Gorky State University, 1977, vol. 12, pp. 22–46.

4.19. Poljak, B.T., Subgradient Methods: A Survey of Soviet Research, in *Nonsmooth Optimization*, IIASA Proceedings Series, Lemarechal, C. and Mifflin, R., Eds., Oxford: Pergamon Press, 1978, vol. 3, pp. 5–29.

4.20. Poljak, B.T., On the Bertsekas' Method for Minimization of Composite Functions, in *Int. Symp. on Systems Optimization and Analysis*, Bensoussan, A. and Lions, J.L., Eds., Springer–Verlag, 1979, vol. 14, pp. 176–184.

4.21. Poljak, B.T., Iterative Algorithms for Singular Minimization Problems, in *Nonlinear Programming*, Mangasarian, O.L., Meyer, R.R., and Robinson, S.M., Eds., Academic Press, 1981, vol. 4, pp. 147–166.

4.22. Tsypkin, Ya.Z. and Polyak, B.T., Identification of Time-Varying Dynamic Plants, in *Itogi Nauki Tekh.* (Results of Science and Technology), Emelyanov, S.V., Ed., Moscow: VINITI, 1987, vol. 21, pp. 68–91.

4.23. Nazin A.V., Polyak B.T., and Tsybakov, A.B., Passive Stochastic Approximation, *Preprint of Institute of Control Sciences, USSR Acad. Sci.*, Moscow, 1990.

4.24. Polyak, B.T., Variational Problems Arising in Statistics, in *Perspectives in Control Theory*, Jakubczyk, B., Malanowski, and Respondek, W., Eds., Birkhauser, 1990, vol. 2, pp. 277–285. (Proc. of the Sielpia Conf., Sielpia, Poland, 1988.)

4.25. Polyak, B.T., Optimization Problems in the Presence of Noise, in *Modern Mathematical Methods of Optimization*, Elster, K., Ed., Academie Verlag, 1993, pp. 53–61.

4.26. Tsypkin, Ya.Z. and Polyak, B.T., Frequency Domain Approach to Robust Stability of Continuous Systems, in *Systems and Control: Topics in Theory and Applications*, Ono, T. and Kozin, F., Eds., Mita Press, 1991, pp. 389–399.

4.27. Polyak, B.T. and Tsypkin, Ya.Z., Robust Stability of Linear Systems, in *Itogi Nauki Tekh.* (Results of Science and Technology), Emelyanov, S.V., Ed., Moscow: VINITI, 1991, vol. 32, pp. 3–31.

4.28. Tsypkin, Ya.Z. and Polyak, B.T., Frequency Domain Criterion for Robust Stability of Polytope of Polynomials, in *Control of Uncertain Dynamic Systems*, Bhattacharyya, S.P. and Keel, L.H., Eds., CRC Press, 1991, pp. 491–499. (Proc. of the Int. Workshop on Robust Control, San Antonio, Texas, March 1999.)

4.29. Polyak, B.T. and Tsybakov, A.B., On Stochastic Approximation with Arbitrary Noise (the KW-case), in *Topics in Nonparametric Estimation*, Khasminskii, R.Z., Ed., AMS, 1992, vol. 12.

4.30. Polyak, B.T., Robustness Analysis for Multilinear Perturbations, in *Robustness of Dynamic Systems with Parameter Uncertainties*, Mansour, M., Balemi, S., and Truöl, W., Eds., Birkhäuser, 1992, pp. 93–104.

4.31. Tsypkin, Ya.Z. and Polyak, B.T., Robust Absolute Stability of Continuous Systems, in *Robustness of Dynamic Systems with Parameter Uncertainties*, Mansour, M., Balemi, S., and Truöl, W., Birkhäuser, 1992, pp. 113–121.

4.32. Tsypkin, Ya.Z. and Polyak, B.T., Robust Stability of Discrete-Time Systems: Frequency Domain Approach, in *Fundamentals of Discrete-Time Systems: A Tribute to Professor Eliahu I. Jury*, Jamshidi, M., Mansour, M., Anderson, B.D.O., and Bose, N.K., Eds., TSI Press, 1993, pp. 163–169.

4.33. Polyak, B.T. and Kogan, J., Necessary and Sufficient Conditions of Robust Stability of Multiaffine Systems, *Research Report 93–06*, Dept. Math., University of Maryland, Baltimore, USA, 1993.

4.34. Barmish, B.R. and Polyak, B.T., The Volumetric Singular Value and Robustness of Feedback Control Systems, *Technical Report ECE-93-9*, University of Wisconsin-Madison, September 1993.

4.35. Polyak, B., Scherbakov, P., and Shmulyian, S., Circular Arithmetic and its Applications to Robustness Analysis, in *Modeling Techniques for Uncertain Systems*, Kurzhanski, A.B., and Veliov, V.M., Eds., Birkhauser, 1994, pp. 229–243.

4.36. Polyak, B.T., Robust Stability of Interval Matrices: A Stochastic Approach, in *Stochastic Programming Methods and Technical Applications*, Marti, K. and Kall, P., Eds., Springer, 1998, pp. 202–207. (Proc. of the 3rd GAMM/IFIP Workshop "Stochastic Optimization: Numerical Methods and Technical Applications," Munich, Germany, June 17–20, 1996.)

4.37. Polyak, B.T. and Shcherbakov, P.S., Numerical Search of Stable or Unstable Element in Matrix or Polynomial Families: A Unified Approach to Robustness Analysis and Stabilization, *Robustness in Identification and Control*, Lect. Notes Control Inf. Sci., vol. 245, Garulli, A., Tesi, A., Vicino, A., Eds., Springer, 1999, pp. 344–358.

4.38. Polyak, B.T., Robust Stability, in *Trudy Instituta Problem Upravleniya RAN* (Proceedings of the Institute of Control Sciences RAS), Moscow: Trapeznikov Institute of Control Sciences RAS, 1999, vol. 5, pp. 36–43.

4.39. Polyak, B.T., Random Algorithms for Solving Convex Inequalities, in *Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications*, Studies in Computational Mathematics, Butnariu, D., Reich, S., and Censor, Y., Eds., Elsevier, 2001, vol. 8, pp. 409–422.

4.40. Nazin, S.A. and Polyak, B.T., Limiting Behavior of Bounding Ellipsoids for State Estimation, in *Nonlinear Control Systems*, Kurzhanski, A. and Fradkov, A., Eds., Elsevier, 2002, vol. 2, pp. 553–558.

4.41. Polyak, B.T., Convexity of the Reachable Set of Nonlinear Systems under L_2 Bounded Controls, *Preprint of the Institute Mittag–Leffler*, no. 02, Spring, 2003.

4.42. Nesterov, Yu. and Polyak, B.T., Cubic Regularization of Newton Scheme and Its Global Performance, *Preprint of the CORE DP 2003/41*, Louvain-la-Neuve, Belgium, 2003.

4.43. Polyak, B.T., Robust Linear Algebra and Robust Aperiodicity, in *Directions in Mathematical Systems Theory and Optimization*, Rantzer, A. and Byrnes, C.I., Eds., Springer-Verlag, 2003, vol. 286, pp. 249–260.

4.44. Polyak, B.T., Randomized Algorithms for Solving Convex Inequalities, in *Stokhasticheskaya optimizatsiya v informatike* (Stochastic Optimization in Computer Science), Granichin, O.N., Ed., St. Petersburg: St. Petersburg State University, 2005, pp. 150–169.

4.45. Polyak, B.T., Early Control Textbooks in Russia (Soviet Union), in *Historic Control Textbook*, Gertler, J., Ed., Elsevier, 2006, pp. 213–232.

4.46. Polyak, B., New Challenges in Nonlinear Control: Stabilization and Synchronization of Chaos, in *Forever Ljung in System Identification*, Glad, T. and Hendeby, G., Lund: Studentlitteratur, 2006, pp. 147–168.

4.47. Polyak, B.T. and Shcherbakov, P.S., A Randomized Method for Solving Semidefinite Programming Problems, in *Stokhasticheskaya optimizatsiya v informatike* (Stochastic Optimization in Computer Science), Granichin, O.N., Ed., St. Petersburg: St. Petersburg State University, 2006, vol. 2, pp. 38–70.

4.48. Polyak, B.T. and Shcherbakov, P.S., A Description of the Signature Constancy Domains of Affine Matrix Functions, in *Prikladnye problemy upravleniya makrosistemami. Trudy ISA RAN* (Applied Problems of Macrosystems Control), Moscow: KomKniga, 2006, vol. 28, pp. 125–147.

4.49. Polyak, B.T., Newton's Method and Its Role in Optimization and Computational Mathematics, in *Prikladnye problemy upravleniya makrosistemami. Trudy ISA RAN* (Applied Problems of Macrosystems Control), Moscow: KomKniga, 2006, vol. 28, pp. 48–66.

4.50. Polyak, B.T., Topunov, M.V., and Shcherbakov, P.S., The Ideology of Invariant Ellipsoids in the Robust Suppression of Bounded Exogenous Disturbances, in *Stokhasticheskaya optimizatsiya v informatike* (Stochastic Optimization in Computer Science), Granichin, O.N., Ed., St. Petersburg: St. Petersburg State University, 2007, vol. 3, pp. 51–84.

4.51. Polyak, B.T. and Shcherbakov, P.S., The *D*-Decomposition Technique in Solving Linear Matrix Inequalities, in *Nelineinye sistemy. Chastotnye i matrichnye neravenstva* (Nonlinear Systems. Frequency Domain and Matrix Inequalities), Gelig, A.Kh., Leonov, G.A., and Fradkov, A.L., Eds., Moscow: Fizmatlit, 2008, pp. 135–154.

4.52. Polyak, B.T., Nazin, S.A., and Khlebnikov, M.V., The Invariant Ellipsoids Technique for Analysis and Design of Linear Control Systems, in *Advances in Mechanics: Dynamics and Control*, Chernousko, F.L., Kostin, G.V., and Saurin, V.V., Eds., Moscow: Nauka, 2008, pp. 239–246. (Proc. of the 14th Int. Workshop on Dynamics and Control, Moscow–Zvenigorod, Russia, May 28–June 2, 2008. Dedicated to Professor Angelo Miele on the occasion of his 85th birthday.)

4.53. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Robust Rejection of Exogenous Disturbances via Invariant Ellipsoids Technique, in *Advances in Mechanics: Dynamics and Control*, Chernousko, F.L., Kostin, G.V., and Saurin, V.V., Eds., Moscow: Nauka, 2008, pp. 247–254. (Proc. of the 14th Int. Workshop on Dynamics and Control, Moscow–Zvenigorod, Russia, May 28–June 2, 2008. Dedicated to Professor Angelo Miele on the occasion of his 85th birthday.)

4.54. Petrikevich, Ya. and Polyak, B., Monte–Carlo Technique for Stabilization of Linear Discrete-Time Systems via Low-Order Controllers, in *Advances in Mechanics: Dynamics and Control*, Chernousko, F.L., Kostin, G.V., and Saurin, V.V., Eds., Moscow: Nauka, 2008, pp. 232–238. (Proc. of the 14th Int. Workshop on Dynamics and Control, Moscow–Zvenigorod, Russia, May 28–June 2, 2008. Dedicated to Professor Angelo Miele on the occasion of his 85th birthday.)

4.55. Polyak, B.T. and Shcherbakov, P.S., The Sets of Reachability and Attraction of Linear Systems with Bounded Control: A Description Using Invariant Ellipsoids, in *Stokhasticheskaya optimizatsiya v informatike* (Stochastic Optimization in Computer Science), Granichin, O.N., Ed., St. Petersburg: St. Petersburg State University, 2008, vol. 4, pp. 3–24.

4.56. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Nonlinear Systems with Bounded or Multiplicative Disturbances, *Problemy ustoichivosti i upravleniya* (Problems of Stability and Control), Moscow: Fizmatlit, 2013, pp. 270–299. (The Collection of Research Papers Dedicated to the 80th Anniversary of Academician V.M. Matrosov.)

4.57. Polyak, B.T. and Shcherbakov, P.S., Why Is the Monte Carlo Method Inefficient in High-Dimensional Problems?, in *Stokhasticheskaya optimizatsiya v informatike* (Stochastic Optimization in Computer Science), Granichin, O.N., Ed., St. Petersburg: St. Petersburg State University, 2014, vol. 10, pp. 89–100.

4.58. Polyak, B.T. and Shcherbakov, P.S., Stability and Performance of Complex Systems Affected by Parametric Uncertainty, in *Encyclopedia of Systems and Control*, Baillieul, J. and Samad, T., Eds., Springer, 2014. Current version: 2021, pp. 2117–2124.

4.59. Polyak, B. and Gryazina, E., Convexity/Nonconvexity Certificates for Power Flow Analysis, in *Advances in Energy System Optimization*, Bertsch, V., Fichtner, W., Heuveline, V., and Leibfried, T., Eds., Springer, 2017, pp. 221–230. (Proc. of the First Int. Symp. on Energy System Optimization.)

4.60. Polyak, B. and Shcherbakov, P., Randomization in Robustness, Estimation, and Optimization, in *Uncertainty in Complex Networked Systems: In Honor of Roberto Tempo*, Basar, T., Ed., Springer, 2018, pp. 181–208.

4.61. Danilova, M., Kulakova, A., and Polyak, B., Non-monotone Behavior of the Heavy Ball Method, in *Difference Equations and Discrete Dynamical Systems with Applications*, Bohner, M., Siegmund, S., Simon Hilscher, R., and Stehlik, P., Eds., Springer, 2020, vol. 312, pp. 213–230. (Proc. of the 24th Int. Conf. on Difference Equations and Applications, Dresden, Germany, May 2018.)

CONFERENCE PROCEEDINGS

5.1. Girsanov, I.V. and Polyak, B.T., Mathematical Methods for Solving the Placement Problem, *Trudy konferentsii "Problemy optimal'nogo upravleniya, planirovaniya i razmeshcheniya proizvodstva"* (Proc. of the Conf. "Problems of Optimal Management, Planning, and Location of Production"), Moscow: Moscow State University, 1963, pp. 288–300.

5.2. Girsanov, I.V., Polyak, B.T., Feigin, E.A., and Platonov, V.M., Calculating the Parameters of Kinetic Equations Based on Experimental Data, *Trudy Vsesoyuznoi konferentsii po khimicheskim reaktoram* (Proc. All-Soviet Conf. on Chemical Reactors), Novosibirsk, 1965, pp. 412–423.

5.3. Polyak, B.T., Some Aspects of Mathematical Theory and Methods for Extremal Problems, *Int. Congr. Math. Abstracts*, Sect. 13, Moscow, 1966, p. 29.

5.4. Polyak, B.T., The Method of Conjugate Gradients, *Trudy II Zimnei shkoly po matematicheskomu programmirovaniyu i smezhnym voprosam* (Proc. Winter School on Mathematical Programming and Related Issues), Drohobych, 1969, vol. 1, pp. 152–201. (Plenary talk.)

5.5. Polyak, B.T., On Some Dual Methods for Solving Constrained Optimization Problems, *Trudy simpoziuma "Problemy tochnosti i effektivnosti vychislitel'nykh metodov"* (Proc. of Symp. "Problems of Accuracy and Efficiency of Computational Methods"), Kiev, 1969, vol. 4, pp. 104–114.

5.6. Polyak, B.T., and Tretyakov, N.V., One Iterative Method for Solving Mathematical Programming Problems and Its Economic Interpretation, *Trudy 1-oi konferentsii po optimal'nomu planirovaniyu i upravleniyu narodnym khozyaistvom* (Proc. of the 1st Conf. on Optimal Planning and Management of the National Economy), Moscow, 1971, pp. 260–264.

5.7. Polyak, B.T., Stochastic Regularized Algorithms, *Suppl. to Preprints of the IFAC Symp. Stoch. Contr.*, Budapest, 1974.

5.8. Polyak, B.T. and Tsypkin, Ya.Z., Potential Capabilities of Adaptation Algorithms, *Trudy II Leningradskogo Simpoziuma po teorii adaptivnykh sistem* (Proc. II Leningrad Symp. on the Theory of Adaptive Systems), Leningrad, 1976, pp. 6–19. (Plenary talk.)

5.9. Polyak, B.T. and Tsypkin, Ya.Z., Interference-Resistant Identification, *Proc. IV IFAC Symp. on Identification and System Parameter Estimation*, 1976, part 1, pp. 190–213. (Plenary talk.)

5.10. Polyak, B.T., Nonlinear Programming Methods in the Presence of Noise, *Proc. of IX Intern. Symp. Math. Progr. "Survey of Math. Progr."*, Budapest, 1976, vol. 2, Prekopa, A., Ed., Budapest: Kiado, 1977, pp. 155–165. (Plenary talk.)

5.11. Polyak, B.T., Use of A Priori Information in Adaptive Algorithms, *Trudy VII Vsesoyuznoi konferentsii po problemam upravleniya* (Proc. VII All-Union Conf. on Control Problems), Minsk, 1977, vol. 1, pp. 87–89.

5.12. Polyak, B.T. and Fedorova, I.E., DAS–method (Descent–Ascent–Saddle Point) for Global Optimization, *Abstracts of the 9th IFIP Conf. on Optimization Techniques*, Warzsawa, Sept. 1979, p. 180.

5.13. Polyak, B.T. and Tsypkin, Ya.Z., Optimal and Robust Methods for Unconstrained Optimization, *Trudy VIII Vsesoyuznoi konferentsii po problemam upravleniya* (Proc. VIII All-Union Conf. on Control Problems), Tallinn, 1980, vol. 2, pp. 261–262. (Plenary talk.)

5.14. Poljak, B.T. and Tsypkin, Ya.Z., Optimal and Robust Methods for Unconditional Optimization, *Proc. of the 8th IFAC World Congress*, Kyoto, August 24–28, 1981, vol. 4, pp. 94–98.

5.15. Nemirovskii, A.S., Polyak, B.T., and Tsybakov, A.B., Nonparametric Identification by the Maximum Likelihood Method, *Trudy Vsesoyuznoi konferentsii "Teoriya adaptivnykh sistem i ee primeneniya"* (Proc. of the All-Union Conf. "Theory of Adaptive Systems and Its Applications"), Moscow, 1983, pp. 160–163.

5.16. Polyak, B.T., Recurrent Methods of Unconstrained Stochastic Optimization, *Proc. Int. Conf. "Stochastic Optimization"*, Kiev, 1984, part II, pp. 55–57.

5.17. Tsypkin, Ya.Z. and Poljak, B.T., Optimal Recurrent Algorithms for Identification of Nonstationary Plants, *Proc. of the 10th IFAC World Congress*, 1987, vol. 10, pp. 318–321.

5.18. Polyak, B.T. and Shcherbakov, P.S., Matrix Estimation and Its Applications to Optimization Problems, *Proc. of the 7th IFAC Workshop on Control Applications of Nonlinear Programming and Optimization*, Tbilisi, 1988, pp. 82–83.

5.19. Polyak, B.T., Variational Problems Arising in Statistics, *Proc. of the Sielpia Conf. "Perspectives in Control Theory"*, Poland, Sep. 19–24, 1988, pp. 60–61. (Plenary talk.)

5.20. Polyak, B.T. and Tsypkin, Y.Z., Frequency Domain Criterion for Robust Stability of a Polytope of Polynomials, *Proc. of the Int. Workshop on Robust Control*, San Antonio, TX, March 1991, p. 52.

5.21. Polyak, B.T. and Tsypkin, Y.Z., Robust Stability of Disk Polynomials: Frequency Domain Approach, *Proc. of the Int. Workshop on Robust Control*, San Antonio, TX, March 1991, p. 53.

5.22. Polyak, B.T. and Tsypkin, Y.Z., Robust Stability: Frequency Domain Approach, *Proc. Symp. on Modelling, Inverse Problems, and Numerical Methods*, Tallinn, Estonia, April 1991.

5.23. Polyak, B. and Tsybakov, A., On Stochastic Approximation with Arbitrary Dependent Noise, *Discussion paper No. 9160*, Center for Operations Research and Econometrics, University Catholique de Louvain, 1991.

5.24. Tsypkin, Ya.Z., Polyak, B.T., Katbab, A., and Jury, E.I., On the Strict and Wide-Sense Stability Robustness of Uncertain Systems — Application of a New Frequency Criterion, *Proc. of the 30th Conf. on Decision and Control*, Brighton, UK, December 11–13, 1991, pp. 37–42.

5.25. Polyak, B.T., Mathematical Programming with Multilinear Functions, *Proc. of the Workshop on Computational Optimization*, Haifa, Israel, March 1992.

5.26. Polyak, B.T., Robustness Analysis: Small Multilinear Perturbations Can Be Treated as Linear Ones, *Proc. of the Int. Workshop on Robust Control*, Monte Verita, Switzerland, April 1992, p. 33.

5.27. Tsypkin, Ya.Z. and Polyak, B.T., Robust Absolute Stability of Continuous Systems, *Proc. of the Int. Workshop on Robust Control*, Monte Verita, Switzerland, April 1992, p. 39.

5.28. Barmish, B.R. and Polyak, B.T., A New Singular Value for Control Problems, *Proc. of the 27th Conf. Inform. Sci. Systems*, Johns Hopkins University, Baltimore, MD, March 1993, p. 195.

5.29. Barmish, B.R. and Polyak, B.T., The Volumetric Singular Value and Robustness of Feedback Control Systems, *Proc. of the 32nd Conf. on Decision and Control*, San Antonio, TX, December 1993, pp. 521–522.

5.30. Abrishamchian, M., Barmish, B.R., and Polyak, B.T., Positive Realness and Coprime Factor Uncertainty Conditions for Reduction of Robust Stabilization Problems to Classical H^∞ Problems, *Proc. of the American Control Conf.*, June 1994, vol. 2, pp. 1413–1415.

5.31. Abrishamchian, M., Barmish, B.R., and Polyak, B.T., Special Weighting and Objective Functions for Robust Synthesis with a Type of Coprime Factor Uncertainty Model, *Proc. of the 33rd Conf. on Decision and Control*, Orlando, FL, December 1994, pp. 2983–2984.

5.32. Nemirovskii, A.S. and Polyak, B.T., Probability for a Random Interval Polynomial to Be Stable, *Proc. Int. Conf. Interval Methods in Science and Engineering*, St. Petersburg, March 1994, pp. 183–184.

5.33. Barmish, B.R. and Polyak, B.T., Stochastic Approach to Robustness in Control Theory, *Proc. of the 7th Int. Conf. on Stochastic Programming*, Naharia, Israel, June 1995.

5.34. Barmish, B.R. and Polyak, B.T., Stochastic Approach to Robustness in Control Theory, *Proc. of the 2nd Russian–Swedish Control Conf.* St. Petersburg, Russia, August 1995, pp. 115–116. (Plenary talk.)

5.35. Polyak, B.T. and Vishnyakov, A.N., Multiplying Disks: Robust Stability of a Cascade Connection, *Proc. of the 3rd European Control Conf.*, Rome, Italy, Sep. 1995, pp. 1406–1411.

5.36. Tsypkin, Ya.Z. and Polyak, B.T., Robustness Radius for Sector Stability of Polynomials, *IFAC Proceedings Volumes*, 1996, vol. 29, no. 1, pp. 3428–3431.

5.37. Barmish, B.R. and Polyak, B.T., A New Approach to Open Robustness Problems Based on Probabilistic Prediction Formulae, *Proc. of the 13th IFAC World Congress*, vol. H., San Francisco, July 1996, pp. 1–6.

5.38. Durieu, C., Polyak, B.T., and Walter, E., Trace versus Determinant in Ellipsoidal Outer-Bounding, with Application to State Estimation, *Proc. of the 13th IFAC World Congress*, vol. I, San Francisco, July 1996, pp. 43–48.

5.39. Polyak, B.T., A Theorem on an Image of a Set under Nonlinear Map and Its Application in Robustness Analysis, *Abstracts of the 4th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"*, Moscow, June 1996.

5.40. Tsypkin, Ya.Z. and Polyak, B.T., Stability and Robust Stability of Nested Transfer Functions, *Proc. of the IFAC Int. Workshop on Robust Control*, Napa, CA, June 1996.

5.41. Durieu, C., Polyak, B.T., and Walter, E., Ellipsoidal State Outer-Bounding for MIMO Systems via Analytical Techniques, *Proc. of the IMACS Multiconf. on Computational Engineering in Systems Applications*, Lille, France, July 1996, pp. 843–848.

5.42. Polyak, B.T., Robust Stability of Interval Matrices: A Stochastic Approach, *Proc. of the 35th CDC*, Kobe, Japan, December 1996, pp. 18–23.

5.43. Polyak, B.T., A Theorem on an Image of a Convex Set under Nonlinear Transformation, *Proc. of the Int. Workshop on Nonsmooth Analysis and Optimization*, Ballarat, Australia, February 1997.

5.44. Polyak, B.T., Quadratic Transformations and Their Use in Control and Optimization, *Proc. of the Workshop "Semidefinite Programming and Large Scale Optimization"*, Haifa, Israel, April 1997.

5.45. Polyak, B.T., Value Sets for Transfer Functions under Parametric Uncertainty, *Proc. of the 2nd IFAC Symp. on Robust Control Design*, Budapest, Hungary, June 1997.

5.46. Durieu, C., Walter, E., and Polyak, B.T., Ellipsoidal Bounding Techniques for Parameter Tracking, *IFAC Proceedings Volumes*, 1997, vol. 30, no. 11, pp. 1673–1678. (Proc. of the 11th IFAC Symp. on System Identification, Kitakyushu, Japan, July 1997.)

5.47. Polyak, B.T. and Halpern, M.E., Robust Stability and Design of Linear Discrete-Time SISO Systems under l_1 Uncertainties, *Proc. of the 36th Conf. on Decision and Control*, vol. 2, San Diego, CA, December 1997, pp. 1564–1567.

5.48. Polyak, B.T., H^∞ Optimization via Low-Order Controllers, *Proc. of the 3rd Russian–Swedish Control Conf.*, Stockholm, Sweden, May 1998.

5.49. Kiselev, O.N. and Polyak, B.T., Computer-Aided H^∞ Design via Low-Order Controllers, *Abstracts of the 5th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"*, Moscow, June 1998, pp. 79–80.

5.50. Polyak, B.T. and Shcherbakov, P.S., Numerical Solution of Hard Problems in Control via Perturbation Theory, *Proc. of the Int. IFAC Workshop "Nonsmooth and Discontinuous Problems of Control and Optimization"*, Chelyabinsk, Russia, June 17–20, 1998, pp. 176–180.

5.51. Polyak, B.T., Convexity of Quadratic Transformations and its Use in Control and Optimization, *Proc. of the 11th Baikal Int. School-Seminar "Optimization Methods and Their Applications"*, Baikal, Russia, July 1998, pp. 205–219.

5.52. Polyak, B.T. and Shcherbakov, P.S., Numerical Search of Stable or Unstable Element in Matrix or Polynomial Families: A Unified Approach to Robustness Analysis and Stabilization, in *Lect. Notes in Control and Inform. Sci.*, Garulli, A. and Tesi, A., Eds., London: Springer, 1999, vol. 245, pp. 344–358. (Proc. of the Workshop "Robustness in Identification and Control", Siena, Italy, August 1998.)

5.53. Vishnyakov, A.N. and Polyak, B.T., Design of Low-Order Controllers for Disturbance Attenuation in Discrete-Time Linear Systems, *Proc. of the DYCOMANS Workshop "Techniques for Supervisory Management Systems"*, Bled, Slovenia, May 1999, pp. 13–16.

5.54. Polyak, B.T., New Control Approaches for Discrete Systems with Bounded Disturbances, *Trudy Mezhdunarodnoi konferentsii, posvyashchennoi 60-letiyu Instituta problem upravleniya* (Proc. of the Int. Conf. Dedicated to the 60th Anniversary of the Institute of Control Sciences), Moscow, June 1999, pp. 111–117. (Plenary talk.)

5.55. Polyak, B.T. and Shcherbakov, P.S., A New Approach to Robustness and Stability of Control Systems via Perturbation Theory, *Proc. of the 14th IFAC World Congress*, Beijing, China, July 1999, vol. C, pp. 13–18.

5.56. Polyak, B.T., New Approaches to Optimal Design for Linear Discrete-Time Systems with Bounded Disturbances, *The 6th St. Petersburg Symp. on Adaptive Systems Theory*, St. Petersburg, September 1999. (Plenary talk.)

5.57. Polyak, B.T., Hidden Convexity in Optimization Problems, *The 7th Workshop on Well-posedness in Optimization and Related Topics*, Gargnano, Italy, September 13–18, 1999. (Plenary talk.)

5.58. Polyak, B.T. and Halpern, M.E., The Use of a New Optimization Criterion for Discrete-Time Feedback Controller Design, *Proc. of the 38th CDC*, Phoenix, Arizona, December 1999, pp. 894–899.

5.59. Polyak, B.T., Random Algorithms for Projections in Feasible and Infeasible Cases, *Proc. of the Conf. "Inherently Parallel Algorithms in Feasibility and Optimization and Their Applications,"* Haifa, Israel, March 2000, pp. 25–26. (Plenary talk.)

5.60. Polyak, B.T. and Tempo, R., Linear-Quadratic Regulator Design under Uncertainty, *Abstracts of the 6th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"*, Moscow, June 2000, pp. 14–15.

5.61. Durieu, C., Walter, E., and Polyak, B., Set-Membership Estimation with the Trace Criterion Made Simpler than with the Determinant Criterion, *IFAC Proceedings Volumes*, 2000, vol. 33, no. 15, pp. 1007–1012. (Proc. of the 12th IFAC Symp. on System Identification – SYSID 2000, Santa Barbara, CA, June 2000.)

5.62. Vishnyakov, A.N. and Polyak, B.T., Design of Low-Order Controllers for Discrete-Time Linear Systems with Nonrandom Disturbances, *Proc. of the 8th IEEE Mediterranean Conf. on Control and Autom.*, Patras, Greece, July 2000.

5.63. Polyak, B.T., History of Mathematical Programming in the USSR: Analyzing the Phenomenon, *The 17th Int. Symp. on Mathematical Programming*, Atlanta, GA, August 2000. (Plenary talk.)

5.64. Polyak, B.T., Random Algorithms for Solving Convex Inequalities, *Proc. of the 17th Int. Symp. on Mathematical Programming*, Atlanta, GA, August 2000.

5.65. Polyak, B.T. and Shcherbakov, P.S., Random Spherical Uncertainty in Estimation and Robustness, *Proc. of the 39th CDC*, Sydney, Australia, December 2000, pp. 3339–3340.

5.66. Polyak, B.T. and Tempo, R., Probabilistic Robust Design with Linear Quadratic Regulators, *Proc. of the 39th CDC*, Sydney, Australia, December 2000, pp. 1037–1042.

5.67. Dabbene, F., Gay, P., and Polyak, B., Inner Ellipsoidal Approximation of Membership Set: A Fast Recursive Algorithm, *Proc. of the 39th CDC*, Sydney, Australia, December 2000, pp. 209–211.

5.68. Halpern, M.E. and Polyak, B.T., Optimal Tracking with Fixed Order Controllers, *Proc. of the 39th CDC*, Sydney, Australia, December 2000, pp. 3908–3913.

5.69. Calafiore, D. and Polyak, B.T., Fast Algorithms for Exact and Approximate Feasibility of Robust LMIs, *Proc. of the 39th CDC*, Sydney, Australia, December 2000, pp. 5035–5040.

5.70. Polyak, B.T., Superstability and Its Use in Feedback, *The 4th Russian–Swedish Control Conf.*, Moscow, May 2001. (Plenary talk.)

5.71. Polyak, B.T., Superstable Control Systems, *Trudy 12-oi Baikal'skoi Mezhdunarodnoi konferentsii "Metody optimizatsii i ikh prilozheniya"* (Proc. of the 12th Baikal Int. Conf. "Optimization Methods and Their Applications"), Irkutsk, June 2001, pp. 209–219. (Plenary talk.)

5.72. Nazin, S.A. and Polyak, B.T., Limiting Behavior of Bounding Ellipsoids for State Estimation, *IFAC Proceedings Volumes*, 2001, vol. 34, no. 6, pp. 553–557. (Proc. of the 5th IFAC Symp. "Nonlinear Control Systems," St. Petersburg, Russia, July 2001.)

5.73. Polyak, B.T., Hidden Convexity in Optimization and Control, *The Workshop "Smooth and Nonsmooth Optimization"*, Rotterdam, the Netherlands, July 2001. (Plenary talk.)

5.74. Polyak, B.T., Superstable Linear Difference and Differential Equations, *The 6th Int. Conf. on Difference Equations and Applications*, Augsburg, Germany, August 2001. (Plenary talk.)

5.75. Polyak, B.T., Robust Linear Algebra and Robust Stability, *Abstracts of the 7th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"*, Moscow, May 2002.

5.76. Polyak, B.T., A Convexity Principle and Its Applications in Linear Algebra, Optimization and Control, *The Conf. "IMPA – 50 Years"*, Rio-de-Janeiro, Brazil, June 2002. (Plenary talk.)

5.77. Polyak, B.T., Robust Linear Algebra with Applications to Real Stability Radius, *Proc. of the IFAC Workshop "Uncertain Dynamical Systems"*, Cascais, Portugal, July 2002.

5.78. Polyak, B.T., Nazin, S.A., Durieu, C., and Walter, E., Ellipsoidal Estimation under Model Uncertainty, *IFAC Proceedings Volumes*, 2002, vol. 35, no. 1, pp. 25–30. (Proc. of the 15th IFAC World Congress, Barcelona, Spain, July 2002, pp. 1090–1095.)

5.79. Polyak, B.T., Sznaier, M., Shcherbakov, P.S., and Halpern, M., Superstable Control Systems, *Proc. of the 15th IFAC World Congress*, Barcelona, Spain, July 2002, pp. 799–804.

5.80. Polyak, B.T., Robust Linear Algebra and Robust Aperiodicity, *The Symp. "New Directions in Mathematical Systems Theory and Optimization"*, Stockholm, November 2002. (Plenary talk.)

5.81. Sznaier, M., Polyak, B.T., Halpern, A., and Lagoa, C., A Superstability Approach to Synthesizing Low Order Suboptimal L-infinity Induced Controllers for LPV Systems, *Proc. of the 41th CDC*, Las Vegas, December 2002.

5.82. Polyak, B.T. and Nazin, S.A., Interval Solutions for Interval Algebraic Equations, *Proc. of the 4th MATHMOD Conf.*, Vienna, February 2003, pp. 973–980.

5.83. Polyak, B.T., Nazin, S.A., Durieu, C., and Walter, E., Guaranteed Ellipsoidal State Estimation for Uncertain MIMO Models, *IFAC Proceedings Volumes*, 2003, vol. 36, no. 16, pp. 1017–1022. (Proc. of the 13th IFAC Symp. on System Identification (SYSID-2003), Rotterdam, Netherlands, July 2003.)

5.84. Polyak, B.T., Newton–Kantorovich Method and Its Global Convergence, *The Kantorovich Memorial "Mathematics and Economics: Old Problems and New Approaches"*, St. Petersburg, Russia, January 2004. (Plenary talk.)

5.85. Polyak, B.T. and Shcherbakov, P.S., Possible Approaches to Solving Difficult Problems of Linear Control Theory, *Trudy 3-i Mezhdunarodnoi konferentsii "Identifikatsiya sistem i zadachi upravleniya"* (Proc. of the 3rd Int. Conf. "System Identification and Control Problems") Moscow, 2004, pp. 6–46. (Plenary talk.)

5.86. Nazin, A.V., Nazin, S.A., and Polyak, B.T., The Convergence of Recurrent Ellipsoidal Estimates for the Reachability Domains of Linear Discrete Dynamic Systems, *Trudy 8-i mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 8th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 2–4, 2004, pp. 125–132.

5.87. Polyak, B.T. and Gryazina, E.N., The Geometry of the D-decomposition, *Trudy 8-i mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 8th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 2–4, 2004.

5.88. Polyak, B.T. and Maslov, V.P., Controlling Chaos via Predictive Control, *Proc. of the Int. Workshop "Dynamical Systems and Control"*, Haifa, Israel, June 2004.

5.89. Polyak, B.T., Newton's Method and Its Use in Optimization, *Proc. of the 22th Euro Summer Institute "Optimization and Data Mining"*, Ankara, Turkey, July 9–25, 2004.

5.90. Polyak, B.T. and Maslov, V.P., Controlling Chaos and Synchronization by Predictive Control, *The 1st Intern. Conf. on Control and Optimization with Industrial Applications*, Baku, Azerbaidjan, May 2005. (Plenary talk.)

5.91. Gryazina, E.N. and Polyak, B.T., On the Root Invariant Regions Structure for Linear Systems, *IFAC Proceedings Volumes*, 2005, vol. 38, no. 1, pp. 90–95. (Proc. of the 16th IFAC World Congress, Prague, Czech Republic, July 2005.)

5.92. Polyak, B.T. and Maslov, V.P., Controlling Chaos by Predictive Control, *Proc. of the 16th IFAC World Congress*, Prague, Czech Republic, July 2005.

5.93. Polyak, B.T. and Nazin, S.A., Interval Technique for Parameter Estimation under Model Uncertainty, *IFAC Proceedings Volumes*, 2005, vol. 38, no. 1, pp. 118–123. (Proc. of the 16th IFAC World Congress, Prague, Czech Republic, July 2005.)

5.94. Polyak, B.T., Newton's Method and Its Role in Optimization, *The 6th Brazilian Workshop on Continuous Optimization*, Goiania, Brazil, July 2005. (Plenary talk.)

5.95. Polyak, B.T., Chaos Control and Stabilization Using Predictive Control, *Trudy 7-i Vserossiiskoi nauchnoi konferentsii "Nelineinye kolebaniya mekhanicheskikh sistem"* (Proc. of the 7th All-Russian Conf. "Nonlinear Oscillations of Mechanical Systems"), Nizhny Novgorod, September 2005.

5.96. Nesterov, Yu. and Polyak, B., Global Performance of the Newton Method, *Proc. of the Workshop "Optimization and Applications"*, January 9–15, 2005. Mathematisches Forschungsinstitut Oberwolfach, Rep. no. 2/2005, pp. 111–11.

5.97. Gryazina, E.N., and Polyak, B.T., Development of the D-decomposition Theory, *Trudy 7-i Vserossiiskoi nauchnoi konferentsii "Nelineinye kolebaniya mekhanicheskikh sistem"* (Proc. of the 7th All-Russian Conf. "Nonlinear Oscillations of Mechanical Systems"), Nizhny Novgorod, September 2005.

5.98. Gryazina, E.N. and Polyak, B.T., Geometry of the Stability Domain in the Parameter Space, *Proc. of the 44th IEEE CDC and ECC*, Seville, Spain, December 2005, pp. 6510–6515.

5.99. Polyak, B.T., Newton's Method and Its Role in Computational Mathematics and Optimization, *7-ya Vserossiiskaya shkola-seminar "Prikladnye problemy upravleniya makrosistemami"* (The 7th All-Russian School–Seminar "Applied Problems of Macrosystems Control"), Apatity, April 2006. (Plenary talk.)

5.100. Polyak, B.T. and Nazin, S.A., Invariant Ellipsoids Technique for Persistent Disturbance Rejection, *Proc. of the 13th IFAC Workshop on Control Applications of Optimization*, Paris–Cachan, France, April 2006, pp. 422–427.

5.101. Polyak, B.T. and Topunov, M.V., Suppressing Bounded Exogenous Disturbances on an Example of the Double Pendulum Problem, *Trudy 9-i mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 9th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 2006, pp. 213–214.

5.102. Polyak, B.T. and Shcherbakov, P.S., The D-decomposition Technique in the Theory of Linear Matrix Inequalities, *Trudy 9-i mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 9th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 2006, pp. 214–216.

5.103. Dabbene, F., Polyak, B.T., and Tempo, R., On the Complete Instability of Interval Polynomials, *Proc. of the 17th Int. Symp. on Mathematical Theory of Networks and Systems*, Kyoto, Japan, July 2006.

5.104. Polyak, B.T. and Gryazina, E.N., Iterations of Perturbed Tent Maps with Applications to Chaos Control, *IFAC Proceedings Volumes*, 2006, vol. 39, no. 8, pp. 125–129. (Proc. of the 1st IFAC Conf. on Analysis and Control of Chaotic Systems (CHAOS06), Reims, France, July 2006.)

5.105. Polyak, B.T., Solving LMIs via Random Search, *Proc. of the 5th Russian–Swedish Control Conf.*, Lund, Sweden, August 2006.

5.106. Polyak, B.T., New Challenges in Nonlinear Control: Stabilization and Synchronization of Chaos, *The Workshop "Forever Ljung in System Identification"*, Linköping, Sweden, September 2006. (Plenary lecture.)

5.107. Dabbene, F., Polyak, B.T., and Tempo, R., On the Complete Instability of Interval Polynomials, *Proc. of the 45th IEEE CDC*, San Diego, CA, December 2006, pp. 3186–3191.

5.108. Polyak, B.T., Nazin, A.V., Topunov, M.V., and Nazin, S.A., Rejection of Bounded Disturbances via Invariant Ellipsoids Technique, *Proc. of the 45th IEEE Conf. on Decision and Control*, San Diego, CA, December 2006, pp. 1429–1434.

5.109. Nazin, S.A. and Polyak, B.T., Suppressing Exogenous Disturbances Based on the Method of Invariant Sets, *Trudy 2-i Vserossiiskoi nauchnoi konferentsii "Teoriya i praktika sistemnoi dinamiki"* (Proc. of the 2nd All-Russian Scientific Conference "The Theory and Practice of System Dynamics"), Apatity, April 2007, pp. 65–66.

5.110. Polyak, B.T. and Shcherbakov, P.S., A Randomized Approach to Solving Semidefinite Programming Problems, *Trudy 2-i Vserossiiskoi nauchnoi konferentsii "Teoriya i praktika sistemnoi dinamiki"* (Proc. of the 2nd All-Russian Scientific Conference "The Theory and Practice of System Dynamics"), Apatity, April 2007, pp. 17–24.

5.111. Polyak, B.T., Boundary Oracle and Its Use in Convex Optimization, *Int. Conf. "Convex Optimization and Applications in Control Theory, Probability and Statistics"*, Luminy, France, April 2007.

5.112. Polyak, B.T., Topunov, M.V., and Shcherbakov, P.S., Robust Rejection of Exogenous Disturbances via Invariant Ellipsoids Technique, *Proc. of XIV Int. Workshop "Dynamics and Control"*, Zvenigorod, May 2007, p. 66.

5.113. Petrikevich, Ya.I. and Polyak, B.T., Monte Carlo Technique for Stabilization of Linear Discrete-Time Systems via Low-Order Controllers, *Proc. of XIV Int. Workshop "Dynamics and Control"*, Zvenigorod, May 2007.

5.114. Polyak, B.T., Nazin, S.A., and Topunov, M.V., The Invariant Ellipsoids Technique for Analysis and Design of Linear Control Systems, *Proc. of XIV Int. Workshop "Dynamics and Control"*, Zvenigorod, May 2007.

5.115. Polyak, B.T. and Gryazina, E.N., New Aspects of D-decomposition, *Trudy 9-i CHetaevskoi Mezhdunarodnoi konferentsii "Analiticheskaya mekhanika, ustoichivost' i upravlenie dvizheniem"* (Proc. of the 9th Chetaev Intern. Conf. "Analytical Mechanics, Stability, and Motion Control"), Irkutsk, June 2007, pp. 141–158. (Plenary talk.)

5.116. Polyak, B.T., Modern Minimization Methods, *2-ya shkola molodykh uchenykh po upravleniyu slozhnymi sistemami* (The 2nd School of Young Researchers on Control of Complex Systems), Voronezh, July 2007. (Plenary lecture.)

5.117. Polyak, B.T. and Shcherbakov, P.S., A Randomized Method for Solving Semidefinite Programs, *Proc. of the 9th IFAC Workshop "Adaptation and Learning in Control and Signal Processing" (ALCOSP'07)*, St. Petersburg, Russia, August 2007.

5.118. Petrikevich, Ya.I., Polyak, B.T., and Shcherbakov, P.S., Fixed-Order Controller Design for SISO Systems Using Monte Carlo Technique, *Proc. of the 9th IFAC Workshop "Adaptation and Learning in Control and Signal Processing" (ALCOSP'07)*, St. Petersburg, Russia, August 2007.

5.119. Polyak, B.T., Shcherbakov, P.S., and Topunov, M.V., Optimal Control of a Mechanical Two-Mass-Spring System Using Invariant Ellipsoids Technique, *Proc. of the 3rd IEEE Conf. "Physics and Control" (PhysCon-2007)*, Potsdam, Germany, September 2007.

5.120. Polyak, B.T. and Shcherbakov, P.S., Stochastic Approach to a Class of Convex Optimization Problems, *Proc. of the 3rd Intern. IEEE Sci. Conf. on Physics and Control (PhysCon-2007)*, Potsdam, Germany, September 2007.

5.121. Polyak, B.T., Modern Control Theory: Results and Prospects, *3-ya Vserossiiskaya molodezhnaya konferentsiya po problemam upravleniya* (The 3rd All-Russian Youth Conference on Control Problems), Moscow, April 2008. (Plenary talk.)

5.122. Polyak, B.T. and Gryazina, E.N., Randomized Methods Based on New Monte Carlo Schemes for Convex Optimization, *Proc. of the 20th EURO Mini Conf. "Continuous Optimization and Knowledge-Based Technologies" (EurOPT-2008)*, Neringa, Lithuania, May 2008, pp. 485–489.

5.123. Polyak, B.T. and Gryazina, E.N., Modern Monte-Carlo Methods for Analysis and Design of Linear Systems, *Trudy 10-i mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 10th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 2008, pp. 243–244.

5.124. Polyak, B.T. and Shcherbakov, P.S., Attraction Domains for Systems with Bounded Controls, *Abstracts of the 10th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"*, Moscow, June 3–6, 2008, pp. 244–245.

5.125. Polyak, B.T., Randomized Methods for Convex Optimization, *Conf. on Nonlinear Analysis and Optimization*, Haifa, Israel, June 2008.

5.126. Polyak, B.T. and Topunov, M.V., Filtering with Nonrandom Noise: Invariant Ellipsoids Technique, *Proc. of the 17th IFAC World Congress*, Seoul, July 2008, pp. 15349–15352.

5.127. Polyak, B.T. and Gryazina, E.N., Hit-and-Run: New Design Technique for Stabilization, Robustness and Optimization of Linear Systems, *IFAC Proceedings Volumes*, 2008, vol. 41, no. 2, pp. 376–380. (Proc. of the 17th IFAC World Congress, Seoul, July 2008, pp. 376–380.)

5.128. Polyak, B.T., Shcherbakov, P.S. and Topunov, M.V., Invariant Ellipsoids Approach to Robust Rejection of Persistent Disturbances, *Proc. of the 17th IFAC World Congress*, Seoul, July 2008, pp. 3976–3981.

5.129. Tremba, A., Calafiore, G., Dabbene, F., et al., RACT: Randomized Algorithms Control Toolbox for MATLAB, *Proc. of the 17th IFAC World Congress*, Seoul, July 6–11, 2008, pp. 390–395.

5.130. Polyak, B.T., Randomized Methods Based on New Monte-Carlo Schemes for Control and Optimization, *Efficient Monte-Carlo: From Variance Reduction to Combinatorial Optimization*, Conf. in Honor of R. Rubinstein, Sonderborg, Denmark, July 2008.

5.131. Polyak, B.T., The History of Mathematical Programming in the USSR: An Analysis Attempt, *Proc. of the 14th Baikal Int. School-Seminar "Optimization Methods and Their Applications"*, Irkutsk, July 2008, pp. 2–20. (Plenary talk.)

5.132. Polyak, B.T., Randomized Methods in Optimization and Control, *33-ya Dal'nevostochnaya matematicheskaya shkola-seminar im. ak. E.V. Zolotova* (The 33rd Far Eastern Mathematical School-Seminar named after Acad. E.V. Zolotov), Vladivostok, August 2008. (Plenary talk.)

5.133. Polyak, B.T. and Khlebnikov, M.V., The Linear Control Problem with Bounded Exogenous Disturbances: A New Approach, *Trudy 8-i Vserossiiskoi nauchnoi konferentsii "Nelineinye kolebaniya mekhanicheskikh sistem"* (Proc. of the 8th All-Russian Conf. "Nonlinear Oscillations of Mechanical Systems"), Nizhny Novgorod, September 2008, vol. 1, pp. 273–278.

5.134. Polyak, B.T. and Khlebnikov, M.V., Suppressing Bounded Exogenous Disturbances Using the Method of Invariant Ellipsoids: Output-Feedback Control, *Tezisy dokladov IX Krymskoi mezhdunarodnoi matematicheskoi shkoly "Metod funktsii Lyapunova i ego prilozheniya"* (Abstracts of IX Crimean Int. Mathematical School "The Method of Lyapunov Functions and Its Applications" (MFL-2008)), Alushta, Ukraine, September 15–20, 2008. Simferopol: Taurida national University, 2008, pp. 140–141.

5.135. Polyak, B.T. and Shcherbakov, P.S., Ellipsoidal Approximations of Reachability and Attraction Domains of Dynamic Systems with Bounded Inputs, *Trudy 15-i Mezhdunarodnoi konferentsii po avtomaticheskomu upravleniyu "Avtomatika-2008"* (Proc. of the 15th Int. Conf. on Automatic Control "Automation-2008"), Odessa, Ukraine, September 22–26, 2008, pp. 451–454.

5.136. Dabbene, F., Polyak, B.T., and Shcherbakov, P.S., A Randomized Cutting Plane Scheme with Geometric Convergence: Probabilistic Analysis and SDP Applications, *Proc. 47th IEEE Conf. on Decision and Control*, Cancun, Mexico, December 2008, pp. 3044–3049.

5.137. Polyak, B.T., Suppressing Bounded Disturbances in Linear and Nonlinear Systems, *Mul'tikonferentsiya "Teoriya i sistemy upravleniya"* (Multi-conference "Control Theory and Systems"), Moscow, January 2009. (Plenary talk.)

5.138. Polyak, B.T., Robustness in Control Systems, *Scientific Session of the Department for Nanotechnology and Information Technology, the Russian Academy of Sciences (ONIT RAS), dedicated to the 80th Anniversary of Academician S.V. Emelyanov*, Moscow, May 2009. (Plenary talk.)

5.139. Polyak, B.T. and Shcherbakov, P.S., Ellipsoidal Approximations to Attraction Domains of Linear Systems with Bounded Control, *Proc. of the American Control Conf. (ACC2009)*, St. Louis, Missouri, June 2009, pp. 5363–5367.

5.140. Polyak, B.T., Some New Randomized Methods for Control and Optimization, *The Workshop in Honor of the Sixtieth Birthday of Bob Barmish*, St. Louis, Missouri, June 2009. (Plenary talk.)

5.141. Polyak, B.T. and Gryazina, E.N., Robust Stabilization via Hit-and-Run Techniques, *Proc. of the IEEE Int. Symp. on Intelligent Control (ISIC-2009)*, Part of the 2009 IEEE Multi-Conf. on Systems and Control, St. Petersburg, Russia, July 2009, pp. 537–541.

5.142. Nazin, A.V. and Polyak, B.T., A Randomized Algorithm for Finding Eigenvector of Stochastic Matrix with Application to PageRank Problem, *Proc. of the IEEE Int. Symp. on Intelligent Control (ISIC-2009)*, Part of the 2009 IEEE Multi-Conf. on Systems and Control, St. Petersburg, Russia, July 2009, pp. 412–416.

5.143. Polyak, B.T. and Khlebnikov, M.V., Suppressing Non-random Bounded Disturbances in Linear Controllable Systems: Output-Feedback Control, *Tezisy dokladov XVI Mezhdunarodnoi konferentsii po avtomaticheskomu upravleniyu "Avtomatika-2009"* (Abstracts of the 16th Int. Conf. on Automatic Control "Automation-2009"), Chernivtsi, Ukraine, September 22–25, 2009. Chernivtsi: Knigi–XXI, 2009, pp. 44–45.

5.144. Nazin, A.V. and Polyak, B.T., Adaptive Randomized Algorithm for Finding Eigenvector of Stochastic Matrix with Application to PageRank, *Proc. of the Joint 48th IEEE Conf. on Decision and Control and 28th Chinese Control Conf.*, Shanghai, December 2009, pp. 127–132.

5.145. Polyak, B.T., New Approaches to Control and Filtering Problems, *7-ya Vserossiiskaya shkola-konferentsiya "Upravlenie bol'shimi sistemami"* (7th All-Russian School-Conference "Large Scale Systems Control"), Perm, May 2010. (Plenary talk.)

5.146. Gryazina, E.N. and Polyak, B.T., Efficient Random Walking, *Trudy 7-i Vserossiiskoi shkoly-konferentsii "Upravlenie bol'shimi sistemami"* (Proc. of the 7th All-Russian School-Conference "Large Scale Systems Control"), Perm, May 2010, pp. 23–24.

5.147. Polyak, B.T. and Khlebnikov, M.V., New Control Methods for Linear Systems with Bounded Exogenous Disturbances, *Trudy XI mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of XI Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 1–4, 2010, pp. 329–330.

5.148. Polyak, B.T., Robustness in Control Systems, *Mezhdunarodnaya Shkola-Seminar "Nelineinyi analiz i ekstremal'nye zadachi"* (Int. School-Seminar "Nonlinear Analysis and Extremal Problems"), Irkutsk, July 2010, pp. 261–273. (Plenary talk.)

5.149. Arzelier, D., Gryazina, E.N., Peaucelle, D., and Polyak, B.T., Mixed LMI/Randomized Methods for Static Output Feedback Control Design, *Proc. of the American Control Conf. (ACC2010)*, Baltimore, MD, July 2010, pp. 4683–4688.

5.150. Polyak, B.T., Simple Examples of Multi-Agent Dynamical Systems: Local Location Control which Guarantees Global Behavior, *The Mini-workshop "Glocal Control" at the IEEE Multi-Conf. on Systems and Control*, Yokohama, September 2010.

5.151. Polyak, B.T. and Gryazina, E.N., Markov Chain Monte Carlo Method Exploiting Barrier Functions with Applications to Control and Optimization, *Proc. of the IEEE Multi-Conf. on Systems and Control*, Yokohama, Japan, September 2010, pp. 1553–1557.

5.152. Polyak, B.T., Uniform Sampling by Random Walks in Convex Sets, *The Bristol Stochastic Approximation Workshop*, Bristol, UK, September 2010. (Plenary talk.)

5.153. Polyak, B.T., l_1-optimization Methods in Control and Filtering, *3-ya Mul'tikonferentsiya po problemam upravleniya* (The 3rd Multiconference on Control Problems), St. Petersburg, October 2010. (Plenary talk.)

5.154. Polyak, B.T. and Gryazina, E.N., Hit-and-Run: Randomized Technique for Control Problems Recasted as Concave Programming, *IFAC Proceedings Volumes*, 2011, vol. 44, no. 1, pp. 2321–2325. (Proc. of the 18th IFAC World Congress, Milan, Italy, August 2011.)

5.155. Polyak, B.T. and Timonina, A.V., PageRank: New Regularizations and Simulation Models, *IFAC Proceedings Volumes*, 2011, vol. 44, no. 1, pp. 11202–11207. (Proc. of the 18th IFAC World Congress, Milan, Italy, August 2011.)

5.156. Polyak, B. and Khlebnikov, M., Design of Dynamic Controller for Rejection of Persistent Disturbances, *Proc. of PHYSCON-2011*, Leon, Spain, September 2011.

5.157. Polyak, B.T., Reducing the Number of Control States or Outputs in Linear Systems, *Proc. of the Russian–Swedish Control Conf.*, St. Petersburg, September 2011.

5.158. Polyak, B.T., L_1-optimization Methods in Control Problems, *Mezhdunarodnaya konferentsiya po avtomaticheskomu upravleniyu "Avtomatika-2011"* (The Int. Conf. on Automatic Control "Automation-2011"), Lviv, Ukraine, September 2011. (Plenary talk.)

5.159. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., ℓ_1-approaches to Designing Low-Order Stabilizing Controllers, *Trudy Mezhdunarodnoi konferentsii po avtomaticheskomu upravleniyu "Avtomatika-2011"* (Proc. of the Int. Conf. on Automatic Control "Automation-2011"), Lviv, Ukraine, September 2011.

5.160. Polyak, B.T., L_1 Techniques in Control Problems, *Proc. of the Conf. "Optimization, Games and Dynamics"*, Paris, France, November 2011.

5.161. Karzan, F., Nemirovski, A.S., Polyak, B., and Juditsky, A., On the Accuracy of l_1-filtering of Signals with Block-sparse Structure, in *Advances in Neural Information Processing Systems (NIPS-2011)*, MIT Press, 2011, pp. 1260–1268.

5.162. Polyak, B.T., L_1 Problems in Control and Numerical Methods for Their Solution, *Proc. of the Conf. "Numerical Analysis and Optimization—Theory and Applications"*, King Fahd University of Petroleum and Minerals, Saudi Arabia, December 2011.

5.163. Polyak, B.T., Non-smooth Optimization Problems Arising in Robust Optimization, Estimation, and Ranking, *3-ya Mezhdunarodnaya konferentsiya "Matematicheskoe modelirovanie, optimizatsiya i informatsionnye tekhnologii"* (3rd Int. Conf. "Mathematical Modeling, Optimization and Information Technologies"), Chisinau, Moldova, March 2012.

5.164. Polyak, B.T., Robust Eigenvector of a Stochastic Matrix with Application to PageRank, *Applied Statistics and Probability Theory Colloquium*, April 2012, Royal Holloway University of London, Egham, UK.

5.165. Polyak, B.T., Robust Eigenvector Problem and Its Application to PageRank, *The 25th European Conf. on Operational Research "EURO 2012"*, *Vilnius*, Lithuania, July 2012. (Plenary talk.)

5.166. Polyak, B.T., L_1 Problems in Control, *The 10th EUROPT Workshop on Advances in Continuous Optimization*, Shiaulai, Lithuania, July 2012. (Plenary talk.)

5.167. Polyak, B.T., Shcherbakov, P.S., and Khlebnikov, M.V., LMI-Based Approaches to Some "Inconvenient" Sparse Optimization Problems, *XIX Mezhdunarodnaya konferentsiya po avtomaticheskomu upravleniyu "Avtomatika-2012"* (XIX Int. Conf. on Automatic Control "Automation-2012"), Kiev, Ukraine, September 2012. (Plenary talk.)

5.168. Polyak, B.T., L_1 Problems in Control and Numerical Methods for Their Solution, *Proc. of the 3rd Int. Conf. "Optimization Methods and Applications" (OPTIMA-2012)*, Costa da Caparica, Portugal, September 2012. (Invited lecture.)

5.169. Juditsky, A. and Polyak, B., Robust Eigenvector of a Stochastic Matrix with Application to PageRank, *Proc. of the 51st IEEE Conf. on Decision and Control*, Maui, Hawaii, December 2012, pp. 3171–3176.

5.170. Polyak, B.T., New Random Sampling: Billiard Walks, *Proc. of the Workshop "Optimization and Statistical Learning"*, Les Houches, France, January 2013.

5.171. Polyak, B.T., Robust Principal Component Analysis, *Proc. of the Workshop "Advances in Predictive Modeling and Optimization"*, Berlin, Germany, May 2013.

5.172. Polyak, B.T., Robust Principal Component Analysis, *Proc. of the Int. Conf. on Robust Statistics*, St. Petersburg, Russia, July 2013. (Keynote lecture.)

5.173. Polyak, B., Khlebnikov, M., and Shcherbakov, P., An LMI Approach to Structured Sparse Feedback Design in Linear Control Systems, *Proc. of the European Control Conf.*, Zurich, Switzerland, July 2013, pp. 833–838.

5.174. Gagloev, A.V. and Polyak, B.T., Comparing Different Methods of Fastest Descent for a Quadratic Function, *Tezisy dokladov 56-i Nauchnoi konferentsii MFTI* (Abstracts of the 56th Scientific Conference of MIPT), Moscow, November 2013, vol. 1, pp. 120–121.

5.175. Polyak, B.T., Control of Linear Systems under Uncertainty: Some Approaches, *Scientific Session of the Department for Nanotechnology and Information Technology, the Russian Academy of Sciences (ONIT RAS), dedicated to the 85th Anniversary of Academician S.V. Emelyanov*, Moscow, May 2014. (Plenary talk.)

5.176. Polyak, B.T., The Theory of Linear Systems Is Still a Source of Interesting Problems, *12-oe Vserossiiskoe soveshchanie po problemam upravleniya* (The 12th All-Russian Meeting on Control Problems), Moscow, June 2014. (Plenary talk.)

5.177. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Sparse Feedback Design in Linear Control Systems, *Trudy 12-go Vserossiiskogo soveshchaniya po problemam upravleniya* (Proc. of the 12th All-Russian Meeting on Control Problems), Moscow, June 2014, pp. 218–229.

5.178. Polyak, B.T. and Tremba, A.A., Analytical Solution of the Linear Differential Equation with Identical Roots of the Characteristic Polynomial, *Trudy 12-go Vserossiiskogo soveshchaniya po problemam upravleniya* (Proc. of the 12th All-Russian Meeting on Control Problems), Moscow, June 2014, pp. 212–217.

5.179. Polyak, B.T. and Gryazina, E.N., Billiard Walk – a New Sampling Algorithm for Control and Optimization, *IFAC Proceedings Volumes*, 2014, vol. 47, no. 3, pp. 6123–6128. (Proc. of the 19th IFAC World Congress, Cape Town, August 2014.)

5.180. Polyak, B.T. and Smirnov, G.V., Large Deviations in Continuous-Time Linear Single-Input Control Systems, *Proc. of the 19th IFAC World Congress*, Cape Town, August 2014, pp. 5586–5591.

5.181. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Estimates of Large Deviations in Continuous-Time Systems, *Trudy 21-i Mezhdunarodnoi konferentsii po avtomaticheskomu upravleniyu "Avtomatika-2014"* (Proc. of the 21st Int. Conf. on Automatic Control "Automation-2014"), Kiev, Ukraine, September 2014, pp. 14–15. (Plenary talk.)

5.182. Kvinto, Ya.I., Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., Some Experiments on Obtaining Sparse Controllers, *11-ya Vserossiiskaya shkola-konferentsiya "Upravlenie bol'shimi sistemami"* (11th All-Russian School-Conference "Large Scale Systems Control"), Arzamas, September 2014, pp. 227–238.

5.183. Polyak, B., Shcherbakov, P., and Khlebnikov, M., Quadratic Image of a Ball: Towards Efficient Description of the Boundary, *Proc. of the 18th Int. Conf. on System Theory, Control and Computing*, Sinaia, Romania, October 2014, pp. 105–112.

5.184. Polyak, B.T., Quadratic Transformations: Convexity vs Nonconvexity, *Proc. of the Workshop "Frontiers of High Dimensional Statistics, Optimization, and Econometrics"*, Moscow, Higher School of Economics, February 2015.

5.185. Polyak, B.T., Shcherbakov, P.S., and Gryazina, E.N., Quadratic Transformations with Applications to Power Systems, *Proc. of the Conf. "Advanced Mathematical Methods for Energy Systems: From Theory to Practice"*, SkolTech, Moscow, June 2015.

5.186. Polyak, B.T., Quadratic Transformations: Convexity vs Nonconvexity, *The 1st IFAC Conf. on Modelling, Identification and Control of Nonlinear Systems (MICNON-2015)*, St. Petersburg, June 2015. (Plenary talk.)

5.187. Polyak, B.T., V.A. Yakubovich and Convexity of Quadratic Transformations, *Proc. of the 1st IFAC Conf. on Modelling, Identification and Control of Nonlinear Systems (MICNON-2015)*, St. Petersburg, June 2015.

5.188. Polyak, B.T. and Gryazina, E.N., Convexity/Nonconvexity Certificates for Power Flow Analysis, *Proc. of the Int. Symp. on Energy System Optimization*, HITS, Heidelberg, Germany, November 2015.

5.189. Polyak, B.T., Quadratic Transformations: Feasibility and Convexity, *The Workshop "Optimization Without Borders"*, Les Houches, France, February 2016.

5.190. Polyak, B.T., Optimization and Asymptotic Stability, *Trudy 13-i mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 13th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 2016, pp. 289–294. (Plenary talk.)

5.191. Barabanov, N., Ortega, R., Grino, R., and Polyak, B., On Existence and Stability of Equilibria of Linear Time-Invariant Systems with Constant Power Loads, *Proc. of the European Control Conf. (ECC 2016)*, Aalborg, Denmark, June 2016.

5.192. Polyak, B.T., Optimization and Asymptotic Stability, *The 7th Int. Conf. "Optimization and Applications"*, Petrovac, Montenegro, September 2016. (Plenary talk.)

5.193. Polyak, B.T., Robust Principal Component Analysis, *The 11th Int. Conf. "Intelligent Data Processing"*, Barcelona, Spain, October 2016. (Plenary talk.)

5.194. Polyak, B.T., Optimization and Asymptotic Stability, *VIII Moscow Int. Conf. on Operations Research (ORM-2016)*, Moscow, October 2016. (Plenary talk.)

5.195. Polyak, B.T., Solving Underdetermined Nonlinear Equations, *Proc. of the Workshop in Optimization to Celebrate Roman Polyak's 80th Birthday*, Technion, Haifa, April 2017.

5.196. Polyak, B.T. and Tremba, A.A., New Newton-like Algorithm for Underdetermined Equations, *Trudy Mezhdunarodnoi konferentsii "Konstruktivnyi negladkii analiz i smezhnye voprosy"* (Proc. Int. Conf. "Constructive Nonsmooth Analysis and Related Issues"), St. Petersburg, Russia, May 2017. Dedicated to the memory of V.F. Demyanov.

5.197. Polyak, B.T., Estimates of Transients in Linear Differential and Difference Equations, *Proc. of the IFAC Workshop "Roberto Tempo" on Uncertain Dynamical Systems*, Banyuls-sur-Mer, France, July 2017.

5.198. Polyak, B. and Khlebnikov, M., Robust Principal Component Analysis: An IRLS Approach, *IFAC-PapersOnLine*, 2017, vol. 50, no. 1, pp. 2762–2767. (Proc. of the 20th IFAC World Congress, Toulouse, France, July 2017.)

5.199. Polyak, B. and Shcherbakov, P., Lyapunov Functions: An Optimization Theory Perspective, *IFAC-PapersOnLine*, 2017, vol. 50, no. 1, pp. 7456–7461. (Proc. of the 20th IFAC World Congress, Toulouse, France, July 2017.)

5.200. Polyak, B. and Khlebnikov, M., Robust Versions of the Principal Component Analysis, *Proc. of the Int. Conf. on Big Data Analytics, Data Mining and Computational Intelligence (BigDaCI 2017)*, Lisbon, Portugal, July 2017, pp. 247–254.

5.201. Polyak, B., Scherbakov, P., and Smirnov, G., Estimates of Transients in Stable Linear Difference Equations, *Abstracts of the 23rd Int. Conf. on Difference Equations and Applications (ICDEA 2017)*, Timisoara, Romania, July 2017, pp. 105–106.

5.202. Polyak, B. and Tremba, A., Norm Variability in Newton Method for Underdetermined Systems of Equations, *Proc. of the 17th Baikal Int. School-Seminar "Optimization Methods and Their Applications"*, Baikal, Russia, August 2017.

5.203. Polyak, B.T., Monte Carlo and Randomization in Optimization Problems, *Trudy Shkoly-konferentsii molodykh uchenykh "Informatsionnye tekhnologii i sistemy – 2017"* (Proc. of the School-Conference of Young Researchers "Information Technologies and Systems – 2017"), Ufa, Russia, September 2017.

5.204. Polyak, B. and Tremba, A., Newton Method with Adaptive Step-size for Under-determined Systems of Equations, *Proc. of the 8th Int. Conf. on Optimization and Applications (OPTIMA-2017)*, Petrovac, Montenegro, October 2017, pp. 475–480. (Plenary talk.)

5.205. Polyak, B.T., Fast Gradient Methods Revisited, *proc. of the 69th Int. Workshop "Variational Analysis and Applications"*, Erice, Italy, September 2018.

5.206. Polyak, B. and Tremba, A., Local and Global Convergence of Frank–Wolfe Method on Stiefel Manifolds for Non-convex Functions, *Proc. of the 8th Int. Conf. "Optimization and Applications" (OPTIMA-2018)*, Petrovac, Montenegro, October 2018.

5.207. Balashov, M., Polyak, B.T., and Tremba, A., The Gradient Projection Algorithm for Smooth Sets and Functions. Nonconvex Case, *Proc. of the Conf. "Quasilinear Equations, Inverse Problems and Their Applications"*, Dolgoprudny, Russia, December 2018.

5.208. Shalbi, L. and Polyak, B.T., Stabilizing Spacecraft at Interior Lagrangian Point with Minimal Fuel Consumption, *Trudy Vserossiiskogo soveshchaniya po problemam upravleniya (VSPU-2019)* (Proc. of the All-Russian Meeting on Control Problems (AMCP-2019)), Moscow, June 2019, pp. 1016–1021.

5.209. Polyak, B.T., Accelerated Optimization Methods Revisited, *Proc. of the Symp. "The Issues of Calculations Optimization" (ISCOPT-XLV)*, Kiev, Ukraine, September 2019.

5.210. Polyak, B.T. and Tremba, A., Sparse Solutions of Optimal Control with L_1-objective, *Proc. of the European Control Conf. (ECC-2020)*, St. Petersburg, Russia, May 2020, pp. 1707–1712.

5.211. Polyak, B.T., Centennary of Yakov Zalmanovich Tsypkin's Birth, *IFAC-PapersOnLine: Preprints of the 21st IFAC World Congress (Virtual)*, Berlin, Germany, July 12–17, 2020, pp. 1401–1404.

5.212. Fradkov, A. and Polyak, B.T., Adaptive and Robust Control in the USSR, *IFAC-PapersOn Line*, 2020, vol. 53, no. 2, pp. 1373–1378. (Proc. of the 21th IFAC World Congress, Berlin, July 2020.)

5.213. Shcherbakov, P., Dabbene, F., and Polyak, B., Solutions of Stable Difference Equations Probably Experience Peak, *IFAC-PapersOnLine*, 2020, vol. 53, no. 2, pp. 4762–4767. (Proc. of the 21th IFAC World Congress, Berlin, July 2020.)

5.214. Polyak, B.T., Accelerated Gradient Methods: History and Properties, *Proc. of the 7th Int. Conf. on Control and Optimization with Industrial Applications (COIA-2020)*, Baku, August 2020, vol. 1, pp. 23–26. (Plenary talk.)

5.215. Polyak, B.T. and Fatkhullin, I., Static Feedback in Linear Control Systems as Optimization Problem, *Proc. of the 11th Int. Conf. "Optimization and Applications" (OPTIMA-2020)*, Petrovac, Montenegro, September 2020. (Invited talk.)

5.216. Khlebnikov, M.V. and Polyak, B.T., Filtering under Arbitrary Bounded Exogenous Disturbances: the Technique of Linear Matrix Inequalities, *Trudy 13-i Mul'tikonferentsii po problemam upravleniya (MKPU-2020)* (Proc. of the 13th Multiconference on Control Problems (MCCP-2020)), St. Petersburg, October 6–8, 2020, St. Petersburg: Concern CSRI Elektropribor, 2020, pp. 291–294. (In memory of N.N. Ostryakov, an outstanding designer of gyroscopic devices.)

5.217. Khlebnikov, M.V., Polyak, B.T., and Shcherbakov, P.S., Sparse Feedback Design in Linear Control Systems, *Trudy konferentsii "Matematicheskaya teoriya upravleniya i ee prilozheniya"* (Proc. of the Conference "Mathematical Theory of Control and Its Applications"), St. Petersburg, October 2020.

5.218. Polyak, B.T., Optimization and Control: Mutual Connections, *Proc. of the Conf. "Optimization without Borders"*, Sochi, Russia, July 2021.

5.219. Khlebnikov, M.V. and Polyak, B.T., Suppressing Exogenous Disturbances as an Optimization Problem, *Trudy XIV Vserossiiskoi Mul'tikonferentsii po problemam upravleniya. Lokal'naya nauchnotekhnicheskaya konferentsiya "Upravlenie v raspredelennykh i setevykh sistemakh"* (Proc. of XIV All-Russian Multiconf. on Control Problems. Local Scientific and Technical Conf. "Control in Distributed and Networked Systems"), Divnomorskoe, Gelendzhik, September 27 – October 2, 2021, pp. 78–81.

5.220. Polyak, B.T., The Unreasonable Effectiveness of Optimization in Applications: Personal Experience, *INFORMS Annual Meeting*, Anaheim, CA, October 2021. (Khachyan prize talk.)

5.221. Shatov, D.V., Polyak, B.T., and Khlebnikov, M.V., PID Controller Design for Tracking Systems by the Quadratic Criterion, *Trudy XVIII Vserossiiskoi shkoly-konferentsii "Upravlenie bol'shimi sistemami"* (Proc. of XVIII All-Russian School-Conference "Large Scale Systems Control"), Chelyabinsk, September 5–8, 2022, pp. 565–568.

5.222. Polyak, B.T. and Khlebnikov, M.V., Feedback Choice in Control Problems as an Optimization Problem, *Trudy XVI mezhdunarodnoi konferentsii "Ustoichivost' i kolebaniya nelineinykh sistem upravleniya"* (Proc. of the 16th Int. Conf. "Stability and Oscillations of Nonlinear Control Systems"), Moscow, June 1–3, 2022, pp. 355–358. (Plenary talk.)

5.223. Polyak, B.T., Feedback Design in Control as Optimization Problem, *Proc. of the 18th IFAC Workshop on Control Applications of Optimization*, Gif-sur-Ivette, France, July 2022.

5.224. Polyak, B.T., The Unreasonable Effectiveness of Optimization in Applications: Personal Experience, *Letnyaya shkola "Obuchenie, ponimanie i optimizatsiya v modelyakh iskusstvennogo intellekta"* (Summer School "Learning, Understanding, and Optimization in Artificial Intelligence Models"), Pushkino, Russia, June 2022.

PREFACES, REVIEWS, PERSONAL COMMUNICATIONS, ETC.

6.1. Polyak, B.T., Reviews in *Novye knigi za rubezhom* (New Books Abroad): 1962, no. 4; 1971, no. 10; 1982, no. 9; 1983, no. 1, no. 2, no. 3, and no. 10; 1984, no. 5; 1985, no. 6; 1986, no. 9; 1987, no. 1, no. 4, and no. 11; 1988, no. 3; 1989, no. 5 and no. 10; 1990, no. 11 and no. 12; 1991, no. 11 and no. 12.

6.2. Dynkin, E.B., Kolgomorov, A.N., Polyak, B.T., and Freidlin, M.I., Igor' Vladimirovich Girsanov, *Theory of Probability & Its Applications*, 1967, vol. 12, no. 3, pp. 482–486.

6.3. Polyak, B.T., Editing the book: *Vychislitel'nye metody i programmirovanie* (Computational Methods and Programming), Moscow: Moscow State University, 1967, vol. 9.

6.4. Polyak, B.T., Editing the Russian translation of the book: Aris, R., *Discrete Dynamic Programming. An Introduction to the Optimization of Staged Processes*, Blaisdell, 1964. The Russian version was published by Mir in 1969.

6.5. Polyak, B.T., Editing the book and writing its preface: Girsanov, I.V., *Lektsii po matematicheskoi teorii ekstremal'nykh zadach* (Lectures on the Mathematical Theory of Extremal Problems), Moscow: Moscow State University, 1970. (An English translation was published by Springer in 1972.)

6.6. Polyak, B.T., Editing the Russian translation of the book: Aoki, M., *Introduction to Optimization Techniques: Fundamentals and Applications of Nonlinear Programming*, Macmillan, 1971. The Russian version was published by Nauka in 1977.

6.7. Polyak, B.T., Editing the book: Vainikko, G.M. and Veretennikov, A.Yu., *Iteratsionnye protsedury v nekorrektnykh zadachakh* (Iterative Procedures in Ill-Posed Problems), Moscow: Nauka, 1986.

6.8. Polyak, B.T. and Tsypkin, Ya.Z., Mira Ch. Chaotic Dynamics, *Avtomat. i Telemekh.*, 1989, no. 5, pp. 188–189.

6.9. Polyak, B.T. and Tsypkin, Ya.Z., International Symposium on Robust Control, *Avtomat. Telemekh.*, 1993, no. 1, pp. 185–187.

6.10. Polyak, B.T., International Symposium "Robustness in Identification and Control" (Siena, Italy, June 30–August 2, 1998), *Avtomat. Telemekh.*, 1999, no. 8, pp. 189–190.

6.11. Polyak, B.T., Yakov Zalmanovich Tsypkin — 19 September 1919 – 2 December 1997 (Obituary), *J. Difference Eq. Appl.*, 1999, vol. 5(3), pp. 217–218.

6.12. Polyak, B.T., He Was a Lucky Person..., *Int. J. of Adapt. Contr. and Signal Proc.*, 2001, vol. 15, no. 2, pp. 113–114.

6.13. Polyak, B.T., Fourth Russian–Swedish Conference on Control, *Autom. Remote Control*, 2001, vol. 62, no. 7, pp. 1220–1222.

6.14. Polyak, B.T., Textbook of a New Type. Review of the Book of B.R. Andrievskii and A.L. Fradkov "Elements of Mathematical Modeling in the MATLAB 5 and Scilab Software Environments" (St. Petersburg: Nauka, 2001), *Autom. Remote Control*, 2001, vol. 62, no. 10, pp. 1756–1757.

6.15. Kurzhanski, A.B., Polyak, B.T., and Chernousko, F.L., Fifteenth International IFAC Congress, *Autom. Remote Control*, 2003, vol. 64, no. 1, pp. 170–173.

6.16. Polyak, B.T., Selected Works of E.S. Pyatnitskii in 3 Volumes. Moscow: Fizmatlit, 2004, vol. 1; 2005, vol. 2, *Autom. Remote Control*, 2005, vol. 66, no. 10, pp. 1701–1702

6.17. Polyak, B.T., Editing the book: *Yakov Zalmanovich Tsypkin (1919–1997)*, Moscow: LKI, 2007; writing two chapters of this book: "Laboratory Named after Ya.Z. Tsypkin" (pp. 86–101) and "He Was a Lucky Person" (pp. 192–194).

6.18. Arutyunov, A.V. and Polyak, B.T., B.S. Mordukhovich. Variational Analysis and Generalized Differentiation. I. Basic Theory, II. Applications, *Autom. Remote Control*, 2009, vol. 70, no. 6, pp. 1086–1087.

6.19. Polyak, B.T. and Gryazina, E.N., First Traditional Russian Youth Summer School "Control, Information, and Optimization," *Autom. Remote Control*, 2009, vol. 70, no. 11, pp. 1923–1925.

6.20. Polyak, B.T., A.S. Poznyak. Advanced Mathematical Tools for Automatic Control Engineers. Vol. I. Deterministic Systems. Elsevier, Amsterdam, 2008, *Autom. Remote Control*, 2009, vol. 70, no. 11, pp. 1926–1927.

6.21. Polyak, B.T., Editing the book and writing its preface: Nesterov, Yu.E., *Vvedenie v vypukluyu optimizatsiyu* (An Introduction to Convex Optimization), Moscow: Moscow Center for Continuous Mathematical Education, 2010.

6.22. Polyak, B.T., Editing the book and writing its preface: Dikin, I.I., *Metod vnutrennikh tochek v lineinom i nelineinom programmirovanii* (Interior Point Method in Linear and Nonlinear Programming), Moscow: URSS, 2010.

6.23. Polyak, B.T., Writing the preface to the 2nd edition of the book: Pervozvanskii, A.A., *Kurs teorii avtomaticheskogo upravleniya* (Course of Automatic Control Theory), St. Petersburg: Lan', 2012.

6.24. Polyak, B.T., Stepanov, O.A., and Fradkov, A.L., 19th IFAC World Congress, *Avtomat. Telemekh.*, 2015, no. 2, pp. 150–156.

6.25. Polyak, B.T., Stepanov, O.A., and Fradkov, A.L., Towards the Results of the 19th IFAC World Congress (August 24–29, 2014, Cape Town, South Africa), *The 28th General Meeting of the Academy of Navigation and Motion Control*, St. Petersburg, October 9, 2014.

6.26. Polyak, B.T., Writing the preface to the monograph: Granichin, O., Volkovich, Z., and Toledano-Kitai, D., *Randomized Algorithms in Automatic Control and Data Mining*, Springer, 2015.

6.27. Polyak, B.T., On the 100th Anniversary of Yakov Zalmanovich Tsypkin's Birth, *Control Engineering Rossiya*, 2019, vol. 80, no. 2, pp. 78–80.

6.28. Polyak, B.T., Optimization, in *Matematicheskaya sostavlyayushchaya* (The Mathematical Component), Andreev, N.N., Konovalov, S.P., and Panyunin, N.M., Moscow: Mathematical Etudes Foundation, 2019.

6.29. Polyak, B.T. and Fradkov, A.L., On the 100th Anniversary of Yakov Zalmanovich Tsypkin's Birth, *The 28th General Meeting of the Academy of Navigation and Motion Control*, St. Petersburg, October 9, 2014.

6.30. Polyak, B.T. and Nazin, A.V., The 100th Birthday of Yakov Zalmanovich Tsypkin, *Avtomat. Telemekh.*, 2019, no. 9, pp. 6–8.

6.31. Polyak, B.T., Writing the preface to the book: Gasnikov, A.V., *Sovremennye chislennye metody optimizatsii: metod universal'nogo gradientnogo spuska* (Modern Numerical Optimization methods: The Method of Universal Gradient Descent), Moscow: Moscow Center for Continuous Mathematical Education, 2020.

6.32. Polyak, B.T., On the 70th Anniversary of K. Aida-zade, in *K. Aida-zade*, Baku: Elm, 2021, pp. 141–143.

At the time of compiling this list, some scientometric indicators of Boris Polyak's publications on Scopus/Google Scholar were as follows: the Hirsch index, 33/64; the number of citations, over 9 100/31 600 (with a constant growth rate since 2004 and an "exponential" one since 2015); the two most cited articles, [3.6] (over 1 600/3 300 citations) and [3.67] (about 1 100/2 300 citations); the number of coauthors, 81.

The compilers of this list are grateful to the readers for possible additions as well as indications of inaccuracies and simple mistakes. In our point of view, for experts in the corresponding fields of science, the list of Boris Polyak's research works well characterizes the epoch and reflects the popularity of various topics and the change of priorities over time. A complete and accurate bibliography will be published in a separate brochure.

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, pp. 479–490.

TOPICAL ISSUE

An Optimal Choice of Characteristic Polynomial Roots for Pole Placement Control Design

V. A. Alexandrov

Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia
e-mail: va.alexandrov@yandex.ru

Received January 25, 2024
Revised March 12, 2024
Accepted March 20, 2024

Abstract—The problem of finding the arrangement of closed-loop control system poles that minimizes an objective function is considered. The system optimality criterion is the value of the H_∞ norm of the frequency transfer function relative to the disturbance with constraints imposed on the system pole placement and the values of the H_∞ norm of the sensitivity function and the transfer function from measurement noise to control. An optimization problem is formulated as follows: the vector of variables consists of the characteristic polynomial roots of the closed loop system with the admissible values restricted to a given pole placement region; in addition to the optimality criterion, the objective function includes penalty elements for other constraints. It is proposed to use a logarithmic scale for the moduli of the characteristic polynomial roots as elements of the vector of variables. The multi-extremality problem of the objective function is solved using the multiple start procedure. A coordinate descent modification with a pair of coordinates varied simultaneously is used for search.

Keywords: control design, transfer function, pole placement, optimization, robust system

DOI: 10.31857/S0005117924050038

1. INTRODUCTION

Rejection of an unmeasurable disturbance is one of the main tasks of control design [1]. On the other hand, the resulting system must satisfy robustness conditions since the plant model used for control design is inaccurate. For linear systems, first of all, the requirements for stability margins must be met [2]. These requirements can be specified as the minimum acceptable stability margins radius [3] or limiting the value of the sensitivity function [4, 5]. The H_∞ norm of the measurement noise sensitivity function can serve as a measure of robustness to unmodeled dynamics [5, 6].

Many control design techniques lead to an optimization problem. For example, the methods of H_∞ optimization [7] and invariant ellipsoids [1] reduce to an optimization procedure for solving a system of linear matrix inequalities. If the variables are the coefficients of a fixed-structure controller, the optimization problem may become non-convex and multi-extremal [8, 9]. The successful results of solving such problems allowed developing similar approaches for tuning PID controllers widely used in practice [10, 11].

For a linear single-input single-output (SISO) system, the following idea of optimization of the closed-loop system pole placement was proposed in [12]: the controller coefficients are found via the standard pole placement procedure, and the roots of the desired characteristic polynomial of the closed loop system are searched using an optimization procedure for specified quality criteria and constraints. The standard global optimization procedure from the MATLAB Global Optimization Toolbox [13] was used in [12]. The value of the H_∞ norm of the transfer function relative to the disturbance was chosen as a quality criterion under given constraints on the values of the H_∞ norms

of the sensitivity function and the transfer function from measurement noise to control. In addition, constraints were imposed on the system pole placement.

This article is devoted to developing an optimization procedure for finding an optimal closed-loop system pole placement that minimizes a given objective function subject to specified constraints; in the corresponding optimization problem, the vector of variables consists of the characteristic polynomial roots of the closed loop system.

2. PROBLEM STATEMENT

Consider a linear SISO system whose structure is presented in Fig. 1. Let the plant be described by the transfer function

$$P(s) = \frac{b(s)}{a(s)} = \frac{b_{n-1}s^{n-1} + \cdots + b_0}{s^n + a_{n-1}s^{n-1} + \cdots + a_0}, \tag{1}$$

where s is the Laplace transform variable; the coefficients $a_i, b_i \in \mathbb{R}$ $(i = 0, \ldots, n-1)$ have known values, and at least one of the coefficients b_i is nonzero; the polynomials $a(s)$ and $b(s)$ are coprime. The frequency response function is obtained for $s = j\omega$, where $\omega \in [0, \infty)$. By assumption, as frequency response functions are used, all system signals (including the unmeasured exogenous disturbance) are integrable and satisfy the restrictions for applying the Fourier transform [2]:

$$\int_{-\infty}^{+\infty} |f(t)|dt < \infty.$$

Suppose that the controller's transfer function has the form

$$C(s) = \frac{d(s)}{c(s)} = \frac{d_{n-1}s^{n-1} + \cdots + d_0}{c_{n-1}s^{n-1} + \cdots + c_0}, \tag{2}$$

where the controller order $(n-1)$ is determined by the order of the plant model (1). A higher-order controller, which can be constructed, e.g., by adding an integral component to the controller, is not considered here. A lower-order controller cannot be constructed by the pole placement technique; see the explanation below.

According to the pole placement method [14, 15], the polynomials $c(s)$ and $d(s)$ of the controller (2) can be obtained by solving the equation

$$a(s)c(s) + b(s)d(s) = \delta(s), \tag{3}$$

where the left-hand side is the characteristic polynomial of system (1), (2) in which $a(s)$ and $b(s)$ are the known polynomials of the plant's transfer function, and $\delta(s)$ is a given desired characteristic polynomial. As is known [14], there exists a unique solution of this equation under the condition $\deg d(s) < \deg a(s)$ or $\deg c(s) < \deg b(s)$. In addition, under the condition $\deg \delta(s) \geqslant 2\deg a(s) - 1$,

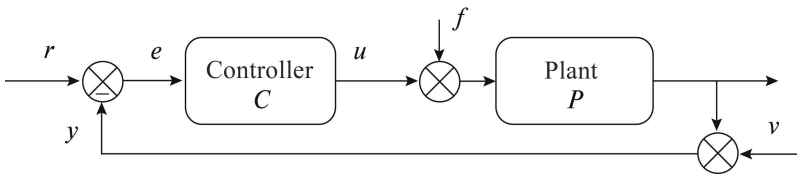

Fig. 1. Closed loop system: y—measured output, ν—measurement noise, r—reference signal, e—control error, u—control, and f—disturbance.

the causality of control is satisfied: $\deg d(s) \leqslant \deg c(s)$. Then, by choosing the desired polynomial $\delta(s)$ of degree $\deg \delta(s) = 2\deg a(s) - 1$, we obtain the solution (2) for which the conditions $\deg d(s) \leqslant \deg c(s)$ and $\deg d(s) < \deg a(s)$ hold. In this case, equation (3) can be solved by compiling a system of $2n$ linear algebraic equations with $2n$ unknowns when equating the coefficients of the left- and right-hand sides of equation (3) at the equal powers of s:

$$\begin{bmatrix} c_{n-1} \\ \dots \\ c_0 \\ d_{n-1} \\ \dots \\ d_0 \end{bmatrix} = W^{-1} \begin{bmatrix} \delta_{2n-1} \\ \dots \\ \delta_0 \end{bmatrix}, \tag{4}$$

where $W \in \mathbb{R}^{2n\times 2n}$ is a matrix obtained from the coefficients a_i, b_i $(i = 0, \dots, n-1)$.

Thus, for any plant (1), one can find a controller of the form (2) ensuring any given characteristic polynomial $\delta(s)$ of degree $(2n-1)$ for the closed loop system. Note that for an unstable plant of order n, there may not exist a controller of order below $(n-1)$ ensuring at least the stability of the system. Therefore, we consider a controller of order $(n-1)$ to ensure not only stability but also other system properties of the system by choosing an appropriate desired characteristic polynomial.

The characteristic polynomial can be represented as

$$\delta(s) = \prod_{i=1}^{n_r} (s + \lambda_i) \prod_{k=1}^{n_c} (s^2 + 2\zeta_k \breve{\omega}_k s + \breve{\omega}_k^2), \tag{5}$$

where $n_r = 2n - 2n_c - 1$ is the number of real roots of the polynomial $\delta(s)$ and n_c is the number of complex conjugate pairs of the roots; the values $\lambda_i, \breve{\omega}_k \in \mathbb{R}$ and $\zeta_k \in [0, 1]$ determine the closed-loop pole placement and the coefficients $\delta_0, \dots, \delta_{2n-2}$ in (4) while $\delta_{2n-1} = 1$. Let $\breve{\omega}_k$ denote the natural frequencies of the system since the notation ω is used for the frequency variable in transfer functions.

In addition to the standard constraints $\lambda_i > 0$, $\breve{\omega}_k > 0$, and $0 < \zeta_k \leqslant 1$, which ensure the stability of the closed loop system, it is possible to specify the supplementary ones

$$0 < \lambda_{\min} \leqslant \lambda_i \leqslant \lambda_{\max}, \quad 0 < \breve{\omega}_{\min} \leqslant \breve{\omega}_k \leqslant \breve{\omega}_{\max}, \quad 0 < \zeta_{\min} \leqslant \zeta_k \leqslant 1 \tag{6}$$

to obtain the desired speed and damping rate of the system and limit the high-frequency components.

Similar to [12], the value of H_∞ norm of the frequency response function relative to the disturbance is taken as the system quality criterion:

$$\|G_{yf}(j\omega)\|_\infty = \sup_\omega \left| \frac{b(j\omega)c(j\omega)}{\delta(j\omega)} \right|. \tag{7}$$

Moreover, the following constraints must be satisfied:

— for the H_∞ norm of the sensitivity function, the inequality

$$\|S(j\omega)\|_\infty = \sup_\omega \left| \frac{a(j\omega)c(j\omega)}{\delta(j\omega)} \right| \leqslant S_{\max} \tag{8}$$

to ensure the required stability margins;

— for the H_∞ norm of the frequency response function relative to the noise, the inequality

$$\|G_{u\nu}(j\omega)\|_\infty = \sup_\omega \left|\frac{a(j\omega)d(j\omega)}{\delta(j\omega)}\right| \leqslant N_{\max} \tag{9}$$

to ensure the robustness of the system in the presence of unmodeled dynamics by limiting the controller gain [5, 6].

Thus, the problem is to find a controller of the form (2) that minimizes the exogenous disturbance effect on the the plant (1) in terms of the norm (7) subject to the constraints (6), (8), and (9) under given values a_i, b_i $(i = 0, \ldots, n-1), \lambda_{\min}, \lambda_{\max}, \breve{\omega}_{\min}, \breve{\omega}_{\max}, \zeta_{\min}, S_{\max}$, and $N_{\max}$. It can be formulated as an optimization problem.

Problem 1. Find

$$\min_{x\in Q} \|G_{yf}(j\omega, x)\|_\infty$$

subject to

$$\begin{aligned} &\|S(j\omega, x)\|_\infty \leqslant S_{\max}, \\ &\|G_{u\nu}(j\omega, x)\|_\infty \leqslant N_{\max}, \end{aligned} \tag{10}$$

where $S_{\max}$ and $N_{\max}$ are given values. The vector of variables $x \in \mathbb{R}^{2n-1}$ has the form

$$x = [\lambda_1, \ldots, \lambda_{n_r}, \breve{\omega}_1, \ldots, \breve{\omega}_{n_c}, \zeta_1, \ldots, \zeta_{n_c}], \tag{11}$$

where n_r and n_c are given values such that $0 \leqslant n_c \leqslant n-1, n_r = 2n - 2n_c - 1$, and n is a known order of the plant (1). The admissible region Q is determined by inequalities (6) with the given parameters $\lambda_{\min}, \lambda_{\max}, \breve{\omega}_{\min}, \breve{\omega}_{\max}$, and $\zeta_{\min}$. In accordance with (7)–(9), the frequency response functions $G_{yf}(j\omega, x), S(j\omega, x)$, and $G_{u\nu}(j\omega, x)$ are constructed from the given polynomials $a(j\omega)$ and $b(j\omega)$ of the plant (1), the polynomial $\delta(j\omega)$ determined for the vector (11) by formula (5), and the controller polynomials $c(j\omega)$ and $d(j\omega)$ whose coefficients are found by solving system (4).

Note that the constraints (6), (8), and (9) may be not satisfied simultaneously; in this case, the set of admissible values will be empty. This issue is not considered here: the constraints are assumed to be consistent. For a particular problem, an iterative process can be carried out in practice to find acceptable values of the constraints for reaching an acceptable value of the objective function.

3. SEARCH FOR THE OPTIMAL ROOTS OF THE CHARACTERISTIC POLYNOMIAL

3.1. Objective Function with Penalties

We use the penalty function method to satisfy the constraints. For the value $\|G(j\omega, x)\|_\infty$, the penalty function $\tilde{G}(x)$ is given by

$$\tilde{G}(x) = \begin{cases} 0 \text{ if } \|G(j\omega, x)\|_\infty \leqslant G_{\max} \\ \ln \dfrac{\|G(j\omega, x)\|_\infty}{G_{\max}} \text{ if } \|G(j\omega, x)\|_\infty > G_{\max}. \end{cases} \tag{12}$$

In this case, the objective function takes the form

$$f(x) = \|G_{yf}(j\omega, x)\|_\infty + \mu_1 \tilde{S}(x) + \mu_2 \tilde{G}_{u\nu}(x), \tag{13}$$

where $\mu_1 > 0$ and $\mu_2 > 0$ are weight coefficients, and $\tilde{S}(x)$ and $\tilde{G}_{u\nu}(x)$ are the penalty functions obtained using (12) for the constraints (10). Note that due to (12), the objective function (13) is non-differentiable at the points where $\|S(j\omega, x)\|_\infty = S_{\max}$ or $\|G_{u\nu}(j\omega, x)\|_\infty = N_{\max}$. Moreover, the functions (7)–(9) may be non-convex and multi-extremal, and their gradients are not written in explicit form.

3.2. Scaling of the Variables

The logarithmic scale is often used to analyze dynamic systems in the frequency domain [2]. Note that the elements λ_i and $\breve{\omega}_k$ of the vector of variables (11) are the natural frequencies of the system. We convert them to a logarithmic scale, thus assigning a greater weight to changes in the roots with a modulus close to zero (slow system dynamics) compared to changes in those with a large modulus (fast system dynamics):

$$\begin{aligned}\tilde{x} &= [\lg\lambda_1,\dots,\lg\lambda_{n_r},\lg\breve{\omega}_1,\dots,\lg\breve{\omega}_{n_c},\zeta_1,\dots,\zeta_{n_c}]\\ &= [\tilde{\lambda}_1,\dots,\tilde{\lambda}_{n_r},\tilde{\omega}_1,\dots,\tilde{\omega}_{n_c},\zeta_1,\dots,\zeta_{n_c}],\end{aligned} \tag{14}$$

where $\tilde{\lambda}_i$ and $\tilde{\omega}_i$ are the common logarithms of the variables λ_i and $\breve{\omega}_i$, respectively. In this case, the constraints (6) take the form

$$\begin{gathered}0<\lg\lambda_{\min}\leqslant\tilde{\lambda}_i\leqslant\lg\lambda_{\max},\\ 0<\lg\breve{\omega}_{\min}\leqslant\tilde{\omega}_k\leqslant\lg\breve{\omega}_{\max},\\ 0<\zeta_{\min}\leqslant\zeta_k\leqslant 1.\end{gathered} \tag{15}$$

To calculate the objective function, the values of the variables must be rescaled to (11) by raising to the tenth power: $\lambda_i=10^{\tilde{\lambda}_i}, i=1,\dots,n_r$, and $\breve{\omega}_i=10^{\tilde{\omega}_i}, i=1,\dots,n_c$. The notations without the subscripts, $\tilde{\lambda},\tilde{\omega}$, and ζ, will be used for the corresponding groups in the vector of variables (14):

$$\begin{aligned}\tilde{\lambda}&=[\tilde{\lambda}_1,\dots,\tilde{\lambda}_{n_r}],\\ \tilde{\omega}&=[\tilde{\omega}_1,\dots,\tilde{\omega}_{n_c}],\\ \zeta&=[\zeta_1,\dots,\zeta_{n_c}].\end{aligned}$$

Accordingly, the vector (14) will be represented as $\tilde{x}=[\tilde{\lambda},\tilde{\omega},\zeta]$.

The dynamics with frequencies exceeding manyfold the minimum natural frequency of the control plant are often neglected during system design. Therefore, the difference between the common logarithms of the admissible values of the moduli of the characteristic polynomial roots usually is not greater than 5. For example, when considering a system with slow dynamics and $\lambda_{\min}=0.001$ and $\lambda_{\max}=1$, we obtain $\lg\lambda_{\min}=-3$ and $\lg\lambda_{\max}=0$; for a system with fast dynamics, $\lg\lambda_{\min}=2$ and $\lg\lambda_{\max}=6$ under the same or similar values for $\lg\breve{\omega}_{\min}$ and $\lg\breve{\omega}_{\max}$. Then the choice of the minimum step for the groups of variables $\tilde{\lambda}$ and $\tilde{\omega}$ is obvious. It follows from practical considerations that a step from 0.0001 to 0.01 will be quite small under such scales. This step is also reasonable for the group ζ, whose elements belong to the range $[\zeta_{\min},1]$.

3.3. Multiple Start

Multiple start is a standard approach to settling the multi-extremality problem of the objective function (13): the search procedure is executed from different initial points. For the problem under consideration, the initial values can be chosen, e.g., using the following rule:

— Choose the number of alternatives n_1,n_2, and n_3 for the groups of variables $\tilde{\lambda},\tilde{\omega}$, and ζ, respectively.

— For the groups $\tilde{\lambda}$ and $\tilde{\omega}$, create alternatives in which the first elements of the groups are uniformly distributed in the admissible range and the remaining elements are uniformly distributed in the

range $[\tilde\lambda_1, \lg\lambda_{\max}]$ or $[\tilde\omega_1, \lg\breve\omega_{\max}]$, respectively:

$$\begin{aligned}
\tilde\lambda_1^{(\ell)} &= \lg\lambda_{\min} + \ell\frac{\lg\lambda_{\max} - \lg\lambda_{\min}}{n_1+1},\ \ell = 1,\dots,n_1,\\
\tilde\lambda_i^{(\ell)} &= \tilde\lambda_1^{(\ell)} + (i-1)\frac{\lg\lambda_{\max} - \tilde\lambda_1^{(\ell)}}{n_r},\ i = 2,\dots,n_r,\\
\tilde\omega_1^{(\ell)} &= \lg\breve\omega_{\min} + \ell\frac{\lg\breve\omega_{\max} - \lg\breve\omega_{\min}}{n_2+1},\ \ell = 1,\dots,n_2,\\
\tilde\omega_i^{(\ell)} &= \tilde\omega_1^{(\ell)} + (i-1)\frac{\lg\breve\omega_{\max} - \tilde\omega_1^{(\ell)}}{n_c},\ i = 2,\dots,n_c.
\end{aligned} \tag{16}$$

— Use the same values for all elements of the group ζ :

$$\zeta_i^{(\ell)} = \begin{cases} \dfrac{1-\zeta_{\min}}{2} & \text{if } n_3 = 1\\ \zeta_{\min} + (\ell-1)\dfrac{1-\zeta_{\min}}{n_3-1} & \text{if } n_3 > 1, \end{cases} \qquad i = 1,\dots,n_c,\ \ell = 1,\dots,n_3. \tag{17}$$

— Create the set of $n_1\cdot n_2\cdot n_3$ initial points by combining all alternatives for each group.

For example, 32 initial points will be obtained if $n_1 = 4, n_2 = 4$, and $n_3 = 2$.

When building another grid of the initial values, one should keep in mind the following: the rearrangement of any elements within the groups $\tilde\lambda$ and $\tilde\omega$ makes no sense because, due to (5), the resulting polynomial $\delta(s)$ will be the same regardless of the order of the elements in the group.

3.4. Search Method

The objective function (13) is generally non-convex, multi-extremal, and non-differentiable at some points; therefore, standard search methods will not necessarily find a global minimum. For the problem under consideration, we use a combined method in which coordinate descent is applied for the group of variables ζ whereas the groups $\tilde\lambda$ and $\tilde\omega$ are merged to execute the search procedure by the pairs of coordinates. The dimension of the vector $[\tilde\lambda, \tilde\omega]$ equals $n_a = n_r + n_c$, and $n_a!/(2(n_a-2)!)$ pairs can be made from the elements of this vector. For $n_a = 10$, we have 45 pairs, which is computationally feasible. For most practical 1D problems, this restriction will be satisfied; for higher-dimension problems, however, some pairs should be discarded. For example, only neighbor elements can be combined into pairs, which gives $(n_a - 1)$ pairs; alternatively, pairs can be formed separately for the groups $\tilde\lambda$ and $\tilde\omega$.

We determine the next point $(k+1)$ after varying a pair of elements i, j $(i = 1,\dots,n_a-1,\ j = i+1,\dots,n_a)$ as follows:

$$\tilde x_{k+1} = \arg\min_{\alpha,\beta} f(\tilde x_k + \alpha e_i + \beta e_j), \tag{18}$$

where e_i and e_j are the vectors with ones for elements i and j, respectively, and zeros for all other elements; α and β are values from some set of variations, e.g.,

$$\alpha, \beta \in \{0,\ 0.001,\ -0.001,\ 0.01,\ -0.01\}. \tag{19}$$

If the result of (18) is $\alpha = \beta = 0$, then a new point has not been obtained. If a new record value of the objective function is reached, then the 1D search procedure can be executed for the corresponding values α and β :

$$\tilde x_{k+1} = \arg\min_{\gamma} f(\tilde x_k + \gamma\alpha e_i + \gamma\beta e_j), \tag{20}$$

where, e.g., $\gamma \in \{0, 10\}$.

Fixed steps are used due to the nonconvexity of the objective function: finding an optimal step in a given direction may be a computationally difficult task.

When varying the elements of the groups $\tilde{\lambda}$ and $\tilde{\omega}$, we take into account that the objective function is independent of the rearrangement of these elements. Therefore, it is possible to fix an order of elements $\tilde{\lambda}_1 \leqslant \tilde{\lambda}_2 \leqslant \ldots \leqslant \tilde{\lambda}_{n_r}$, $\tilde{\omega}_1 \leqslant \tilde{\omega}_2 \leqslant \ldots \leqslant \tilde{\omega}_{n_c}$ and, in addition to the bounds (15), use neighbor elements as bounds as well. For example, in the case $n_r > 2$,

$$\begin{aligned} &\tilde{\lambda}_1 \in [\lg \lambda_{\min}, \tilde{\lambda}_2], \\ &\tilde{\lambda}_i \in [\tilde{\lambda}_{i-1}, \tilde{\lambda}_{i+1}],\ 1 < i < n_r, \\ &\tilde{\lambda}_{n_r} \in [\tilde{\lambda}_{n_r-1}, \lg \lambda_{\max}]. \end{aligned} \tag{21}$$

After the search procedure (18) for all pairs $(i = 1, \ldots, n_a - 1,\ j = i + 1, \ldots, n_a)$, we execute coordinate descent for the group ζ :

$$\tilde{x}_{k+1} = \arg\min_{\eta} f(\tilde{x}_k + \eta e_i),\ i = 1, \ldots, n_c, \tag{22}$$

where η is the set of fixed steps and e_i is the vector with one for element $(i + n_r + n_c)$ and zeros for the other elements. For example, the set of steps can be

$$\eta \in \{0.001, -0.001, 0.01, -0.01, 0.05, -0.05\}. \tag{23}$$

The elements of the group ζ are varied within the specified bounds: $\zeta_i \in [\zeta_{\min}, 1]$.

Thus, Problem 1 is solved using the following algorithm for $n_a > 1$.

Algorithm 1.

1. Choose the penalty weight coefficients μ_1 and μ_2 for the objective function (13) and set the search threshold ε.

2. Generate a grid of initial points as described in subsection 3.3 and take the first initial point.

3. Calculate the value of the objective function at the initial point, $f_{\min}^{(\ell)}$.

4. Take a pair of elements from the groups of variables $\tilde{\lambda}$ and $\tilde{\omega}$.

5. Execute (18) through the exhaustive search procedure over the set (19).

6. If a new record value of the objective function is obtained, execute (20) in the obtained direction and go to the new point.

7. Take the next pair of elements from the groups of variables $\tilde{\lambda}$ and $\tilde{\omega}$ and revert to Step 5. If the exhaustive search procedure for the pairs is completed, proceed to Step 8.

8. If $n_c > 0$, take an element of the group ζ. Otherwise, proceed to Step 11.

9. Execute (22) through the exhaustive search procedure over the set (23).

10. Take the next element from the group ζ and revert to Step 9. If the exhaustive search procedure within the group ζ is completed, proceed to Step 11.

11. If the record value of the objective function $\hat{f}$ yielded by Steps 4–10 is less than $f_{\min}^{(\ell)} - \varepsilon$, replace the value $f_{\min}^{(\ell)}$ with $\hat{f}$ and revert to Step 4 with the corresponding new point. Otherwise, remember the objective function value $\min(f_{\min}^{(\ell)}, \hat{f})$ and the corresponding point $\tilde{x}$, take the next initial point, and revert to Step 3. If the search procedure for all initial points obtained in Step 2 is completed, proceed to Step 12.

12. Find the minimum among the objective function values obtained for all initial points and the corresponding point $\tilde{x}$. Complete the search procedure.

Additional search stages can be embedded in this algorithm if the objective function value does not decrease in Step 11: 1) increase the weight coefficients μ_1 and μ_2 and continue the search procedure from the resulting point; 2) continue the search procedure with smaller values of the set of variations (19) for α and β.

4. EXAMPLES

4.1. Underwater Vehicle Position Control

The transfer functions for local coordinate system positioning were identified in [16]. In this section, we consider control design for the coordinate z with the identified transfer function

$$P_z(s) = \frac{0.018}{s(0.98s+1)}. \tag{24}$$

The two-degree-of-freedom (2DOF) PID controller presented in [16] allows setting a desired transfer function of the closed loop system. For this example, we take the desired transfer function

$$P_m(s) = \frac{1}{(0.98s+1)(0.5s+1)}. \tag{25}$$

The denominator of the transfer function (25) must be included in the desired characteristic polynomial of the closed loop system when designing a 2DOF controller. Then there are only two roots left for variation. Assume that they form a complex conjugate pair of roots of the characteristic polynomial. In this case, the controller coefficients

$$C(s) = \frac{d_2s^2 + d_1s + d_0}{s(c_1s + c_0)} \tag{26}$$

are obtained from the equation

$$s^2(0.98s+1)(c_1s+c_0) + 0.018(d_2s^2 + d_1s + d_0) = (0.98s+1)(0.5s+1)(s^2 + 2\zeta\breve{\omega} + \breve{\omega}^2).$$

This example illustrates the search procedure for the variables ζ and $\breve{\omega}$. Since $n_a = 1$ here, we use coordinate descent instead of Algorithm 1.

Let the following bounds be specified:

$$\breve{\omega}_{\min} = 0.6,\ \breve{\omega}_{\max} = 20,\ \zeta_{\min} = 0.8,\ S_{\max} = 1.7,\ N_{\max} = 150. \tag{27}$$

Weight coefficients should be assigned for the penalty functions of the objective function (13). These coefficients are chosen so that the constraints have priority over disturbance rejection. Note that the penalty functions are included in (13) as the ratio of the H_∞ norm to its admissible

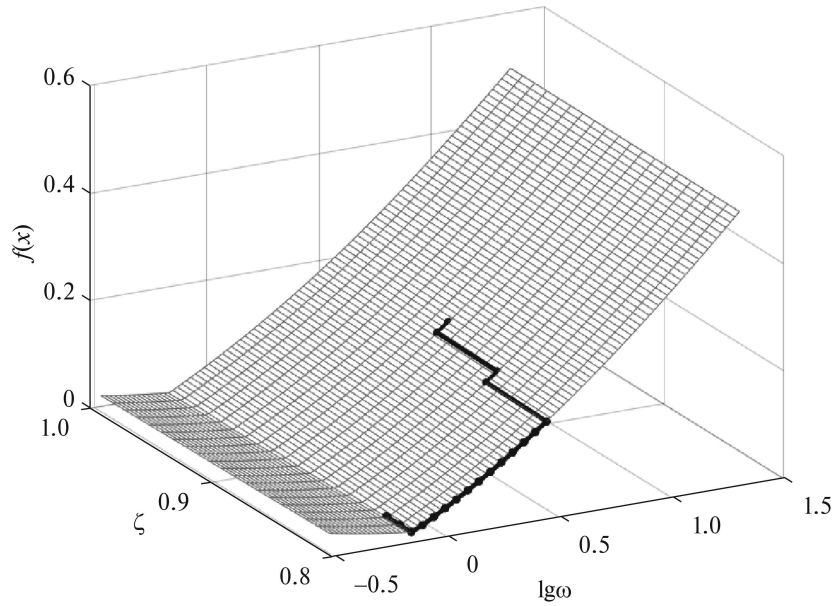

Fig. 2. Coordinate descent: $\tilde{x} = [\tilde{\omega}, \zeta]$.

maximum value whereas the H_∞ norm of the frequency response function relative to the disturbance is used in absolute units. Therefore, to choose the weight coefficients, it is necessary to estimate the value $\|G_{yf}(j\omega)\|_\infty$. For example, for the minimum values from the admissible region $\breve{\omega} = 0.6$, $\zeta = 0.8$, we obtain $\|G_{yf}(j\omega)\|_\infty = 0.0239$. Then the values $\mu_1 = 1$ and $\mu_2 = 0.1$ can be taken. The set of steps (23) is used for both variables.

Figure 2 shows the surface of the objective function on a grid with steps of 0.02 for $\tilde{\omega}$ and 0.01 for ζ within the given constraints. Also, this figure presents the objective function values in each step of the coordinate descent procedure with the initial point

$$\tilde{x}_0 = \left[\frac{\lg \breve{\omega}_{\max} + \lg \breve{\omega}_{\min}}{2}, \frac{1 + \zeta_{\min}}{2}\right] = [0.5396, 0.9].$$

The minimum point is $\breve{\omega} = 0.6928, \zeta = 0.821$, and the corresponding values are

$$\|S(j\omega)\|_\infty = 1.27, \quad \|G_{u\nu}(j\omega)\|_\infty = 149.97, \quad \|G_{yf}(j\omega)\|_\infty = 0.0206.$$

4.2. Controller for a Two-Mass System

Consider the benchmark problem presented in [17], i.e., a robust control design for two trolleys joined by a spring. For this problem, the pole placement optimization method was used to build a controller satisfying the speed and robustness requirements of the system [12]. Note that the standard global optimization procedure from the MATLAB Global Optimization Toolbox [13] was applied therein to find the optimal roots of the characteristic polynomial. In this subsection, we use Algorithm 1 to solve the same problem.

Let the transfer function relative to control be

$$P(s) = \frac{1}{s^2(s^2 + 2)}. \tag{28}$$

In this plant, control and disturbance are applied at different points, and the open-loop transfer function relative to the disturbance is known:

$$P_f(s) = \frac{s^2 + 1}{s^2(s^2 + 2)}. \tag{29}$$

In this case, the H_∞ norm of the closed-loop frequency transfer function relative to the disturbance differs from (7) and is calculated as

$$\|G_{yf}(j\omega)\|_\infty = \sup_\omega \left|\frac{b_f(j\omega)c(j\omega)}{\delta(j\omega)}\right|, \tag{30}$$

where $b_f(j\omega)$ is the numerator polynomial of the transfer function (29).

Similar to [12], we design a controller of the form (2) with $n = 4$ under the following bounds and constraints:

$$\lambda_{\min} = \breve{\omega}_{\min} = 0.1, \quad \lambda_{\max} = \breve{\omega}_{\max} = 100, \quad \zeta_{\min} = 0.7, \quad S_{\max} = 1.665, \quad N_{\max} = 100. \tag{31}$$

We choose the desired structure of the characteristic polynomial (5) with $n_r = 1$ and $n_c = 3$ and the weight coefficients $\mu_1 = \mu_2 = 100$ for the penalty functions in (13). Let the threshold for varying the objective function be $\varepsilon = 10^{-6}$. We form twenty-four initial points for multiple start by choosing $n_1 = 4$, $n_2 = 3$, and $n_3 = 2$ and using (16) for the groups $\tilde{\lambda}$ and $\tilde{\omega}$ as well as the following alternatives for the group ζ : 1) all elements equal $\zeta_{\min}$; 2) all elements equal one.

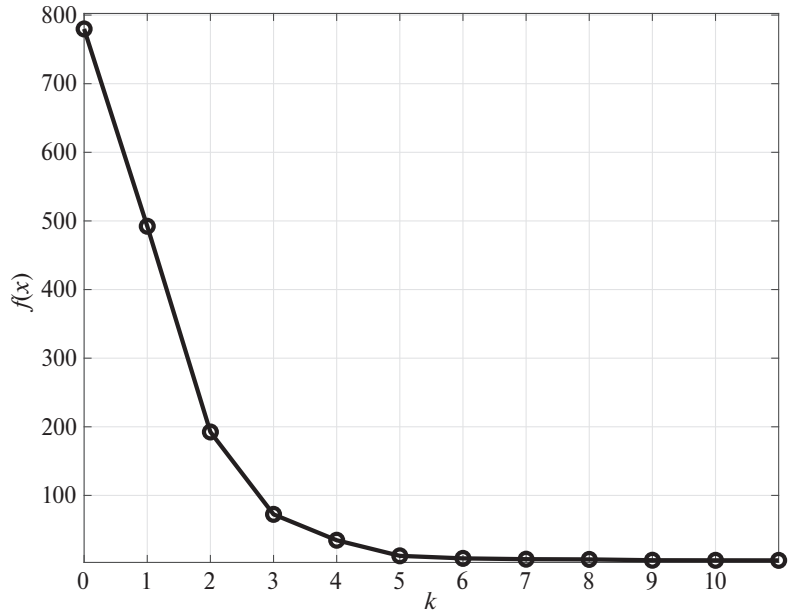

Fig. 3. The values of the objective function $f(x)$ at each iteration of the search algorithm.

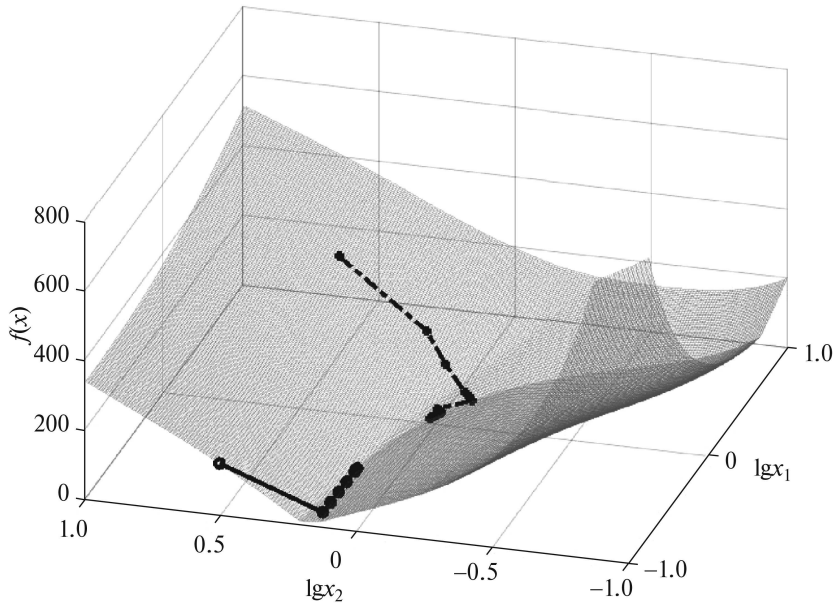

Fig. 4. The values of the objective function $f(\tilde{\lambda}_1, \tilde{\omega}_1)$.

The resulting minimum point of the objective function (13) is

$$x_{\min} = [0.3417, 1.4138, 1.4145, 3.6593, 0.701, 0.700, 0.700], \tag{32}$$

for which

$$\|S(j\omega)\|_\infty = 1.665, \quad \|G_{uv}(j\omega)\|_\infty = 99.96, \quad \|G_{yf}(j\omega)\|_\infty = 5.296. \tag{33}$$

The minimum was found in twenty iterations from an initial point. Figure 3 shows the graph of the record values of the objective function. Other six points of multiple start yielded $\|G_{yf}(j\omega)\|_\infty < 6$ under the valid constraints. The remaining initial points led to local minima with the invalid constraint $\|S(j\omega)\|_\infty \leqslant S_{\max}$ or a higher value of $\|G_{yf}(j\omega)\|_\infty$. Only two of the twenty-four initial

points of multiple start resulted in the same local minimum; in the rest cases, the search procedure was completed at different points.

Figure 4 shows the surface of the objective function calculated for the vector $\tilde{x}$ with only the first two elements being varied on a grid (and the rest equaling the obtained values (32)) and search alternatives for these two elements from the initial points $[0.5, 0.5]$ and $[-1, 0.5]$. Obviously, the search procedure converged to different local minima with values of the objective function equal to 19.8 and 5.3, respectively. In other words, the objective function in this example has a complicated ravine surface even in the simplified case with two variables.

The same example with the same constraints was solved by several methods in [12]. The controller with $\|G_{yf}(j\omega)\|_\infty = 5.301$, the result almost coinciding with (33), was obtained using systune, the fixed-structure control system tuning procedure [18] of the MATLAB Robust Control Toolbox. The solution by the pole placement optimization method using the standard global optimization procedure was implemented in [12]; the resulting controller rejects the disturbance slightly worse, ensuring the value $\|G_{yf}(j\omega)\|_\infty = 6.64$.

Thus, the search algorithm proposed in this article found a better solution than the standard global optimization procedure. The solution obtained by systune is practically not improved, which suggests its global minimum character.

5. CONCLUSIONS

The control design problem using the pole placement method has been considered, and an algorithm has been developed to find the desired poles based on the specified system quality criteria and constraints. The value of the H_∞ norm of the frequency transfer function relative to the disturbance has been selected as the quality criterion of the system, and the maximum admissible values of the H_∞ norms of the sensitivity function and the frequency transfer function relative to the measurement noise have been set as the constraints. The resulting search algorithm can be used for other criteria and constraints. In this case, only the penalty components (12) in the objective function (13) will be changed. Note that in the example of subsection 4.1, the controller structure differs from (2) since an integral component has been added to the controller. Thus, the scope of application of the developed approach is not restricted to systems with the controller (2): it covers all controller structures that can be obtained by the pole placement method. Also, for the sake of simplicity, an exogenous disturbance has been applied along with the control action in the system structure. Indeed, the real transfer function relative to the disturbance is often unknown; in this case, such a simplification of the system structure still allows considering the effect of the disturbance in the system. If the plant's transfer function relative to the disturbance is known (see the example of subsection 4.2), it should be used when forming the transfer function of the closed loop system relative to the disturbance.

The advantages of the proposed search method are due to considering the properties of the characteristic polynomial roots. The logarithmic scale taken for the moduli of the characteristic polynomial roots provides the following benefits. First, it serves to reasonably choose the increment of the variables in the search procedure. Second, it allows one to form a limited set of initial points for the multiple start procedure. The search algorithm with a pair of simultaneously varied elements finds the minimum for an objective function with a complicated surface. Thus, the known features of the vector of variables in the problem under consideration have been utilized to develop an effective constrained minimization algorithm for a non-convex multi-extremal objective function.

FUNDING

The research presented in Sections 2 and 3 was supported by the Russian Science Foundation (project no. 23-29-00588, https://rscf.ru/en/project/23-29-00588/).

REFERENCES

1. Polyak, B.T., Khlebnikov, M.V., and Shcherbakov, P.S., *Upravlenie lineinymi sistemami pri vneshnikh vozmushcheniyakh: tekhnika lineinykh matrichnykh neravenstv* (Control of Linear Systems Subject to Exogenous Disturbances: Linear Matrix Inequalities Technique), Moscow: LENAND, 2014.
2. Besekerskii, V.A. and Popov, E.P., *Teoriya sistem avtomaticheskogo regulirovaniya* (The Theory of Automatic Regulation Systems), Moscow: Nauka, 1966.
3. Aleksandrov, A.G., Stability Margins of the Systems of Optimal and Modal Control, *Autom. Remote Control*, 2007, vol. 68, no. 8, pp. 1296–1308.
4. Chestnov, V.N., Design of Controllers of Multidimensional Systems with a Given Radius of Stability Margins Based on the H_∞–optimization Procedure, *Autom. Remote Control*, 1999, vol. 60, no. 7, pp. 986–993.
5. Astrom, K.J. and Murray, R.M., *Feedback Systems: an Introduction for Scientists and Engineers*, New Jersey: Princeton University Press, 2008.
6. Alexandrov, V.A., Chestnov, V.N., and Shatov, D.V., Stability Margins for Minimum-Phase SISO Plants: A Case Study, *Proc. Eur. Control Conf.*, 2020, St. Petersburg, pp. 2068–2073.
7. Gahinet, P. and Apkarian, P., A Linear Matrix Inequality Approach to H_∞-control, *Int. J. Robust Nonlin. Control*, 1994, vol. 4, no. 4, pp. 421–448.
8. Apkarian, P. and Noll, D., Nonsmooth H_∞ Synthesis, *IEEE Trans. Autom. Control*, 2006, vol. 51, no. 1, pp. 71–86.
9. Polyak, B.T. and Khlebnikov, M.V., Static Controller Synthesis for Peak-to-Peak Gain Minimization as an Optimization Problem, *Autom. Remote Control*, 2021, vol. 82, no. 9, pp. 1530–1553.
10. Polyak, B.T. and Khlebnikov, M.V., New Criteria for Tuning PID Controllers, *Autom. Remote Control*, 2022, vol. 83, no. 11, pp. 1724–1741.
11. Shatov, D.V., Synthesis of Parameters of Proportionally-Integral and Proportionally-Integral-Differential Controllers for Stationary Linear Objects with Nonzero Initial Conditions, *J. Comput. Syst. Sci. Int.*, 2023, vol. 62, pp. 17–26.
12. Alexandrov, V.A., Pole Placement Optimization for SISO Control System, *Autom. Remote Control*, 2021, vol. 82, no. 6, pp. 1013–1029.
13. Ugray, Z., Lasdon, L., Plummer, J., Glover, F., Kelly, J., and Marti, R., Scatter Search and Local NLP Solvers: A Multistart Framework for Global Optimization, *INFORMS J. Computing*, 2007, vol. 19, no. 3, pp. 328–340.
14. Astrom, K.J. and Wittenmark, B., *Computer-Controlled Systems. Theory and Design*, Upper Saddle River, NJ: Prentice Hall, 1984.
15. Gaiduk, A.R., *Teoriya i metody analiticheskogo sinteza sistem avtomaticheskogo upravleniya (polinomial'nyi podkhod)* (Theory and Methods for the Analytic Synthesis of Automatic Control Systems (A Polynomial Approach)), Moscow: Fizmatlit, 2012.
16. Alexandrov, V., Shatov, D., Abramenkov, A., and Abdulov, A., Position Control of Maneuverable Underwater Vehicle Based on Model Identification, *Proc. of the 5th International Conference on Control Systems, Mathematical Modeling, Automation and Energy Efficiency (SUMMA)*, 2023, Lipetsk, pp. 76–81.
17. Wie, B. and Bernstein, D.S., A Benchmark Problem for Robust Control Design, *Proc. Amer. Control Conf.*, San Diego, CA, 1990, pp. 961–962.
18. Apkarian, P., Gahinet, P., and Buhr, C., Multi-Model, Multi-Objective Tuning of Fixed-Structure Controllers, *Proc. Eur. Control Conf.*, 2014, Strasbourg, pp. 856–861.

This paper was recommended for publication by P.S. Shcherbakov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, pp. 491–511.

TOPICAL ISSUE

On Some Problems with Multivalued Mappings

M. V. Balashov[*,a], **K. Z. Biglov**[*,b], **and A. A. Tremba**[*,**,c]

[*] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
[**] *Moscow Institute of Physics and Technology, Dolgoprudny, Russia*
e-mail: [a] *balashov73@mail.ru,* [b] *biglov.kz@phystech.edu,* [c] *atremba@ipu.ru*

Received January 25, 2024
Revised March 12, 2024
Accepted March 20, 2024

Abstract—We consider some problems with a set-valued mapping, which can be reduced to minimization of a homogeneous Lipschitz function on the unit sphere. Latter problem can be solved in some cases with a first order algorithm—the gradient projection method. As one of the examples, the case when set-valued mapping is the reachable set of a linear autonomous controlled system is considered. In several settings, the linear convergence is proven. The methods used in proofs follow those introduced by B.T. Polyak for the case where Lezanski–Polyak–Lojasiewicz condition holds. Unlike algorithms that use approximation of the reachable set, the proposed algorithms depend far less on dimension and other parameters of the problem. Efficient error estimation is possible. Numerical experiments confirm the effectiveness of the considered approach. This approach can also be applied to various set-theoretical problems with general set-valued mappings.

Keywords: gradient projection method, set-valued integral, strong convexity, supporting set, Lipschitz condition, nonsmooth analysis

DOI: 10.31857/S0005117924050047

1. INTRODUCTION

Let $\mathbb{R}^n$ be a real Euclidean space with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Define the ball $\mathcal{B}_r(a) = \{x \in \mathbb{R}^n \,:\, \|x-a\| \leqslant r\}$, $(a \in \mathbb{R}^n,\ r > 0)$ and the unit sphere $\mathcal{S}_1 = \partial\mathcal{B}_1(0)$. Denote by $\operatorname{int}\mathcal{N}$ and $\partial\mathcal{N}$ the interior and the boundary of a set $\mathcal{N} \subset \mathbb{R}^n$, respectively. Recall that the supporting function for a closed convex set $\mathcal{N} \subset \mathbb{R}^n$ and vector $p \in \mathbb{R}^n$ is $s(p,\mathcal{N}) = \sup_{x\in\mathcal{N}}(p,x)$ and the supporting subset is $\mathcal{N}(p) = \{x \in \mathcal{N} \,:\, (p,x) = s(p,\mathcal{N})\}$. The set $\mathcal{N}(p)$ is called the supporting element if it is a singleton. For a convex compact set $\mathcal{N}$ the set $\mathcal{N}(p)$ is the subdifferential (in the sense of convex analysis) of the supporting function $s(p,\mathcal{N})$ at the point p. Let $P_{\mathcal{N}}x$ be the metric projection of a point $x \in \mathbb{R}^n$ onto a closed convex set $\mathcal{N}$.

Let $\mathcal{N} \subset \mathbb{R}^n\backslash\{0\}$ be a convex compact set and $f(p) = s(p,\mathcal{N})$. Consider the problem

$$\min_{\|p\|=1} f(p) = J. \tag{1}$$

It is obvious that the solution of problem (1) is a unit vector p_0 such that $p_0 = -z_0/\|z_0\|$, $P_{\mathcal{N}}0 = \{z_0\}$ and $J = (p_0, z_0)$. Also $z_0 \in \mathcal{N}(p_0)$. Thus finding the projection of zero $z_0 = P_{\mathcal{N}}0$ is equivalent to the problem (1). The general projection problem can be solved the same way as $P_{\mathcal{N}}x = x + P_{\mathcal{N}+(-x)}0$.

There are many ways to solve the problem of projecting a point onto a convex closed set $\mathcal{N}$, that depend on how the set $\mathcal{N}$ is defined. If the set $\mathcal{N}$ is a polyhedron, then it can be solved with the help of quadratic programming: $\min\|x\|^2$ under conditions $(p_i,x) \leqslant s(p_i,\mathcal{N})$, where $\{p_i\}$ is the set

of unit normals to $\mathcal{N}$. Method of alternating projections under the transversality condition can be found in [1, Section 8.5]. In [2], the author considers properties of projector operators. They also consider convergence of an iterative projection/reflection algorithm for finding points that achieve a local minimum distance between two closed convex sets or one closed convex set and a closed prox-regular set. Usefulness of conditional gradient-like methods for determining projections onto convex sets was considered in [3]. In [4], the authors proposed an iterative algorithm for metric projection of a point onto a level set of a quadric function. Some algorithms for finding the Bregman projection of a point onto a closed convex set can be found in [5].

The best rate of convergence for the algorithms considered in the papers above is linear. Besides that, in many cases, the considered algorithms do not allow one to obtain an efficient computational procedure.

Further we shall assume that we know supporting function $s(p, \mathcal{N})$ and supporting subset $\mathcal{N}(p)$. "We know" means that we can efficiently compute $s(p, \mathcal{N})$ and $\mathcal{N}(p)$ for any vector $p \in \mathbb{R}^n \backslash \{0\}$.

Suppose that $\mathcal{M} \subset \mathbb{R}^n$ is a convex compact set and $\mathcal{R}(\cdot) : [0, T] \to 2^{\mathbb{R}^n}$, $\mathcal{R}(0) = \{0\}$, is a set-valued mapping with convex compact values that is continuous in Hausdorff metric. Consider a few problems that can be solved in the framework of statement (1).

Problem (P1). For given $t \geqslant 0$, find the distance between sets $\mathcal{R}(t)$ and $\mathcal{M}$, i.e. the value of $\rho(\mathcal{R}(t), \mathcal{M}) = \inf_{x \in \mathcal{R}(t),\ y \in \mathcal{M}} \|x - y\|$. Find minimal $t \geqslant 0$, so that $\rho(\mathcal{R}(t), \mathcal{M}) = 0$.

Problem (P2). For given $t \geqslant 0$, check whether the inclusion $\mathcal{R}(t) \subset \mathcal{M}$ holds. Find maximal $t \geqslant 0$, so that $\mathcal{R}(t) \subset \mathcal{M}$.

Problem (P3). For given $t \geqslant 0$, check whether the inclusion $\mathcal{R}(t) \supset \mathcal{M}$ holds. Find minimal $t \geqslant 0$, so that $\mathcal{R}(t) \supset \mathcal{M}$.

Problems (P1)–(P3) can be stated for an arbitrary set-valued continuous mapping with convex compact images $\mathcal{R}(t)$ and a convex compact set $\mathcal{M}$. Consider a particular case of a set-valued integral of the form

$$\mathcal{R}(t) = \int_0^t \mathcal{F}(s)\, ds, \tag{2}$$

where $\mathcal{F}$ is a set-valued mapping with convex compact values. By default we shall assume that $0 \in \mathcal{F}(s)$ for all $s \geqslant 0$. The last integral is treated as the Aumann integral [6]

$$\int_0^t \mathcal{F}(s)\, ds = \left\{ \int_0^t u(s)\, ds \ : \ u(s) \in \mathcal{F}(s) \text{—a measurable selector} \right\}.$$

By the Lyapunov theorem on vector measures [7] the value of the integral is convex and compact. From formula (2) and the inclusion $0 \in \mathcal{F}(s)$ for all $s \in [0, t]$ we conclude that $\{\mathcal{R}(t)\}_{t \geqslant 0}$ is increasing: $\mathcal{R}(t_1) \subset \mathcal{R}(t_2)$ for all $0 \leqslant t_1 \leqslant t_2$. It is also possible to consider a set $\mathcal{M}(t)$ depending on t.

The support function and supporting subset for integral (2) can be calculated easily: for a unit vector p and any $t \geqslant 0$ we get

$$s(p, \mathcal{R}(t)) = s\left(p, \int_0^t \mathcal{F}(s)\, ds\right) = \int_0^t s\left(p, \mathcal{F}(s)\right)\, ds, \qquad \mathcal{R}(t)(p) = \int_0^t \mathcal{F}(s)(p)\, ds. \tag{3}$$

Another class of sets for which we know the supporting function and the supporting element are finite sums of linear images of some fixed sets $\mathcal{M}$ with known $s(p, \mathcal{M})$ and $\mathcal{M}(p)$, e.g. ellipsoids. Suppose that $\mathcal{R}(t) = \sum_{k=1}^m A_k(t) B_1(0)$, where $A_k(t)$ are continuous nondegenerate matrices for all

$t \geqslant 0$. Then

$$s(p, \mathcal{R}(t)) = \sum_{k=1}^{m} s(p, A_k(t)B_1(0)) = \sum_{k=1}^{m} \|A_k^T(t)p\|, \qquad \mathcal{R}(t)(p) = \sum_{k=1}^{m} \frac{A_k(t)A_k^T(t)p}{\|A_k^T(t)p\|}. \tag{4}$$

Note that a finite sum of ellipsoids is, in general, not an ellipsoid.

Our most important example is the reachable set of an autonomous linear controlled system, which is described by a differential inclusion

$$x'(t) \in Ax(t) + \mathcal{U},\ x(0) = 0, \qquad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}, \tag{5}$$

where $\mathcal{U} \subset \mathbb{R}^n$ is a compact, $0 \in \mathcal{U}$. The reachable set (all points to which the system can arrive at the given moment of time) can be represented in the form

$$\mathcal{R}(t) = \int_0^t e^{As}\mathcal{U}\, ds. \tag{6}$$

The most important strengthening of the convexity condition is the concept of strong convexity with radius R. The set in $\mathbb{R}^n$ is strongly convex with radius R if it can be represented as an intersection of closed balls of radius R [8, 9]. This property can also be defined via the modulus of convexity [10]. In [8], the authors proved that the set-valued integral (2) is strongly convex if the multifunction $\mathcal{F}(s)$ has strongly convex values. In [11], the local strong convexity in certain sense was proved for integral (2) with $\mathcal{F}(s) = A(s)U$, where $A(s)$ is a certain class of smooth matrices and U is a polyhedron. In [12], the second order approximation in time of a Runge-Kutta type scheme for discretization of strongly convex differential inclusions was considered.

Various problems with set-valued integrals can be solved with the help of approximation of values of the integrals. In [13], the authors describe different methods to construct an approximation of the reachable set of a controlled system, see Table 1 therein. One of the most general and effective methods is based on the supporting function (it is also called hyperplane method), see, for example, [14]. We can consider an outer polyhedral approximation for $\mathcal{M}$ of the form

$$\{x \in \mathbb{R}^n\ :\ (p, x) \leqslant s(p, \mathcal{M}), \quad \forall p \in \mathbb{G}\}, \tag{7}$$

where $\mathbb{G} \subset \mathbb{R}^n$ is a finite grid of unit vectors and solve the problem for the approximation. The disadvantage of this approach is that a reasonable approximation can be obtained only in a space of low dimensions $2 \leqslant n < 5$, see [15].

There are also different approaches using special approximations, e.g. with zonotopes [16] or ellipsoidal technique [17]. The latter technique sometimes permits to describe the reachable set locally.

In the present paper we think $\mathcal{R}(t)$, $\mathcal{M}$, $\mathcal{N}$ to be either the value of a set-valued integral or a finite sum of ellipsoids. We shall show how to reduce different problems, e.g. (P1)–(P3), with such sets to the problem (1). The function $f(p)$ in (1) turns out to be the supporting function of some convex compact set $\mathcal{N}$, which depends on $\mathcal{R}(t)$ and $\mathcal{M}$. Lezanski-Polyak-Lojasiewicz (LPL) condition [18, formula (4.6)] is proven in problem (1), from which a linear convergence rate for gradient projection algorithm is obtained. The supporting function $f(p)$ and its gradient can be computed, e.g. using formula (3) for a set-valued integral or by (4) for sum of ellipsoids. With the supporting function and its gradient we obtain an efficient calculation scheme. We also consider a local condition of strong convexity: for some $R > 0$ for the solution p_0 of (1) the inclusion $\mathcal{N} \subset B_R(\mathcal{N}(p_0) - Rp_0)$ holds. Under this condition the problem can be solved with the help of the gradient projection method with a fixed step-size or with Armijo's step-size. We prove a linear rate of convergence for all algorithms and consider various examples.

There is another way to solve (1) using the conditional gradient (CG) method: take the function $g(x) = \frac{1}{2}\|x\|^2$, a starting point $x_1 \in \mathcal{N}$ and iterations $\overline{x}_k = \arg\max_{x\in\mathcal{N}}(-g'(x_k), x)$, $x_{k+1} \in \operatorname{Arg\,min}_{x\in[x_k,\overline{x}_k]} g(x)$. Note that, to ensure the linear convergence of this algorithm, strong convexity of $\mathcal{N}$ is usually required [18, Therorem 6.1, 5].

1.1. Notation and Auxiliary Results

Recall that for sets $\mathcal{M}$ and $\mathcal{N}$ from $\mathbb{R}^n$ we have $\mathcal{M}+\mathcal{N} = \{x+y \,:\, x\in\mathcal{M},\ y\in\mathcal{N}\}$ and $\mathcal{M} \stackrel{*}{-} \mathcal{N} = \{x \,:\, x+\mathcal{N}\subset\mathcal{M}\} = \bigcap_{x\in\mathcal{N}}(\mathcal{M}-x)$. These operations are called the Minkowski sum and difference of sets $\mathcal{M}$ and $\mathcal{N}$.

Denote by $\varrho(x,\mathcal{M}) = \inf_{y\in\mathcal{M}}\|x-y\|$ the distance from a point x to a set $\mathcal{M}$.

The *Hausdorff distance* on the space of convex compacts in $\mathbb{R}^n$ can be defined like this: for any convex compact sets $\mathcal{M},\mathcal{N}\subset\mathbb{R}^n$

$$h(\mathcal{M},\mathcal{N}) = \max_{\|p\|=1} |s(p,\mathcal{M}) - s(p,\mathcal{N})|.$$

Define $[a]_-$: $[a]_- = |a|$ for $a\leqslant 0$ and $[a]_- = 0$ for $a>0$. Then $[\min_{\|p\|=1}(s(p,\mathcal{M})-s(p,\mathcal{N}))]_-$ is the halfdistance from $\mathcal{N}$ to $\mathcal{M}$ and it is equal to $\max_{x\in\mathcal{N}}\varrho(x,\mathcal{M})$.

Suppose that the set $\mathcal{R}(t)$ (6) depends on parameter t. Then we shall denote the *supporting set* for a vector p by $\mathcal{R}(t)(p)$. From the Aumann's or Riemann's definition of the integral for any matrix $J\in\mathbb{R}^{m\times n}$ we have $J\mathcal{R}(t) = \int_0^t Je^{As}\mathcal{U}\,ds$. In particular, for any vector $p\in\mathbb{R}^n$

$$\mathcal{R}(t)(p) = \int_0^t (e^{As}\mathcal{U})(p)\,ds.$$

A set $\mathcal{M}\subset\mathbb{R}^n$ is *strongly convex with radius* $R>0$ if we can represent $\mathcal{M}$ as intersection of some collection of closed Euclidean balls with radius R. For any strongly convex set $\mathcal{M}$ with radius $R>0$ there exists another strongly convex set $\mathcal{N}$ with radius R such that $\mathcal{M}+\mathcal{N} = \mathcal{B}_R(0)$ [8, 19]. Strong convexity of a compact convex set $\mathcal{M}$ with radius R is equivalent to the Lipschitz condition for the supporting element $\mathcal{M}(p)$ on the unit sphere: for all $\|p\| = \|q\| = 1$ we have $\|\mathcal{M}(p)-\mathcal{M}(q)\|\leqslant R\|p-q\|$ [8].

We shall say that a convex set $\mathcal{M}\subset\mathbb{R}^n$ is *uniformly smooth with constant* $r>0$ if we have $\mathcal{M} = \mathcal{M}_0 + \mathcal{B}_r(0)$, where $\mathcal{M}_0\subset\mathbb{R}^n$ is a convex compact set. For more details see [20, Definition 2.1].

Let $\mathcal{S}_0\subset\mathbb{R}^n$ be a smooth manifold without boundary, $\overline{x}\in\mathcal{S}_0$, $\varepsilon>0$. For a differentiable function $f:\mathcal{S}_0+\operatorname{int}\mathcal{B}_\varepsilon(0)\to\mathbb{R}$ define $\mathcal{S} = \mathcal{S}(f,\overline{x}) = \{x\in\mathcal{S}_0 : f(x)\leqslant f(\overline{x})\}$. Assume $\mathcal{S}$ to be a smooth manifold with the boundary $\partial\mathcal{S}\subset\{x\in\mathcal{S}_0 : f(x) = f(\overline{x})\}$. We shall say that *the Lezanski–Polyak–Lojasiewicz (LPL)* condition holds on $\mathcal{S}$ [18; 21, Section 3.2] with a constant $\mu>0$ if $\Omega = \operatorname{Arg\,min}_{x\in\mathcal{S}} f(x)\neq\emptyset$ and for all $x\in\mathcal{S}$ the following inequality holds

$$\|P_{\mathcal{T}_x} f'(x)\|^2 \geqslant \mu(f(x)-f(\Omega)). \qquad (\star)$$

Here $\mathcal{T}_x$ is the tangent subspace to the manifold $\mathcal{S}$ at the point $x\in\mathcal{S}$, $P_{\mathcal{T}_x}$ is the orthogonal projector onto $\mathcal{T}_x$, $f'(x)$ is the Frechet gradient of the function f at the point $x\in\mathcal{S}$.

Lemma 1. *For any nonzero vectors $p,q\in\mathbb{R}^n$ we have* $\left\|\frac{p}{\|p\|}-\frac{q}{\|q\|}\right\| \leqslant \frac{\|p-q\|}{\sqrt{\|p\|\,\|q\|}}$,

Proposition 1 [8]. *Suppose that a set-valued mapping $\mathcal{F}:[0,t]\to 2^{\mathbb{R}^n}$ is continuous in the Hausdorff metric and has strongly convex images $\mathcal{F}(s)$ with radius $R(s)$ for all $s\in[0,t]$, that is integrable at $[0,t]$. Then the integral $\mathcal{P} = \int_0^t F(s)\,ds$ is strongly convex with radius $R = \int_0^t R(s)\,ds$.*

It should be mentioned that the set-valued integral can be strongly convex even when $\mathcal{F}(s)$ is not. itself. For example, this situation typically takes place for the reachable set $\mathcal{R}(t)$ of system (5) in dimension $n=2$ [22]. Nevertheless, the reachable set in dimensions $n\geqslant 3$ is often not strongly convex.

Let us look at an elementary example of a system mentioned in (5) (a similar system is considered in Example 1 below). Let the control set be a segment: $\mathcal{U} = \operatorname{co}\{\pm v\}$. Define an analytic function $g_p(s) = (p, e^{As}v)$. The supporting set $\mathcal{R}(t)(p)$ is a singleton, provided that $g_p(s) \not\equiv 0$. This is guaranteed by full rank conditions

$$\operatorname{span}\{A^i v\}_{i=0}^{n-1} = \mathbb{R}^n \quad \Leftrightarrow \quad \operatorname{span}\mathcal{R}(t) = \mathbb{R}^n.$$

Since g_p is analytic, the equation $g_p = 0$ has a finite number of roots in $[0, t]$. The supporting element can be written down as

$$\mathcal{R}(t)(p) = \int_0^t e^{As}v \times \operatorname{sign} g_p(s)\, ds = \sum_{i=0}^{k} \epsilon_i \int_{s_i(p)}^{s_{i+1}(p)} e^{As}v\, ds, \tag{8}$$

where $s_i(p)$, $i = \overline{1,k}$ are the roots of $g_p(s)$, $s_0 = 0$, $s_{k+1} = t$, $\epsilon_i = \pm 1$ is equal to sign of $g_p(s)$ when $s \in [s_i, s_{i+1}]$. Therefore, the behaviour of supporting element is defined by dependence of roots of analytic function $g_p(s)$ on parameter p. If all roots are simple and lie on interval $(0, t)$, then it follows from implicit function theorem that the support element depends smoothly on p in the neighbourhood. Therefore, the supporting element is locally Lipschitz. On the other hand, g_p can have roots with multiplicity greater than one belonging to $[0, t]$. In this case the supporting element is typically not locally Lipschitz, which means that the strong convexity fails. This is illustrated in the example below. However, it is easy to show the set of vectors p, such that g_p has non-simple zeros on $[0, t]$, has measure zero on the unit sphere. Some generalizations of this approach to set-valued integrals can be seen in [11].

Note that if all eigenvalues of A are real, then the number of switchings in optimal control $u(t) = \mathcal{U}(e^{A^T(T-t)}p) = v \times \operatorname{sgn} g_p(T-t)$ is no greater than $n-1$, it is a special case of Feldbaum theorem, see [23, Theorem 2.11]. In the examples below, we consider a dynamical system defined by $\dot{x} = Ax + Bu,\ u \in \mathcal{U},\ t \in [0, T]$. The optimal control that guides the system to the support element $\mathcal{R}(t)(p)$ is [24]:

$$u(t) = \mathcal{U}(B^\top e^{A^\top (T-t)}), \quad t \in [0, T]. \tag{9}$$

Consider the system

$$\dot{x} = Ax + Bu, \quad x(0) = 0, \quad u \in \mathbb{R}: \quad |u| \leqslant 1, \quad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

Following what was said above, let $g_p(s) = (p, e^{As}B) = \frac{1}{2}e^{-s}\left(p_1 s^2 + 2p_2 s + 2p_3\right)$.

Let $p_0 = \frac{1}{3}(2, -2, 1)$, note that $g_{p_0}(s) = \frac{1}{3}e^{-s}(s-1)^2$ has a multiple root $s = 1$. We are interested in behaviour of supporting element near p_0. Remember that $f(s) \asymp g(s)$, $s \to 0$, if $f(s) = O(g(s))$ and $g(s) = O(f(s))$, $s \to 0$. Define for $\varepsilon \in (0, 1)$ a unit vector $q = q(\varepsilon) = \frac{(2, -2, 1-\varepsilon)}{\sqrt{9 - 2\varepsilon + \varepsilon^2}}$. It is easy to see that $\|p - q(\varepsilon)\| \asymp \varepsilon$, $\varepsilon \to 0$, and to find the roots $g_{q(\varepsilon)} =: s_{1,2}(\varepsilon) = 1 \pm \sqrt{\varepsilon}$. Then for $t > 1 + \sqrt{\varepsilon}$ we can write down the supporting element in the following way:

$$\mathcal{R}(t)(p) - \mathcal{R}(t)(q) = \int_{1-\sqrt{\varepsilon}}^{1+\sqrt{\varepsilon}} e^{-s}(s^2,\ 2s,\ 2)^\top\, ds,$$

$$\|\mathcal{R}(t)(p) - \mathcal{R}(t)(q)\| \geqslant \int_{1-\sqrt{\varepsilon}}^{1+\sqrt{\varepsilon}} 2e^{-s}\, ds \asymp \sqrt{\varepsilon},\ \varepsilon \to 0.$$

Therefore, the supporting element fails to be Lipschitz in a neighbourhood of p_0, so the reachable set $\mathcal{R}(t)$ is not strongly convex.

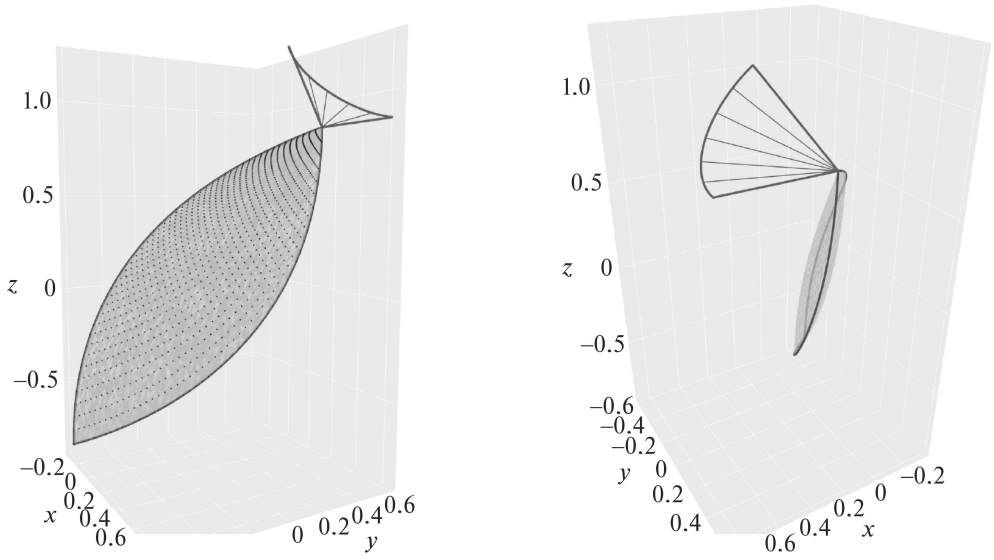

Fig. 1. Attainable set of (10) and normal vectors, where the supporting element is not locally Lipschitz, $t = 2$.

The reachable set for $t = 2$ can be seen at Fig. 1. Normal vectors, for which the supporting element is not locally Lipschitz, can be seen in the upper part of the figure. They lie on the boundary of the normal cone at the tip of the set. Moreover, it is evident that the reachable set is structured like a CW complex. This structure appears as a result of (8), since the supporting element can be determined by positions and multiplicities of the roots of $g_p(s)$ on $[0, t]$ and the sign of g_p around the left end of the segment. If the system has a matrix with real eigenvalues, then the overall multiplicity of roots of g_p is not greater that $n - 1$. It can be shown, that in this case an arbitrary configuration of roots substituted into (8) produces a point from $\partial Rs(t)$. Evaluating (8) on sets of roots with different overall multiplicities allows us to extract curvilinear edges and faces from the reachable set. Some generalization of the above arguments can be seen in [11].

Lemma 2. *Suppose that $A_1 = J^{-1}AJ$ is the Jordan form of the matrix A from system (5), $\mathcal{U}_1 = J^{-1}\mathcal{U}$, where $J \in \mathbb{R}^{n\times n}$ is the transfer matrix. If the set $\mathcal{R}_1(t) = \int_0^t e^{A_1 s}\mathcal{U}_1\, ds$ is strongly convex with radius r, then $\mathcal{R}(t) = \int_0^t e^{As}\mathcal{U}\, ds$ is also strongly convex with radius $R = r\alpha^2/\beta$, where $\alpha = \|J\| = \max_{\|h\|=1} \|Jh\|$, $\beta = \min_{\|h\|=1} \|Jh\|$.*

Note that by [25, Theorem 3] any ellipsoid

$$\mathcal{N} = \left\{ x \in \mathbb{R}^n \ : \ \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2} \leqslant 1 \right\}, \quad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0,$$

is strongly convex with radius $R = \frac{\lambda_1^2}{\lambda_n}$.

Lemma 3. *Suppose that in system (5) $\mathcal{U}$ is uniformly smooth with constant $r > 0$. Then $\mathcal{R}(t)$ (6) is uniformly smooth with constant $r_0 = r \int\limits_0^t \frac{\lambda_n^2(s)}{\lambda_1(s)}\, ds$, where $\lambda_1(s) \geqslant \ldots \geqslant \lambda_n(s) > 0$ are the semiaxes of the ellipsoid $e^{As}\mathcal{B}_1(0)$.*

Note that by the proof of Lemma 3 any ellipsoid

$$\mathcal{N} = \left\{ x \in \mathbb{R}^n \ : \ \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2} \leqslant 1 \right\}, \quad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0,$$

is uniformly smooth with constant $r = \frac{\lambda_n^2}{\lambda_1}$.

In particular, Lemmas 2 and 3 show that it is enough to consider system (5) with the Jordan form of the matrix A.

The next proposition estimates the rate of decrease for a Lipschitz differentiable function per step of the gradient projection method.

Proposition 2 [26, Lemma 2]. *Consider the problem* $\min_{\mathcal{M}} f(x)$ *in* $\mathbb{R}^n$. *Suppose that* $\mathcal{M}$ *is a closed set,* f' *is a Lipschitz function with constant* L_1. *Fix* $0 < \lambda \leqslant \frac{1}{L_1}$. *Assume that* $x_0 \in \mathcal{M}$ *and* $y_0 \in P_{\mathcal{M}}(x_0 - \lambda f'(x_0))$. *Then*

$$f(x_0) - f(y_0) \geqslant \frac{1}{2}\left(\frac{1}{\lambda} - L_1\right)\|x_0 - y_0\|^2.$$

For the validity of the previous formula the Lipschitz condition for f' with constant L_1 is essential on the segment $[x_0, y_0]$, see the proof of [27, Proposition 2.2].

1.2. Additional Assumptions on $\mathcal{R}(s)$

When solving problems (P1)–P(3) we will require some additional assumptions on sets we work with. Here we will enumerate all of them, we will only need some of them for each problem.

(1) $\mathcal{R}(s)$ is strongly convex with radius $R_T > 0$ for all $s \in [0, T]$.
(2) $\mathcal{M}$ is uniformly smooth with constant $r > 0$: $\mathcal{M} = \mathcal{M}_0 + \mathcal{B}_r(0)$, also
  (a) $\mathcal{M}_0$ us strongly convex with constant $R_0 > 0$.
  (b) $r > R_T$.
(3) $\mathcal{M}$ is strongly convex with constant $R_0 > 0$.
(4) $\mathcal{U}$ is uniformly smooth with constant $r_{\mathcal{U}} > 0$: $\mathcal{U} = \mathcal{U}_0 + \mathcal{B}_{r_{\mathcal{U}}}(0)$.
(5) $r(t) > R_0$, where $r(t) = r_{\mathcal{U}} \int_0^t \frac{\lambda_n^2(s)}{\lambda_1(s)}$ and $\lambda_1(s) \geqslant \ldots \geqslant \lambda_n(s)$ are the semiaxes of the ellipsoid $e^{As}\mathcal{B}_1(0)$.

The first assumption is fulfilled if, for example, the set $e^{As}\mathcal{U}$ is strongly convex with radius $R(s) > 0$. Then from proposition 1 and linearity of the integral it follows that

$$\mathcal{R}(T) = \int_0^T e^{As}\mathcal{U}\,ds = \int_0^t e^{As}\mathcal{U}\,ds + \int_t^T e^{As}\mathcal{U}\,ds = \mathcal{R}(t) + \int_t^T e^{As}\mathcal{U}\,ds,$$

then we obtain that the set

$$\mathcal{R}(t) = \bigcap \left\{ \mathcal{R}(T) - x \ : \ x \in \int_t^T e^{As}\mathcal{U}\,ds \right\}$$

is strongly convex with radius $R_T = \int_0^T R(s)\,ds$ for all $t \in [0, T]$.

1.3. Structure of the Paper

In Sections 2–4 we formulate sufficient conditions and prove results about linear convergence of the gradient projection method for a particular optimization problem with supporting functions to which problems (P1)–(P3) are reduced. This solves problems for a fixed $t \in [0, T]$.

In Section 5 we discuss how we can find the starting point p_1 for the iteration process. Estimates of the probability of finding p_1 using random search are given.

In Section 6 we discuss the results of numerical experiments. Here we also consider an algorithm for finding the optimal t in problems (P1)–(P3).

2. PROBLEM (P1)

Assumptions: 1, 2(a).

For all $t \in [0,T]$ consider the set $\mathcal{N}(t) = \mathcal{R}(t) + (-\mathcal{M}_0)$. The set $\mathcal{N}(t)$ is strongly convex with radius $R = R_T + R_0$ as a sum of strongly convex sets [19]. The equality $\mathcal{R}(t) \cap \mathcal{M} = \emptyset$ can be reformulated as follows: the distance from zero to $\mathcal{N}(t)$ is more than $r > 0$. If the last assertion is true, then $0 \notin \mathcal{R}(t) + (-\mathcal{M})$ and otherwise $0 \in \mathcal{R}(t) + (-\mathcal{M})$. Using the supporting function we can check the inclusion as follows: for the function $f(p) = s(p, \mathcal{N}(t)) = s(p, \mathcal{R}(t)) + s(p, -\mathcal{M}_0)$ find

$$\min_{\|p\|=1} f(p) = J. \tag{11}$$

If $J < -r$, then the distance from zero to the set $\mathcal{N}(t)$ is greater than r. If $J \geqslant -r$, then the distance from zero to the set $\mathcal{N}(t)$ is no greater than r and hence $0 \in \mathcal{R}(t) + (-\mathcal{M})$. Note that

$$f'(p) = \mathcal{R}(t)(p) + (-\mathcal{M}_0)(p) = \int_0^t (e^{As}\mathcal{U})(p)\, ds + (-\mathcal{M}_0)(p). \tag{12}$$

Theorem 1. *Fix $\varepsilon \in (0,1)$. Suppose that in (11) $J < 0$. Then under above mentioned assumptions the function f in (11) satisfies the LPL condition on the manifold $\mathcal{S} = \{p \in \mathcal{S}_1 : f(p) \leqslant 0\}$ with constant $\mu = |J|$. Also the function f has Lipschitz continuous gradient on the set $\{p \in \mathbb{R}^n : 1 - \varepsilon \leqslant \|p\| \leqslant 1 + \varepsilon\}$ with Lipschitz constant $L_1 = \frac{R}{1-\varepsilon} = \frac{R_T + R_0}{1-\varepsilon}$.*

Consider the following iteration process

$$p_1 \in \mathcal{S} \text{ (i.e. } f(p_1) \leqslant 0), \quad p_{k+1} = P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)), \qquad \lambda \in \left(0, \frac{1}{L_1}\right]. \tag{13}$$

If $p_k \in \mathcal{S}$, then $p_{k+1} \in \mathcal{S}$. Indeed, by Proposition 2

$$f(p_k) - f(p_{k+1}) \geqslant \frac{1}{2}\left(\frac{1}{\lambda} - L_1\right)\|p_k - p_{k+1}\|^2 \geqslant 0, \quad f(p_{k+1}) \leqslant f(p_k) \leqslant 0.$$

Consider the point $p_k - \lambda f'(p_k)$. We have

$$\|p_k - \lambda f'(p_k)\| \geqslant (p_k, p_k - \lambda f'(p_k)) = 1 - \lambda(p_k, f'(p_k)) = 1 - \lambda f(p_k) \geqslant 1.$$

Theorem 2. *Suppose that the function f is Lipschitz continuous with constant $L = \|\mathcal{N}(t)\|$, the function f' is Lipschitz continuous on S_1 with constant $R = R_T + R_0$. Suppose that $J < 0$. Put $L_1 = 2R$.*

Fix $\lambda \in (0, \min\{\frac{1}{L_1}, \frac{1}{2L}\})$. Then algorithm (13) converges to a point of minimum $p_0 \in \mathcal{S}_1$ at a linear rate:

$$f(p_{k+1}) - f(p_0) \leqslant q(f(p_k) - f(p_0)),$$
$$\|p_{k+1} - p_k\| \leqslant q^{k/2}\sqrt{2\lambda(f(p_1) - f(p_0))},$$
$$q = 1 - \frac{\lambda|J|}{2L\lambda + 2} \in (0,1).$$

The next example shows that the sharpness condition of the type $\exists \alpha > 0$ that $f(p) - f(p_0) \geqslant \alpha\|p - p_0\|$ for all $p \in \mathcal{S}$ does not hold.

Consider $L > r > 0$, $\|p_0\| = 1$ and the set $\mathcal{N} = \mathcal{B}_r(-Lp_0)$. Then for all $p \in \mathcal{S}_1$ we have

$$s(p, \mathcal{N}) - s(p_0, \mathcal{N}) = L(1 - (p, p_0)) = \frac{L}{2}\|p - p_0\|^2.$$

Remark 1. The above results can be proven under more local assumptions. Instead of the strong convexity assumption 1 of $\mathcal{R}(T)$ with radius R_T we can require the fulfillment for all $p \in \mathcal{S}$ the supporting principle for the set $\mathcal{R}(t)$: there exists $R_T > 0$ with

$$\mathcal{R}(t) \subset B_{R_T}(\mathcal{R}(t)(p) - R_T p), \quad \forall p \in \mathcal{S}. \tag{14}$$

Assumption 2(a) concerning $\mathcal{M}$ must be met.

In this situation the set $Z(t) = \mathcal{R}(t) + (-\mathcal{M}_0)$ satisfies the supporting principle for all $p \in \mathcal{S}$ with radius $R = R_T + R_0$:

$$Z(t) \subset B_R(\mathcal{N}(t)(p) - Rp), \quad \forall p \in \mathcal{S}.$$

For any $p, q \in \mathcal{S}$ we get $\|\mathcal{N}(t)(p) - Rp - \mathcal{N}(t)(q)\|^2 \leqslant R^2$, $\|\mathcal{N}(t)(q) - Rq - \mathcal{N}(t)(p)\|^2 \leqslant R^2$ and $\|\mathcal{N}(t)(p) - \mathcal{N}(t)(q)\|^2 \leqslant 2R(p, \mathcal{N}(t)(p) - \mathcal{N}(t)(q))$,

$$\|\mathcal{N}(t)(q) - \mathcal{N}(t)(p)\|^2 \leqslant 2R(q, \mathcal{N}(t)(q) - \mathcal{N}(t)(p)) = 2R(-q, \mathcal{N}(t)(p) - \mathcal{N}(t)(q)),$$

hence $\|\mathcal{N}(t)(p) - \mathcal{N}(t)(q)\| \leqslant R\|p - q\|$. Keeping in mind that for any $p, q \in \mathcal{S}$ the small arc of the circle of radius 1 with center 0 and endpoints p, q belongs to $\mathcal{S}$, we can repeat proofs of Theorems 1 and 2 for the considered case. In the generalization of Theorem 1 we should take $p, q \in \mathbb{R}^n$ with $\frac{p}{\|p\|}$, $\frac{q}{\|q\|} \in \mathcal{S}$, i.e. Lipschitz condition will be proved on the set $\left\{p \in \mathbb{R}^n : 1 - \varepsilon \leqslant \|p\| \leqslant 1 + \varepsilon, \ \frac{p}{\|p\|} \in \mathcal{S}\right\}$.

3. PROBLEM (P2)

Assumptions: 1, 2(b), 3.

Fix $\varepsilon \in (0, r - R_T)$. Consider ε-neighbourhood $\mathcal{R}_\varepsilon(t) = \mathcal{R}(t) + \mathcal{B}_\varepsilon(0)$ of the set $\mathcal{R}(t)$. Inclusion $\mathcal{R}(t) \subset \mathcal{M}$ means that

$$\max_{x \in \mathcal{R}_\varepsilon(t)} \varrho(x, \mathcal{M}) \leqslant \varepsilon$$

and otherwise, if $\max_{x \in \mathcal{R}_\varepsilon(t)} \varrho(x, \mathcal{M}) > \varepsilon$, then $\mathcal{R}(t) \not\subset \mathcal{M}$. Using supporting functions we can formulate an equivalent problem: for the function $f(p) = s(p, \mathcal{M}) - s(p, \mathcal{R}_\varepsilon(t))$ find minimum

$$\min_{\|p\|=1} f(p) = J. \tag{15}$$

If $J \geqslant -\varepsilon$ then $\mathcal{R}(t) \subset \mathcal{M}$ and if $J < -\varepsilon$ then $\mathcal{R}(t) \not\subset \mathcal{M}$.

Let $\mathcal{S} = \{p \in \mathcal{S}_1 \ : \ f(p) \leqslant 0\}$. Suppose that $p_0 \in \mathcal{S}_1$ is a solution of (15).

Assume that $\mathcal{S} \neq \emptyset$. Consider an iteration process

$$p_1 \in \mathcal{S}, \qquad p_{k+1} = P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)). \tag{16}$$

Theorem 3. *Suppose that under assumptions of Section 3 we have $J < 0$ in problem (15). Let $r_0 = r - R_T - \varepsilon > 0$, $L = \|\mathcal{M} \stackrel{*}{-} \mathcal{R}_\varepsilon(t)\| > 0$. Then for any $p_1 \in \mathcal{S}$ and $0 < \lambda \leqslant \min\{r_0^2/R_0^3, 1/(2L), 1/(2R_0)\}$ iterations (16) converge at a linear rate to the solution p_0:*

$$\|p_{k+1} - p_0\| \leqslant q\|p_k - p_0\|, \qquad q = \sqrt{1 - \frac{2r_0^2}{R_0}\lambda + R_0^2\lambda^2} \in (0, 1).$$

Remark 2. As in Section 2, we can prove the above results under more local assumptions. Instead of the Assumption 1 on strong convexity of $\mathcal{R}(s)$ for all $s \in [0, T]$ with radius R_T we can require the fulfillment for all $p \in \mathcal{S}$ of the supporting principle for the set $\mathcal{R}(t)$: there exists $R_T > 0$ such that for a number $\varepsilon \in (0, r - R_T)$ we have

$$\mathcal{M}(p) - \mathcal{R}(t)(p) + \mathcal{R}(t) \subset B_{R_T}(\mathcal{M}(p) - R_T p) \subset B_{r-\varepsilon}(\mathcal{M}(p) - (r - \varepsilon)p) \subset \mathcal{M}, \quad \forall p \in \mathcal{S}. \tag{17}$$

Assumptions 2(b), 3 concerning $\mathcal{M}$ must be met.

In the considered situation we have

$$\mathcal{M}(p) - \mathcal{R}(t)(p) - \varepsilon p + \mathcal{R}_\varepsilon(t) \subset \mathcal{M}, \quad \forall p \in \mathcal{S} \tag{18}$$

and hence $f'(p) = \mathcal{M}(p) - \mathcal{R}(t)(p) - \varepsilon \times p = \mathcal{M}(p) - \mathcal{R}_\varepsilon(t)(p) = \left(\mathcal{M} \stackrel{*}{-} \mathcal{R}_\varepsilon(t)\right)(p)$ for all $p \in \mathcal{S}$ because $f'(p) \in \mathcal{M} \stackrel{*}{-} \mathcal{R}_\varepsilon(t)$ and $(p, f'(p)) = s(p, \mathcal{M} \stackrel{*}{-} \mathcal{R}_\varepsilon(t))$ for all $p \in \mathcal{S}$. Indeed, fix $p \in \mathcal{S}$. From the inclusion $f'(p) + \mathcal{R}_\varepsilon(t) \subset \mathcal{M}$ we get $f'(p) \in \mathcal{M} \stackrel{*}{-} \mathcal{R}_\varepsilon(t)$. On the other hand $(p, f'(p)) + s(p, \mathcal{R}_\varepsilon(t)) = s(p, \mathcal{M})$ and thus $(p, f'(p)) = s(p, M) - s(p, \mathcal{R}_\varepsilon(t)) \geqslant \operatorname{co}(s(p, M) - s(p, \mathcal{R}_\varepsilon(t)) = s(p, \mathcal{M} \stackrel{*}{-} \mathcal{R}_\varepsilon(t))$.

The next steps repeat the proof of Theorem 3.

4. PROBLEM (P3)

Assumptions: 1, 3, 4, 5.

Note that by Lemma 3 the set $\mathcal{R}(t)$ is uniformly smooth with constant $r(t)$ and hence $R_T \geqslant r(t)$.

Fix $\varepsilon \in (0, r(t) - R_0)$. Consider ε-neighbourhood $\mathcal{M}_\varepsilon = \mathcal{M} + \mathcal{B}_\varepsilon(0)$ of the set $\mathcal{M}$. Inclusion $\mathcal{R}(t) \supset \mathcal{M}$ means that

$$\max_{x \in \mathcal{M}_\varepsilon} \varrho(x, \mathcal{R}(t)) \leqslant \varepsilon$$

and otherwise, if $\max_{x \in \mathcal{M}_\varepsilon} \varrho(x, \mathcal{R}(t)) > \varepsilon$, then $\mathcal{R}(t) \not\supset \mathcal{M}$. On the base of supporting functions we can formulate the next equivalent problem: for the function $f(p) = s(p, \mathcal{R}(t)) - s(p, \mathcal{M}_\varepsilon) = s(p, \mathcal{R}(t)) - s(p, \mathcal{M}) - \varepsilon\|p\|$ find minimum

$$\min_{\|p\|=1} f(p) = J. \tag{19}$$

If $J \geqslant -\varepsilon$ then $\mathcal{R}(t) \supset \mathcal{M}$ and if $J < -\varepsilon$ then $\mathcal{R}(t) \not\supset \mathcal{M}$.

As usual, $\mathcal{S} = \{p \in \mathcal{S}_1 \ : \ f(p) \leqslant 0\}$. Suppose that $p_0 \in \mathcal{S}_1$ is a solution of (19).

Assume that $\mathcal{S} \neq \emptyset$. Consider an iteration process

$$p_1 \in \mathcal{S}, \qquad p_{k+1} = \mathcal{P}_{\mathcal{S}_1}(p_k - \lambda f'(p_k)). \tag{20}$$

Theorem 4. *Suppose that under assumptions of Section 4 we have $J < 0$ in problem (19). Let $r = r(t) - R - \varepsilon > 0$, $L = \|\mathcal{R}(t) \stackrel{*}{-} \mathcal{M}_\varepsilon\|$. Then for any $p_1 \in \mathcal{S}$ and $0 < \lambda \leqslant \min\{r^2/R_T^3, 1/(2L), 1/(2R_T)\}$ iterations (20) converges at a linear rate to the solution p_0:*

$$\|p_{k+1} - p_0\| \leqslant q\|p_k - p_0\|, \qquad q = \sqrt{1 - \frac{2r^2}{R_T}\lambda + R_T^2\lambda^2} \in (0, 1).$$

Remark 3. As in Section 3, we can also prove the above results under more local assumptions. Instead of the strong convexity Assumption 3 of $\mathcal{M}$ with radius R_0 we can require the fulfillment for all $p \in \mathcal{S}$ of the supporting condition for the set $\mathcal{M}$: there exists $R_0 > 0$ such that

$$\mathcal{M} \subset B_{R_0}(\mathcal{M}(p) - R_0 p), \quad \forall p \in \mathcal{S}. \tag{21}$$

Assumptions 1, 4 and 5 must be met.

5. CHOOSING THE INITIAL POINT

We choose p_1 using random search: in problems (P1)–(P3) we sample a random vector $p_1 \in \mathcal{S}_1$ from a uniform distribution and check the inequality $f(p_1) \leqslant 0$. If it fails, we choose another random vector $p_1 \in \mathcal{S}_1$ and so on. In the present section we estimate the probability $\mathbb{P}(\{f(p_1) \leqslant 0\})$ to find an appropriate vector p_1. As an example, let us consider (P1) for fixed $t > 0$. Recall, that

$J<0$ is the solution of problem (11). By assumptions for (P1), the set $\mathcal{N}(t)$ is strongly convex with radius $R>0$. Denote $z_0=P_{\mathcal{N}(t)}0$, $p_0=-z_0/\|z_0\|$. For a set $\mathcal{M}\subset\mathbb{R}^n$ define $\operatorname{cone}\mathcal{M}$ to be the (convex) conic hull of the set $\mathcal{M}$, i.e. $\operatorname{cone}\mathcal{M}=\{\sum_{i=1}^n\lambda_i x_i\ :\ x_i\in\mathcal{M},\ \lambda_i\geqslant 0\}$. For a pair of points $x,y\in\mathbb{R}^n$, $x\neq y$, define the ray $[x,y)=\{x+t(y-x)\ :\ t\geqslant 0\}$.

Let $D>0$ and $H=\{x\in\mathbb{R}^n\colon (p_0,x-z_0)=0\}$. Suppose that $\mathcal{K}=\operatorname{cone}(H\cap B_D(z_0))\supset\operatorname{cone}\mathcal{N}(t)$. For example, D can be the diameter of the set $\mathcal{N}(t)$, i.e. $D=\sup_{x,y\in\mathcal{N}(t)}\|x-y\|$.

The set $\mathcal{K}$ is a cone of revolution with axis $[0,z_0)$. The angle between the axis and a generatrix is equal to α, $\tan\alpha=\frac{D}{|J|}$. The polar set $\mathcal{K}^-=\{p\in\mathbb{R}^n\ :\ (p,q)\leqslant 0\quad\forall q\in\mathcal{K}\}$ is also a cone of revolution with axis $[0,-z_0)$ and the angle between the axis and a generatrix is equal to $\beta=\frac{1}{2}\pi-\alpha$, thus $\cos\beta=\frac{D}{\sqrt{D^2+J^2}}$.

By the definition of $\mathcal{K}$ we have for any $p_1\in\mathcal{S}_1\cap\mathcal{K}^-$ that $f(p_1)\leqslant 0$. Denote $\mathcal{S}_{\mathrm{cap}}=\mathcal{S}_1\cap\mathcal{K}^-$ and $\mathcal{S}_0=\mathcal{K}^-\cap H_0$, here $H_0=\{x\in\mathbb{R}^n\ :\ (p_0,x)=\cos\beta\}$. Note that $\mathcal{S}_0=H_0\cap B_{r_0}(\cos\beta\times p_0)$ with $r_0=\sin\beta=\frac{|J|}{\sqrt{D^2+J^2}}$. $(n-1)$-Lebesgue's measure $\mu_{n-1}\mathcal{S}_0\leqslant\mu_{n-1}\mathcal{S}_{\mathrm{cap}}$ and thus

$$\mathbb{P}(\{f(p_1)\leqslant 0\})\geqslant\frac{\mu_{n-1}\mathcal{S}_{\mathrm{cap}}}{\mu_{n-1}\mathcal{S}_1}\geqslant\frac{\mu_{n-1}\mathcal{S}_0}{\mu_{n-1}\mathcal{S}_1}=\frac{r_0^{n-1}}{n}\frac{V_{n-1}}{V_n}=\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{|J|}{\sqrt{D^2+J^2}}\right)^{n-1},$$

$V_n=\frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ is the volume of a unit ball in $\mathbb{R}^n$.

Suppose now that $B_r(z_0-rp_0)\subset\mathcal{N}(t)$ for some $r>0$. Then consider a cone of revolution $\mathcal{K}=\operatorname{cone}B_r(z_0-rp_0)\subset\operatorname{cone}\mathcal{N}(t)$ with axis $[0,z_0)$. The angle between the axis and a generatrix of $\mathcal{K}$ is equal to α, $\sin\alpha=\frac{r}{r+|J|}$. Define a polar cone $\mathcal{K}^-\supset(\operatorname{cone}\mathcal{N}(t))^-$ with the angle β between the axis $[0,-z_0)$ and a generatrix, $\cos\beta=\frac{r}{r+|J|}$. We have for any $p_1\in\mathcal{S}_1$ with $f(p_1)\leqslant 0$ that $p_1\in\mathcal{S}_{\mathrm{cap}}$, as previously $\mathcal{S}_{\mathrm{cap}}=\mathcal{S}_1\cap\mathcal{K}^-$. Define $\mathcal{S}_0^1=\mathcal{K}^-\cap H_1$ with $H_1=\{x\in\mathbb{R}^n\ :\ (p_0,x)=1\}$. From the elementary planimetry it is easy to see that $\mathcal{S}_0^1=H_1\cap B_{r_1}(p_0)$, $r_1=\tan\beta=\frac{\sqrt{2r|J|+|J|^2}}{r}$. Then $\mu_{n-1}\mathcal{S}_0^1\geqslant\mu_{n-1}\mathcal{S}_{\mathrm{cap}}$ and

$$\mathbb{P}(\{f(p_1)\leqslant 0\})\leqslant\frac{\mu_{n-1}\mathcal{S}_{\mathrm{cap}}}{\mu_{n-1}\mathcal{S}_1}\leqslant\frac{\mu_{n-1}\mathcal{S}_0^1}{\mu_{n-1}\mathcal{S}_1}=\frac{r_1^{n-1}}{n}\frac{V_{n-1}}{V_n}=\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{\sqrt{2r|J|+|J|^2}}{r}\right)^{n-1}.$$

Finally for a set $\mathcal{N}(t)$ of diameter D that is also uniformly smooth with constant r we have

$$\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{|J|}{\sqrt{D^2+J^2}}\right)^{n-1}\leqslant\mathbb{P}(\{f(p_1)\leqslant 0\})\leqslant\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{\sqrt{2r|J|+|J|^2}}{r}\right)^{n-1}.\tag{22}$$

Similarly with the right estimate in (22) for an R-strongly convex set $\mathcal{N}(t)$ one can prove that

$$\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{\sqrt{2R|J|+|J|^2}}{R+|J|}\right)^{n-1}\leqslant\mathbb{P}(\{f(p_1)\leqslant 0\}).$$

This estimate shows that $\mathbb{P}(\{f(p_1)\leqslant 0\})\asymp|J|^{n-1}$ when $J\to 0$. In our consideration $|J|$ is of the order $\varepsilon>0$ and in this case the left inequality in (22) gives a more reasonable estimate because in most examples the value of D is much less than R.

The estimated probability can be very small and strongly influences calculations when either $|J|$ is close to zero or n is large. In our experiments in the examples below for n in range $3\leqslant n\leqslant 12$ we found p_1 in a few dozens attempts at most (for problems (P1), (P2)). Sometimes we needed about 1000 attempts to find the vector p_1 in problem (P3). One of the reasons is that $D>0$ in the above estimate can be chosen to be significantly smaller than the diameter $\mathcal{N}(t)$, since we only need the fulfillment of the inclusion $\operatorname{cone}(H\cap B_D(z_0))\supset\operatorname{cone}\mathcal{N}(t)$.

Sometimes we can choose p_1 deterministically, see the Algorithm from Section 9.

The step size λ for solving problems (P1)–(P3) can be chosen using the Armijo rule. Its detailed description can be found in [29].

6. MODELING AND EXAMPLES

Some of the considered examples are low-dimensional ($n = 3$) for ease of interpretation by a reader. As shown in the following, convergence rates for such examples and for examples of higher dimension are the same.

6.1. Problem (P1). Example 1

In this example we calculate the point of time at which the reachable set $\mathcal{R}$ first intersects the target set $\mathcal{M}$.

Consider the system

$$\dot{x} = Ax + Bu, \quad x(0) = 0, \quad u \in \mathbb{R}: \quad |u| \leqslant 1, \quad A = \begin{bmatrix} -1.3 & 1 & 0 \\ 0 & -1.3 & 1 \\ 0 & 0 & -1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{23}$$

The target set is $\mathcal{M} = \mathcal{M}_0 + B_r(0)$, where $\mathcal{M}_0$ is the ball $B_{0.2}(0.7, -0.3, 0.35)$, $r = 0.5$. Recall that $f(p)$ in problem (11) depends on t, i.e. $f(p,t) = s(p, \mathcal{R}(t)) + s(p, -\mathcal{M}_0)$.

We first consider the auxiliary problem of finding the distance between sets $\mathcal{R}(t)$ and $\mathcal{M}$ for $t = 1$, with initial condition $p_1 = (0{,}03123620,\ -0{,}72453809,\ 0{,}68852659)$, $f(p_1, 1) = -0{,}05270947$.

Figure 2,a: Convergence of the gradient projection algorithm for the auxiliary problem $\min_{\|p\|=1} f(p,t)$ for $t = 1$. Approximation of the convergence rate is $f(p_k, 1) - f(p_0, 1) \approx 0.2486 \times 0.83043^k$. The found solution is $p_0 = (0.87540058,\ -0.46926876,\ 0.11602002)$ with $f(p_0, 1) = -0.573989$.

The reachable set and the point closest to the target set are depicted on Fig. 3.

When searching for the minimal time at which intersection occurs, we only know the search interval $[0, T]$, but not the starting point p_1 for arbitrary moment of time from the interval. There are two different strategies. The first one is to randomly find $p_1 \in \mathcal{S}_1$ with $f(p_1, t) < 0$ for a given t and increase t by a small amount. However, due to the time-related nature of (P1) there is a better algorithm. This algorithm involves keeping track of suitable p, $f(p,t) < 0$, while increasing the time.

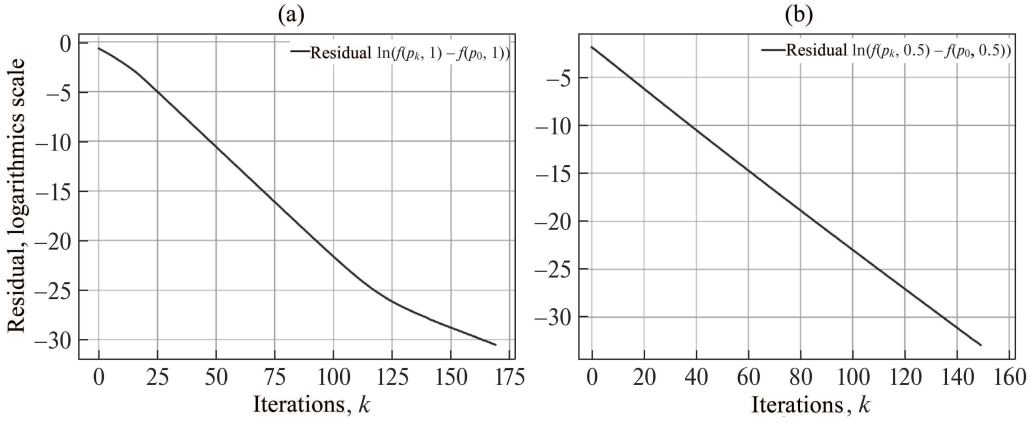

Fig. 2. Convergence of gradient projection algorithm with step size $\lambda = 0.1$. (a) Problem (P1), Example 1, (b) problem (P1), Example 2.

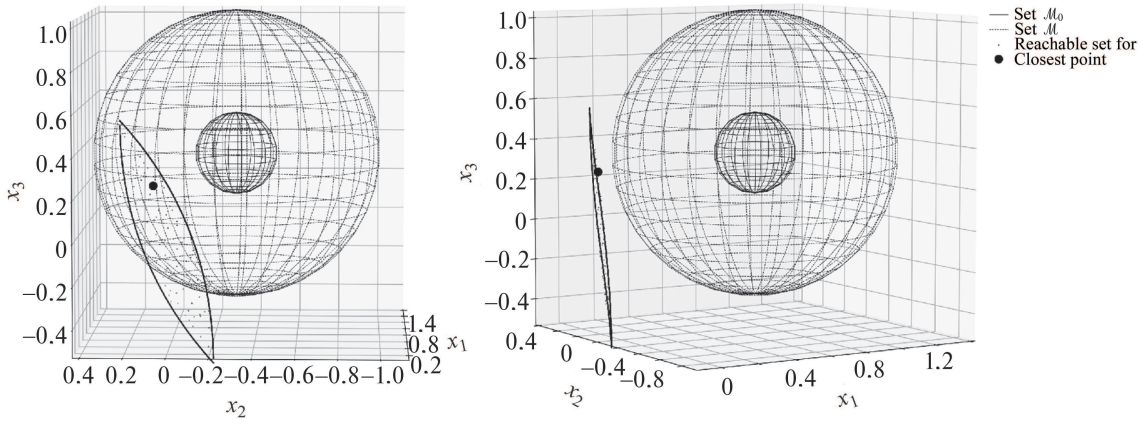

Fig. 3. The point of the reachable set $\mathcal{R}(t)$ $(t = 1)$ closest to the target set found by gradient projection algorithm (problem (P1), Example 1).

Algorithm for problem (P1) (finding minimal time)

Data: $T > 0$, $f(p,t)$, $r > 0$, tolerance $\varepsilon_{\text{tol}} > 0$, bounds $t_{\text{lower}} = 0, t_{\text{upper}} = T$, time step $\Delta_t > 0$.

(1) Put $t \leftarrow 0$ and find initial p_1 satisfying $f(p_1, 0) < 0$ first. Then run the gradient projection method which gives $p(0) = \arg\min_{\|p\|=1} f(p,0) : f(p(0),0) < 0$.

(2) Put $t_{\text{test}} = \min\{t + \Delta_t, t_{\text{upper}}\}$.
If $f(p(t), t_{\text{test}}) \geqslant 0$, then set $\Delta_t \leftarrow \Delta_t/2$ and repeat this step.
If $f(p(t), t_{\text{test}}) < 0$, then proceed to Step (3).

(3) Run the gradient projection method (13) for function $f(p, t_{\text{test}})$ with initial point $p_1 = p(t)$. It results in p_0 and $J = f(p_0, t_{\text{test}}) = \min_{\|p\|=1} f(p, t_{\text{test}}) < 0$.

(4) If $J > -r + \varepsilon_{\text{tol}}$, then the reachable set intersects the set $\mathcal{M}$. Update $t_{\text{upper}} \leftarrow t_{\text{test}}$, $\Delta_t \leftarrow \frac{1}{2}\min\{\Delta_t, t_{\text{upper}} - t_{\text{lower}}\}$ and proceed to Step (2) with the same t and $p(t)$. Otherwise continue with Step (5).

(5) If $J < -r - \varepsilon_{\text{tol}}$, then the reachable set has yet to reach the set $\mathcal{M}$. Update $t_{\text{lower}} \leftarrow t_{\text{test}}$, $\Delta_t \leftarrow \min\{2\Delta_t, \frac{t_{\text{upper}} - t_{\text{lower}}}{2}\}$.
Also update $t \leftarrow t_{\text{test}}, p(t) \leftarrow p_0$ and continue with Step (2).
Otherwise finish with Step (6).

(6) A solution is found within given tolerance: $|J + r| \leqslant \varepsilon_{\text{tol}}$. Return $t_0 = t_{\text{test}}$ as the optimal time for problem (P1), and p_0.

Notes: the algorithm performs bisection-like search on the time interval $[0, T]$. Probability of finding suitable p_1 at Step (1) may be estimated using results from Section 5. However, it can be found non-randomly at Step (1) if we can somehow find a unit separation vector $p_1 \in \mathbb{R}^n$ such that $(p_1, x) \leqslant 0$ for all $x \in -\mathcal{M}_0$. Further at each Step (2), the initial value p_1 of the gradient projection algorithm is chosen non-randomly. At Step (5), the time step is doubled for faster search. The algorithm may also operate if the value T is unknown (i.e. $t_{\text{upper}} = \infty$), but for $t_{\text{upper}} > T$ convergence conditions for the gradient projection algorithm may be violated. Nevertheless the invariance $t_{\text{lower}} \leqslant t_{\text{test}} \leqslant t_{\text{upper}}$ is satisfied.

The algorithm stops when we obtain J with a given tolerance ε_{tol}, in all examples here and below $\varepsilon_{\text{tol}} = 10^{-7}$ and at the final stage $t_{\text{upper}} - t_{\text{lower}} \sim 10^{-6}$. We also can stop the algorithm with a given precision with respect to the time t: e.g. when $t_{\text{upper}} - t_{\text{lower}} \leqslant \varepsilon_{time}$ we finish calculations and take $t \in [t_{\text{lower}}, t_{\text{upper}}]$. Here $\varepsilon_{time} > 0$ is an admissible time error.

For system (23) Algorithm converges in 21 steps. The optimal time is 2.7383842,

$$p_0 = (0.77091811, -0.60777697, 0.19050571).$$

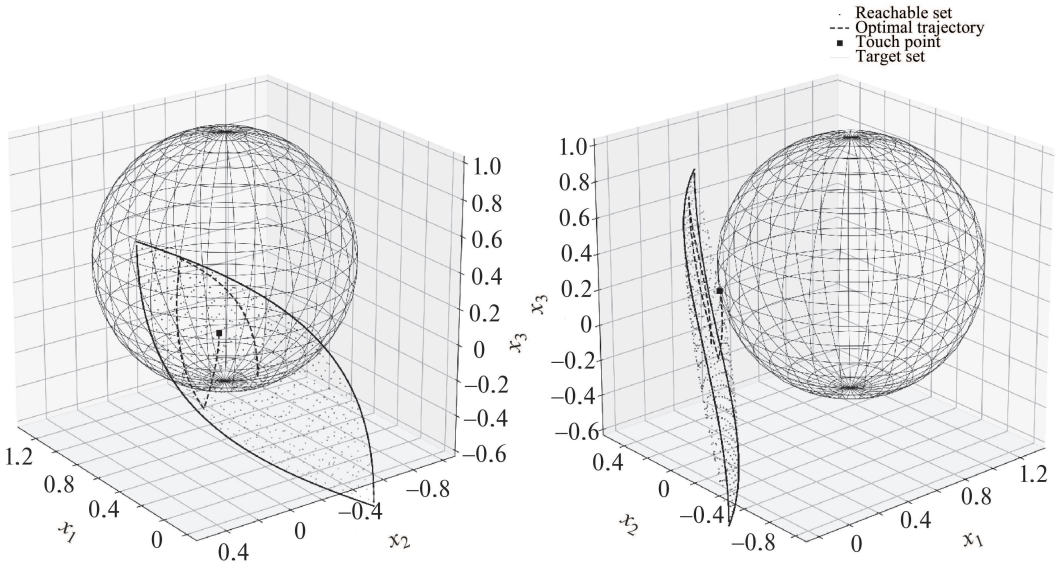

Fig. 4. Attainable set at the moment of intersection and the optimal trajectory (problem (P1), Example 1).

Figure 4 depicts the reachable set and the target set at the moment when they intersect. The optimal trajectory (as in 9) with two switches can also be seen. Also see [24].

As was shown in Introduction, the reachable set $\mathcal{R}(t)$ of system (23) is not strongly convex. For $\mathcal{U} = B \times [-1, 1]$ and $t > 0$ we have $s(p, \mathcal{R}(t)) = \int_0^t e^{-1.3s} |p_1 \frac{s^2}{2} + p_2 s + p_3|\, ds$ for any $p = (p_1, p_2, p_3) \in \mathcal{S}_1$. For the solution $p_0 = (0.77091811, -0.60777697, 0.19050571)$ and $t = 2.73838\ldots$ we have the roots $s_1(p_0) < s_2(p_0)$ of the equation $p_1 \frac{s^2}{2} + p_2 s + p_3 = 0$ for $p = p_0$. By the inverse function theorem the roots $\mathcal{S}_1 \ni p \to s_i(p)$, $i = 1, 2$, of the equation $p_1 \frac{s^2}{2} + p_2 s + p_3 = 0$ are analytic in some neighbourhood of the point $p_0 \in \mathcal{S}_1$. In other words, there exists a number $\gamma > 0$ such that the functions

$$\mathcal{S}_1 \cap B_\gamma(p_0) \ni p \to s_i(p), \quad i = 1, 2,$$

are Lipschitz continuous with some constant $L > 0$. Moreover, we can choose the number $\gamma > 0$ so that the first components of p and q are strictly positive and $\max\{s_1(p), s_1(q)\} \leqslant \min\{s_2(p), s_2(q)\}$ for all $p, q \in \mathcal{S}_1 \cap B_\gamma(p_0)$.

Fix a pair of points $p, q \in \mathcal{S}_1 \cap B_\gamma(p_0)$. Put $M = \max\limits_{s \in [0,t]} \|e^{As}\|$. Then $|s_i(p) - s_i(q)| \leqslant L\|p - q\|$ for $i = 1, 2$ and for the supporting elements, using the estimate $\|\mathcal{U}(e^{A^T s} p) - \mathcal{U}(e^{A^T s} q)\| \leqslant 2$, we have

$$\|\mathcal{R}(t)(p) - \mathcal{R}(t)(q)\| = \sum_{i=1}^{2} \left\| \int_{s_i(p)}^{s_i(q)} e^{As} (\mathcal{U}(e^{A^T s} p) - \mathcal{U}(e^{A^T s} q))\, ds \right\| \leqslant 4ML\|p - q\|.$$

Thus the part of surface $\{\mathcal{R}(t)(p) \ : \ p \in \mathcal{S}_1 \cap B_\gamma(p_0)\}$ is a part of a strongly convex set with radius $R = 4ML$. In the present example it's enough for convergence of the gradient projection algorithm at time t. The same situation takes place for a time less than t.

6.2. Problem (P1), Example 2

Consider an example in $\mathbb{R}^{12}$.
$A = \operatorname{diag}(-0.3, -0.8, -1, -0.7, -0.71, -0.52, -0.37, -0.05, -0.25, -0.89, -0.99, -0.2)$, $\mathcal{U} = B_1(0)$. The target set is $\mathcal{M} = \mathcal{M}_0 + B_r(0)$, where $\mathcal{M}_0$ is the ball $B_{0.4}(0.3 \times \mathbf{1})$ ($\mathbf{1} = (1, 1, \ldots, 1)$), $r = 0.2$, step-size $\lambda = 0.1$.

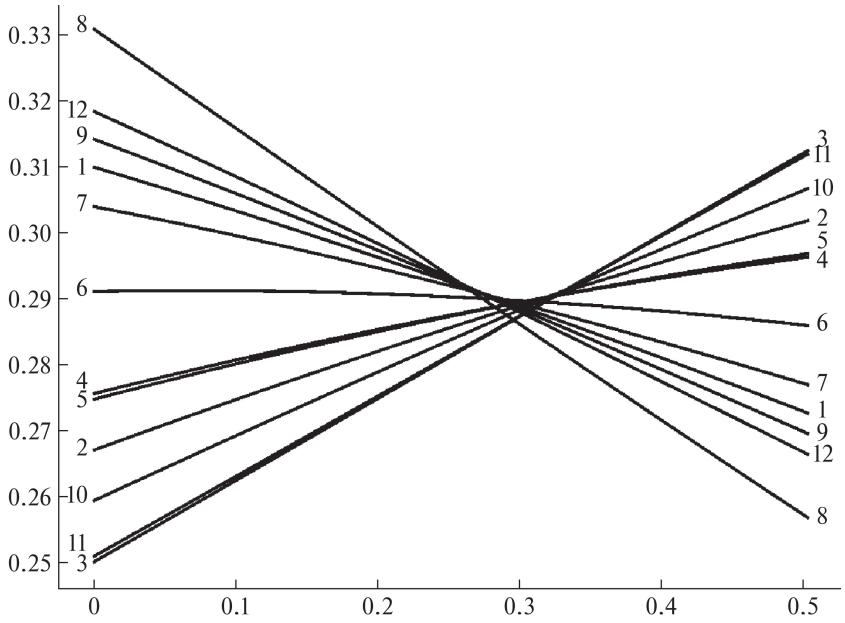

Fig. 5. The kth components u_k of the optimal control (problem (P1), Example 2).

Figure 2,b: convergence of the gradient projection algorithm for the auxiliary problem $\min_{\|p\|=1} f(p,t)$ for the time $t = 0.5$ and the initial condition

$$p_1 = (0.02046203,\ 0.24278712,\ 0.2199823,\ 0.33539534,\ 0.11750331,\ 0.07584814,\ 0.44196329,\ 0.14159412,\ 0.08314335,\ 0.32560626,\ 0.49401057,\ 0.43339861)$$

with $f(p_1, 0.5) = -0.047713028083805786$.

Approximation of convergence rate is $f(p_k, 0.5) - f(p_0, 0.5) \approx 0.1218 \times 0.8122^k$.

The optimal value is

$$p_0 = (0.2730037,\ 0.30197686,\ 0.3125336,\ 0.29647251,\ 0.29702965,\ 0.28619273,\ 0.27727228,\ 0.2572461,\ 0.26991497,\ 0.30680235,\ 0.31202019,\ 0.26679398)$$

with $f(p_0, 0.5) = -0.2023841828091369$.

Algorithm converges in 21 steps to the point

$$p_0 = (0.27281666,\ 0.3021221,\ 0.31280135,\ 0.29655398,\ 0.29711758,\ 0.28615572,\ 0.27713348,\ 0.25688441,\ 0.26969324,\ 0.30700357,\ 0.31228196,\ 0.26653741)$$

and the optimal time is 0.503150463104248.

Figure 5 illustrates the optimal control (per components, each line means one of 12 components).

6.3. Problem (P2). Example 3

The reachable set (as in (23)) is touching the target set from the inside.

The target set is the ellipsoid $\mathcal{M} = \{x : (x-c)^T Q (x-c) \leqslant R^2\}$, with

$$Q = \begin{bmatrix} 4.5 & -1.2 & -1.6 \\ -1.2 & 6.8 & -2.3 \\ -1.6 & -2.3 & 8 \end{bmatrix}, \quad c = \begin{bmatrix} -3.4 \\ -3.8 \\ 0.3 \end{bmatrix}, \quad R = 12.$$

Recall that $f(p,t) = s(p, \mathcal{M}) - s(p, \mathcal{R}_\varepsilon(t))$, here we take $\varepsilon = 0.05$, step-size $\lambda = 0.2$.

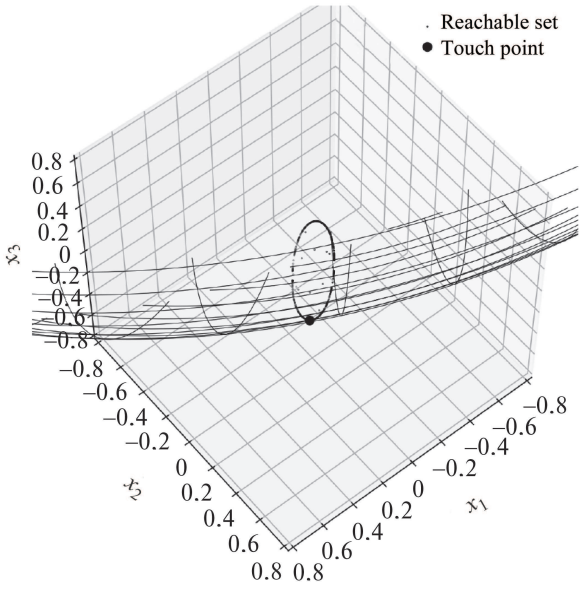

Fig. 6. Solution of problem (P2), Example 3.

Figure 6: For system (23) a similar bisection algorithm converges in 19 steps (i.e. $|J+\varepsilon| \leqslant \varepsilon_{\text{tol}} = 10^{-7}$). The optimal time is $t = 1{,}64610733$, $p_0 = (0{,}36800454,\ 0{,}72705740 - 0{,}57962073)$.

6.4. Problem (P2). Example 4. Homothete Inside the Target Set

We solve problem (P2) for a homothete, i.e. the problem is stated as

$$\max_{t\geqslant 0} t: \quad t\mathcal{R} \subset \mathcal{M}. \tag{24}$$

Define $\mathcal{M} = B_{10}(0)$, i.e. the ball centered at 0 of raduis 10. The set $\mathcal{R}$ is a strongly convex segment with endpoints $[-0.1, 3, 2.05884573]$, $[-1.9, 3, -1.05884573]$ and radius of strong convexity $R = 3$, i.e. $\mathcal{R}$ is the intersection of all closed balls of radius $R = 3$ containing the endpoints.

The supporting element for a unit vector $p = (p_1, \ldots, p_n)$ for a strongly convex segment with endpoints $[-ae_1, ae_1]$ and radius of strong convexity $R > a$ is equal to $Rp - \frac{\sqrt{R^2-a^2}}{\sqrt{1-p_1^2}}(I - e_1e_1^T)p$ if $\arctan\left(\frac{p_1}{\sqrt{1-p_1^2}}\right) < \arcsin\left(\frac{a}{R}\right)$, otherwise it is equal to $\text{sign}(p_1)ae_1$. We shall consider the homothety $t\mathcal{R}$, with parameter $\varepsilon = 0.1$ in the definition of f in (15), and step-size $\lambda = 0.2$.

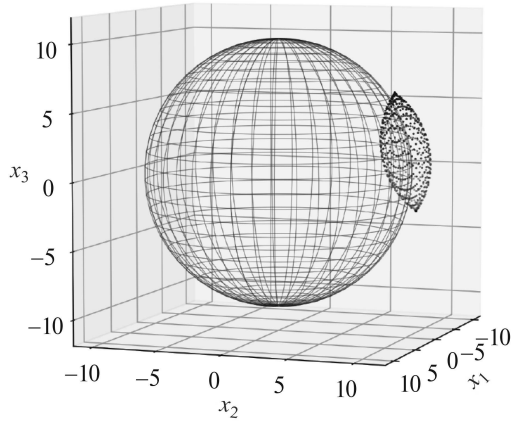

Fig. 7. Problem (P2), Example 4. The homothete is not contained inside $\mathcal{M}$ when $t = 3$.

For $t = 3$ the set $t\mathcal{R}$ is not contained in $\mathcal{M}$ (see Fig. 7). An algorithm, similar to one for problem (P2), in 21 steps gives the optimal value $t_0 = 2.62904820$ and $p_0 = (-0.3425777, 0.93398621, 0.10153957)$ (i.e. $|J + \varepsilon| \leqslant \varepsilon_{\text{tol}} = 10^{-7}$).

6.5. Problem (P3). Example 5

Consider an example in $\mathbb{R}^{10}$
$A = \operatorname{diag}(0.1, 0.75, 0.8, 0.81, 0.82, 0.95, 1.0, 1.0, 1.05, 1.1)$, $\mathcal{U} = B_1(0)$. The target set is $\mathcal{M} = B_{0.1}(0.1 \times \mathbf{1})$, $(\mathbf{1} = (1, 1, \ldots, 1))$, $\varepsilon = 0.1$, step-size $\lambda = 0.1$.

We need 21 runs of the gradient projection algorithm to get the solution point

$$p_0 = (0.44643102,\ 0.32328081,\ 0.3153902,\ 0.3138356,\ 0.31228874,\ 0.29286442,\ 0.28572048,\ 0.28572048,\ 0.27875066,\ 0.27195027)$$

and the optimal time is $t_0 = 0.35823087$.

7. CONCLUSION

In this paper we used a minimization Problem 1 to propose effective solution methods for several other problems (P1)—(P3) that involve distances and inclusions between sets. Linear convergence of proposed algorithms is proven. Several examples are given to prove the effectiveness of proposed solutions.

APPENDIX

A.1. PROOF OF LEMMA 1

Multiply both sides of the inequality by $\sqrt{\|p\|\,\|q\|}$ and take the square.

A.2. PROOF OF LEMMA 3

By the equality $e^{As} = Je^{A_1 s}J^{-1}$ we get

$$\mathcal{R}(t) = \int_0^t Je^{A_1 s}J^{-1}\mathcal{U}\,ds = \int_0^t Je^{A_1 s}\mathcal{U}_1\,ds = J\mathcal{R}_1(t).$$

The result follows from [25, Theorem 3].

A.3. PROOF OF LEMMA 3

We have $\mathcal{U} = \mathcal{U}_0 + \mathcal{B}_r(0)$. Then $\mathcal{R}(t) = \mathcal{R}_0(t) + r\int_0^t e^{As}\mathcal{B}_1(0)\,ds$,

$$\mathcal{R}_0(t) = \int_0^t e^{As}\mathcal{U}_0\,ds.$$

It is enough to prove that the ellipsoid $e^{As}\mathcal{B}_1(0)$ is uniformly smooth with constant $r(s) = \frac{\lambda_n^2(s)}{\lambda_1(s)}$. Consider orthonormal basis where the ellipsoid $e^{As}\mathcal{B}_1(0)$ has a canonical form

$$\mathcal{N} = \left\{x \in \mathbb{R}^n \ :\ \sum_{k=1}^n \frac{x_k^2}{\lambda_k^2} \leqslant 1\right\}, \quad \lambda_k = \lambda_k(s).$$

Then the matrix $L = \operatorname{diag}\{\lambda_1, \ldots, \lambda_n\}$ gives $L\mathcal{B}_1(0) = \mathcal{N}$. The ellipsoid $\mathcal{V} = \{x \ :\ \sum_{k=1}^n \lambda_k^2 x_k^2 \leqslant 1\}$ is strongly convex with radius $\rho = \lambda_1/\lambda_n^2$. Hence there exists another compact convex set $\mathcal{P}$ with $\mathcal{V} + \mathcal{P} = \mathcal{B}_\rho(0)$ and, taking in mind that $L\mathcal{V} = \mathcal{B}_1(0)$, we have

$$L\mathcal{V} + L\mathcal{P} = L\mathcal{B}_\rho(0) = \rho L\mathcal{B}_1(0) = \rho\mathcal{N} \quad \Leftrightarrow \quad \frac{1}{\rho}\mathcal{B}_1(0) + \frac{1}{\rho}\mathcal{P} = \mathcal{N}.$$

Thus the set $\mathcal{N}$ is uniformly smooth with constant $\frac{1}{\rho} = \lambda_n^2/\lambda_1$.

A.4. PROOF OF THEOREM 1

Let I be the identity matrix. Assume that $p_0 \in \mathcal{S}_1$ is the solution of problem (1). From the necessary condition of extremum $f(p_0) = (p_0, f'(p_0)) = -\|f'(p_0)\|$. Then $P_{\mathcal{T}_p} = I - pp^T$ for any $p \in \mathcal{S}_1$ and $\|(I - pp^T)f'(p)\|^2 = \|f'(p)\|^2 - f^2(p)$. Hence for all $p \in \mathcal{S}$ we get

$$\|f'(p)\|^2 - f^2(p) = (\|f'(p)\| - f(p))(\|f'(p)\| + f(p_0) + f(p) - f(p_0)).$$

From the inequality $f(p) \leqslant 0$ and the fact that the supporting element $f'(p_0) = \mathcal{N}(t)(p_0)$ has minimal norm, we have $\|f'(p)\| - f(p) \geqslant \|f'(p)\| \geqslant \|f'(p_0)\| = |J|$. It remains to note that $\|f'(p)\| + f(p_0) = \|f'(p)\| - \|f'(p_0)\| \geqslant 0$.

For any vectors $p, q \in \mathbb{R}^n$, $1 - \varepsilon \leqslant \|p\|, \|q\| \leqslant 1 + \varepsilon$, by Lemma 1 we obtain that $\left\| \frac{p}{\|p\|} - \frac{q}{\|q\|} \right\| \leqslant \frac{\|p-q\|}{\sqrt{\|p\|\,\|q\|}}$. Fix such p, q. Then by Lipschitz continuity of the supporting element $f'(\xi) = \mathcal{N}(t)(\xi)$ on the unit sphere with Lipschitz constant R and by the equality $f'(\xi) = f'(\xi/\|\xi\|)$, for all $\xi \neq 0$, we get

$$\|f'(p) - f'(q)\| \leqslant R \left\| \frac{p}{\|p\|} - \frac{q}{\|q\|} \right\| \leqslant \frac{R\|p-q\|}{\sqrt{\|p\|\,\|q\|}} \leqslant \frac{R}{1-\varepsilon}\|p-q\|. \quad \square$$

A.5. PROOF OF THEOREM 2

Define $q_k = p_k - \lambda f'(p_k)$, $\|q_k\| \geqslant 1 - \lambda\|f'(p_k)\| \geqslant 1 - \lambda L \geqslant \frac{1}{2}$. By $\|p_k\| = \|p_{k+1}\| = 1$, Lemma 1 and from the inequality

$$\|p_{k+1} - p_k\| = \|P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)) - p_k\| \leqslant \frac{\|p_k - q_k\|}{\sqrt{\|p_k\|\,\|q_k\|}} \leqslant \lambda\sqrt{2}\|f'(p_k)\| \leqslant \lambda\sqrt{2}L \leqslant \frac{1}{\sqrt{2}}$$

we get $[p_k, p_{k+1}] \subset \{p \in \mathbb{R}^n \,:\, \frac{1}{2} \leqslant \|p\| \leqslant \frac{3}{2}\}$. By Theorem 1 f' is Lipschitz continuous on the segment $[p_k, p_{k+1}]$ with constant $L_1 = R/(1 - \frac{1}{2}) = 2R$.

We also have the LPL condition for the function f on the set $\mathcal{S}$ by Theorem 1 with $\mu = |J|$.

Fix λ from the proposition and $\ell = \frac{1}{\lambda} \geqslant L_1$. Put $z_k = \|\ell p_k - f'(p_k)\| - (p_k, p_k - f'(p_k)) \geqslant 0$,

$$z_k = \frac{\|(I - p_k p_k^T)f'(p_k)\|^2}{\|\ell p_k - f'(p_k)\| + (p_k, p_k - f'(p_k))} \geqslant \frac{\|(I - p_k p_k^T)f'(p_k)\|^2}{2\|\ell p_k - f'(p_k)\|}. \tag{A.1}$$

We have

$$\|p_{k+1} - p_k\|^2 = 2 - 2\frac{(p_k, \ell p_k - f'(p_k))}{\|\ell p_k - f'(p_k)\|} = \frac{2z_k}{\|\ell p_k - f'(p_k)\|}$$

and from the Lipschitz property of f' on the segment $[p_k, p_{k+1}]$ with constant L_1

$$\begin{aligned} f(p_{k+1}) - f(p_k) &\leqslant (f'(p_k), p_{k+1} - p_k) + \frac{L_1}{2}\|p_{k+1} - p_k\|^2 \\ &= (p_k, L_1 p_k - f'(p_k)) - \left(L_1 p_k - f'(p_k), \frac{\ell p_k - f'(p_k)}{\|\ell p_k - f'(p_k)\|}\right) \\ &= \left(\ell p_k - f'(p_k) + (L_1 - \ell)p_k, p_k - \frac{\ell p_k - f'(p_k)}{\|\ell p_k - f'(p_k)\|}\right), \end{aligned}$$

$$f(p_{k+1}) - f(p_k) \leqslant -z_k + (L_1 - \ell)\left(p_k, p_k - \frac{\ell p_k - f'(p_k)}{\|\ell p_k - f'(p_k)\|}\right) = -z_k + \frac{L_1 - \ell}{\|\ell p_k - f'(p_k)\|} z_k \leqslant -z_k.$$

From (A.1) and from the LPL condition with $\mu = |J|$ we obtain that

$$f(p_{k+1}) - f(p_k) \leqslant -\frac{\|(I - p_k p_k^T)f'(p_k)\|^2}{2\|\ell p_k - f'(p_k)\|} \leqslant -\frac{|J|}{2\|\ell p_k - f'(p_k)\|}(f(p_k) - f(p_0)).$$

Define $\varphi(p) = f(p) - f(p_0)$ for all $p \in \mathcal{S}_1$. From the estimate $\|\ell p_k - f'(p_k)\| \leqslant \ell + \|f'(p_k)\| \leqslant \ell + L$ we have

$$\varphi(p_{k+1}) \leqslant \left(1 - \frac{|J|}{2\ell + 2L}\right) \varphi(p_k) = q\varphi(p_k)$$

and $q \in (0,1)$ because $|J| = \varrho(0, \mathcal{N}(t)) \leqslant \|\mathcal{N}(t)\| = L$.

For the points $\{p_k\}$ we have (note that $\|p_k - \lambda f'(p_k)\| \geqslant 1$)

$$\|p_{k+1} - p_k\|^2 \leqslant \frac{2z_k}{\|\ell p_k - f'(p_k)\|} \leqslant \frac{2\lambda(f(p_k) - f(p_{k+1}))}{\|p_k - \lambda f'(p_k)\|} \leqslant 2\lambda\varphi(p_k).$$

A.6. PROOF OF THEOREM 3

Consider $f(p)$:

$$f(p) = s(p, \mathcal{M}_0) + r\|p\| - s(p, \mathcal{R}_\varepsilon(t)).$$

The set $\mathcal{R}_\varepsilon(t)$ is strongly convex with radius $R_T + \varepsilon < r$. Hence there exists another convex compact set $\mathcal{N}(t)$ with $\mathcal{R}_\varepsilon(t) + \mathcal{N}(t) = \mathcal{B}_{R_T+\varepsilon}(0)$ and $r\|p\| - s(p, \mathcal{R}_\varepsilon(t)) = (r - R_T - \varepsilon)\|p\| + s(p, \mathcal{N}(t))$. Thus for all $p \in \mathbb{R}^n$

$$f(p) = s(p, \mathcal{M}_0) + (r - R_T - \varepsilon)\|p\| + s(p, \mathcal{N}(t)) = s(p, \mathcal{M}_0 + \mathcal{N}(t) + \mathcal{B}_{r-R_T-\varepsilon}(0))$$

and the function $f(p)$ is the supporting function of the set $\mathcal{N}(t) = \mathcal{M} \dot{-}^* \mathcal{R}_\varepsilon(t) = \mathcal{M}_0 + \mathcal{N}(t) + \mathcal{B}_{r-R_T-\varepsilon}(0)$. The latter set is strongly convex with radius R_0 and uniformly smooth with constant $r_0 = r - R_T - \varepsilon > 0$. The function f' is Lipschitz on the set $\mathcal{S}_1$ with constant R_0 and as in the proof of Proposition 2 $[p_k, p_{k+1}] \subset \{p \in \mathbb{R}^n : \frac{1}{2} \leqslant \|p\| \leqslant \frac{3}{2}\}$. Thus for any point p from the segment $[p_k, p_{k+1}]$ we have $\|p\| \geqslant \frac{1}{2}$ and for any $p, q \in [p_k, p_{k+1}]$ by Lemma 1

$$\|f'(p) - f'(q)\| = \left\| f'\left(\frac{p}{\|p\|}\right) - f'\left(\frac{q}{\|q\|}\right)\right\| \leqslant R_0 \left\| \frac{p}{\|p\|} - \frac{q}{\|q\|}\right\| \leqslant R_0 \frac{\|p - q\|}{\sqrt{\|p\|\,\|q\|}} \leqslant 2R_0\|p - q\|,$$

i.e. f' is Lipschitz on any segment $[p_k, p_{k+1}]$ with constant $2R_0$. From the Lipschitz property of f' and Proposition 2 $f(p_k) \leqslant 0$ for all k.

$$\|p_{k+1} - p_0\|^2 = \|P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)) - P_{\mathcal{S}_1}(p_0 - \lambda f'(p_0))\|^2,$$

$\|p_k - \lambda f'(p_k)\| \geqslant 1$, $\|p_0 - \lambda f'(p_0)\| \geqslant 1$, i.e. $p_k - \lambda f'(p_k) \notin \operatorname{int} B_1(0)$, $p_0 - \lambda f'(p_0) \notin \operatorname{int} B_1(0)$ and thence

$$\|p_{k+1} - p_0\|^2 \leqslant \|p_k - p_0 + \lambda(f'(p_k) - f'(p_0))\|^2$$
$$\leqslant \|p_k - p_0\|^2 - 2\lambda(p_k - p_0, f'(p_k) - f'(p_0)) + \lambda^2\|f'(p_k) - f'(p_0)\|.$$

From the strong convexity of the set $\mathcal{N}(t)$ with radius R_0 we have $\|f'(p_k) - f'(p_0)\| \leqslant R_0\|p_k - p_0\|$. Also by the strong convexity of the set $\mathcal{N}(t)$ with radius R_0 we have [28, Theorem 2.1 (h)] $(p_k - p_0, f'(p_k) - f'(p_0)) \geqslant \frac{1}{R_0}\|f'(p_k) - f'(p_0)\|^2$ and by the uniform smoothness of the set $\mathcal{N}(t)$ with constant r_0 [28, Definition 3.2, Theorem 3.6]

$$(p_k - p_0, f'(p_k) - f'(p_0)) \geqslant \frac{1}{R_0}\|f'(p_k) - f'(p_0)\|^2 \geqslant \frac{r_0^2}{R_0}\|p_k - p_0\|^2.$$

Thus $\|p_{k+1} - p_0\|^2 \leqslant q^2\|p_k - p_0\|^2$.

A.7. PROOF OF THEOREM 4

Repeat the proof of Theorem 3. In particular, the function $f(p)$ is the supporting function for the set $\mathcal{R}(t) \dot{-}^* \mathcal{M}_\varepsilon = \mathcal{R}(t) \dot{-}^* \mathcal{M} \dot{-}^* \mathcal{B}_\varepsilon(0)$. The last set is strongly convex with constant R_T and uniformly smooth with constant r. □

FUNDING

Theorems 1, 3, 4 and Section 5 were obtained by M.V. Balashov under the support of the Russian Science Foundation grant no. 22-11-00042, `https://rscf.ru/project/22-11-00042/` in Institute of Control Sciences. Theorem 2 and numerical results of Section 6 were obtained by A.A. Tremba under the support of the Russian Science Foundation grant no. 21-71-30005, `https://rscf.ru/project/21-71-30005/`.

REFERENCES

1. Ioffe, A.D., Metric regularity — a survey Part I and II, *J. Aust. Math. Soc.*, 2016, vol. 101, pp. 188–243, pp. 376–417.
2. Luke, D.R., Finding best approximation pairs relative to a convex and prox-regular set in a Hilbert space, *SIAM J. Optim.*, 2008, vol. 19, no. 2, pp. 714–739.
3. Grünewälder, S., Compact convex projections, *J. Mach. Learn. Res.*, 2018, vol. 18, no. 2019, pp. 1–43.
4. Sosa, W. and Raupp, F.M.P, An algorithm for projecting a point onto a level set of a quadratic function, *Optimization*, 2022, vol. 71, no. 1, pp. 71–89.
5. Bregman, L.M., Censor, Y., Reich, S., and Zepkowitz-Malachi, Y., Finding the projection of a point onto the intersection of convex sets via projections onto half-spaces, *J. Approx. Theory.*, 2003, vol. 124, no. 2, pp. 194–218.
6. Aumann, R., Integrals of set-valued functions, *J. Math. Anal. Appl.*, 1965, vol. 12, no. 1, pp. 1–12.
7. Liapounoff, A.A., Sur les fonctions-vecteurs complètement additives, *Izv. Akad. Nauk SSSR Ser. Mat*, 1940, vol. 4, no. 6, pp. 465–478.
8. Frankowska, H. and Olech, C., R-convexity of the integral of the set-valued functions. *Contributions to analysis and geometry (Baltimore, Md., 1980)*, Baltimore, Md.: Johns Hopkins Univ. Press, 1981, pp. 117–129.
9. Vial, J.-Ph., Strong and Weak Convexity of Sets and Functions, *Math. Oper. Res.*, 1983, vol. 8, no. 2, pp. 231–259.
10. Balashov, M.V. and Repovs, D., Uniformly convex subsets of the Hilbert space with modulus of convexity of the second order, *J. Math. Anal. Appl.*, 2011, vol. 377, no. 2, pp. 754–761.
11. Veliov, V.M., On the convexity of integrals of multivalued mappings: application in control theory, *J. Optim. Theor. Appl.*, 1987, vol. 54, no. 3, pp. 541–563.
12. Veliov, V.M., Second order discrete approximations to strongly convex differential inclusions, *Syst. Control Lett.* 1989, vol. 13, no. 3, pp. 263–269.
13. Althoff, M., Frehse, G., and Girard, A., Set propagation techniques for reachability analysis, *Annu. Rev. Control Robot. Auton. Syst.*, 2021, vol. 4, pp. 369–395.
14. Le Guernic, C. and Girard, A., Reachability analysis of linear systems using support functions, *Nonlinear Anal. Hybrid Syst.*, 2010, vol. 4, pp. 250–262.
15. Gruber, P.M., Approximation of convex bodies / *Convexity and Its Applications.*, Basel: Birkhäuser, 1983, pp. 131–162.
16. Serry, M. and Reissig, G., Over-approximating reachable tubes of linear time-varying systems, *IEEE Trans. Automat. Control.*, vol. 67, no. 1, pp. 443–450.
17. Kurzhanski, A.B. and Varaiya, P., Dynamics and control of trajectory tubes, theory and computation, *Ser. Systems and Control: Foundations and Applications*, Birkhauser/Springer, 2014.
18. Levitin, E.S. and Polyak, B.T., Constrained minimization methods, *U.S.S.R. Comput. Math. Math. Phys.*, 1966, vol. 6, no. 5, pp. 1–50.

19. Balashov, M.V. and Polovinkin, E.S., *M*-strongly convex subsets and their generating sets, *Sb. Math.*, 2000, vol. 191, no. 1, pp. 25–60.
20. Cannarsa, P. and Frankowska, H., Interior sphere property of attainable sets and time optimal control problems, *ESAIM Control Optim. Calc. Var.*, 2006, vol. 12, no. 2, pp. 350–370.
21. Balashov, M.V., Polyak, B.T., and Tremba, A.A., Gradient projection and conditional gradient methods for constrained nonconvex minimization, *Numer. Funct. Anal. Optim.*, 2020, vol. 41, no. 7, pp. 822–849.
22. Balashov, M.V., Strong convexity of reachable sets of linear systems, *Sb. Math.*, 2022, vol. 213, no. 5, pp. 30–49.
23. Boltyansky, V.G., *Matematicheskie metody optimalnogo upravleniya* (Mathematical methods of optimal control), Moscow: Nauka, 1969.
24. Tremba, A.A, *Vychisleniye mnozhestva dostizhimosti lineynyh statsionarnyh sistem s pomoshyu opornoy funktsii i opornyh elementov* (Calculation of reachable set of a linear stationary system using supporting function and supporting elements), *Materials of XVI International Conference on Stability and Oscillations in Nonlinear Control Systems (Pyatnitskiy's conference)*, Moscow: IPU RAN, 2022, pp. 437–441.
25. Polovinkin, E.S., Strongly convex analysis, *Sb. Math.*, 1996, vol. 187, no. 2, pp. 259–286.
26. Bolte, J., Sabach, Sh., and Teboulle, M., Proximal alternating linearized minimization for nonconvex and nonsmooth problems, *Math. Program.*, 2014, vol. 146, pp. 459–494.
27. Balashov, M.V. and Tremba, A.A., Error bound conditions and convergence of optimization methods on smooth and proximally smooth manifolds, *Optimization.*, 2022, vol. 71, no. 3, pp. 711–735.
28. Ivanov, G.E. and Goncharov, V.V., Strong and weak convexity of closed sets in a Hilbert space, *Operations Research, Engineering, and Cyber Security. Springer Optimization and Its Applications.* Springer, 2017, vol. 113, pp. 259–297.
29. Balashov, M.V. and Kamalov, R.A., The gradient projection method with Armijo's step size on manifolds, *Comput. Math. Math. Phys.*, 2021, vol. 61, no. 11, pp. 1776–1786.
30. Tremba, A.A., Computing reachability set with support function and support points: Python code repository, `https://github.com/atremba/lti-reachability-set`

This paper was recommended for publication by P.S. Shcherbakov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, pp. 512–531.

TOPICAL ISSUE

Approximation-Based Approach to Adaptive Control of Linear Time-Varying Systems

A. Glushchenko[*,a] **and K. Lastochkin**[*,b]

[*] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
e-mail: [a] *aiglush@ipu.ru,* [b] *lastconst@yandex.ru*

Received January 25, 2024
Revised March 12, 2024
Accepted March 20, 2024

Abstract—An adaptive state-feedback control system is proposed for a class of linear time-varying systems represented in the controller canonical form. The adaptation problem is reduced to the one of Taylor series-based first approximations of the ideal controller parameters. The exponential convergence of identification and tracking errors of such an approximation to an arbitrarily small and adjustable neighbourhood of the equilibrium point is ensured if the condition of the regressor persistent excitation with a sufficiently small time period is satisfied. The obtained theoretical results are validated via numerical experiments.

Keywords: adaptive control, time-varying parameters, parametric error, persistent excitation, identification

DOI: 10.31857/S0005117924050054

1. INTRODUCTION

Starting from the 1960s, the subject of adaptive control has been one of the central ones for Laboratory No. 7 of V.A. Trapeznikov Institute of Control Sciences of RAS. Its founder, academician Yakov Zalmanovich Tsypkin, made a significant contribution to research on adaptation and learning problems and proposed a unified approach to their solution based on stochastic approximation methods. Using it, in particular, the problems of identification and parameter estimation were successfully solved. Subsequently, Boris Theodorovich Polyak proposed optimal and robust pseudogradient adaptation algorithms and strictly analysed their convergence rate [1, 2]. These studies have largely become the foundations of the adaptive control theory, which, having started with linear systems with time-invariant parameters, is gradually being generalised to wider classes of plants. One such class will be discussed in this study.

One of the subjects of adaptive control theory is the problem of the time-invariant reference model tracking by a time-varying plant with zero steady-state error. Despite more than 65 years of efforts, this problem still lacks a universal practical solution, which motivates researchers all over the world to design new approaches and tools.

Conventional adaptive control algorithms are applicable to linear systems with quasi-time-invariant parameters. When they are applied to control linear time-varying systems, an uncompensated summand occurs in the derivative of the Lyapunov function, which is proportional to the rate of the unknown parameters change. As a result, instead of the convergence of the tracking error to zero, only its boundedness inside some ball with non-adjustable boundary is guaranteed. In [4], based on the speed-gradient method, these results are generalised to the problem of a time-varying reference model tracking by a nonlinear time-varying system. In [5, 6], various composite adaptive laws are proposed, which are claimed to reduce the steady-state error value in case the

regressor persistent excitation condition is met. In [7], a *congelation of variables* method is proposed, which allows one to damp the above-mentioned uncompensated summand with the help of not always suitable for practice high-gain feedback, thus ensuring asymptotic convergence of the tracking error to zero. The alternative approach [8] also provides asymptotic stability, but uses a high gain in the adaptive law instead of the control one. Considering the method of majorizing functions [9, 10], the high gain is also used in the adaptive law, but, in contrast to [8], only dissipativity of the closed-loop system is guaranteed. In [11] an adaptive control system is proposed that provides exponential convergence of the tracking error to zero for systems represented in the controllable canonical form with time-varying parameters that are described by known exosystems with unknown initial conditions. In [12] it is proposed to reduce the problem of adaptive control of time-varying mechanical systems to the identification of the piecewise-constant parameters of the polynomial obtained by local expansion of the system time-varying parameters into a Taylor series of an arbitrary order. In [13, 14], based on the parametric identification methods, an approach to adaptive-optimal output feedback control of time-varying linear systems is developed under the assumption that the plant parameters are known time-varying functions of time.

The disadvantages of the described above and other known approaches to solve the time-varying system adaptive control problems can be classified as follows:

1) application of high-gain in the control or adaptive law (sliding-mode control, high values of the parameters, nonlinear damping signals, etc.) [7–10, 12];
2) the necessity to meet the parametric identifiability conditions [5, 6, 11, 13, 14];
3) the dimensionality of the identification/adaptation problem to be solved is enlarged by taking into account the coefficients of the physical laws of the system parameters change or approximation polynomials [11–14].

A more complete state-of-the-art understanding of the time-varying systems adaptive control problem can be obtained from the statement sections of the cited studies [4–15]. In this paper, a new approximation-based adaptive control method, which exploits the parameter identification theory, is proposed for time-varying systems.

The motivation is to investigate the applicability conditions of the recently proposed algorithm [16] that identifies time-varying parameters of a linear regression equation to solve the time-varying linear system control problem. According to [16], the problem of time-varying parameters identification is reduced to the one of estimation of their piecewise-constant approximation. As follows from the theoretical conclusions of [16], unlike many existing methods of time-varying parameters identification, the algorithm from [16] allows one to ensure convergence of the time-varying parameters identification error to a region, which can be arbitrarily reduced by decreasing the Taylor series expansion time interval in case the regressor is persistently exciting over a sufficiently small period of excitation T_s. In this study, the approach is proposed to be used to control a class of linear systems with time-varying parameters. To that end:

1) a non-adaptive control law is proposed for a time-varying system, which feedback and forward parameters are calculated only via the first (piecewise-constant) approximation of the system time-varying parameters;
2) in case the control law from 1) is applied, the convergence conditions of the tracking error to an arbitrarily small neighbourhood of zero are obtained;
3) based on the results from [16], the law to estimate the parameters of the controller from 1) is proposed, which allows one to ensure the convergence of the tracking error to an arbitrarily small neighbourhood of zero in case the regressor is persistently exciting with a sufficiently small period of excitation.

Considering the above-given literature review, the obtained approximation-based approach to design adaptive control systems for time-varying plants is close to [12]. However, unlike in [12],

firstly, the time-varying parameters are approximated only by the first summand of Taylor series, which reduces the computational complexity and does not increase the dimensionality of the identification problem, and secondly, the step of the obtained estimates interpolation is not needed. In comparison with other existing solutions [4–14], the proposed algorithm of adaptive control of time-varying linear systems has the following advantages (+) and disadvantages (–):

(+) high gain and damping components are not used in both control and adaptive laws;
(+) the function of the time-varying parameters change is not required to be known;
(+) no *a priori* information about the system parameters is used;
(–) the repressor persistent excitation condition is required to be met to achieve even asymptotic convergence of the tracking error to a neighbourhood of zero;
(–) the value of the steady-state tracking error can be reduced only if the period T_s of the regressor persistent excitation is small enough;
(–) violation of the parametric identifiability condition (the regressor persistent excitation) may result in instability of the closed-loop system.

In general, although the proposed solution does not overcome all the shortcomings of the existing approaches, it expands the set of adaptive control methods for the time-varying systems, and therefore, in the authors' opinion, it is of interest.

Notation and Definitions

The following notation is adpoted: $f(t) \in \mathbb{R}^{n\times m}$ means a value of a function $f : \left[t_0^+, +\infty\right) \to \mathbb{R}^{n\times m}$ at the time point t, where $t_0^+ \geqslant 0$ is an initial time instant; for a vector $a \in \mathbb{R}^n$ the notation $||a||$ is the Euclidean norm; the minimum and maximum eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ are denoted as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The abbreviation exp stands for the exponential stability.

The definitions of finite and persistent excitation are used to prove theorems and propositions.

Definition 1. A signal $\omega(t) \in \mathbb{R}^n$ is finitely exciting over a time range $[t_1, t_2] \subset \left[t_0^+, \infty\right)$ if there exists $\alpha > 0$ such that the following inequality holds:

$$\int_{t_1}^{t_2} \omega(\tau)\,\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_n. \tag{1.1}$$

Definition 2. A signal $\omega(t) \in \mathbb{R}^n$ is persistently exciting if for all $t \geqslant t_0^+ \geqslant 0$ there exist $T_s > 0$ and $\alpha > 0$ such that the following inequality holds:

$$\int_{t}^{t+T_s} \omega(\tau)\,\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_n. \tag{1.2}$$

Set of signals, for which condition (1.1) or (1.2) is met, we denote as FE or PE, respectively. A signal $\omega(t)$ is persistently exciting if $\omega \in \mathrm{PE}$, and it is finitely exciting if $\omega \in \mathrm{FE}$.

The main result of the study utilises the Taylor formula with integral remainder. The conditions of existence of such equation are defined in the following lemma [17].

Lemma 1. *Let (t_1, t_2) be an open time interval, and $f(t) \in \mathbb{R}$ be a p-times continuously differentiable function of time t, then for any pair of time instants t and a from (t_1, t_2) it holds that*

$$f(t) = f(a) + \frac{t-a}{1!}f^{(1)}(a) + \ldots + \frac{(t-a)^p}{p!}f^{(p)}(a) + \int_a^t \frac{(t-\zeta)^p}{p!}f^{(p+1)}(\zeta)\,d\zeta. \tag{1.3}$$

2. PROBLEM STATEMENT

We consider continuous linear systems with time-varying parameters

$$\begin{aligned}\dot{x}(t) &= A(t)x(t) + B(t)u(t) = A_0 x(t) + e_n\left(a^{\mathrm{T}}(t)\,x(t) + b(t)\,u(t)\right)\\ &= A_0 x(t) + e_n\Phi^{\mathrm{T}}(t)\,\Theta(t), \quad x\left(t_0^+\right) = x_0,\end{aligned} \tag{2.1}$$

where

$$A(t) = A_0 + e_n a^{\mathrm{T}}(t), \quad B(t) = e_n b(t),$$

$$A_0 = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ 0_{1\times n} & \end{bmatrix}, \quad e_n = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \quad \begin{aligned}\Phi^{\mathrm{T}}(t) &= \begin{bmatrix} x^{\mathrm{T}}(t) & u(t) \end{bmatrix},\\ \Theta(t) &= \begin{bmatrix} a^{\mathrm{T}}(t) & b(t) \end{bmatrix}^{\mathrm{T}},\end{aligned}$$

$x(t) \in \mathbb{R}^n$ is a state vector with unknown initial conditions x_0, $u(t) \in \mathbb{R}$ stands for a control signal, $A(t) \in \mathbb{R}^{n\times n}$ denotes an unknown matrix of the system under consideration, $B(t) \in \mathbb{R}^n$, $\Theta(t) \in \mathbb{R}^{n+1}$ are unknown vectors, $A_0 \in \mathbb{R}^{n\times n}$ stands for a Frobenius matrix, $e_n \in \mathbb{R}^n$ is the nth Euclidean basis vector. The pair $(A(t), B(t))$ is completely controllable for all $t \geqslant t_0^+$. The controllability condition for the system (2.1) can be validated via application of, for example, a criterion given in [18].

A salient feature of the class of systems (2.1) is the fact that control and uncertainty signals are in the same equation. Such systems are called the ones with matched uncertainty, and they are widely met in practice. For example, the Euler angles dynamics of a rigid body, assuming its symmetry, is described by a second-order system with matched uncertainty. Another good example of a control problem with matched uncertainties is the control of a manipulator state using the Euler–Lagrange formalism.

The following assumption is adopted with respect to the unknown parameters $\Theta(t)$.

Assumption 1. *The parameters $\Theta(t)$ and their first and second derivatives are continuous and bounded*

$$\|\Theta(t)\| \leqslant \Theta_{\max}, \quad \left\|\dot{\Theta}(t)\right\| \leqslant \dot{\Theta}_{\max}, \quad \left\|\ddot{\Theta}(t)\right\| \leqslant \ddot{\Theta}_{\max},$$

where the upper bounds $\Theta_{\max}$, $\dot{\Theta}_{\max}$ and $\ddot{\Theta}_{\max}$ exist, but they are unknown.

The required control quality for the closed-loop system that includes the system (2.1) and the controller is defined with the help of the reference model with time-invariant parameters

$$\dot{x}_{ref}(t) = A_0 x_{ref}(t) + e_n\left(b_{ref} r(t) + a_{ref}^{\mathrm{T}} x_{ref}(t)\right), \; x_{ref}\left(t_0^+\right) = x_{0ref}, \tag{2.2}$$

where $x_{ref}(t) \in \mathbb{R}^n$ is a reference model state vector with known initial conditions x_{0ref}, $r(t) \in \mathbb{R}$ denotes a reference signal, $A_{ref} = A_0 + e_n a_{ref}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ stands for a Hurwitz reference model state matrix, b_{ref} is a reference model high frequency gain.

We assume that the reference model (2.2) is chosen in such a way that the matching conditions are met, *i.e* the state vector of (2.1) can ideally track the one of (2.2).

Assumption 2. *There exist parameters $k_x(t) \in \mathbb{R}^{1\times n}$ and $k_r(t) \in \mathbb{R}$ such that the following equations hold*

$$a_{ref}^{\mathrm{T}} - a^{\mathrm{T}}(t) = b(t)\,k_x(t), \quad b_{ref} = b(t)\,k_r(t).$$

This assumption is necessary and sufficient condition for the existence of a control signal $u(t)$ that ensures for all $t \geqslant t_0^+$ that the equations of the system (2.1) coincide with those of the chosen reference model (2.2). The assumption is ensured to be satisfied by choosing a reference model

in the form of (2.2), by consideration of a class of systems with a time-invariant sign of the high-frequency gain $b(t)$ and by a completely controllable pair $(A(t), B(t))$. It should be noted that Assumption 2 imposes the following constraint on the system (2.1): $\operatorname{sgn}(b(t)) = \text{const}$,[1] and hence jointly Assumptions 1 and 2 require boundedness of $b_{\max} \geqslant |b(t)| \geqslant b_{\min} > 0$.

The aim is to design an adaptive control law $u(t)$, which, if $\Phi \in \mathrm{PE}$, ensures exponential convergence (exp) of the error $e_{ref}(t) = x(t) - x_{ref}(t)$ into the goal set

$$\lim_{t\to\infty} \|e_{ref}(t)\| \leqslant \Delta_{e_{ref}} \ (\exp), \tag{2.3}$$

in such a way that there exists some parameter of the adaptive control procedure, from which value the steady-state error $\Delta_{e_{ref}} > 0$ depends.

3. PRELIMINARY RESULTS AND TRANSFORMATIONS

An effective solution of the control problem of a linear system with unknown piecewise-constant parameters has been obtained recently in [19]. In this section the problem of adaptive control of a system (2.1) with time-varying parameters will be transformed into the one of control of a system with the piecewise-constant parameters. To this end, first of all, we show that the stated goal (2.3) is achievable with the help of the non-adaptive control law with known ideal parameters, which uses only piecewise-constant approximations of the time-varying parameters of the system (2.1) in its feedback and feedforward summands.

Taking into account Assumption 1, the error equation between the plant (2.1) and the reference model (2.2) is written as

$$\begin{aligned}\dot{e}_{ref}(t) &= A_{ref}e_{ref}(t) + e_n b(t)\left[u(t) - k_x(t)x(t) - k_r(t)r(t)\right] \\ &= A_{ref}e_{ref}(t) + e_n b(t)\left[u(t) - \mathcal{K}^{\mathrm{T}}(t)\omega(t)\right],\end{aligned} \tag{3.1}$$

where

$$e_{ref}(t) = x(t) - x_{ref}(t), \quad \omega(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & r^{\mathrm{T}}(t)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+1},$$
$$\mathcal{K}(t) = \begin{bmatrix} k_x(t) & k_r(t)\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{(n+1)\times 1}.$$

The disturbance $\mathcal{K}^{\mathrm{T}}(t)\omega(t)$ is going to be represented as a sum of two terms: with the piecewise-constant and time-varying parameters. To that end, a growing sequence is introduced

$$t_i^+ = T\left\lfloor \frac{t - t_0^+}{T} \right\rfloor, \quad i \in \mathbb{N},$$

where $t_{i+1}^+ - t_i^+ = T > 0$, $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$ is a function to round down to the closest integer.

As, owing to Assumptions 1 and 2, the parameters $\mathcal{K}(t)$ are differentiable, then, following the Taylor equation (1.3), it can be written for the neighbourhood T of the time instant t_i^+:

$$\mathcal{K}(t) = \mathcal{K}\left(t_i^+\right) + \underbrace{\int_{t_i^+}^{t} \dot{\mathcal{K}}(\zeta)\, d\zeta}_{\delta_{\mathcal{K}0}(t)}, \tag{3.2}$$

where $\mathcal{K}\left(t_i^+\right) = \mathcal{K}_i$ are values of the parameters $\mathcal{K}(t)$ at the time instant t_i^+, $\|\delta_{\mathcal{K}0}(t)\| \leqslant \dot{\mathcal{K}}_{\max} T$ is the reminder of the zeroth order ($p = 0$, see (1.3)).

[1] Otherwise there exists a time instant $t_a \geqslant t_0^+$ at which $b(t_a) = 0$, and equations from Assumption 2 have no solution in the general case ($b_{ref} \neq 0$, $a_{ref} - a(t_a) \neq 0_n$).

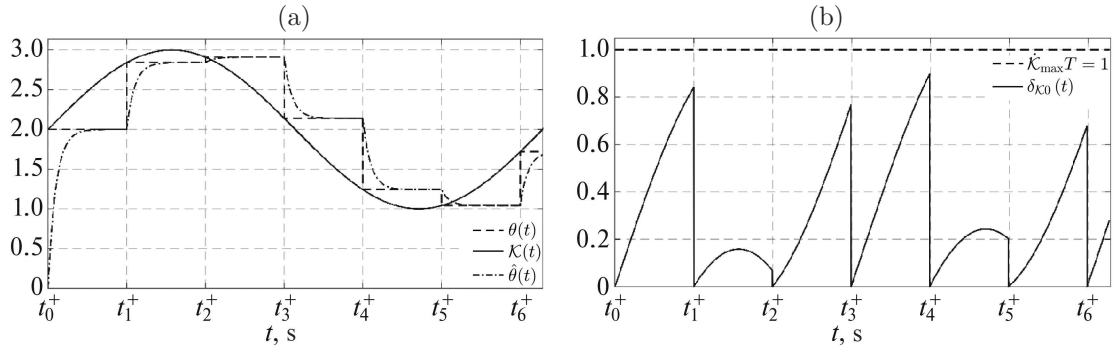

Fig. 1. Graphical illustration of relationship between $\mathcal{K}(t)$, $\theta(t)$ and $\hat{\theta}(t)$.

Owing to (3.2), for each time range $\left[t_i^+, t_i^+ + T\right)$ the time-varying parameters $\mathcal{K}(t)$ can be approximated by their value $\mathcal{K}_i$ at the beginning of such time range. Then the sequence of such values $\{\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_i\}$ together with the sequence of the switching time instants $\{t_0^+, t_1^+, \ldots, t_i^+\}$ define the piecewise-constant signal, which is the first approximation of the time-varying parameters $\mathcal{K}(t)$ for all $t \geqslant t_0^+$:

$$\theta(t) = \mathcal{K}_i = \mathcal{K}_0 + \sum_{q=1}^{i} \Delta_q^\theta h\left(t - t_q^+\right), \tag{3.3}$$

where $\Delta_q^\theta = \mathcal{K}_q - \mathcal{K}_{q-1}$ is the amplitude of the parameters $\mathcal{K}(t)$ change over the time range $\left[t_i^+, t_{i+1}^+\right]$, $h : [t_0^+, \infty) \to \{0, 1\}$ stands for the Heaviside function.

For all $t \geqslant t_0^+$, equation (3.3) allows one to write the time-varying parameters as a sum $\mathcal{K}(t) = \theta(t) + \delta_{\mathcal{K}0}(t)$, which results in the required representation of the disturbance

$$\dot{e}_{ref}(t) = A_{ref} e_{ref}(t) + e_n b(t) \left[u(t) - \theta^{\mathrm{T}}(t)\,\omega(t) - \delta_{\mathcal{K}0}^{\mathrm{T}}(t)\,\omega(t)\right]. \tag{3.4}$$

Equation (3.4) motivates to introduce the following implementable continuous non-adaptive control law

$$u(t) = \hat{\theta}^{\mathrm{T}}(t)\,\omega(t), \tag{3.5a}$$

$$\dot{\hat{\theta}}(t) = -\gamma_1\left(\hat{\theta}(t) - \theta(t)\right) = -\gamma_1\tilde{\theta}(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0, \tag{3.5b}$$

where $\hat{\theta}(t)$ stands for the result of the parameters $\theta(t)$ filtration, and $\gamma_1 > 0$ denotes the filter parameter.

Considering a particular case $\mathcal{K}(t) = \sin(t) + 2$ and $T = 1$, the relationship between the parameters $\mathcal{K}(t)$, $\theta(t)$ and $\hat{\theta}(t)$ is explained in Figs. 1a and 1b. For the same example, Fig. 1b demonstrates the approximation error $\delta_{\mathcal{K}0}(t)$ and its upper bound $\dot{\mathcal{K}}_{\max}T = 1$.

The conditions, under which the stated goal is achieved by application of the law (3.5a) + (3.5b), are presented in the following proposition.

Proposition 1. *If the condition $i \leqslant i_{\max} < \infty$ is met, then there exists $T_{\min} > 0$ such that for all $0 < T < T_{\min}$ the control law* (3.5) *ensures that the stated goal* (2.3) *is achieved.*

Proof of proposition is postponed to Appendix.

According to Proposition 1, in order to solve the stated problem (2.3), it is sufficient to use piecewise-constant approximations $\theta(t)$ of the time-varying parameters of the disturbance $\mathcal{K}^{\mathrm{T}}(t)\omega(t)$ to calculate the parameters of the control law (3.5a). Thus the adaptive control problem for a class

of systems with unknown time-varying parameters (2.1) is reduced to the one of identification of the unknown piecewise-constant parameters $\theta(t)$. To solve this problem, it is natural to be based jointly on approaches previously developed in [16, 19].

Remark 1. The condition $i \leqslant i_{\max} < \infty$ is required for formal proof of proposition 1 and is not restrictive for practical scenarios.

4. MAIN RESULT

Following the method of exponentially stable adaptive control of systems with piecewise-constant parameters [19], for indirect implementation of (3.5), we first obtain a regression equation relating the parameters $\theta(t)$ to the signals calculated on the basis of the measurable vector $\Phi(t)$. The result of such a parameterisation can be formulated as a proposition.

Proposition 2. *Using the state of the stable filter* ($l > 0$) *with resetting at some time instants* t_i^+

$$\begin{gathered}\dot{\overline{\Phi}}(t) = -l\overline{\Phi}(t) + \Lambda^{\mathrm{T}}\left(t, t_i^+\right)\Phi(t), \quad \overline{\Phi}\left(t_i^+\right) = 0_{2(n+1)}, \\ \Lambda\left(t, t_i^+\right) = \left[I_{n+1} \quad \left(t - t_i^+\right) I_{n+1}\right] \in \mathbb{R}^{(n+1)\times 2(n+1)},\end{gathered} \tag{4.1}$$

normalization procedure

$$\begin{gathered}\overline{z}_n(t) = n_s(t)\, e_n^{\mathrm{T}}\left[x(t) - l\overline{x}(t) - A_0\overline{x}(t)\right], \\ \overline{\varphi}_n^{\mathrm{T}}(t) = n_s(t)\,\overline{\varphi}(t) = n_s(t)\left[\overline{\Phi}^{\mathrm{T}}(t) \quad e^{-l\left(t-t_i^+\right)}\right], \\ n_s(t) = \frac{1}{1 + \overline{\varphi}^{\mathrm{T}}(t)\,\overline{\varphi}(t)}, \quad \overline{x}(t) = \left[I_{n\times n} \quad 0_{n\times(n+2)}\right]\overline{\Phi}(t),\end{gathered} \tag{4.2}$$

extension ($\sigma > 0$)

$$\dot{z}(t) = e^{-\sigma\left(t-t_i^+\right)}\overline{\varphi}_n(t)\,\overline{z}_n^{\mathrm{T}}(t), \quad z\left(t_i^+\right) = 0_{2n+3}, \tag{4.3a}$$

$$\dot{\varphi}(t) = e^{-\sigma\left(t-t_i^+\right)}\overline{\varphi}_n(t)\,\overline{\varphi}_n^{\mathrm{T}}(t), \quad \varphi\left(t_i^+\right) = 0_{(2n+3)\times(2n+3)}, \tag{4.3b}$$

mixing

$$Y(t) := \operatorname{adj}\{\varphi(t)\}\, z(t), \quad \Delta(t) := \det\{\varphi(t)\}, \tag{4.4}$$

elimination

$$\begin{gathered}z_a(t) = Y^{\mathrm{T}}(t)\,\mathfrak{L}_a, \quad z_b(t) = Y^{\mathrm{T}}(t)\,\mathfrak{L}_b, \\ \mathfrak{L}_a = \left[I_{n\times n} \quad 0_{n\times(n+3)}\right]^{\mathrm{T}} \in \mathbb{R}^{(2n+3)\times n}, \quad \mathfrak{L}_b = \left[0_{1\times n} \quad 1 \quad 0_{1\times(n+2)}\right]^{\mathrm{T}} \in \mathbb{R}^{(2n+3)\times 1},\end{gathered} \tag{4.5}$$

substitution

$$\mathcal{Y}(t) := \left[\Delta(t)\, a_{ref}^{\mathrm{T}} - z_a(t) \quad \Delta(t)\, b_{ref}\right]^{\mathrm{T}}, \quad \mathcal{M}(t) := z_b(t), \tag{4.6}$$

and smoothing ($k > 0$)

$$\dot{\Upsilon}(t) = -k\left(\Upsilon(t) - \mathcal{Y}(t)\right), \quad \Upsilon\left(t_0^+\right) = 0_{n+1}, \tag{4.7a}$$

$$\dot{\Omega}(t) = -k\left(\Omega(t) - \mathcal{M}(t)\right), \quad \Omega\left(t_0^+\right) = 0, \tag{4.7b}$$

we have a perturbed regression equation

$$\Upsilon(t) = \Omega(t)\,\theta(t) + w(t), \tag{4.8}$$

where the signals $\Upsilon(t), \Omega(t)$ *are calculated via* $\Phi(t)$ *and additionally:*

a) *if* $\Phi \in \mathrm{PE} \Rightarrow \overline{\varphi}_n \in \mathrm{PE}$ *with the period* $T_s < T$, *then there exists* $T_{\min} > 0$ *such that for all* $0 < T < T_{\min}$ *and* $t \geqslant t_0^+ + T_s$ *it holds that*

$$0 < \Omega_{\mathrm{LB}} \leqslant \Omega(t) \leqslant \Omega_{\mathrm{UB}}.$$

b) *if* $i \leqslant i_{\max} < \infty$, *then for all* $t \geqslant t_0^+ + T_s$ *it holds that*

$$\|w(t)\| \leqslant w_{1\max} e^{-\gamma_1 \left(t - t_0^+ - T_s\right)} + w_{2\max}(T),$$
$$\lim_{T \to 0} w_{2\max}(T) = 0.$$

Proof of proposition and definition of $w(t)$ *are given in Appendix.*

The parameterisation (4.1)–(4.8) uses the procedures proposed to solve the problem of adaptive control of systems with piecewise-constant parameters [19]. The difference is that the time-varying matrix $\Lambda\left(t, t_i^+\right)$ is used in (4.1) and the states of the filters (4.1) and (4.3) are reset at known, rather than algorithmically detectable, time instants.

Here we briefly explain the purpose of the procedures in use. Having the measurable signals $\Phi(t)$ at hand, the application of the filter (4.1) allows one to obtain a regression equation with measurable regressor and regressand with respect to parameters $\overline{\vartheta}(t) = \left[\Theta_i{}^{\mathrm{T}} \quad \dot{\Theta}_i^{\mathrm{T}} \quad e_n^{\mathrm{T}} x\left(t_i^+\right)\right]^{\mathrm{T}}$, where $\Theta\left(t_i^+\right) = \Theta_i$, $\dot{\Theta}\left(t_i^+\right) = \dot{\Theta}_i$ are the values of the system parameters $\Theta(t)$ and the rate of their change at the time instant t_i^+. The normalisation (4.2) ensures that all signals used in further procedures belong to L_∞ space. The extension and mixing procedures (4.3), (4.4) allow one to transform the vector regressor $\overline{\varphi}_n(t) \in \mathbb{R}^{2n+3}$ into a scalar one $\Delta(t) \in \mathbb{R}$. Owing to $\Delta(t) \in \mathbb{R}$, the elimination (4.5) separates the regression equation under consideration into two ones with respect to the parameters of the piecewise-constant approximation of $a(t)$ and $b(t)$. By substitution (4.6) of (4.5) into the matching conditions (see assumption 2), we transform the equations with respect to approximation of the system parameters into the ones with respect to approximation $\theta(t)$ of the perturbation parameters. Smoothing (4.7a), (4.7b) allows one to ensure sufficient smoothness of the signals $\Upsilon(t)$ and $\Omega(t)$.

Having at hand the regression equation (4.8) that regressor and regressand are based only on measurarble signals $\Phi(t)$, we can indirectly implement the law (3.5) and guarantee the achievement of the goal (2.3).

Theorem 1. *Let* $\Phi \in \mathrm{PE} \Rightarrow \overline{\varphi}_n \in \mathrm{PE}$ *with the period* $T_s < T$, *Assumptions 1–2 be met, then there exists* $T_{\min} > 0$ *such that for all* $0 < T < T_{\min}$ *the control law* (3.5a) *with the adaptive law*

$$\begin{gathered}\dot{\hat{\theta}}(t) = -\gamma(t)\,\Omega(t)\left(\Omega(t)\,\hat{\theta}(t) - \Upsilon(t)\right) \\ = -\gamma(t)\,\Omega^2(t)\,\tilde{\theta}(t) + \gamma(t)\,\Omega(t)\,w(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0, \\ \gamma(t) = \begin{cases} 0, & \text{if } \Omega(t) < \rho \in (0, \Omega_{\mathrm{LB}}], \\ \dfrac{\gamma_1}{\Omega^2(t)} & \text{otherwise,} \end{cases}\end{gathered} \tag{4.9}$$

in case $i \leqslant i_{\max} < \infty$ *for* $\xi(t) = \left[e_{ref}^{\mathrm{T}}(t) \quad vec^{\mathrm{T}}\left(\tilde{\theta}(t)\right)\right]^{\mathrm{T}}$, *ensures that:*

1) $\forall t \geqslant t_0^+ \quad \xi(t) \in L_\infty$,

2) $\lim\limits_{t \to \infty} \|\xi(t)\| \leqslant \Delta_\xi(T) \ \ (\exp), \ \ \lim\limits_{T \to 0} \Delta_\xi(T) = 0.$

Proof of theorem is presented in Appendix.

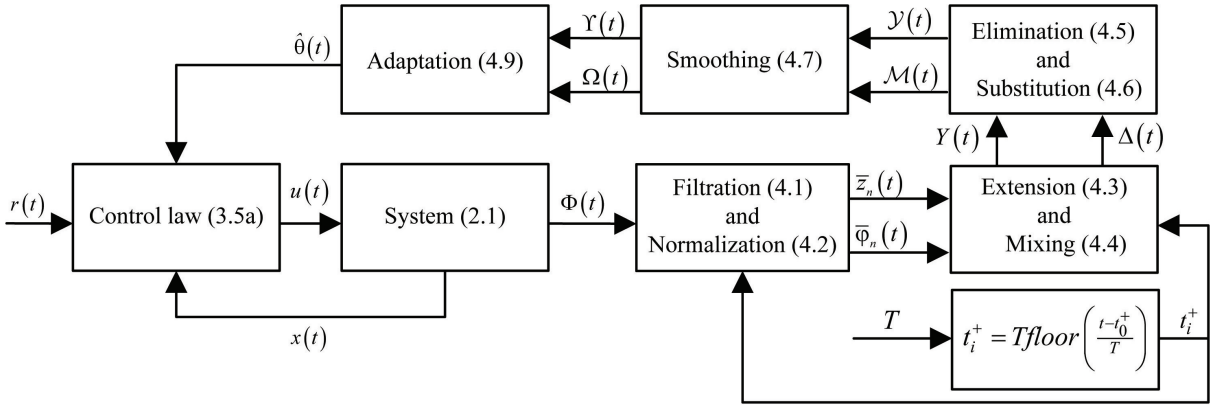

Fig. 2. Block diagram of proposed adaptive control system.

The block diagram of the obtained algorithm for adaptive control of systems with time-varying unknown parameters (2.1) is presented in Fig. 2.

Thus, the developed control system consists of a control law (3.5a), an adaptive law (4.9), a set of procedures (4.1)–(4.7) to process the measurable signals. In contrast to existing adaptive control methods [4–14], the proposed approach does not require any *a priori* information about the system parameters $a(t)$ and $b(t)$, does not use high-gain in control or adaptive laws, guarantees global exponential convergence of the error $\xi(t)$ to the bounded neighbourhood of the equilibrium, which can be adjusted by the parameter T.

Remark 2. The feature of the proposed solution is the relationship between the steady-state error $\Delta_\xi(T)$, the length of the Taylor series expansion interval T and the period of the regressor persistent excitation T_s. The problem is that the parameter T cannot be made smaller than the value of the regressor excitation period T_s. However, for a fixed period T_s and a minimum possible $T < T_{\min}$ such that $T - T_s > 0$, the error $\xi(t)$ may be bounded in an unacceptably large neighbourhood of the equilibrium point $\Delta_\xi(T)$. Therefore, in order to reduce the steady-state error, it is necessary, first of all, to ensure a persistent excitation of the regressor with a sufficiently small period T_s, which in practice can be achieved by addition of a high-frequency or random test signal to the reference $r(t)$.

5. NUMERICAL EXPERIMENTS

In Matlab/Simulink numerical experiments have been conducted for the proposed adaptive system using the explicit Euler solver with a constant step time of $\tau_s = 10^{-3}$ s.

The system (2.1) was considered with $n = 2$. The initial conditions, the parameters of the system and reference model (2.2) were chosen as

$$\begin{gathered} x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathrm{T}}, \quad b(t) = 3 + \cos(0.4t)\sin(0.1t), \quad a_{ref}^{\mathrm{T}}(t) = \begin{bmatrix} -8 & -4 \end{bmatrix}, \\ a^{\mathrm{T}}(t) = \begin{bmatrix} 2 + \sin(0.1t) & 1 + 5\left(1 - e^{\frac{-1}{25}t}\right) \end{bmatrix}, \quad b_{ref} = 8. \end{gathered} \tag{5.1}$$

First, we verified the preliminary conclusions made in Proposition 1. We picked $\gamma_1 = 50$ as the filter constant (3.5b), and defined the reference as $r(t) = 10$. Figure 3 presents the comparison of the error $e_{1ref}(t)$ for different T.

The obtained results validated the conclusions made in Proposition 1. Indeed, a decrease of T resulted in a decrease of the steady-state value of the tracking error $e_{ref}(t)$ when the control law (3.5a) with (3.5b) was applied. Having checked proposition 1, we proceeded to verify the main result.

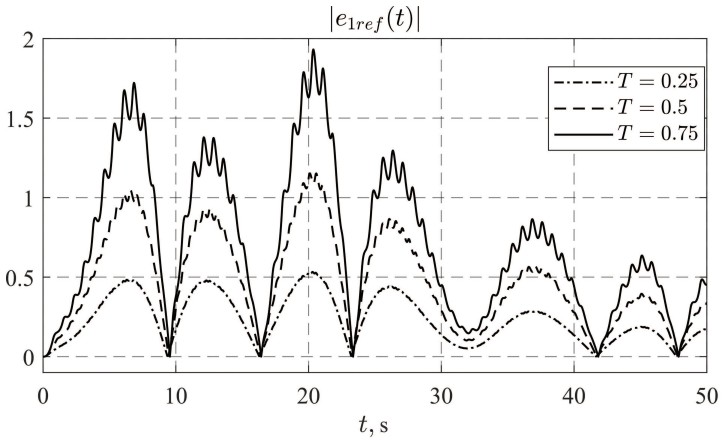

Fig. 3. Behavior of $|e_{1ref}(t)|$ for different T.

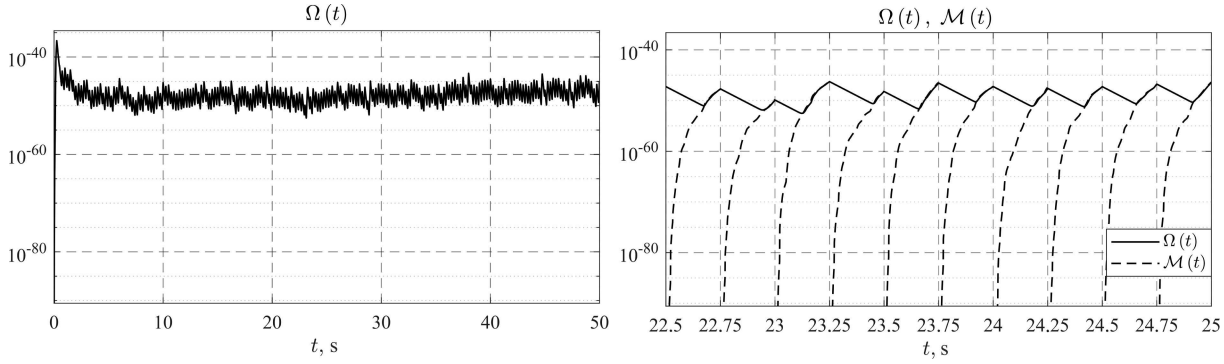

Fig. 4. Behavior of regressors $\mathcal{M}(t)$ and $\Omega(t)$.

The parameters of the filters (4.1), (4.3), (4.7) and the adaptive law (4.9) were chosen as

$$l = 10, \quad \sigma = \frac{0.05}{T}, \quad k = 50, \quad \rho = 10^{-72}, \quad \gamma_1 = 100, \quad T = 0.25,$$

the reference was picked as $r(t) = 1 + r_d(t)$ for $r_d(t) \sim \mathcal{N}(0,\ 10^{-2})$. A random signal $r_d(t)$ was added to a unity reference signal to ensure that $\overline{\varphi}_n \in \mathrm{PE}$ for the closed-loop system (3.1).

Figure 4 depicts the behavior of the regressors $\mathcal{M}(t)$ and $\Omega(t)$ on the logarithmic scale.

It follows from the obtained results that despite the fact that the filters (4.1) and (4.3) were reset every T seconds, the regressor $\Omega(t)$ (unlike $\mathcal{M}(t)$) was globally bounded away from zero starting from some time instant, which confirms the theoretical conclusions made in statement (a) of Proposition 2. Figure 4 demonstrates the importance of the smoothing procedure (4.7), which, as can be seen, allows one to (*i*) average the values of the regressor $\mathcal{M}(t)$ over the period T, and (*ii*) avoid discontinuities caused by the reinitialisation of the filters (4.1) and (4.3).

Figure 5 shows the behavior of (a) the state $x(t)$ when the control law (3.5a) with (3.5b) and with (4.9) is used, (b) the estimates of $\hat{\theta}_{\mathrm{i}}(t)$ and the true parameters $\theta_i(t) + 1$ shifted by one for clarity of illustration, (c) the control signal (3.5a) with (4.9).

Figure 6 compares the values of the integral control quality index of tracking $e_{ref}(t)$ and parametric $\tilde{\theta}(t)$ errors for different values of T.

The simulation results illustrate the conclusions of Propositions 1, 2 and theorem. The goal (2.3) is achieved, and the steady-state values of the errors $e_{ref}(t)$ and $\tilde{\theta}(t)$ are directly proportional to the parameter T.

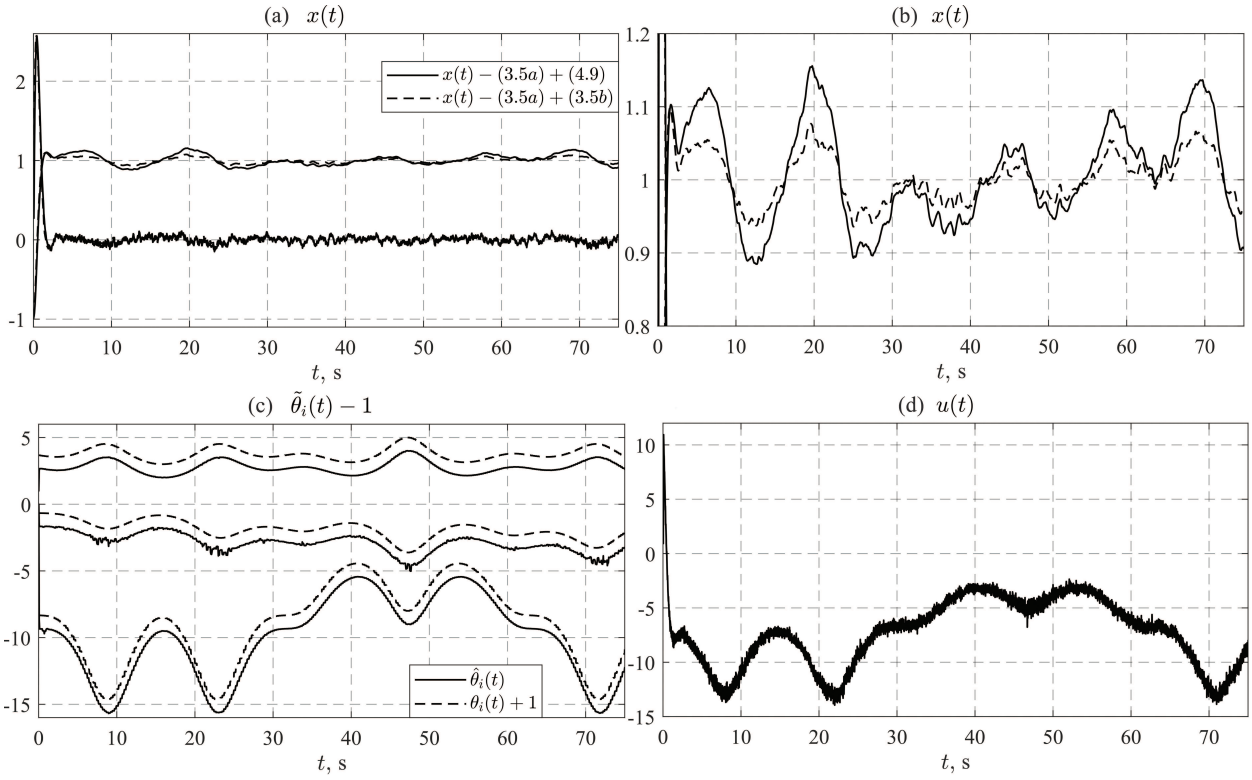

Fig. 5. Behavior of (a)–(b) state $x\,(t)$ when control law (3.5a) used with (3.5b) and with (4.9), (c) estimates $\hat{\theta}_i\,(t)$ and ideal parameters $\theta_i\,(t) + 1$ shifted by one for clarity of illustration, (d) control signal (3.5a) with (4.9).

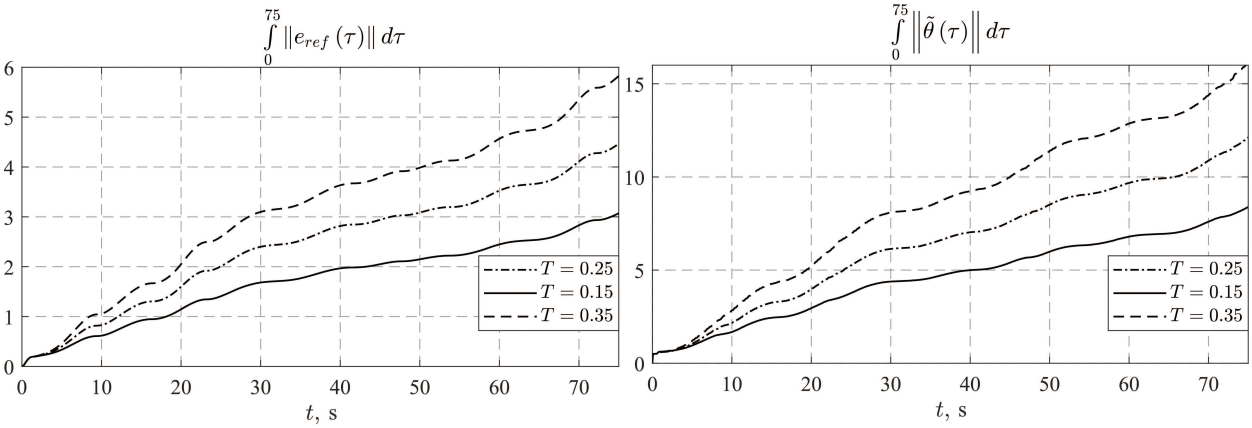

Fig. 6. Comparison of integral control quality indexes.

6. CONCLUSION

The problem of tracking of a linear time-invariant reference model by a linear time-varying system is solved. It is proposed to approximate the unknown time-varying parameters of the ideal control law by piecewise-constant parameters. Parametric identification methods proposed in [16, 19] are combined to identify these piecewise-constant parameters. The resulting adaptive control system requires persistent excitation of the regressor with a sufficiently small period to achieve the control goal, but it does not require *a priori* information about the unknown parameters of the system.

FUNDING

This research was in part financially supported by Grants Council of the President of the Russian Federation (project MD-1787.2022.4).

APPENDIX

Proof of Proposition 1. The proof of the proposition is divided into two steps. At the first one we analyse the properties of the parametric error $\tilde{\theta}(t)$, at the second one — the properties of the tracking error $e_{ref}(t)$.

Step 1. Owing to proposition 1 from [19], if $i \leqslant i_{\max} < \infty$, then for the differential equation

$$\dot{\tilde{\theta}}(t) = -\gamma_1 \tilde{\theta}(t) - \dot{\theta}(t), \quad \tilde{\theta}\left(t_0^+\right) = \hat{\theta}_0 - \theta\left(t_0^+\right),$$

the following upper bound holds

$$\left\|\tilde{\theta}(t)\right\| \leqslant \beta_{\max} e^{-\gamma_1\left(t-t_0^+\right)}, \quad \beta_{\max} > 0, \tag{A.1}$$

where $\dot{\theta}(t) = \sum\limits_{q=1}^{i} \Delta_q^{\theta} \delta\left(t - t_q^+\right)$, and $\delta : [t_0^+; \infty) \to \{0, \infty\}$ is the Dirac function.

Step 2. The following quadratic form is introduced:

$$\begin{gathered} V_{e_{ref}} = e_{ref}^{\mathrm{T}} P e_{ref} + \frac{a_0^2}{\gamma_1} e^{-2\gamma_1\left(t-t_0^+\right)}, \quad H = \operatorname{blockdiag}\left\{P, \ \frac{a_0^2}{\gamma_1}\right\}, \\ \underbrace{\lambda_{\min}(H)}_{\lambda_{\mathrm{m}}} \left\|\overline{e}_{ref}\right\|^2 \leqslant V\left(\left\|\overline{e}_{ref}\right\|\right) \leqslant \underbrace{\lambda_{\max}(H)}_{\lambda_M} \left\|\overline{e}_{ref}\right\|^2, \end{gathered} \tag{A.2}$$

where $\overline{e}_{ref}(t) = \left[e_{ref}^{\mathrm{T}}(t) \quad e^{-\gamma_1\left(t-t_0^+\right)}\right]^{\mathrm{T}}$, $P = P^{\mathrm{T}} > 0$ is the solution of the below-given Lyapunov equation in case $\lambda_{\min}(Q) > 2$:

$$A_{ref}^{\mathrm{T}} P + P A_{ref} = -Q, \quad Q = Q^{\mathrm{T}} > 0.$$

The derivative of the quadratic form (A.2) is written as:

$$\begin{aligned} \dot{V}_{e_{ref}} &= e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}} P + P A_{ref}\right) e_{ref} - 2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} + 2e_{ref}^{\mathrm{T}} P e_n b \tilde{\theta}^{\mathrm{T}} \omega + 2e_{ref}^{\mathrm{T}} P e_n b \delta_{\theta_0}^{\mathrm{T}} \omega \\ &= -e_{ref}^{\mathrm{T}} Q e_{ref} - 2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} + 2e_{ref}^{\mathrm{T}} P e_n b \tilde{\theta}^{\mathrm{T}}\left(\omega_{e_{ref}} + \omega_r\right) + 2e_{ref}^{\mathrm{T}} P e_n b \delta_{\theta_0}^{\mathrm{T}}\left(\omega_{e_{ref}} + \omega_r\right) \\ &\leqslant -\lambda_{\min}(Q)\left\|e_{ref}\right\|^2 - 2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} \\ &\quad + 2\lambda_{\max}(P) b_{\max} \|e_{ref}\|^2 \left\|\tilde{\theta}\right\| + 2\lambda_{\max}(P)\overline{\omega}_r b_{\max}\left\|e_{ref}\right\| \left\|\tilde{\theta}\right\| \\ &\quad + 2\lambda_{\max}(P) b_{\max} \dot{\mathcal{K}}_{\max} T \|e_{ref}\|^2 + 2\lambda_{\max}(P) b_{\max} \overline{\omega}_r \dot{\mathcal{K}}_{\max} T \left\|e_{ref}\right\|, \end{aligned} \tag{A.3}$$

where

$$\|\omega(t)\| \leqslant \underbrace{\left\|\left[e_{ref}(t) \quad 0\right]\right\|}_{\left\|\omega_{e_{ref}}(t)\right\| = \left\|e_{ref}(t)\right\|} + \underbrace{\left\|\left[x_{ref}(t) \quad r(t)\right]\right\|}_{\|\omega_r(t)\| \leqslant \overline{\omega}_r} \leqslant \|e_{ref}(t)\| + \overline{\omega}_r.$$

Having applied Young's inequality twice:

$$\begin{aligned} 2\lambda_{\max}(P)\overline{\omega}_r b_{\max}\left\|e_{ref}\right\| \left\|\tilde{\theta}\right\| &\leqslant \|e_{ref}\|^2 + \lambda_{\max}^2(P)\overline{\omega}_r^2 b_{\max}^2 \left\|\tilde{\theta}\right\|^2, \\ 2\lambda_{\max}(P) b_{\max}\overline{\omega}_r \dot{\mathcal{K}}_{\max} T \left\|e_{ref}\right\| &\leqslant \lambda_{\max}^2(P) b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2 + \|e_{ref}\|^2, \end{aligned} \tag{A.4}$$

equation (A.3) is rewritten as:

$$\begin{aligned}\dot{V}_{e_{ref}} \leqslant \left[-\lambda_{\min}(Q) + 2\lambda_{\max}(P) b_{\max}\left(\left\|\tilde{\theta}\right\| + \dot{\mathcal{K}}_{\max} T\right) + 2\right] \|e_{ref}\|^2 \\ -2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} + \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \left\|\tilde{\theta}\right\|^2 + \lambda_{\max}^2(P)\, b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2.\end{aligned} \tag{A.5}$$

As the parametric error $\tilde{\theta}(t)$ converges to zero exponentially (A.2), then, if $\lambda_{\min}(Q) > 2$, then there definitely exists a time instant $t_{e_{ref}} \geqslant t_0^+$ and constants $T_{\min} > 0,\ a_0 > \lambda_{\max}(P)\,\overline{\omega}_r b_{\max}\beta_{\max}$ such that for all $t \geqslant t_{e_{ref}}$ and $0 < T < T_{\min}$ it holds that

$$\begin{gathered}-\lambda_{\min}(Q) + 2\lambda_{\max}(P) b_{\max}\left(\beta_{\max} e^{-\gamma_1\left(t_{e_{ref}} - t_0^+\right)} + \dot{\mathcal{K}}_{\max} T\right) + 2 = -c_1 < 0, \\ \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \beta_{\max}^2 - 2a_0^2 = -c_2 < 0.\end{gathered} \tag{A.6}$$

Then the upper bound of the derivative (A.5) for all $t \geqslant t_{e_{ref}}$ is written as

$$\dot{V}_{e_{ref}} \leqslant -\eta_{\overline{e}_{ref}} V_{e_{ref}} + \lambda_{\max}^2(P)\, b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2, \tag{A.7}$$

where $\eta_{\overline{e}_{ref}} = \min\left\{\frac{c_1}{\lambda_{\max}(P)}, \frac{c_2\gamma_1}{a_0^2}\right\}$.

The solution of the differential inequality (A.7) for all $t \geqslant t_{e_{ref}}$ is obtained as

$$V_{e_{ref}}(t) \leqslant e^{-\eta_{\overline{e}_{ref}}}\left(t - t_{e_{ref}}\right) V_{e_{ref}}\left(t_{e_{ref}}\right) + \frac{\lambda_{\max}^2(P)\, b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2}{\eta_{\overline{e}_{ref}}}. \tag{A.8}$$

Tending time to infinity for (A.8) and considering expression for $V_{e_{ref}}$, it is concluded that (2.3) holds, which completes the proof.

Proof of Proposition 2. Owing to assumption 2 and following (3.2)–(3.3), we apply the Taylor formula (1.3) to the parameters $\Theta(t)$ to obtain:

$$\Theta(t) = \Theta\left(t_i^+\right) + \overbrace{\dot{\Theta}\left(t_i^+\right)\left(t - t_i^+\right) + \underbrace{\int_{t_i}^{t} (t-\zeta)\ddot{\Theta}(\zeta)\, d\zeta}_{\delta_1(t)}}^{\delta_0(t)}, \tag{A.9}$$

where $\Theta\left(t_i^+\right) = \Theta_i$, $\dot{\Theta}\left(t_i^+\right) = \dot{\Theta}_i$ are the values of the system parameters $\Theta(t)$ and the rate of their change at the time instant t_i^+, $\|\delta_1(t)\| \leqslant 0.5\ddot{\Theta}_{\max} T^2$ denotes the bounded reminder of the first order ($p = 1$), $\|\delta_0(t)\| \leqslant \dot{\Theta}_{\max} T$ is the bounded reminder of the zeroth order ($p = 0$).

Equation (A.9) is rewritten in the matrix form

$$\Theta(t) = \Lambda\left(t,\ t_i^+\right)\vartheta(t) + \delta_1(t), \tag{A.10}$$

where $\vartheta(t) = \left[\Theta_i{}^{\mathrm{T}} \quad \dot{\Theta}_i^{\mathrm{T}}\right]^{\mathrm{T}} \in \mathbb{R}^{2(n+1)}$.

The substitution of (A.10) into (2.1) yields

$$\dot{x}(t) = A_0 x + e_n\left(\Phi^{\mathrm{T}}(t)\,\Lambda\left(t, t_i^+\right)\vartheta(t) + \Phi^{\mathrm{T}}(t)\,\delta_1(t)\right). \tag{A.11}$$

The expression $x(t) - l\overline{x}(t)$ is differentiated to obtain

$$\dot{x}(t) - l\dot{\overline{x}}(t) = -l\left(x(t) - l\overline{x}(t)\right) + A_0 x + e_n\left(\Phi^{\mathrm{T}}(t)\,\Lambda\left(t, t_i^+\right)\vartheta(t) + \Phi^{\mathrm{T}}(t)\,\delta_1(t)\right). \tag{A.12}$$

The solution of (A.12) is written as

$$x(t) - l\overline{x}(t) = e^{-l\left(t-t_i^+\right)}x(t_i) + A_0\overline{x}(t) + \int_{t_i^+}^{t} e^{-l(t-\tau)}e_n\Phi^{\mathrm{T}}(\tau)\,\Lambda\left(\tau, t_i^+\right)\vartheta(\tau)\,d\tau$$
$$+\int_{t_i^+}^{t} e^{-l(t-\tau)}e_n\Phi^{\mathrm{T}}(\tau)\,\delta_1(\tau)\,d\tau = A_0\overline{x}(t) + e_n\overline{\varphi}(t)\,\overline{\vartheta}(t) + e_n\underbrace{\int_{t_i^+}^{t} e^{-l(t-\tau)}\Phi^{\mathrm{T}}(\tau)\,\delta_1(\tau)\,d\tau}_{\varepsilon_0(t)}, \tag{A.13}$$

where $\overline{\vartheta}(t) = \left[\vartheta^{\mathrm{T}}(t) \;\; e_n^{\mathrm{T}}x\left(t_i^+\right)\right]^{\mathrm{T}} \in \mathbb{R}^{2n+3}$, and the third equality is not violated since the reset of the filter states (4.1) and the change of parameters occur synchronously at a known time instant t_i^+, i.e. $\overline{\vartheta}(t) = \text{const}$ for all $t \in \left[t_i^+, t_i^+ + T\right)$.

Equation (A.13) is substituted into (4.2) to obtain

$$\overline{z}_n(t) = n_s(t)\,e_n^{\mathrm{T}}\left[x(t) - l\overline{x}(t) - A_0\overline{x}(t)\right] = \overline{\varphi}_n^{\mathrm{T}}(t)\,\overline{\vartheta}(t) + \overline{\varepsilon}_0(t), \tag{A.14}$$

where $\overline{z}_n(t) \in \mathbb{R}$, $\overline{\varphi}_n(t) \in \mathbb{R}^{2n+3}$ and the perturbation $\overline{\varepsilon}_0(t) \in \mathbb{R}$ is bounded as follows (see definitions of $\Phi(t)$ and $\overline{\varphi}_n(t)$):

$$\|\overline{\varepsilon}_0(t)\| = \left\| n_s(t)\int_{t_i^+}^{t} e^{-l(t-\tau)}\Phi^{\mathrm{T}}(\tau)\,\delta_1(\tau)\,d\tau \right\| \leqslant \left\|\overline{\varphi}_n^{\mathrm{T}}(t)\right\| 0.5\ddot{\Theta}_{\max}T^2. \tag{A.15}$$

Owing to the multiplication of the regression equation (A.14) by $n_s(t)$, the regressor $\overline{\varphi}_n^{\mathrm{T}}(t)$, the regressand $\overline{z}_n(t)$ and the perturbation $\overline{\varepsilon}_0(t)$ are bounded. In addition, according to the upper bound (A.15), the perturbation $\overline{\varepsilon}_0(t)$ can be reduced by decreasing the parameter T. Therefore, further on we will use the definition $\overline{\varepsilon}_0(t) := \overline{\varepsilon}_0(t, T)$ and imply that any perturbation obtained by transformation of $\overline{\varepsilon}_0(t, T)$ can also be reduced by a reduction of T.

Having applied (4.3) and multiplicated $z(t)$ by $\operatorname{adj}\{\varphi(t)\}$, we have (commutativity of the filter (4.3a) is not violated as its reinitialization and parameters change happen synchronously at a known time instant t_i^+, *i.e.* $\overline{\vartheta}(t) = \text{const}$ for all $t \in [t_i^+, t_i^+ + T)$)

$$\begin{gathered} Y(t) := \operatorname{adj}\{\varphi(t)\}\,z(t) = \Delta(t)\,\overline{\vartheta}(t) + \overline{\varepsilon}_1(t, T), \\ \operatorname{adj}\{\varphi(t)\}\,\varphi(t) = \det\{\varphi(t)\}\,I_{2(n+1)+1} = \Delta(t)\,I_{2(n+1)+1}, \\ \overline{\varepsilon}_1(t, T) = \operatorname{adj}\{\varphi(t)\}\int_{t_i^+}^{t} e^{-\sigma\left(\tau-t_i^+\right)}\overline{\varphi}_n(\tau)\,\overline{\varepsilon}_0(\tau, T)\,d\tau, \end{gathered} \tag{A.16}$$

where $Y(t) \in \mathbb{R}^{2n+3}$, $\Delta(t) \in \mathbb{R}$, $\overline{\varepsilon}_1(t,\ T) \in \mathbb{R}^{2n+3}$.

Owing to $\Delta(t) \in \mathbb{R}$, the elimination (4.5) allows one to obtain the following from (A.16)

$$\begin{aligned} z_a(t) &= Y^{\mathrm{T}}(t)\,\mathfrak{L}_a = \Delta(t)\,\vartheta_a^{\mathrm{T}}(t) + \overline{\varepsilon}_1^{\mathrm{T}}(t,\ T)\,\mathfrak{L}_a, \\ z_b(t) &= Y^{\mathrm{T}}(t)\,\mathfrak{L}_b = \Delta(t)\,\vartheta_b(t) + \overline{\varepsilon}_1^{\mathrm{T}}(t,\ T)\,\mathfrak{L}_b, \end{aligned} \tag{A.17}$$

where $z_a(t) \in \mathbb{R}^{1\times n}$, $z_b(t) \in \mathbb{R}$, and $\vartheta_a(t)$, $\vartheta_b(t)$ are the first order approximations of the parameters $a(t)$ and $b(t)$, respectively (components of the vector Θ_i).

In case Assumption 2 is met, following the definition of the signal $\mathcal{K}(t)$, the first order approximations $\theta_x(t)$ and $\theta_r(t)$ of the parameters $k_x(t)$ and $k_r(t)$, respectively, satisfy the equations

$$a_{ref}^{\mathrm{T}} - \vartheta_a^{\mathrm{T}}(t) = \vartheta_b(t)\,\theta_x(t),\ \ b_{ref} = \vartheta_b(t)\,\theta_r(t). \tag{A.18}$$

where $\theta(t) = \begin{bmatrix}\theta_x(t) & \theta_r(t)\end{bmatrix}^{\mathrm{T}}$.

Each equation from (A.18) is multiplied by $\Delta(t)$. Equations (A.17) are substituted into the obtained result to have equation (4.6):

$$\begin{gathered}\mathcal{Y}(t) = \mathcal{M}(t)\,\theta(t) + d(t,T),\\ \mathcal{Y}(t) := \begin{bmatrix}\Delta(t)\,a_{ref}^{\mathrm{T}} - z_a(t) & \Delta(t)\,b_{ref}\end{bmatrix}^{\mathrm{T}},\\ \mathcal{M}(t) := z_b(t),\\ d(t,T) := -\begin{bmatrix}\overline{\varepsilon}_1^{\mathrm{T}}(t,T)\,\mathfrak{L}_a + \overline{\varepsilon}_1^{\mathrm{T}}(t,T)\,\mathfrak{L}_b\theta_r(t) & \overline{\varepsilon}_1^{\mathrm{T}}(t,T)\,\mathfrak{L}_b\theta_r(t)\end{bmatrix}^{\mathrm{T}},\end{gathered} \tag{A.19}$$

where $\mathcal{Y}(t) \in \mathbb{R}^{n+1},\ \mathcal{M}(t) \in \mathbb{R},\ d(t,\,T) \in \mathbb{R}^{n+1}$.

Owing to (A.19), the solution of (4.7a) is written as

$$\Upsilon(t) = \int_{t_0^+}^{t} e^{\int_t^\tau k d\tau}\,\mathcal{M}(\tau)\,\theta(\tau)\,d\tau + \int_{t_0^+}^{t} e^{\int_t^\tau k d\tau}\,d(\tau,\,T)\,d\tau \pm \Omega(t)\,\theta(t) = \Omega(t)\,\theta(t) + w(t), \tag{A.20}$$

where

$$w(t) = \Upsilon(t) - \Omega(t)\,\theta(t).$$

Equation (A.20) completes the proof of the fact that equation (4.8) can be obtained via procedure (4.1)–(4.7).

In order to prove statement (a), the regressor $\Omega(t)$ is represented as:

$$\begin{aligned}&\Omega(t) = \Omega_1(t) + \Omega_2(t),\\ &\dot{\Omega}_1(t) = -k\,(\Omega_1(t) - \Delta(t)\,\vartheta_b(t)),\ \ \Omega_1\left(t_0^+\right) = 0,\\ &\dot{\Omega}_2(t) = -k\left(\Omega_2(t) - \overline{\varepsilon}_1^{\mathrm{T}}(t,T)\,\mathfrak{L}_b\right),\ \ \Omega_2\left(t_0^+\right) = 0.\end{aligned} \tag{A.21}$$

As $k > 0$ and the perturbation $\overline{\varepsilon}_1(t,T)$ is bounded, then $\Omega_2(t)$ is bounded, moreover, for all $t \geqslant t_0^+$ the following holds

$$|\Omega_2(t)| \leqslant \Omega_{2\max}(T), \tag{A.22}$$

and there exists a limit $\lim_{T\to 0}\Omega_{2\max}(T) = 0$ for the upper bound as, following (A.15)–(A.19), the value of $\overline{\varepsilon}_1(t,T)$ can be arbitrarily reduced by reduction of T.

The next aim is to analyze $\Omega_1(t)$. The solution of the first differential equation from (A.21) is written for all $t \in \left[t_i^+ + T_s,\, t_{i+1}^+\right)$ as

$$\Omega_1(t) = \phi\left(t, t_i^+ + T_s\right)\Omega_1\left(t_i^+ + T_s\right) + \int_{t_i^+ + T_s}^{t} \phi(t,\tau)\,\Delta(\tau)\,\vartheta_b(\tau)d\tau, \tag{A.23}$$

where $\phi(t,\tau) = e^{-\int_\tau^t k d\tau}$.

The upper bound is required for the signal $\Omega_1(t)$ over the time range under consideration. To this end, we need bounds for $\Delta(t)$, and, in its turn, the ones for $\varphi(t)$.

As, according to the premises of the proposition, $\overline{\varphi}_n \in \mathrm{PE}$ for $T_s < T$, then $\overline{\varphi}_n \in \mathrm{FE}$ over $\left[t_i^+, t_i^+ + T_s\right]$ (this fact can be validated by substitution of $t = t_i^+$ into (1.2)). Then for all $t \in \left[t_i^+ + T_s, t_{i+1}^+\right)$ the following lower bound holds for the regressor $\varphi(t)$

$$\begin{aligned}\varphi(t) &= \int_{t_i^+}^{t} e^{-\sigma\left(\tau - t_i^+\right)} \overline{\varphi}_n(\tau)\, \overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \\ &\geqslant \int_{t_i^+}^{t_i^+ + T_s} e^{-\sigma\left(\tau - t_i^+\right)} \overline{\varphi}_n(\tau)\, \overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \\ &\geqslant e^{-\sigma\left(t_{i+1}^+ - t_i^+\right)} \int_{t_i^+}^{t_i^+ + T_s} \overline{\varphi}_n(\tau)\, \overline{\varphi}_n^{\mathrm{T}}(\tau)\, d\tau \geqslant \alpha e^{-\sigma\left(t_{i+1}^+ - t_i^+\right)} I_{n+1}.\end{aligned} \tag{A.24}$$

On the other hand, as $\|\overline{\varphi}_n(t)\|^2 \leqslant \overline{\varphi}_n^{\max}$, then there exists an upper bound

$$\varphi(t) \leqslant \overline{\varphi}_n^{\max} \int_{t_i^+}^{t} e^{-\sigma\left(\tau - t_i^+\right)} d\tau \leqslant \overline{\varphi}_n^{\max} \frac{1 - e^{-\sigma\left(t - t_i^+\right)}}{\sigma} \leqslant \sigma^{-1} \overline{\varphi}_n^{\max}, \tag{A.25}$$

and, therefore, for all $t \in \left[t_i^+ + T_s, t_{i+1}^+\right)$ it holds that $\Delta_{UB} \geqslant \Delta(t) \geqslant \Delta_{LB} > 0$.

Taking into consideration that, following Assumptions 1 and 2, $b_{\max} \geqslant |b(t)| \geqslant b_{\min} > 0$, and $\vartheta_b(t)$ is the approximation of first order of $b(t)$, then the following holds for the multiplication $\Delta(t)\, \vartheta_b(t)$

$$\forall t \in \left[t_i^+ + T_s, t_{i+1}^+\right) \quad \Delta_{UB} b_{\max} \geqslant |\Delta(t)\, \vartheta_b(t)| \geqslant \Delta_{LB} b_{\min} > 0. \tag{A.26}$$

Having applied (A.21) and (A.26) and considered that $0 \leqslant \phi(t, \tau) \leqslant 1$, the following estimates hold for $\Omega_1(t)$

$$\begin{gathered}\forall t \in \left[t_0^+, t_0^+ + T_s\right] \quad \Omega_1(t) \equiv 0, \\ \forall i \geqslant 1 \;\; \forall t \in \left[t_i^+ + T_s, t_{i+1}^+\right] \quad \Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right) \Delta_{UB} b_{\max} \geqslant \Omega_1(t) \\ \geqslant \phi\left(t_{i+1}^+, t_i^+ + T_s\right) \left(\Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right) \Delta_{LB} b_{\min}\right) > 0,\end{gathered} \tag{A.27}$$

from which we have

$$\begin{gathered}\forall t \geqslant t_0 + T_s \quad \Omega_{1\max} \geqslant \Omega_1(t) \geqslant \Omega_{1\min} > 0, \\ \Omega_{1\max} = \min_{\forall i \geqslant 1} \left\{\phi\left(t_{i+1}^+, t_i^+ + T_s\right) \left(\Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right) \Delta_{LB} b_{\min}\right)\right\}, \\ \Omega_{1\min} = \max_{\forall i \geqslant 1} \left\{\Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right) \Delta_{UB} b_{\max}\right\}.\end{gathered} \tag{A.28}$$

Then, using (A.28) and (A.23), the bounds for the regressor $\Omega(t)$ are written

$$\forall t \geqslant t_0 + T_s \;\; \Omega_{1\max} + \Omega_{2\max}(T) \geqslant |\Omega(t)| \geqslant \Omega_{1\min} - \Omega_{2\max}(T), \tag{A.29}$$

and, therefore, considering $\lim_{T\to 0}\Omega_{2\max}(T)=0$, there exists $T_{\min}>0$ such that for all $0<T<T_{\min}$ and $t\geqslant t_0+T_s$ the following inequality holds

$$\Omega_{\mathrm{UB}}\geqslant\Omega(t)\geqslant\Omega_{\mathrm{LB}}>0, \tag{A.30}$$

which was to be proved in statement (a).

In order to prove the statement (b), the disturbance $w(t)$ is differentiated with (A.20) and (4.7) at hand

$$\begin{aligned}\dot{w}(t)&=\dot{\Upsilon}(t)-\dot{\Omega}(t)\theta(t)-\Omega(t)\dot{\theta}(t)\\&=-k(\Upsilon(t)-\mathcal{Y}(t))+k(\Omega(t)-\mathcal{M}(t))\theta(t)-\Omega(t)\dot{\theta}(t)\\&=-k(\Upsilon(t)-\mathcal{M}(t)\theta(t)-d(t,T))+k(\Omega(t)-\mathcal{M}(t))\theta(t)-\Omega(t)\dot{\theta}(t)\\&=-k(\Upsilon(t)-\Omega(t)\theta(t))-\Omega(t)\dot{\theta}(t)+kd(t,T)\\&=-kw(t)-\Omega(t)\dot{\theta}(t)+kd(t,T),\quad w\left(t_0^+\right)=0_{n+1}.\end{aligned} \tag{A.31}$$

The solution of (A.31) is represented as:

$$\begin{aligned}&w(t)=w_1(t)+w_2(t),\\&\dot{w}_1(t)=-kw_1(t)-\Omega(t)\dot{\theta}(t),\quad w_1\left(t_0^+\right)=0_{n+1},\\&\dot{w}_2(t)=-kw_2(t)+kd(t,T),\quad w_2\left(t_0^+\right)=0_{n+1}.\end{aligned} \tag{A.32}$$

As for the first differential equation from (A.32), in Proposition 2 from [19] it is proved (up to notation) that the following inequality holds

$$\|w_1(t)\|\leqslant w_{1\max}\phi\left(t,t_0^++T_s\right), \tag{A.33}$$

when $i\leqslant i_{\max}<\infty$.

As $k>0$ and the disturbance $d(t,T)$ is bounded, then $w_2(t)$ is also bounded, and consequently, the following inequality holds

$$\|w_2(t)\|\leqslant w_{2\max}(T), \tag{A.34}$$

where the limit $\lim_{T\to 0}w_{2\max}(T)=0$ holds, as the input of the second differential equation from (A.32) depends only from the value of $d(t,T)$, which, in its turn, according to (A.15)–(A.19), can be reduced arbitrarily by reduction of T. The combination of the inequalities (A.33) and (A.34) in accordance with (A.32) completes the proof of proposition.

Proof of Theorem 1. Proof of theorem is similar to the above-given proof of Proposition 1.

Step 1. For all $t\geqslant t_0^++T_s$ the solution of the differential equation (4.9) is written as

$$\begin{aligned}\tilde{\theta}(t)=\phi\left(t,t_0^++T_s\right)\tilde{\theta}\left(t_0^++T_s\right)&+\int\limits_{t_0^++T_s}^{t}\phi(t,\tau)\frac{\gamma_1 w(\tau)}{\Omega(\tau)}d\tau\\&-\int\limits_{t_0^++T_s}^{t}\phi(t,\tau)\sum_{q=1}^{i}\Delta_q^{\theta}\delta\left(\tau-t_q^+\right)d\tau,\end{aligned} \tag{A.35}$$

where $\phi(t,\tau)=e^{-\int\limits_{\tau}^{t}\gamma_1 d\tau}$.

Then, following the proof of Theorem 1 from [19], if $i \leqslant i_{\max} < \infty$, then the boundedness of the parametric error (A.35) can be shown:

$$\begin{aligned}\left\|\tilde{\theta}\left(t\right)\right\| &\leqslant \beta_{\max} e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} + \frac{\gamma_1 w_{1\max}}{\Omega_{LB}} \int\limits_{t_0^+ + T_s}^{t} \phi\left(t,\tau\right)\phi\left(\tau, t_0^+ + T_s\right) d\tau \\ + \frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}} &\int\limits_{t_0^+ + T_s}^{t} \phi\left(t,\tau\right) d\tau \leqslant \left(\beta_{\max} + \frac{2w_{1\max}}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} + \frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}}.\end{aligned} \tag{A.36}$$

Step 2. The following quadratic form is introduced for all $t \geqslant t_0^+ + T_s$:

$$\begin{gathered}V_{e_{ref}} = e_{ref}^{\mathrm{T}} P e_{ref} + \frac{4a_0^2}{\gamma_1} e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_s\right)}, \quad H = \operatorname{blockdiag}\left\{P, \ \frac{4a_0^2}{\gamma_1}\right\}, \\ \underbrace{\lambda_{\min}\left(H\right)}_{\lambda_{\mathrm{m}}} \left\|\overline{e}_{ref}\right\|^2 \leqslant V\left(\left\|\overline{e}_{ref}\right\|\right) \leqslant \underbrace{\lambda_{\max}\left(H\right)}_{\lambda_M} \left\|\overline{e}_{ref}\right\|^2, \\ \overline{e}_{ref}\left(t\right) = \left[e_{ref}^{\mathrm{T}}\left(t\right) \quad e^{-\frac{\gamma_1}{4}\left(t-t_0^+-T_s\right)}\right]^{\mathrm{T}}.\end{gathered} \tag{A.37}$$

Similar to proof of Proposition 1, the derivative of (A.37) is written as

$$\begin{aligned}\dot{V}_{e_{ref}} &\leqslant \left[-\lambda_{\min}\left(Q\right) + 2\lambda_{\max}\left(P\right) b_{\max}\left(\left\|\tilde{\theta}\right\| + \dot{\mathcal{K}}_{\max} T\right) + 2\right] \left\|e_{ref}\right\|^2 \\ &+ \lambda_{\max}^2\left(P\right)\overline{\omega}_r^2 b_{\max}^2 \left\|\tilde{\theta}\right\|^2 + \lambda_{\max}^2\left(P\right) b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2 - 2a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_s\right)}.\end{aligned} \tag{A.38}$$

As for all $t \geqslant t_0^+ + T_s$ the parametric error $\tilde{\theta}\left(t\right)$ meets the inequality (A.36), then, considering

$$\begin{aligned}\left\|\tilde{\theta}\left(t\right)\right\|^2 &\leqslant \left(\beta_{\max} + \frac{2w_{1\max}}{\Omega_{LB}}\right)^2 e^{-\gamma_1\left(t-t_0^+-T_0\right)} + \left(\frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}}\right)^2 \\ &+ 2\left(\beta_{\max} + \frac{2w_{1\max}}{\Omega_{LB}}\right)\frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}} e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} \\ &\leqslant \left(\beta_{\max} + \frac{2w_{1\max}}{\Omega_{LB}}\right)\left(\beta_{\max} + \frac{2\left(w_{1\max} + \gamma_1 w_{2\max}\left(T\right)\right)}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} + \left(\frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}}\right)^2 \\ &= \overline{\beta}_{\max} e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_0\right)} + \left(\frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}}\right)^2\end{aligned}$$

the upper bound of (A.38) is written as follows:

$$\begin{aligned}\dot{V}_{e_{ref}} &\leqslant \Bigg[-\lambda_{\min}\left(Q\right) + 2 + 2\lambda_{\max}\left(P\right) b_{\max} \\ &\times \left(\left(\beta_{\max} + \frac{2w_{1\,\max}}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_s\right)} + \frac{\gamma_1 w_{2\,\max}\left(T\right)}{\Omega_{LB}} + \dot{\mathcal{K}}_{\max} T\right)\Bigg] \left\|e_{ref}\right\|^2 \\ &+ \lambda_{\max}^2\left(P\right)\overline{\omega}_r^2 b_{\max}^2 \overline{\beta}_{\max} e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_s\right)} + \lambda_{\max}^2\left(P\right)\overline{\omega}_r^2 b_{\max}^2 \left(\frac{\gamma_1 w_{2\max}\left(T\right)}{\Omega_{LB}}\right)^2 \\ &+ \lambda_{\max}^2\left(P\right)\overline{\omega}_r^2 b_{\max}^2 \dot{\mathcal{K}}_{\max}^2 T^2 - 2a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^+-T_s\right)}.\end{aligned} \tag{A.39}$$

There definitely exists a time instant $t_{e_{ref}} \geqslant t_0^+ + T_s$ and constants $T \to 0$, $a_0 > \lambda_{\max}(P)\overline{\omega}_r b_{\max}\overline{\beta}_{\max}^{\frac{1}{2}}$ such that for all $t \geqslant t_{e_{ref}}$ it holds that

$$\begin{gathered} -\lambda_{\min}(Q) + 2 + 2\lambda_{\max}(P) b_{\max}\left(\left(\beta_{\max} + \frac{2w_{1\max}}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t_{e_{ref}} - t_0^+ - T_s\right)}\right. \\ \left. + \frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}} + \dot{\mathcal{K}}_{\max} T\right) = -c_1 < 0, \\ \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \overline{\beta}_{\max} - 2a_0^2 = -c_2 < 0. \end{gathered} \tag{A.40}$$

Then the upper bound for the derivative (A.39) for all $t \geqslant t_{e_{ref}}$ is obtained as

$$\dot{V}_{e_{ref}} \leqslant -\eta_{\overline{e}_{ref}} V_{e_{ref}} + \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 + \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \dot{\mathcal{K}}_{\max}^2 T^2, \tag{A.41}$$

where $\eta_{\overline{e}_{ref}} = \min\left\{\frac{c_1}{\lambda_{\max}(P)},\ \frac{c_2\gamma_1}{4a_0^2}\right\}$.

The solution of the differential inequality (A.41) for all $t \geqslant t_{e_{ref}}$ is written as

$$\begin{gathered} V_{e_{ref}}(t) \leqslant e^{-\eta_{\overline{e}_{ref}}\left(t - t_{e_{ref}}\right)} V_{e_{ref}}\left(t_{e_{ref}}\right) \\ + \frac{1}{\eta_{\overline{e}_{ref}}}\left(\lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 + \lambda_{\max}^2(P)\, b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2\right), \end{gathered} \tag{A.42}$$

which completes the proof of statement (ii) of theorem.

Step 3. Owing to (A.36) and (A.42), the error $\tilde{\theta}(t)$ is bounded for all $t \geqslant t_0^+ + T_s$, and the error $e_{ref}(t)$ — for all $t \geqslant t_{e_{ref}}$. Then, to prove the statement (i), we need to show that $\tilde{\theta}(t)$ is bounded over $\left[t_0^+, t_0^+ + T_s\right)$, and $e_{ref}(t)$ is bounded over $\left[t_0^+, t_{e_{ref}}\right)$.

In the conservative case, the inequality $\Omega(t) \leqslant \Omega_{LB}$ is satisfied over $\left[t_0^+, t_0^+ + T_s\right)$, whence, owing to $\dot{\tilde{\theta}}(t) = 0_{n+1}$, if Assumption 1 is met, it follows that the parametric error $\tilde{\theta}(t) = \hat{\theta}\left(t_0^+\right) - \theta(t)$ is bounded over $\left[t_0^+, t_0^+ + T_s\right)$ and, as a consequence, for all $t \geqslant t_0^+$.

Considering the time range $\left[t_0^+, t_{e_{ref}}\right)$ and taking into account the notation from (A.3), (A.18), the error equation (3.1) is written in the following form:

$$\dot{e}_{ref}(t) = \left(A_{ref} + e_n b(t)\left(\hat{\theta}_x(t) - k_x(t)\right)\right) e_{ref}(t) + e_n b(t)\left(\hat{\theta}^{\mathrm{T}}(t) - \mathcal{K}^{\mathrm{T}}(t)\right)\omega_r(t),$$

which, as it has been proved that $\tilde{\theta}(t)$ is bounded for all $t \geqslant t_0^+$ and Assumptions 1 and 2 are met, allows one, using Theorem 3.2 from [20], to make the conclusion that 1) $e_{ref}(t)$ is bounded over $\left[t_0^+, t_{e_{ref}}\right)$, 2) $\xi(t) \in L_\infty$ for all $t \geqslant t_0^+$.

REFERENCES

1. Polyak, B.T. and Tsypkin, Ya.Z., Optimal Pseudogradient Adaptation Algorithms, *Autom. Remote Control*, 1981, vol. 41, pp. 1101–1110.

2. Polyak, B.T. and Tsypkin, Ya.Z., Robust Pseudogradient Adaptation Algorithms, *Autom. Remote Control*, 1981, vol. 41, no. 10, pp. 1404–1409.

3. Ioannou, P. and Sun, J., *Robust Adaptive Control*, New York: Dover, 2013.

4. Fradkov, A.L., Lyapunov-Bregman functions for speed-gradient adaptive control of nonlinear time-varying systems, *IFAC-PapersOnLine*, 2022, vol. 55, no. 12, pp. 544–548.

5. Goel, R. and Roy, S.B., Composite adaptive control for time-varying systems with dual adaptation, *arXiv preprint arXiv:2206.01700*, 2022, pp. 1–6.

6. Na, J., Xing, Y., and Costa-Castelló, R., Adaptive estimation of time-varying parameters with application to roto-magnet plant, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2018, vol. 51, no. 2, pp. 731–741.

7. Chen, K. and Astolfi, A., Adaptive control for systems with time-varying parameters, *IEEE Transactions on Automatic Control*, 2020, vol. 66, no. 5, pp. 1986–2001.

8. Patil, O.S., Sun, R., Bhasin, S., and Dixon, W.E., Adaptive control of time-varying parameter systems with asymptotic tracking, *IEEE Transactions on Automatic Control*, 2022, vol. 67, no. 9, pp. 4809–4815.

9. Putov, V.V., Methods of Adaptive Control Systems Design for Nonlinear Time-Varying Dynamic Plants with Functional-Parametric Uncertainty, *Thesis ... Dr. of Technical Sciences*, SPbGETU 'LETI'. SPb., 1993, 590 p. In Russian.

10. Putov, V.V., Polushin, I.G., Lebedev, V.V., and Putov, A.V., Generalisation of the majorizing functions method for the problems of adaptive control of nonlinear dynamic plants, *Izvestiya SPbGETU LETI*, 2013, no. 8, pp. 32–37. In Russian.

11. Glushchenko, A. and Lastochkin, K., Exponentially Stable Adaptive Control. Part III. Time-Varying Plants, *Autom. Remote Control*, 2023, vol. 84, no. 11, pp. 1232–1247.

12. Pagilla, P.R. and Zhu, Y., Adaptive control of mechanical systems with time-varying parameters and disturbances, *J. Dyn. Sys., Meas., Control*, 2004, vol. 126, no. 3, pp. 520–530.

13. Quoc, D.V., Bobtsov, A.A., Nikolaev, N.A., and Pyrkin, A.A., Stabilization of a linear non-stationary system under conditions of delay and additive sinusoidal perturbation of the output, *Journal of Instrument Engineering*, 2021, vol. 64, no. 2, pp. 97–103. In Russian.

14. Dat, V.Q. and Bobtsov, A.A., Output Control by Linear Time-Varying Systems using Parametric Identification Methods, *Mekhatronika, Avtomatizatsiya, Upravlenie*, 2020, vol. 21, no. 7, pp. 387–393. In Russian.

15. Grigoryev, V.V., Design of control equations for variable parameter systems, *Autom. Remote Control*, 1983, vol. 44, no. 2, pp. 189–194.

16. Glushchenko, A. and Lastochkin, K., Robust Time-Varying Parameters Estimation Based on I-DREM Procedure, *IFAC-PapersOnLine*, 2022, vol. 55, no. 12, pp. 91–96.

17. Dieudonne, J., *Foundations of Modern Analysis*, New York, Academic Press, 1960.

18. Leiva, H. and Siegmund, S., A necessary algebraic condition for controllability and observability of linear time-varying systems, *IEEE Transactions on Automatic Control*, 2003, vol. 48, no. 12, pp. 2229–2232.

19. Glushchenko, A.I. and Lastochkin, K.A., Exponentially Stable Adaptive Control. Part II. Switched Systems, *Autom. Remote Control*, 2023, vol. 84, no. 3, pp. 260–291.

20. Khalil, H., *Nonlinear Systems*, 3rd ed., Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

This paper was recommended for publication by P.S. Shcherbakov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, pp. 532–543.

TOPICAL ISSUE

Investigation of Feasible and Marginal Operating Regimes of Electric Power Systems

E. N. Gryazina[*,a] **and D. Y. Baluev**[*,b]

[*] *Skolkovo Institute of Science and Technology, Moscow, Russia*
e-mail: [a] *E.Gryazina@skoltech.ru,* [b] *Dmitrii.Baluev@skoltech.ru*

Received September 28, 2023
Revised November 27, 2023
Accepted December 21, 2023

Abstract—The paper is devoted to the analysis of the feasibility domain of electric power systems. The problems of calculating feasible and marginal regimes of power systems, analyzing the geometry of the feasibility domain, and generating samples in this region are considered. Parallels are drawn with the works of B.T. Polyak on the analysis of the image of a quadratic map, modification of the Newton method and the development of methods for generating asymptotically uniform samples in areas with complex geometry. Particular attention is paid to Newton's method with the transversality condition (TENR), its application for constructing a boundary oracle procedure and utilization for generating samples in the power system feasibility domain.

Keywords: admissible domain of power systems, power flow equations, quadratic mapping image, Newton's method, sampling

DOI: 10.31857/S0005117924050064

1. INTRODUCTION

The development of theory, models and methods for calculating optimal and marginal operating regimes of the power system remains a challenging and relevant research topic due to the widespread distribution of distributed renewable energy sources, changing patterns of electrical energy consumption and digital transformation in the energy sector. Control of modern power systems requires fast and reliable methods for estimating static stability margins, which are characterized by the distance to the boundary of the feasibility domain. In addition, the growing integration of distributed renewable energy sources is prompting a reassessment of the criteria for optimal grid operating regimes, which shifts frequently operated regimes closer to the boundary. The concept of a feasibility domain for electric power system — a region in the multidimensional space of nodal power injections (the right-hand side of power flow equations) such that this system of equations has at least one real solution — bridges between various problems in energy sector.

Further in the introduction, we discuss the links between the analysis of feasible and marginal operating regimes of a power system with the image of a quadratic mapping, D-partition, Newton's method and its modifications, as well as methods for generating asymptotically uniform samples in areas with complex geometry.

1.1. Image of Quadratic Mapping

The steady-state operation regimes are described by power flow equations, reflecting fundamental Ohm's and Kirchhoff's laws they provide the relation between complex voltages and power at the nodes of power system (1)–(2). These equations are quadratic with respect to variable V. In papers [1, 2] B.T. Polyak proposed sufficient conditions for convexity of the image of quadratic

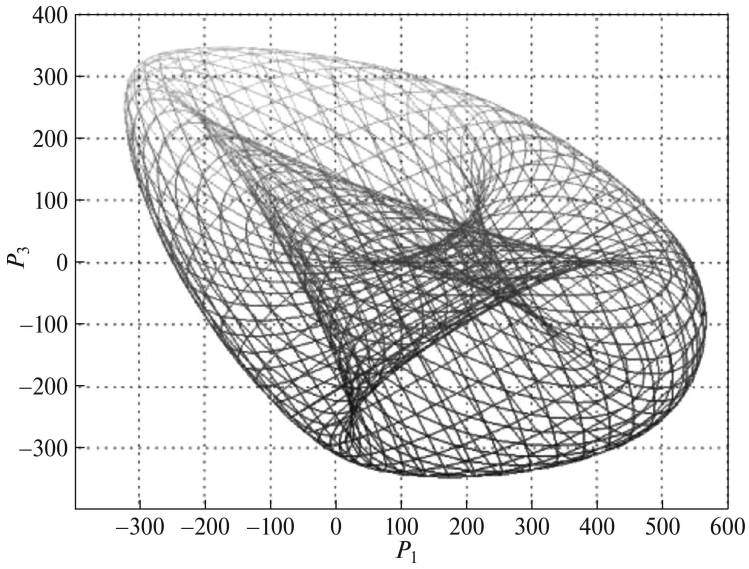

Fig. 1. Convex region of feasible regimes from [4].

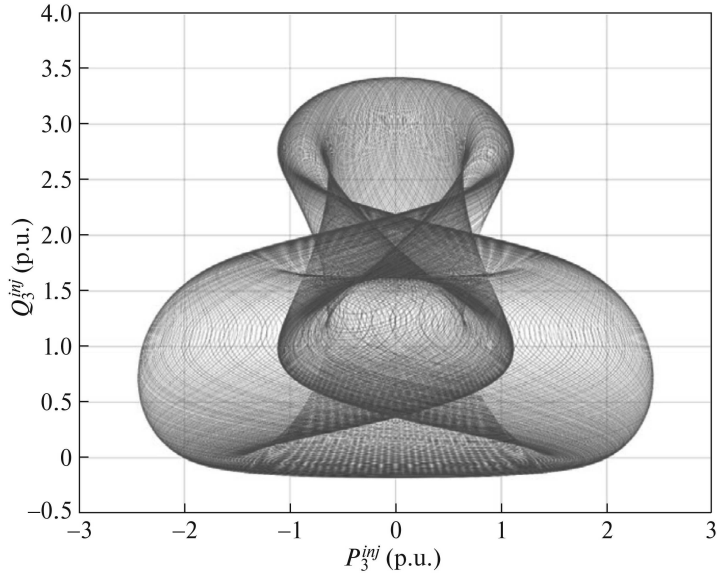

Fig. 2. Non-convex region of feasible regimes from [5].

mapping $f : \mathbb{R}^n \to \mathbb{R}^m$ at $m = 2, 3$, later in [3] a randomized approach for certifying convexity/nonconvexity of the image of quadratic mapping was proposed.

Indeed, feasibility domain of power system has a pretty complicated geometry. Moreover, this complexity holds regardless the number of nodes, one may observe complicated structure for the systems of 3–5 nodes ("nodes" are typical called "buses" in power systems analysis). In 1 presents the cross-section of the feasibility domain for 3 bus system. The area is convex, though it's demonstrated in [4] that it looses convexity for perturbed right-had side of power flow equation.

Examples of more exotic feasibility domains can be found in [5], one of cross-sections of this type for 5-bus system is presented in Fig. 2.

Looking at Figs. 1–2, it is straightforward to recognize a complex internal structure of the feasibility domain. Due to the nonlinearity of the system of power flow equations there are internal bifurcation curves, such that crossing them corresponds to either a change in the number of solutions or the disappearance of solutions, and therefore the admissible state of the system. These

equations can have multiple isolated solutions, representing either a stable or unstable equilibrium of a dynamic power system model. The presence of multiple solutions was previously ignored by researchers; their efforts were mainly focused on identifying only one real solution rather than on all isolated solutions. The work [6] is apparently the first to study the phenomenon of multiplicity of solutions to the control system and propose a method for constructing all critical points of the region of admissible modes. Critical points are understood as points of bifurcation curves, the intersection of which changes the number of solutions to the system of power flow equations. Such an analysis of the feasibility domain has much in common with D-decomposition method developed in the works of B.T. Polyak [7, 8]. For linear dynamical system D-decomposition curve splits the parameter space into regions with different numbers of stable roots of characteristic polynomial, while bifurcation curve (or surface in higher dimensional spaces) separates the regions wit different number of solutions to the power flow equations in thee feasibility domain.

1.2. The Role of Newton Method

Traditionally, Newton–Raphson method is applied for calculating steady state regimes. The introduction of additional variables characterizing stability margins makes the system underdetermined. In this case, the problem of calculating regimes appears in the context of B.T. Polyak's papers [9, 10].

One of the most promising methods for fast calculations of marginal regimes is Newton's method with the transversality condition (TENR, Transversality Enforced Newton–Raphson) [11], where, in addition to an additional variables, the condition of degeneracy of the Jacobian matrix is added.

The TENR method is conceptually similar to, but mathematically different from, traditional methods based on the standard Newton method. In TENR, the standard system of power flow equations is complemented by the transversality condition. This constraint regularizes the initially degenerate system at the marginal point and ensures the convergence of Newton's method. In addition, TENR allows the steady state calculation to take into account any technical constraints, which can be represented either as equalities or inequalities. From a computational point of view, a key advantage of TENR is its simple form of writing transversality conditions, which does not require explicit tracking and initialization of zero eigenvectors of the Jacobian. This simplification results in a smaller system of nonlinear equations and also allows for easier initialization of the algorithm.

The TENR method has a number of advantages: the algorithm is numerically stable in the immediate vicinity of the boundary, as well as at the feasibility boundary; it weakly depends on the starting point; decomposition of the Jacobian matrix by singular values has been implemented, which allows us to analyze the sensitivity of the power system and identify the most "effective nodes" for applying control actions. Based on TENR, it is possible to solve the problem of estimating the transfer capability margins [12], as well as online assessment of voltage stability margins [13]. The method has been tested on a number of IEEE benchmark systems as well as on a model of the power system of the Russian Far East [14].

1.3. Sampling in Feasibility Domain

Knowledge about the feasibility domain geometry of power system and its boundaries allows us to make fast estimation of the stability margins and to calculate optimal emergency control actions. The challenge of ensuring reliable and secure real-time operation of power systems is increasing as the current operating regime rapidly changes due to uncertainties associated with increased renewable generation, less predictable demand and various unexpected circumstances. Therefore, to avoid any undesirable system behavior or large-scale power outage, real-time evaluation of voltage stability margins is required. Such an assessment is a challenging task that requires significant

computational resources, mainly due to the constantly changing state of operation. Both during the planning and operational stage, safe operation of the network requires voltage stability, which is the ability of the power system to maintain acceptable voltage levels on all buses after exposure to disturbances [15].

Modern power systems are more vulnerable in terms of stability because they operate close to the boundary of the feasibility region. Voltage instability occurs in electrical networks when the operating mode approaches the point of collapse or the point of saddle-node bifurcation, after which the real solution to the steady-state equations vanishes or the number of solutions to the system of steady-state equations changes. You can clarify your description of the feasibility domain via sampling, i.e. generating parameters of feasible modes. Such parameter sets are also useful for tuning machine learning algorithms. One of the directions of B.T. Polyak's research was the development of methods for generating asymptotically uniformly distributed samples in complex domains [16, 17].

This paper provides a detailed description of the TENR method as the most effective tool for calculating the marginal states of a power system, and also shows how to use TENR to build a boundary oracle procedure to generate samples in the feasibility domain.

The paper is organized as follows: Section 2 presents the problem formulation. Section 3 describes the TENR method and discusses the strategy for choosing the optimal step size for its implementation. Section 4 is devoted to the problem of generating samples in the feasibility domain of power system. Section 5 provides numerical examples illustrating the effectiveness of the TENR method both for calculating marginal regimes (boundary points of the feasibility domain) and for generating samples.

2. PROBLEM STATEMENT

Power system marginal states (marginal operating regimes) assessment is closely related to the power flow analysis (so-called regime). A regime is a state of the power system that can be characterized by quantitative indicators: power, voltage, current, phase angles of the EMF vectors, and others. A regime can be categorized as transient or steady-state, depending on the rate of their change. A steady-state is one in which the parameters remain constant over the considered time interval or change relatively slowly [18]. Since a regime has quantitative characteristics, it can be calculated and evaluated. The calculation of the steady-state regime (power flow analysis) involves determining all parameters of the steady-state regime given the known system parameters (circuit diagrams, line impedance, etc.) and some specified regime parameters [19]. The set of equations based on the equivalent circuits of the power system, as well as Ohm's and Kirchhoff's laws, constitutes the mathematical model of the steady-state regimes of power systems.

In the theory of electrical systems, there are numerous available mathematical models, each with its advantages and disadvantages. In this work, the model used is the system of power balance equations presented in a rectangular form. The voltage is represented as a complex: $\hat{V}_i = V_i^r + jV_i^m \in \mathbb{C}$. G_{ij} and B_{ij} are the real and imaginary parts of the complex admittance $\hat{Y}_{ij} = G_{ij} + jB_{ij} \in \mathbb{C}$. The system consists of n buses, where $\mathcal{N} = \{1, 2, \ldots, n\}$ is the set of buses excluding the balancing (slack) bus $\mathcal{S}$; the set of PQ buses (load buses) is denoted as $\mathcal{L}$; and the set of PV buses (generator buses) is denoted as $\mathcal{G}$. For each $i \in \mathcal{N}$, the values of the nodal active power injections can be computed as follows [18, 20, 21]:

$$\sum_{k=1}^{n} \left\{ V_i^r (G_{ik} V_k^r - B_{ik} V_k^m) + V_i^m (G_{ik} V_k^m + B_{ik} V_k^r) \right\} = P_i(x) - \lambda (P_{\text{gen},i} - P_{\text{load},i}), \tag{1}$$

where the vector $\mathbf{x} \in \mathbb{R}^n$ is a set of variables (the magnitude of voltages and phase angles at the buses for each bus in the system). Similarly, for each $i \in \mathcal{L}$, one can write the equation for the

nodal reactive power injections:

$$\sum_{k=1}^{n}\Big\{V_i^m(G_{ik}V_k^r - B_{ik}V_k^m) - V_i^r(G_{ik}V_k^m + B_{ik}V_k^r)\Big\} = Q_i(x) - \lambda(Q_{\mathrm{gen},i} - Q_{\mathrm{load},i}). \tag{2}$$

Subscripts "gen" and "load" denote the levels of generation and load at the buses, respectively. The parameter λ is a coefficient used to "stress" the system, meaning that loads are gradually increased. When $\lambda = \lambda_{\max}$, the system reaches its marginal state. Unlike the representation in polar coordinates, the formulation in Cartesian coordinates requires an additional set of equations to account for the voltage limitations at the PV buses. Thus, for each $i \in \mathcal{G}$,

$$(V_i^r)^2 + (V_i^m)^2 - |\hat{V}_i|^2_{\mathrm{ref}} = 0, \tag{3}$$

where $|\hat{V}_i|_{\mathrm{ref}}$ is the reference voltage magnitude at the specific bus.

The standard system of power flow equations can be generally expressed as follows:

$$\mathcal{F}(x, \lambda) = 0, \tag{4}$$

where $\mathcal{F}$ represents k nonlinear equations, including both power balance equations (such as in (1) and (2)) and various technological constraints presented as equalities. The parameter λ is the loading coefficient that characterizes the system's proximity to the steady-state equation solvability boundary.

The finding of marginal states means to find such $\lambda_{\max}$ that the solution of the system (4) exists for all $0 \leqslant \lambda \leqslant \lambda_{\max}$ but do not exist when $\lambda > \lambda_{\max}$.

From a mathematical perspective, finding of marginal states involves solving the system of equations (4) under the condition that the Jacobian matrix is singular:

$$g(x) = \det\ \nabla_x \mathcal{F}(x, \lambda) = 0. \tag{5}$$

It follows that, to find the marginal state (stability boundary), it is necessary to solve the system of equations (4) together with the additional condition that accounts for the singularity of the Jacobian matrix (5).

3. TRANSVERSALITY ENFORCED NEWTON–RAPHSON METHOD

To solve the system of power flow equations, numerical iterative methods must be employed, which improve the approximation of the initial variables with each iteration. One of the most common and accessible methods is the Newton–Raphson method.

It should be noted that the classical Newton–Raphson method has several drawbacks, including convergence dependence on the chosen initial conditions and poor convergence in close proximity to the stability boundary. The reason for this is the poor conditioning of the Jacobian matrix. Therefore, the standard Newton method provides a consistently underestimated assessment of the power system's stability margin. If the numerical method remains stable at the feasibility boundary, the distance to the marginal state can be determined more accurately.

There is a method that addresses these issues — the TENR method.

During the marginal state analysis, when λ reaches its maximum value $\lambda_{\max}$, the Jacobian matrix of the power flow system becomes singular. Under these circumstances, the Newton method's computational step $J^{-1}\mathcal{F}(x)$ increases, keeping the classical method numerically unstable. As a result, the method may fail to converge or require too many iterations to achieve a result. In the TENR method, an additional condition that accounts for the Jacobian matrix's singularity at the feasibility boundary is added to the base system of equations, with λ also treated as a variable. Thus, the solution domains of the original and the augmented systems of equations coincide.

Within the TENR method, the condition accounting for the Jacobian's singularity at the stability boundary is called the transversality condition $g(\mathbf{x})$. Numerous possible variations of the condition $g(\mathbf{x})$ are available, as presented in [11]. The least computationally expensive approach is based on singular value decomposition.

In general form, the system of equations for finding the limit modes can be written as follows:

$$\begin{aligned} \mathcal{F}(x,\lambda) &= 0, \\ g(x) &= 0 \end{aligned} \tag{6}$$

The system (6) can be numerically solved using the standard Newton method. Undertaking a linearization of equation utilizing the first-order Taylor series within the realms of x and λ results in:

$$\begin{bmatrix} \mathcal{F} \\ g \end{bmatrix} + \begin{pmatrix} \nabla_{\mathbf{x}}\mathcal{F} & \nabla_{\lambda}\mathcal{F} \\ (\nabla_{\mathbf{x}} g)^{\top} & 0 \end{pmatrix} \begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{7}$$

In the TENR method, the so-called extended Jacobian matrix $\mathcal{J}(x,\lambda)$ is used in the calculations:

$$\mathcal{J}(x,\lambda) = \begin{pmatrix} \nabla_{x}\mathcal{F} & \nabla_{\lambda}\mathcal{F} \\ (\nabla_{\mathbf{x}} g)^{\top} & 0 \end{pmatrix}. \tag{8}$$

The increments of the unknowns Δx and $\Delta\lambda$ are determined as follows:

$$\begin{bmatrix} \Delta x \\ \Delta\lambda \end{bmatrix} = -\begin{pmatrix} \nabla_{\mathbf{x}}\mathcal{F} & \nabla_{\lambda}\mathcal{F} \\ (\nabla_{\mathbf{x}} g)^{\top} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \mathcal{F} \\ g \end{bmatrix}. \tag{9}$$

Using the calculated increments of the variables, the values of the variables at the next step are determined as follows:

$$\mathcal{N}(x,\lambda) := \begin{bmatrix} x \\ \lambda \end{bmatrix} - \alpha \left[\begin{pmatrix} \nabla_{x}\mathcal{F} & \nabla_{\lambda}\mathcal{F} \\ (\nabla_{x} g)^{\top} & 0 \end{pmatrix}^{-1} \begin{bmatrix} \mathcal{F} \\ g \end{bmatrix} \right]. \tag{10}$$

The parameter α determines the step size in the Newton method, which must be chosen to be sufficiently small. The calculation is performed iteratively until the established convergence criterion is reached:

$$\|\mathcal{N}^{(\kappa)}(x,\lambda) - \mathcal{N}^{(\kappa-1)}(x,\lambda)\| \leqslant \epsilon, \quad \kappa = 1, 2, \ldots, \tag{11}$$

where κ is the iteration counter and ϵ is the desired calculation accuracy.

Newton–Raphson method has quadratic convergence if the initial design point is chosen in close proximity to the actual solution. However, the Newton–Raphson method may diverge. To prevent such situations, it is necessary to optimally select the Newton iteration step size. The methodology for choosing the optimal step size is presented below.

3.1. Optimal Step-Size Strategy

The Newton–Raphson method is highly sensitive to the initial approximation. In some cases, an improper choice of the initial guess can lead to a large number of iterations or the method may fail to converge altogether. To ensure faster convergence and global convergence with any reasonable initial approximation, the system of equations should be supplemented with a damping coefficient α, as introduced in (10). One of the most efficient and computationally simple methods is to adjust α at each iteration.

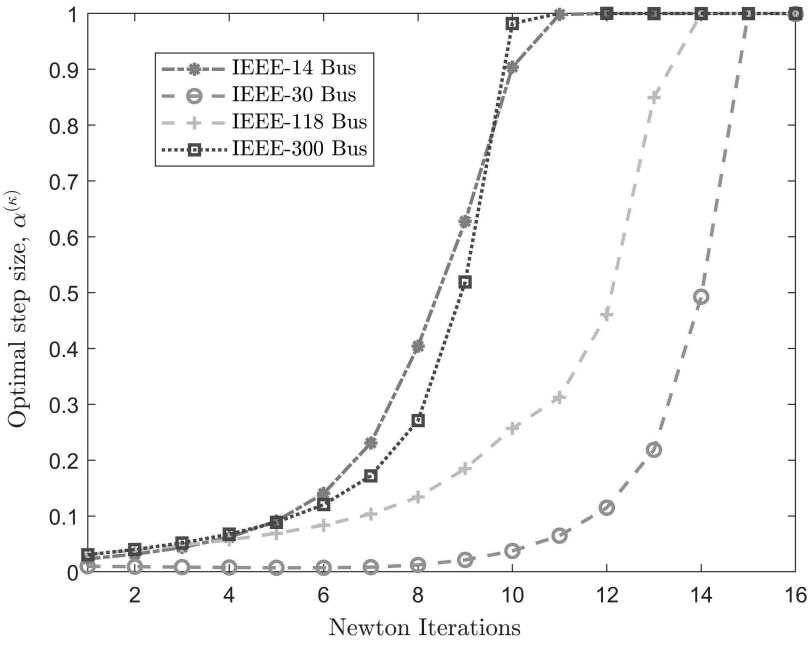

Fig. 3. Optimal step size $\alpha^{(\kappa)}$ selection for each iteration of the TENR method for IEEE test systems.

The original system of equations can be written in a compact form as:

$$\mathcal{H}(z) = \begin{cases} \mathcal{F}(x,\lambda) & = 0 \\ g(x) & = 0. \end{cases} \tag{12}$$

The vector z represents the set of variables $z = [x, \lambda]^\top$. In the Cartesian formulation, the equations considered in (4) are a set of quadratic equations, while the transversality condition $g(x) = 0$ is a linear equation, especially when the transversality condition is expressed through the singular value decomposition of the Jacobian. Therefore, for the point $z^{(\kappa)}$ and the vector $\Delta z^{(\kappa)}$, the system (12) can be approximated by a second-order Taylor expansion as

$$\mathcal{H}(z^{(\kappa)} + \Delta z^{(\kappa)}) \approx \mathcal{H}(z^{(\kappa)}) + [\nabla_z \mathcal{H}(z^{(\kappa)})]\ \Delta z^{(\kappa)} + \frac{1}{2}(\Delta z^{(\kappa)})^\top [\nabla_{zz} \mathcal{H}(z^{(\kappa)})]\ \Delta z^{(\kappa)}. \tag{13}$$

Since all the equations in (4) are quadratic, and $g(x)$ is linear, (13) holds exactly. The optimal step size $\alpha^{(\kappa)}$ in the direction of the vector $\Delta z^{(\kappa)}$ is found by solving the following minimization problem:

$$\mathcal{H}(\alpha) = \left\{ \mathcal{H}(z^{(\kappa)}) + \alpha[\nabla_z \mathcal{H}(z^{\kappa})]\ \Delta z^{(\kappa)} + \frac{\alpha^2}{2}(\Delta z^{(\kappa)})^\top [\nabla_{zz} \mathcal{H}(z^{(\kappa)})]\ \Delta z^{(\kappa)} \right\}, \tag{14}$$

$$\alpha^{(\kappa)} = \underset{\alpha}{\operatorname{argmin}}\ \frac{1}{2}\|\mathcal{H}(\alpha)\|_2^2. \tag{15}$$

The optimization problem (15) can be solved explicitly by applying the first-order optimality condition.

Figure 3 shows the values of $\alpha^{(\kappa)}$ at each iteration of the TENR method for the IEEE test systems consisting of 14, 30, 118, and 300 buses. These test cases include the standard problem of determining the marginal state under initial conditions, where all bus voltage magnitudes are 1, and the corresponding angles are 0. It can be observed that the initially proposed step size strategy leads to small values of α. However, as the algorithm approaches the solution, the step size gradually increases. This behaviour can be explained by the fact that, in the initial iterations, the first-

order Taylor approximation of the equations in (12) poorly satisfies the equality. As the algorithm progresses, this approximation becomes more accurate, leading to an increase in the step size α.

4. SAMPLING PARAMETERS OF FEASIBLE REGIMES

For regions with complicated geometry (non-convex, represented by nonlinear equations), which certainly includes the region of feasible power system modes, a working method for obtaining asymptotically uniform samples is based on the use of a version of the Monte Carlo method, namely Markov Chain Monte Carlo (MCMC) [22]. One of the most famous and effective MCMC-type algorithm is called Hit-and-Run (HR), originally proposed in [23], and later rediscovered and analyzed in detail in [24]. Unfortunately, even for simple ill-posed domains (for example, level sets of ill-conditioned functions), the HR method does not work, or at least is computationally inefficient [25].

The variety of applications as well as drawbacks of existing random walk methods open up wide scope for improvement of random walk algorithms. In particular, in the works of B.T. Polyak, it was presented an attempt to use barrier functions (well known in the analysis of interior point methods for convex optimization) and combine them with random walks based on Markov chains. As a result the Barrier Monte Carlo method [26] was proposed, whose mixing properties in some cases turned out to be preferable to the HR method. However, the complexity of each iteration remained quite high (in particular, at each iteration it is required to calculate $\left(\nabla^2 F(x)\right)^{-1/2}$, where $F(x)$ is the barrier function for the region Q). Moreover, this approach cannot accelerate the convergence of the distribution of the resulting points to a uniform one for areas similar to simplexes. Finally, in [27] the idea of Billiard Walk was presented, and theorems on the asymptotic uniformity of generated samples have been proved for the convex and non-convex cases. In contrast to the Ball Walk method, where each subsequent point is selected uniformly random at the intersection of the ball centered at the current point and the region under consideration, and the Hit-and-Run method, where the next point is randomly selected uniformly on a random chord drawn through the current point, the Billiard Walk method is based on a billiard trajectory of random length, released from the current point in a random direction.

The Hit-and-Run method and its improved modification, the Billiard Walk method, provide a useful tool for generating samples in the feasibility domain. The only requirement for a domain is that it must have a boundary oracle procedure and, in the case of a Billiard Walk, a way to recover the normal to the boundary.

Let us describe the application of the Hit-and-Run method and the boundary oracle procedure necessary for its implementation for the power system feasibility domain. The generated samples are located in the multidimensional space of nodal power injections $S_i = P_i + jQ_i$, $i = 1, \ldots, n$, which includes active power P_i and reactive power Q_i.

1. Choose initial regime S^0, $k = 0$. It can be arbitrary feasible point or so-called flat start: $V_i = 1$, $P_i = 0$, $Q_i = 0$; $i = 1, \ldots, n$.
2. Generate random direction d^k, which is uniform random on the unit sphere in R^{2n}. Components of d correspond to increments of active and reactive powers in the righ-had side of the equations (1)–(2).
3. Calculate marginal states in the directions d^k and $-d^k$ as well as corresponding $\overline{\lambda}$, $\underline{\lambda}$ via TENR method.
4. Update $k = k + 1$ and specify the next sample as $S^k = S^{k-1} + td$, where scalar t is uniform random in $[-\underline{\lambda}, \overline{\lambda}]$.
5. Save S^k and correspondng regime parameters. Go to Step 2.

For implementation of Billiard Walk algorithm eigenvector corresponding to zero eigenvalue of the Jacobian should be used as a normal to the boundary.

5. NUMERICAL EXPERIMENTS

Let's illustrate the effectiveness of the TENR method and its modifications for stability margin assessment and generating samples in the power system feasibility domain. Several power systems models from the IEEE collection [28, 29], widely used in academic research, were chosen as examples. The TENR method is integrated into the open-source software package PESOL [30].

5.1. Determination of Marginal States

The accuracy comparison of stability margin assessment was conducted between TENR and three of the most common limit state estimation methods integrated into various software packages: Continuation Power Flow (CPF), Power System Analysis Toolbox (PSAT), and MATPOWER. The comparison results of the λ values, characterizing the stability margins, are presented in Table.

Comparison of the stability margin obtained using the TENR method with analogues (without taking into account voltage constraints)

IEEE-scheme	λ_{TENR}	λ_{CPF}	λ_{PSAT}	λ_{MAT}
9 buses	1.486	1.486	1.481	1.483
14 buses	3.061	3.061	3.059	3.056
30 buses	1.958	1.957	1.959	1.838
57 buses	0.893	0.892	0.891	0.890
118 buses	2.188	2.187	2.187	2.184
300 buses	0.430	0.430	0.429	0.425

The comparison results between the TENR method and its direct competitors show that the stability margin calculated using TENR is not lower than the values obtained using other methods in all considered cases. For some cases, TENR indicated a slightly higher actual stability margin than other methods. The main advantage of TENR is the calculation speed and scalability (calculation of power systems with thousands of buses). A detailed comparison of the calculation speed of TENR with its direct competitors is presented in [14].

5.2. Generation of Parameters for Marginal States

In this work, a five-bus scheme, shown in Fig. 4, is considered as an example. This system is a modified example first presented in [31]: Bus 1 is the slack bus with voltage $\hat{V}_1 = 1.0$. The adopted description in power engineering of complex voltage in polar form $|V|e^{j\delta}$ is used here, where the

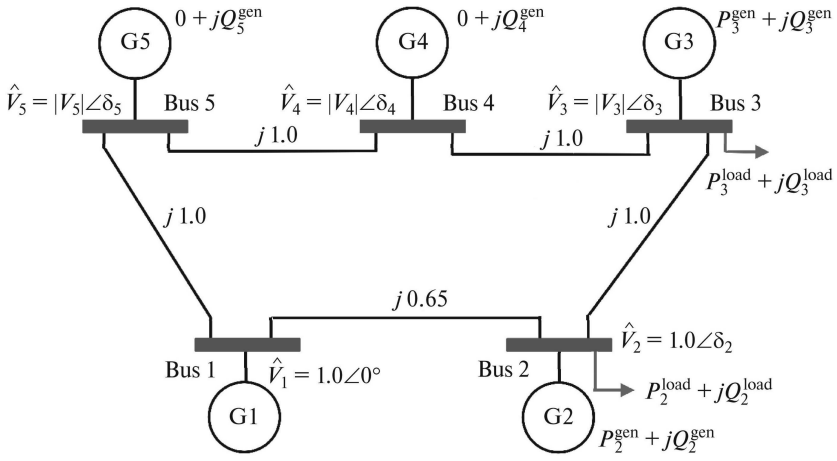

Fig. 4. 5-bus power system test case.

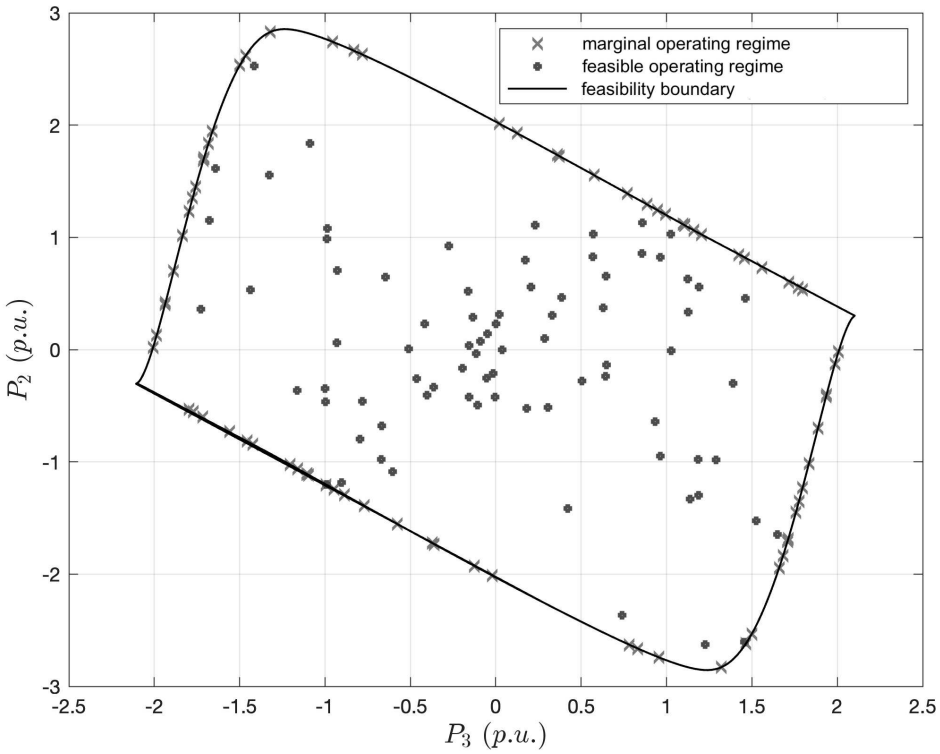

Fig. 5. Cross-section of the feasible region by P_2–P_3 parameter plane with a fixed value $Q_3 = 2$ $p.u.$ (80 feasible points).

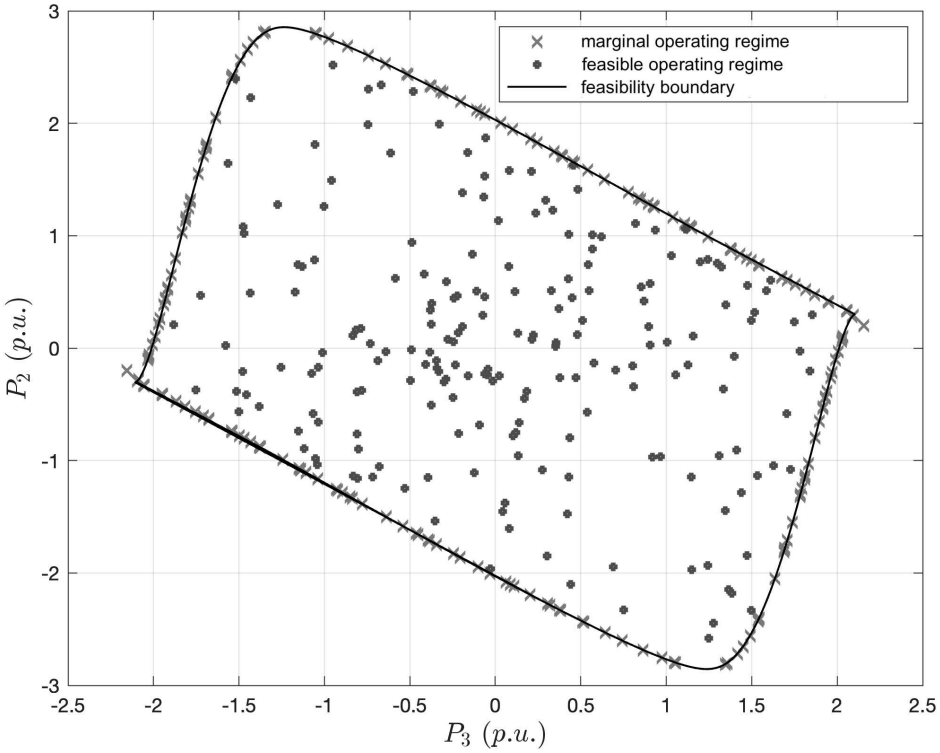

Fig. 6. Cross-section of the feasible region by P_2–P_3 parameter plane with a fixed value $Q_3 = 2$ $p.u.$ (200 feasible points).

phase angle is presented following the magnitude in the form $V\angle\delta$. The other buses in the system are PV buses with a fixed voltage value of 1.0 (p.u.), except for Bus 3, which is a PQ bus with a complex voltage value of $\hat{V}_3 = |V_3|e^{j\delta_3}$. It is also assumed that synchronous compensators with zero active power are installed at Buses 4 and 5. Consequently, the solution space is limited to the parameters P_2, P_3, Q_3.

Consider a cross-section of the feasibility region by the P_2–P_3 parameter plane, with the remaining parameters of the right-hand side of the power flow equations fixed as indicated above, $Q_3 = 2$. The results of generating 80 and 200 feasible regimes are shown in Figs. 5 and 6. The dots correspond to the internal points of the feasible operating region, while the crosses represent the limit operating regimes. The figures show that 200 generated feasible operating modes are sufficient for solving practical optimization problems, and the limit modes fairly densely cover the boundary of the permissible region.

6. CONSLUSION

This paper describes the main difficulties encountered in calculating the parameters of critical regimes in power systems and presents the TENR method, which currently appears to be the most effective method for marginal state assessment. Moreover, for the first time, the use of the TENR method for constructing a boundary oracle procedure and generating points within the feasible operating region has been presented and tested.

Surprisingly, the tasks of analyzing feasible and marginal states in power systems draw their solutions from the works of B.T. Polyak. His results on the convexity of the image of quadratic mappings, modifications of the Newton method, and detailed descriptions of random walk schemes for generating points in regions with complex geometry have proven extremely useful for power engineering. The authors do not doubt that researchers will discover many more such connections and bridges in the future.

FUNDING

This work was supported by the Russian Science Foundation, project no. 22-19-00773.

REFERENCES

1. Polyak, B., Convexity of Quadratic Transformations and Its Use in Control and Optimization, *J. Optim. Theor. Appl.*, 1998, vol. 99, pp. 553–583.
2. Polyak, B., Convexity of Nonlinear Image of a Small Ball with Applications to Optimization, *Set-Valued Analysis*, 2001, vol. 9, pp. 159–168.
3. Polyak, B. and Gryazina, E., Convexity/Nonconvexity Certificates for Power Flow Analysis, *Advances in Energy System Optimization: Proceedings of the First International Symposium on Energy System Optimization*, 2017, pp. 221–230.
4. Ayuev, B., Davydov, V., and Erokhin, P., Fast and Reliable Method of Searching Power System Marginal States, *IEEE Transactions on Power Systems*, 2016, vol. 31, no. 6, pp. 4525–4533.
5. Ali, M., Gryazina, E., Dymarsky, A., and Vorobev, P., Calculating Voltage Feasibility Boundaries for Power System Security Assessment, *Int. J. Elect. Power & Energy Syst.*, 2023, vol. 146, art. no. 108739.
6. Ali, M., Ali, M.H., Gryazina, E., and Terzija, V., Calculating Multiple Loadability Points in the Power Flow Solution Space, *Int. J. Elect. Power & Energy Syst.*, 2023, vol. 148, art. no. 108915.
7. Gryazina, E. and Polyak, B., On the Root Invariant Regions Structure for Linear Systems, *IFAC Proceedings Volumes*, 2005, vol. 38, no. 1, pp. 90–95.
8. Gryazina, E. and Polyak, B., Stability Regions in the Parameter Space: D-Decomposition Revisited, *Automatica*, 2006, vol. 42, no. 1, pp. 13–26.
9. Polyak, B. and Tremba, A., Sparse Solutions of Optimal Control via Newton Method for Under-Determined Systems, *J. Global Optim.*, 2020, vol. 76, no. 3, pp. 613–623.

10. Polyak, B. and Tremba, A., New Versions of Newton Method: Step-Size Choice, Convergence Domain and Under-Determined Equations, *Optim. Method. and Soft.*, 2020, vol. 35, no. 6, pp. 1272–1303.
11. Ali, M., Dymarsky, A., and Turitsyn, K., Transversality Enforced Newton–Raphson Algorithm for Fast Calculation of Maximum Loadability, *IET Generat. Transmis. Distribut.*, 2018, vol. 12, no. 8, pp. 1729–1737.
12. Ali, M., Gryazina, E., and Turitsyn, K., Fast Calculation of the Transfer Capability Margins, *IEEE Milan PowerTech*, 2019, pp. 1–6.
13. Ali, M., Gryazina, E., Khamisov, O., and Sayfutdinov, T., Online Assessment of Voltage Stability Using Newton-Corrector Algorithm, *IET Generat. Transmis. Distribut.*, 2020, vol. 14, no. 19, pp. 4207–4216.
14. Baluev, D., Ali, M., and Gryazina, E., State of the Art Approach for Comprehensive Power System Security Assessment-Real Case Study, *Int. J. Elect. Power & Energy Syst.*, 2024, vol. 155, art. no. 109594.
15. Sauer, P.W. and Pai, M.A., Power System Steady-State Stability and the Load-Flow Jacobian, *IEEE Transact. Power Syst.*, 1990, vol. 5, no. 4, pp. 1374–1383.
16. Tremba, A., Calafiore, G., Dabbene, F., Gryazina, E., Polyak, B., Shcherbakov, P., and Tempo, R., RACT: Randomized Algorithms Control Toolbox for MATLAB, *IFAC Proceedings Volumes*, 2008, vol. 41, no. 2, pp. 390–395.
17. Polyak, B. and Gryazina, E., Randomized Methods Based on New Monte Carlo Schemes for Control and Optimization, *Ann. Oper. Res.*, 2011, vol. 189, pp. 343–356.
18. Venikov, V.A., *Perekhodnye elektromekhanicheskie protsessy v elektricheskikh sistemakh* (Transient Electromechanical Processes in Electrical Systems), Moscow: Vyssh. shk., 1985.
19. Ayuev, B.I., Davydov, V.V., Erohin, P.M., and Nejumin, V.G., *Vychislitel'nye modeli potokoraspredeleniya v elektricheskikh sistemakh: monografiya* (Computational Models of Power Flow in Electrical Systems: Monograph), Moscow: Flinta, 2008.
20. Zhdanov, P.S., *Voprosy ustoichivosti elektricheskikh sistem* (Issues of Stability in Electrical Systems), Moscow: Energiya, 1979.
21. Idel'chik, V.I., *Elektricheskie sistemy i seti* (Electrical Systems and Networks), Moscow: Energoatomizdat, 1989.
22. Diaconis, P., The Markov Chain Monte Carlo Revolution, *Bull. Amer. Math. Soc.*, 2009, vol. 46, no. 2, pp. 179–205.
23. Turchin, V.F., On the Computation of Multidimensional Integrals by the Monte-Carlo Method, *Theor. Probab. Appl.*, 1971, vol. 16, no. 4, pp. 720–724.
24. Smith, R., Efficient Monte Carlo Procedures for Generating Points Uniformly Distributed over Bounded Regions, *Oper. Res.*, 1984, vol. 32, no. 6, pp. 1296–1308.
25. Polyak, B. and Shcherbakov, P., Why Does Monte Carlo Fail to Work Properly in High-Dimensional Optimization Problems?, *Theor. Probab. Appl.*, 2017, vol. 173, pp. 612–627.
26. Polyak, B. and Gryazina, E., Markov Chain Monte Carlo Method Exploiting Barrier Functions with Applications to Control and Optimization, *2010 IEEE International Symposium on Computer-Aided Control System Design*, 2010, pp. 1553–1557.
27. Gryazina, E. and Polyak, B., Random Sampling: Billiard Walk Algorithm, *Eur. J. Oper. Res.*, 2014, vol. 238, no. 2, pp. 497–504.
28. Vittal, V., Martin, D., Chu, R., Fish, J., Giri, J.C., Tang, C.K., Villaseca, F.E., and Farmer, R.G., Transient Stability Test Systems for Direct Stability Methods, *IEEE Transact. Power Syst.*, 1992, vol. 7, no. 1, p. 37.
29. Working Group, Common Format for Exchange of Solved Load Flow Data, *IEEE Transact. Power Apparat. Syst.*, 1973, no. 6, pp. 1916–1925.
30. Ali, M., Baluev, D., Ali, M.H., and Gryazina, E., A Novel Open Source Power Systems Computational Toolbox, *North American Power Symposium (NAPS)*, 2021, pp. 1–6.
31. Lesieutre, B.C. and Hiskens, I.A., Convexity of the Set of Feasible Injections and Revenue Adequacy in FTR Markets, *IEEE Transact. Power Syst.*, 2005, vol. 20, no. 4, pp. 1790–1798.

This paper was recommended for publication by P.S. Shcherbakov, a member of the Editorial Board

ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2024, Vol. 85, No. 5, pp. 544–548.

TOPICAL ISSUE

Iterative Methods with Self-Learning for Solving Nonlinear Equations

Yu. S. Popkov[*,**]

[*] *Federal Research Center "Computer Science and Control," Russian Academy of Sciences, Moscow, Russia*
[**] *Trapeznikov Institute of Control Sciences, Russian Academy of Sciences, Moscow, Russia*
e-mail: popkov@isa.ru

Received January 25, 2024
Revised March 12, 2024
Accepted March 20, 2024

Abstract—This paper is devoted to the problem of solving a system of nonlinear equations with an arbitrary but continuous vector function on the left-hand side. By assumption, the values of its components are the only a priori information available about this function. An approximate solution of the system is determined using some iterative method with parameters, and the qualitative properties of the method are assessed in terms of a quadratic residual functional. We propose a self-learning (reinforcement) procedure based on auxiliary Monte Carlo (MC) experiments, an exponential utility function, and a payoff function that implements Bellman's optimality principle. A theorem on the strict monotonic decrease of the residual functional is proven.

Keywords: nonlinear equation, iterative methods, reinforcement, Monte Carlo experiment

DOI: 10.31857/S0005117924050076

1. INTRODUCTION

The vast majority of applied problems lead to the need to solve nonlinear equations. Parameterized iterative methods are a classical tool yielding approximate solutions of nonlinear equations under definite conditions [1–4]. These conditions are certain properties of the functions (convexity, concavity, differentiability, etc.) included in the equations and interval *sufficient* parametric conditions ensuring the convergence of the corresponding iterative method.

The increasing complexity of functions narrows the set of their classes where such properties can be verified and the verification results can be used in suitable iterative methods. On the other hand, interval conditions on the parameters of iterative methods significantly depend on the properties of functions that are generally unverifiable.

To find a way out of this situation, the ideas of reinforcement learning can be applied to the iterative computational process to determine the values of its parameters using statistical Monte Carlo (MC) experiments and a game-theoretic mathematical model [5, 6]. The essence of this branch of machine learning is to train an object (model, algorithm, etc.) by interacting not with a "teacher" (supervised learning) but with an "environment," using the trial-and-error method, followed by rewards or penalties for its results.

This approach was employed in clustering and recognition problems, apparently because of the ability to calculate the so-called feature characteristics in the form of "distances" between objects. The distance matrix was adopted to arrange some "rewards" or "penalties" when tuning the algorithm parameters. The algorithms considered were neural networks [7] and game-theoretic

models implementing the principle of competition between neural network nodes [8]. In particular, the advantage was given to the nodes with the minimum distance between objects at each step of the algorithm.

Subsequently, reinforcement learning based on the automata models of interaction between an object (agent) and an environment was simulated in game-theoretic terms (strategies, utility functions, and payoffs) and was actively developed [9]. Many algorithms appeared, differing in the models and amounts of a priori information about the environment, algorithms for choosing strategies, and procedures for designing utility functions [10–13].

An important component of reinforcement learning procedures is MC experiments intended to simulate agent's strategies [14]. They are used to average a fixed number of current rewards with their discounting. The resulting function depends on the state of the environment and the agent's strategy; it is taken as a utility function (an analog of the objective function in supervised learning procedures) and is sequentially maximized during the learning process [15, 16] using Bellman's optimality principle [17] in combination with stochastic approximation [18].

This paper considers the problem of numerically solving a system of nonlinear equations using a parameterized iterative procedure whose convergence depends on the parameter values. To determine these values, we develop a reinforcement learning procedure based on a game-theoretic model.

The problem addressed below was actively discussed and shaped under the influence of Boris Polyak. The author was fortunate to work and be friends with him for many years. The blessed memory of Boris will always be the author's compass in life.

2. PROBLEM STATEMENT

Consider the nonlinear equation

$$\mathbf{f}(\mathbf{x}) = \mathbf{1}, \quad (\mathbf{f}, \mathbf{x}, \mathbf{1}) \in R^n. \tag{1}$$

By assumption, the only a priori information available about the function $\mathbf{f}$ is the values of its components $f_i(\mathbf{x}^{(k)})$, $i = \overline{1,n}$; $k = 1, \ldots$.

We introduce the residual functional

$$J(\mathbf{x}) = \|\mathbf{f}(\mathbf{x}) - \mathbf{1}\|^2 \geqslant 0. \tag{2}$$

The absolute minimum of this functional is zero. In the general case, it is not unique, i.e., there exists a finite set $\mathbb{X} = \{\mathbf{x}_*^{(1)}, \ldots, \mathbf{x}_*^{(r)}\}$ of points at which the residual functional vanishes. In this situation, any solution from the set $\mathbb{X}$ will be considered suitable.

An approximate solution of this equation is determined using an iterative Markov-type procedure. In this procedure, the approximate solution $\mathbf{x}^{(p+1)}$ at step $(p+1)$ is set equal to the value of the operator $\mathcal{B}[\mathbf{x}^{(p)}, \mathbf{a}^{(p)}]$ of the iterative procedure at step p, depending on the values of the components of the function $\mathbf{f}(\mathbf{x}^{(p)})$ and the parameter vector $\mathbf{a}^{(p)} \in \mathcal{A} \subset R^r$ that adjusts the qualitative properties of the iterative process:

$$\mathbf{x}^{(p+1)} = \mathcal{B}[\mathbf{f}(\mathbf{x}^{(p)}), \mathbf{a}^{(p)}]. \tag{3}$$

Qualitative properties often include *convergence*, *the rate of convergence*, and *accuracy*. The conditions for ensuring these properties are formulated in terms of the vector $\mathbf{a}$ and interval inequalities depending on the properties of the operator $\mathcal{B}$ and those of the function $\mathbf{f}(\mathbf{x})$.

However, the function $\mathbf{f}(\mathbf{x})$ may have an arbitrary structure, making it impossible to postulate or reveal its properties. As a result, these inequalities become analytically unverifiable.

To find a way out of this situation, we utilize the ideas of *reinforcement*, which are actively used in various implementations in modern machine learning procedures. In this case, reinforcement is intended to determine, at each step of the iterative procedure, a suitable parameter vector $\mathbf{a}$ via an appropriate self-learning procedure.

We propose a game-theoretic model to calculate the suitable parameters $\mathbf{a}$ of the iterative procedure (3). This model operates in the intervals between steps p and $(p+1)$ and simulates the behavior of an *agent*, i.e., a strategy for varying the parameters $\mathbf{a}$ depending on the response quality of an *environment*. The quality is characterized by a conditional *payoff*, a function that depends on the values of the residual functional and its decrement.

3. STRUCTURE OF THE REINFORCEMENT PROCEDURE

Consider the original problem (1) and find its solution by minimizing the residual functional

$$J(\mathbf{x}) \Rightarrow \min, \quad \mathbf{x} \in R^n. \tag{4}$$

In some applications, it may be useful to transform problem (1). We introduce the new variables

$$z_i = \frac{1}{1+\exp(-b_i x_i)}, \quad x_i = \frac{1}{b_i} \ln \frac{z_i}{1-z_i}, \quad i = \overline{1,n}.$$

Then problem (1) takes the form

$$J(\mathbf{z}) = \|\mathbf{\Psi}(\mathbf{z})\| \Rightarrow \min, \ \mathbf{z} \in Z_+^n = [\mathbf{0},\mathbf{1}], \quad \mathbf{\Psi}(\mathbf{z}) = \mathbf{f}(\mathbf{z}) - \mathbf{1}.$$

Problem (1) will be solved using the iterative procedure (3) under the assumption that the parameters $\mathbf{a}$ are of interval type: $\mathbf{a} \in [\mathbf{a}^-, \mathbf{a}^+]$.

To determine the values of the components of the vector $\mathbf{a}$ in the procedure (3), we employ the *reinforcement* technology, implementing it in the interval between steps p and $(p+1)$ of the iterative procedure (3).

This technology is based on a game-theoretic model simulating the game of an *agent* with an *environment*. The agent generates *strategies* (actions) that cause changes in the environment. The magnitude of these changes is characterized by a *utility function*. The value of a *payoff function* depends on the success of the agent's strategy and its utility for the environment.

In the interval between the successive steps of the iterative procedure, a statistical simulation is carried out via a given number M of MC experiments that simulate the agent's strategies, i.e., the values of the components of the vector $\mathbf{a}^{(p,k)}$, where $k = \overline{1,M}$.

As the agent's actions, we will consider the vector

$$\mathbf{x}^{(p,k+1)} = \mathcal{B}[\mathbf{f}(\mathbf{x}^{(p,k)}), \mathbf{a}^{(p,k)}], \quad p = \text{fix}, \ k = \overline{1,M}. \tag{5}$$

In this problem, the environment is the residual functional $J(\mathbf{x} \,|\, \mathbf{a})$. The MC-simulated actions of the agent yield a sequence of M residuals,

$$J(\mathbf{x}^{(p,1)} \,|\, \mathbf{a}^{(p,1)}), \ldots, J(\mathbf{x}^{(p,M)} \,|\, \mathbf{a}^{(p,M)}), \tag{6}$$

and their decrements,

$$u^{(p,k)}(\mathbf{a}^{(p,k)}) = J(\mathbf{x}^{(p,k+1)} \,|\, \mathbf{a}^{(p,k)}) - J(\mathbf{x}^{(p,k)} \,|\, \mathbf{a}^{(p,k-1)}), \quad k = \overline{1,M}. \tag{7}$$

We introduce a *utility function* to characterize the response quality of the environment measured by the decrement:

$$\varphi(u^{(p,k)}(\mathbf{a}^{(p,k)})) = \alpha \, \exp[u^{(p,k)}(\mathbf{a}^{(p,k)})]. \tag{8}$$

The quality of the agent's strategies is assessed in terms of a *payoff function* that characterizes the dependence of the agent's payoff on its strategy. Choosing an appropriate payoff function seems to be a creative task [2] involving some enumeration. Several general properties of this function can be declared. It is a continuous and bounded function of the following form:

$$Q(\mathbf{a}^{(p,k)}) = \begin{cases} l(u^{(p,k)}(\mathbf{a}^{(p,k)})) & \varphi(u^{(p,k)}(\mathbf{a}^{(p,k)})) \leqslant 1 \\ 0 & \varphi(u^{(p,k)}(\mathbf{a}^{(p,k)})) > 1, \end{cases} \tag{9}$$

with the function

$$l(u^{(p,k)}(\mathbf{a}^{(p,k)})) = \begin{cases} \alpha\,\varphi(u^{(p,k)}(\mathbf{a}^{(p,k)})), & 0 \geqslant u^{(p,k)}(\mathbf{a}^{(p,k)}) \geqslant -U, \\ 0, & u^{(p,k)}(\mathbf{a}^{(p,k)}) \geqslant 0, \end{cases} \tag{10}$$

where U is the limit of the decrement's magnitude.

MC experiments yield a value set of the payoff functions. Following the concept of reinforcement as applied to the iterative procedure (3), we determine the optimal value of the parameter $\mathbf{a}^{(p+1)}$ by the rule

$$\mathbf{a}^{(p+1)} = \mathbf{a}^{(p)} + \beta \arg \max_{1 \leqslant j \leqslant M} Q(\mathbf{a}^{(p,k_j)}). \tag{11}$$

If the agent chooses its strategy by the rule (11), in view of (9), we have

$$J(\mathbf{a}^{(p+1)}) < J(\mathbf{a}^{(p)}). \tag{12}$$

Thus, the following result has been established for the properties of the residual sequence in the iterative reinforcement procedure (8)–(11):

Theorem 1. *Assume that:*

a) The only a priori information available about the function $\mathbf{f}(\mathbf{x})$ *in* (1) *is the values of its components* $f_i(\mathbf{x}^{(k)})$, $i = \overline{1,n}$.

b) The parameters of the iterative procedure $\mathbf{a}$ *are chosen by the rule* (12), (11), (9).

Then the iterative procedure (5) *with reinforcement* (8)–(11) *generates a strictly monotonically decreasing sequence of the residual functionals* $J(\mathbf{x})$ (4).

This theorem is not a convergence theorem for the iterative procedure in the mathematical sense (convergence to one of the solutions). However, it is known that this solution corresponds to a zero residual value. The theorem states that the sequence of residuals is strictly monotonically decreasing. Since the calculation error is finite and can be specified, the final value of the parameters $\mathbf{a}$ obtained when reaching this error can be taken as a solution.

4. CONCLUSIONS

This paper has been devoted to the problem of solving a system of nonlinear equations with continuous functions on the left-hand sides. By assumption, the only a priori information available about these functions is their values. To find solutions under such conditions, an iterative procedure with parameters has been used: by tuning their values, it is possible to ensure the convergence of the procedure in some sense.

It has been proposed to employ the ideas of reinforcement, which are being rather actively developed in the theory and practice of machine learning. A self-learning procedure has been designed in which a given number of MC experiments are carried out at each iteration step to simulate the agent's strategy (the values of the parameters of the iterative procedure). In this procedure, the environment is the residual functional (5), and its response to the agent's actions is

the decrement (6). For an acceptable evolution of the iterative process, the decrement magnitude must decrease. The decrement has been characterized by an exponential utility function so that smaller decrement magnitudes correspond to larger values of the utility function. The agent's actions, i.e., the implemented parameters of the iterative procedure, have been assessed in terms of a payoff function whose morphology considers both the state of the environment and the degree of success of the agent's actions.

It has been proven that, due to this self-learning procedure, the iterative reinforcement algorithm generates a strictly monotonically decreasing sequence of residual functionals.

REFERENCES

1. Krasnosel'skii, M.A., Vainikko, G.M., Zabreiko, P.P., Rutitski, Ja.B., and Stecenko, V.Ja., *Approximated Solutions of Operator Equations*, Groningen: Walters-Noordhoff, 1972.
2. Bakhvalov, N.S., Zhidkov, N.P., and Kobel'kov, G.M., *Chislennye metody* (Numerical Methods), Moscow: Binom, 2003.
3. Polyak, B.T., *Introduction to Optimization*, Optimization Software, 1987.
4. Strekalovsky, A.S., *Elementy nevypukloi optimizatsii* (Elements of Nonconvex Optimization), Novosibirsk: Nauka, 2003.
5. Lyle, C., Rowland, M., Dabney, W., Kwiatkowska, M., and Gal, Y., Learning Dynamics and Generalization in Deep Reinforcement Learning, *Proceedings of the 39th International Conference on Machine Learning (PMLR)*, 2022, vol. 162, pp. 14560–14581.
6. Wang, C., Yaun, S., and Ross, K.W., On the Convergence of the Monte Carlo Exploring Starts Algorithm for Reinforcement Learning, *Proceedings of the International Conference on Learning Representations (ICLR)*, 2022.
7. Wasserman, Ph.D., *Neural Computing: Theory and Practice*, Coriolis Group, 1989.
8. Kohonen, T., *Self-organizing Maps*, Berlin–Heidelberg: Springer, 1995.
9. Mnih, V., Kavukcuoglu, K., Silver, D., Rusu, A.A., Veness, J., Bellemare, M.G., Graves, A., Riedmiller, M., and Fidjeland, A., Human-Level Control through Deep Reinforcement Learning, *Nature*, 2015, vol. 518, no. 7540, pp. 529–533.
10. Sutton, R.S. and Barto, A.G., *Introduction to Reinforcement Learning*, Cambridge: MIT Press, 1998.
11. Russel, S.J. and Norvig, P., *Artificial Intelligence: A Modern Approach*, 3rd ed., Upper Saddle River: Prentice Hall, 2010.
12. van Hasselt, H., Reinforcement Learning in Continuous State and Action Spaces, in *Reinforcement Learning: State-of-the-Art*, Wiering, M. and van Otterio, M., Eds., 2012, Springer, pp. 207–257.
13. Ivanov, S., Reinforcement Learning Textbook, *ArXiv*, 2022. `https://doi.org/10.48550/arXiv.2201.09746`.
14. Bozinovski, S., Crossbar Adaptive Array: The First Connectionist Network That Solved the Delayed Reinforcement Learning Problem, in *Artificial Neural Nets and Genetic Algorithms*, Proc. Int. Conf., Portoroz, Slovenia, Dobnikar, A., Steele, N.C., Pearson, D.W., and Albrecht, R.F., Eds., Springer, 1999, pp. 320–325.
15. Watkins, C. and Dayan, P., Q-learning, *Machine Learning*, 1992, vol. 8, no. 3–4, pp. 279–292.
16. van Hasselt, H., Guez, A., and Silver, D., Deep Reinforcement Learning with Double Q-learning, *Proc. AAAI Conf. Artificial Intelligence*, 2016, vol. 30, no. 1, pp. 2094–2100.
17. Bellman, R., *Dynamic Programming*, Princeton: Princeton University Press, 1957.
18. Robbins, H. and Monro, S., A Stochastic Approximation Method, *The Annals of Mathematical Statistics*, 1951, vol. 22, no. 3, pp. 400–407.

This paper was recommended for publication by P.S. Shcherbakov, a member of the Editorial Board