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TOPICAL ISSUE

Approximation-Based Approach to Adaptive Control of Linear Time-Varying Systems

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Abstract—An adaptive state-feedback control system is proposed for a class of linear time-varying systems represented in the controller canonical form. The adaptation problem is reduced to the one of Taylor series-based first approximations of the ideal controller parameters. The exponential convergence of identification and tracking errors of such an approximation to an arbitrarily small and adjustable neighbourhood of the equilibrium point is ensured if the condition of the regressor persistent excitation with a sufficiently small time period is satisfied. The obtained theoretical results are validated via numerical experiments.

Keywords: adaptive control, time-varying parameters, parametric error, persistent excitation, identification

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1. INTRODUCTION

Starting from the 1960s, the subject of adaptive control has been one of the central ones for Laboratory No. 7 of V.A. Trapeznikov Institute of Control Sciences of RAS. Its founder, academician Yakov Zalmanovich Tsypkin, made a significant contribution to research on adaptation and learning problems and proposed a unified approach to their solution based on stochastic approximation methods. Using it, in particular, the problems of identification and parameter estimation were successfully solved. Subsequently, Boris Theodorovich Polyak proposed optimal and robust pseudogradient adaptation algorithms and strictly analysed their convergence rate [1, 2]. These studies have largely become the foundations of the adaptive control theory, which, having started with linear systems with time-invariant parameters, is gradually being generalised to wider classes of plants. One such class will be discussed in this study.

One of the subjects of adaptive control theory is the problem of the time-invariant reference model tracking by a time-varying plant with zero steady-state error. Despite more than 65 years of efforts, this problem still lacks a universal practical solution, which motivates researchers all over the world to design new approaches and tools.

Conventional adaptive control algorithms are applicable to linear systems with quasi-time-invariant parameters. When they are applied to control linear time-varying systems, an uncompensated summand occurs in the derivative of the Lyapunov function, which is proportional to the rate of the unknown parameters change. As a result, instead of the convergence of the tracking error to zero, only its boundedness inside some ball with non-adjustable boundary is guaranteed. In [4], based on the speed-gradient method, these results are generalised to the problem of a time-varying reference model tracking by a nonlinear time-varying system. In [5, 6], various composite adaptive laws are proposed, which are claimed to reduce the steady-state error value in case the regressor persistent excitation condition is met. In [7], a *congelation of variables* method is proposed, which allows one to damp the above-mentioned uncompensated summand with the help of not always suitable for practice high-gain feedback, thus ensuring asymptotic convergence of the tracking error to zero. The alternative approach [8] also provides asymptotic stability, but uses a high gain in the adaptive law instead of the control one. Considering the method of majorizing functions [9, 10], the high gain is also used in the adaptive law, but, in contrast to [8], only dissipativity of the closed-loop system is guaranteed. In [11] an adaptive control system is proposed that provides exponential convergence of the tracking error to zero for systems represented in the controllable canonical form with time-varying parameters that are described by known exosystems with unknown initial conditions. In [12] it is proposed to reduce the problem of adaptive control of time-varying mechanical systems to the identification of the piecewise-constant parameters of the polynomial obtained by local expansion of the system time-varying parameters into a Taylor series of an arbitrary order. In [13, 14], based on the parametric identification methods, an approach to adaptive-optimal output feedback control of time-varying linear systems is developed under the assumption that the plant parameters are known time-varying functions of time.

The disadvantages of the described above and other known approaches to solve the time-varying system adaptive control problems can be classified as follows:

1) application of high-gain in the control or adaptive law (sliding-mode control, high values of the parameters, nonlinear damping signals, etc.) [7–10, 12];
2) the necessity to meet the parametric identifiability conditions [5, 6, 11, 13, 14];
3) the dimensionality of the identification/adaptation problem to be solved is enlarged by taking into account the coefficients of the physical laws of the system parameters change or approximation polynomials [11–14].

A more complete state-of-the-art understanding of the time-varying systems adaptive control problem can be obtained from the statement sections of the cited studies [4–15]. In this paper, a new approximation-based adaptive control method, which exploits the parameter identification theory, is proposed for time-varying systems.

The motivation is to investigate the applicability conditions of the recently proposed algorithm [16] that identifies time-varying parameters of a linear regression equation to solve the time-varying linear system control problem. According to [16], the problem of time-varying parameters identification is reduced to the one of estimation of their piecewise-constant approximation. As follows from the theoretical conclusions of [16], unlike many existing methods of time-varying parameters identification, the algorithm from [16] allows one to ensure convergence of the time-varying parameters identification error to a region, which can be arbitrarily reduced by decreasing the Taylor series expansion time interval in case the regressor is persistently exciting over a sufficiently small period of excitation T_s. In this study, the approach is proposed to be used to control a class of linear systems with time-varying parameters. To that end:

1) a non-adaptive control law is proposed for a time-varying system, which feedback and forward parameters are calculated only via the first (piecewise-constant) approximation of the system time-varying parameters;
2) in case the control law from 1) is applied, the convergence conditions of the tracking error to an arbitrarily small neighbourhood of zero are obtained;
3) based on the results from [16], the law to estimate the parameters of the controller from 1) is proposed, which allows one to ensure the convergence of the tracking error to an arbitrarily small neighbourhood of zero in case the regressor is persistently exciting with a sufficiently small period of excitation.

Considering the above-given literature review, the obtained approximation-based approach to design adaptive control systems for time-varying plants is close to [12]. However, unlike in [12], firstly, the time-varying parameters are approximated only by the first summand of Taylor series, which reduces the computational complexity and does not increase the dimensionality of the identification problem, and secondly, the step of the obtained estimates interpolation is not needed. In comparison with other existing solutions [4–14], the proposed algorithm of adaptive control of time-varying linear systems has the following advantages (+) and disadvantages (–):

(+) high gain and damping components are not used in both control and adaptive laws;
(+) the function of the time-varying parameters change is not required to be known;
(+) no *a priori* information about the system parameters is used;
(–) the repressor persistent excitation condition is required to be met to achieve even asymptotic convergence of the tracking error to a neighbourhood of zero;
(–) the value of the steady-state tracking error can be reduced only if the period T_s of the regressor persistent excitation is small enough;
(–) violation of the parametric identifiability condition (the regressor persistent excitation) may result in instability of the closed-loop system.

In general, although the proposed solution does not overcome all the shortcomings of the existing approaches, it expands the set of adaptive control methods for the time-varying systems, and therefore, in the authors' opinion, it is of interest.

Notation and Definitions

The following notation is adpoted: $f(t) \in \mathbb{R}^{n\times m}$ means a value of a function $f : \left[t_0^+,\ +\infty\right) \to \mathbb{R}^{n\times m}$ at the time point t, where $t_0^+ \geqslant 0$ is an initial time instant; for a vector $a \in \mathbb{R}^n$ the notation $||a||$ is the Euclidean norm; the minimum and maximum eigenvalues of a matrix $A \in \mathbb{R}^{n\times n}$ are denoted as $\lambda_{\min}(A)$ and $\lambda_{\max}(A)$, respectively. The abbreviation exp stands for the exponential stability.

The definitions of finite and persistent excitation are used to prove theorems and propositions.

Definition 1. A signal $\omega(t) \in \mathbb{R}^n$ is finitely exciting over a time range $[t_1,\ t_2] \subset \left[t_0^+, \infty\right)$ if there exists $\alpha > 0$ such that the following inequality holds:

$$\int\limits_{t_1}^{t_2} \omega(\tau)\,\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_n. \tag{1.1}$$

Definition 2. A signal $\omega(t) \in \mathbb{R}^n$ is persistently exciting if for all $t \geqslant t_0^+ \geqslant 0$ there exist $T_s > 0$ and $\alpha > 0$ such that the following inequality holds:

$$\int\limits_{t}^{t+T_s} \omega(\tau)\,\omega^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha I_n. \tag{1.2}$$

Set of signals, for which condition (1.1) or (1.2) is met, we denote as FE or PE, respectively. A signal $\omega(t)$ is persistently exciting if $\omega \in \mathrm{PE}$, and it is finitely exciting if $\omega \in \mathrm{FE}$.

The main result of the study utilises the Taylor formula with integral remainder. The conditions of existence of such equation are defined in the following lemma [17].

Lemma 1. *Let $(t_1,\ t_2)$ be an open time interval, and $f(t) \in \mathbb{R}$ be a p-times continuously differentiable function of time t, then for any pair of time instants t and a from $(t_1,\ t_2)$ it holds that*

$$f(t) = f(a) + \frac{t-a}{1!} f^{(1)}(a) + \ldots + \frac{(t-a)^p}{p!} f^{(p)}(a) + \int\limits_a^t \frac{(t-\zeta)^p}{p!} f^{(p+1)}(\zeta)\,d\zeta. \tag{1.3}$$

2. PROBLEM STATEMENT

We consider continuous linear systems with time-varying parameters

$$\begin{aligned}\dot{x}(t) = A(t)x(t) + B(t)u(t) = A_0 x(t) + e_n \left(a^{\mathrm{T}}(t)\, x(t) + b(t)\, u(t)\right) \\ = A_0 x(t) + e_n \Phi^{\mathrm{T}}(t)\, \Theta(t), \quad x\left(t_0^+\right) = x_0,\end{aligned} \tag{2.1}$$

where

$$A(t) = A_0 + e_n a^{\mathrm{T}}(t), \quad B(t) = e_n b(t),$$

$$A_0 = \begin{bmatrix} 0_{(n-1)\times 1} & I_{n-1} \\ \multicolumn{2}{c}{0_{1\times n}} \end{bmatrix}, \quad e_n = \begin{bmatrix} 0_{(n-1)\times 1} \\ 1 \end{bmatrix}, \quad \begin{aligned} \Phi^{\mathrm{T}}(t) = \left[x^{\mathrm{T}}(t) \quad u(t)\right], \\ \Theta(t) = \left[a^{\mathrm{T}}(t) \quad b(t)\right]^{\mathrm{T}}, \end{aligned}$$

$x(t) \in \mathbb{R}^n$ is a state vector with unknown initial conditions x_0, $u(t) \in \mathbb{R}$ stands for a control signal, $A(t) \in \mathbb{R}^{n\times n}$ denotes an unknown matrix of the system under consideration, $B(t) \in \mathbb{R}^n$, $\Theta(t) \in \mathbb{R}^{n+1}$ are unknown vectors, $A_0 \in \mathbb{R}^{n\times n}$ stands for a Frobenius matrix, $e_n \in \mathbb{R}^n$ is the nth Euclidean basis vector. The pair $(A(t), B(t))$ is completely controllable for all $t \geqslant t_0^+$. The controllability condition for the system (2.1) can be validated via application of, for example, a criterion given in [18].

A salient feature of the class of systems (2.1) is the fact that control and uncertainty signals are in the same equation. Such systems are called the ones with matched uncertainty, and they are widely met in practice. For example, the Euler angles dynamics of a rigid body, assuming its symmetry, is described by a second-order system with matched uncertainty. Another good example of a control problem with matched uncertainties is the control of a manipulator state using the Euler–Lagrange formalism.

The following assumption is adopted with respect to the unknown parameters $\Theta(t)$.

Assumption 1. *The parameters $\Theta(t)$ and their first and second derivatives are continuous and bounded*

$$\|\Theta(t)\| \leqslant \Theta_{\max}, \quad \left\|\dot{\Theta}(t)\right\| \leqslant \dot{\Theta}_{\max}, \quad \left\|\ddot{\Theta}(t)\right\| \leqslant \ddot{\Theta}_{\max},$$

where the upper bounds $\Theta_{\max}$, $\dot{\Theta}_{\max}$ and $\ddot{\Theta}_{\max}$ exist, but they are unknown.

The required control quality for the closed-loop system that includes the system (2.1) and the controller is defined with the help of the reference model with time-invariant parameters

$$\dot{x}_{ref}(t) = A_0 x_{ref}(t) + e_n \left(b_{ref} r(t) + a_{ref}^{\mathrm{T}} x_{ref}(t)\right), \; x_{ref}\left(t_0^+\right) = x_{0ref}, \tag{2.2}$$

where $x_{ref}(t) \in \mathbb{R}^n$ is a reference model state vector with known initial conditions x_{0ref}, $r(t) \in \mathbb{R}$ denotes a reference signal, $A_{ref} = A_0 + e_n a_{ref}^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ stands for a Hurwitz reference model state matrix, b_{ref} is a reference model high frequency gain.

We assume that the reference model (2.2) is chosen in such a way that the matching conditions are met, *i.e* the state vector of (2.1) can ideally track the one of (2.2).

Assumption 2. *There exist parameters $k_x(t) \in \mathbb{R}^{1\times n}$ and $k_r(t) \in \mathbb{R}$ such that the following equations hold*

$$a_{ref}^{\mathrm{T}} - a^{\mathrm{T}}(t) = b(t)\, k_x(t), \quad b_{ref} = b(t)\, k_r(t).$$

This assumption is necessary and sufficient condition for the existence of a control signal $u(t)$ that ensures for all $t \geqslant t_0^+$ that the equations of the system (2.1) coincide with those of the chosen reference model (2.2). The assumption is ensured to be satisfied by choosing a reference model in the form of (2.2), by consideration of a class of systems with a time-invariant sign of the high-frequency gain $b(t)$ and by a completely controllable pair $(A(t), B(t))$. It should be noted that Assumption 2 imposes the following constraint on the system (2.1): $\mathrm{sgn}(b(t)) = \mathrm{const}$,[1] and hence jointly Assumptions 1 and 2 require boundedness of $b_{\max} \geqslant |b(t)| \geqslant b_{\min} > 0$.

The aim is to design an adaptive control law $u(t)$, which, if $\Phi \in \mathrm{PE}$, ensures exponential convergence (exp) of the error $e_{ref}(t) = x(t) - x_{ref}(t)$ into the goal set

$$\lim_{t\to\infty} \|e_{ref}(t)\| \leqslant \Delta_{e_{ref}} \ (\exp), \tag{2.3}$$

in such a way that there exists some parameter of the adaptive control procedure, from which value the steady-state error $\Delta_{e_{ref}} > 0$ depends.

3. PRELIMINARY RESULTS AND TRANSFORMATIONS

An effective solution of the control problem of a linear system with unknown piecewise-constant parameters has been obtained recently in [19]. In this section the problem of adaptive control of a system (2.1) with time-varying parameters will be transformed into the one of control of a system with the piecewise-constant parameters. To this end, first of all, we show that the stated goal (2.3) is achievable with the help of the non-adaptive control law with known ideal parameters, which uses only piecewise-constant approximations of the time-varying parameters of the system (2.1) in its feedback and feedforward summands.

Taking into account Assumption 1, the error equation between the plant (2.1) and the reference model (2.2) is written as

$$\begin{aligned} \dot{e}_{ref}(t) &= A_{ref}e_{ref}(t) + e_n b(t)\left[u(t) - k_x(t)x(t) - k_r(t)r(t)\right] \\ &= A_{ref}e_{ref}(t) + e_n b(t)\left[u(t) - \mathcal{K}^{\mathrm{T}}(t)\omega(t)\right], \end{aligned} \tag{3.1}$$

where

$$e_{ref}(t) = x(t) - x_{ref}(t), \quad \omega(t) = \begin{bmatrix} x^{\mathrm{T}}(t) & r^{\mathrm{T}}(t) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{n+1},$$
$$\mathcal{K}(t) = \begin{bmatrix} k_x(t) & k_r(t) \end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{(n+1)\times 1}.$$

The disturbance $\mathcal{K}^{\mathrm{T}}(t)\omega(t)$ is going to be represented as a sum of two terms: with the piecewise-constant and time-varying parameters. To that end, a growing sequence is introduced

$$t_i^+ = T\left\lfloor \frac{t - t_0^+}{T} \right\rfloor, \quad i \in \mathbb{N},$$

where $t_{i+1}^+ - t_i^+ = T > 0$, $\lfloor . \rfloor : \mathbb{R} \to \mathbb{Z}$ is a function to round down to the closest integer.

As, owing to Assumptions 1 and 2, the parameters $\mathcal{K}(t)$ are differentiable, then, following the Taylor equation (1.3), it can be written for the neighbourhood T of the time instant t_i^+:

$$\mathcal{K}(t) = \mathcal{K}\left(t_i^+\right) + \underbrace{\int_{t_i^+}^{t} \dot{\mathcal{K}}(\zeta)\, d\zeta}_{\delta_{\mathcal{K}0}(t)}, \tag{3.2}$$

where $\mathcal{K}\left(t_i^+\right) = \mathcal{K}_i$ are values of the parameters $\mathcal{K}(t)$ at the time instant t_i^+, $\|\delta_{\mathcal{K}0}(t)\| \leqslant \dot{\mathcal{K}}_{\max}T$ is the reminder of the zeroth order ($p = 0$, see (1.3)).

[1] Otherwise there exists a time instant $t_a \geqslant t_0^+$ at which $b(t_a) = 0$, and equations from Assumption 2 have no solution in the general case ($b_{ref} \neq 0$, $a_{ref} - a(t_a) \neq 0_n$).

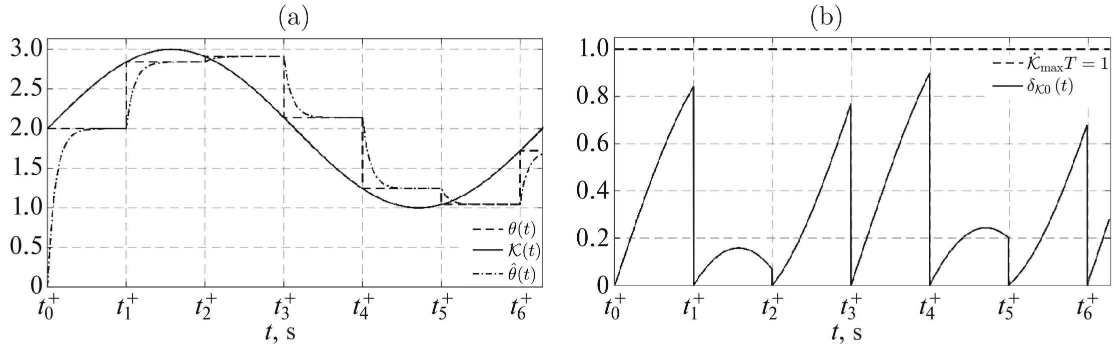

Fig. 1. Graphical illustration of relationship between $\mathcal{K}(t)$, $\theta(t)$ and $\hat{\theta}(t)$.

Owing to (3.2), for each time range $\left[t_i^+, t_i^+ + T\right)$ the time-varying parameters $\mathcal{K}(t)$ can be approximated by their value $\mathcal{K}_i$ at the beginning of such time range. Then the sequence of such values $\{\mathcal{K}_0, \mathcal{K}_1, \ldots, \mathcal{K}_i\}$ together with the sequence of the switching time instants $\{t_0^+, t_1^+, \ldots, t_i^+\}$ define the piecewise-constant signal, which is the first approximation of the time-varying parameters $\mathcal{K}(t)$ for all $t \geqslant t_0^+$:

$$\theta(t) = \mathcal{K}_i = \mathcal{K}_0 + \sum_{q=1}^{i} \Delta_q^\theta h\left(t - t_q^+\right), \tag{3.3}$$

where $\Delta_q^\theta = \mathcal{K}_q - \mathcal{K}_{q-1}$ is the amplitude of the parameters $\mathcal{K}(t)$ change over the time range $\left[t_i^+, t_{i+1}^+\right]$, $h : [t_0^+, \infty) \to \{0, 1\}$ stands for the Heaviside function.

For all $t \geqslant t_0^+$, equation (3.3) allows one to write the time-varying parameters as a sum $\mathcal{K}(t) = \theta(t) + \delta_{\mathcal{K}0}(t)$, which results in the required representation of the disturbance

$$\dot{e}_{ref}(t) = A_{ref}e_{ref}(t) + e_n b(t)\left[u(t) - \theta^{\mathrm{T}}(t)\omega(t) - \delta_{\mathcal{K}0}^{\mathrm{T}}(t)\omega(t)\right]. \tag{3.4}$$

Equation (3.4) motivates to introduce the following implementable continuous non-adaptive control law

$$u(t) = \hat{\theta}^{\mathrm{T}}(t)\omega(t), \tag{3.5a}$$

$$\dot{\hat{\theta}}(t) = -\gamma_1\left(\hat{\theta}(t) - \theta(t)\right) = -\gamma_1\tilde{\theta}(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0, \tag{3.5b}$$

where $\hat{\theta}(t)$ stands for the result of the parameters $\theta(t)$ filtration, and $\gamma_1 > 0$ denotes the filter parameter.

Considering a particular case $\mathcal{K}(t) = \sin(t) + 2$ and $T = 1$, the relationship between the parameters $\mathcal{K}(t)$, $\theta(t)$ and $\hat{\theta}(t)$ is explained in Figs. 1a and 1b. For the same example, Fig. 1b demonstrates the approximation error $\delta_{\mathcal{K}0}(t)$ and its upper bound $\dot{\mathcal{K}}_{\max}T = 1$.

The conditions, under which the stated goal is achieved by application of the law (3.5a) + (3.5b), are presented in the following proposition.

Proposition 1. *If the condition $i \leqslant i_{\max} < \infty$ is met, then there exists $T_{\min} > 0$ such that for all $0 < T < T_{\min}$ the control law* (3.5) *ensures that the stated goal* (2.3) *is achieved.*

Proof of proposition is postponed to Appendix.

According to Proposition 1, in order to solve the stated problem (2.3), it is sufficient to use piecewise-constant approximations $\theta(t)$ of the time-varying parameters of the disturbance $\mathcal{K}^{\mathrm{T}}(t)\omega(t)$ to calculate the parameters of the control law (3.5a). Thus the adaptive control problem for a class of systems with unknown time-varying parameters (2.1) is reduced to the one of identification of the unknown piecewise-constant parameters $\theta(t)$. To solve this problem, it is natural to be based jointly on approaches previously developed in [16, 19].

Remark 1. The condition $i \leqslant i_{\max} < \infty$ is required for formal proof of proposition 1 and is not restrictive for practical scenarios.

4. MAIN RESULT

Following the method of exponentially stable adaptive control of systems with piecewise-constant parameters [19], for indirect implementation of (3.5), we first obtain a regression equation relating the parameters $\theta(t)$ to the signals calculated on the basis of the measurable vector $\Phi(t)$. The result of such a parameterisation can be formulated as a proposition.

Proposition 2. *Using the state of the stable filter ($l > 0$) with resetting at some time instants t_i^+*

$$\begin{gathered}\dot{\overline{\Phi}}(t) = -l\overline{\Phi}(t) + \Lambda^{\mathrm{T}}\left(t, t_i^+\right)\Phi(t), \quad \overline{\Phi}\left(t_i^+\right) = 0_{2(n+1)},\\ \Lambda\left(t, t_i^+\right) = \begin{bmatrix} I_{n+1} & \left(t - t_i^+\right) I_{n+1}\end{bmatrix} \in \mathbb{R}^{(n+1)\times 2(n+1)},\end{gathered} \tag{4.1}$$

normalization procedure

$$\begin{gathered}\overline{z}_n(t) = n_s(t)\, e_n^{\mathrm{T}}\left[x(t) - l\overline{x}(t) - A_0\overline{x}(t)\right],\\ \overline{\varphi}_n^{\mathrm{T}}(t) = n_s(t)\,\overline{\varphi}(t) = n_s(t)\begin{bmatrix}\overline{\Phi}^{\mathrm{T}}(t) & e^{-l\left(t - t_i^+\right)}\end{bmatrix},\\ n_s(t) = \frac{1}{1 + \overline{\varphi}^{\mathrm{T}}(t)\,\overline{\varphi}(t)}, \quad \overline{x}(t) = \begin{bmatrix} I_{n\times n} & 0_{n\times(n+2)}\end{bmatrix}\overline{\Phi}(t),\end{gathered} \tag{4.2}$$

extension ($\sigma > 0$)

$$\dot{z}(t) = e^{-\sigma\left(t - t_i^+\right)}\overline{\varphi}_n(t)\,\overline{z}_n^{\mathrm{T}}(t), \quad z\left(t_i^+\right) = 0_{2n+3}, \tag{4.3a}$$

$$\dot{\varphi}(t) = e^{-\sigma\left(t - t_i^+\right)}\overline{\varphi}_n(t)\,\overline{\varphi}_n^{\mathrm{T}}(t), \quad \varphi\left(t_i^+\right) = 0_{(2n+3)\times(2n+3)}, \tag{4.3b}$$

mixing

$$Y(t) := \operatorname{adj}\{\varphi(t)\}\, z(t), \quad \Delta(t) := \det\{\varphi(t)\}, \tag{4.4}$$

elimination

$$\begin{gathered} z_a(t) = Y^{\mathrm{T}}(t)\,\mathfrak{L}_a, \quad z_b(t) = Y^{\mathrm{T}}(t)\,\mathfrak{L}_b,\\ \mathfrak{L}_a = \begin{bmatrix} I_{n\times n} & 0_{n\times(n+3)}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{(2n+3)\times n}, \quad \mathfrak{L}_b = \begin{bmatrix} 0_{1\times n} & 1 & 0_{1\times(n+2)}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{(2n+3)\times 1},\end{gathered} \tag{4.5}$$

substitution

$$\mathcal{Y}(t) := \begin{bmatrix}\Delta(t)\, a_{ref}^{\mathrm{T}} - z_a(t) & \Delta(t)\, b_{ref}\end{bmatrix}^{\mathrm{T}}, \quad \mathcal{M}(t) := z_b(t), \tag{4.6}$$

and smoothing ($k > 0$)

$$\dot{\Upsilon}(t) = -k\left(\Upsilon(t) - \mathcal{Y}(t)\right), \quad \Upsilon\left(t_0^+\right) = 0_{n+1}, \tag{4.7a}$$

$$\dot{\Omega}(t) = -k\left(\Omega(t) - \mathcal{M}(t)\right), \quad \Omega\left(t_0^+\right) = 0, \tag{4.7b}$$

we have a perturbed regression equation

$$\Upsilon(t) = \Omega(t)\,\theta(t) + w(t), \tag{4.8}$$

where the signals $\Upsilon(t), \Omega(t)$ are calculated via $\Phi(t)$ and additionally:

a) *if* $\Phi \in \mathrm{PE} \Rightarrow \overline{\varphi}_n \in \mathrm{PE}$ *with the period* $T_s < T$*, then there exists* $T_{\min} > 0$ *such that for all* $0 < T < T_{\min}$ *and* $t \geqslant t_0^+ + T_s$ *it holds that*

$$0 < \Omega_{\mathrm{LB}} \leqslant \Omega(t) \leqslant \Omega_{\mathrm{UB}}.$$

b) *if* $i \leqslant i_{\max} < \infty$*, then for all* $t \geqslant t_0^+ + T_s$ *it holds that*

$$\|w(t)\| \leqslant w_{1\max} e^{-\gamma_1\left(t - t_0^+ - T_s\right)} + w_{2\max}(T),$$
$$\lim_{T\to 0} w_{2\max}(T) = 0.$$

Proof of proposition and definition of $w(t)$ *are given in Appendix.*

The parameterisation (4.1)–(4.8) uses the procedures proposed to solve the problem of adaptive control of systems with piecewise-constant parameters [19]. The difference is that the time-varying matrix $\Lambda\left(t, t_i^+\right)$ is used in (4.1) and the states of the filters (4.1) and (4.3) are reset at known, rather than algorithmically detectable, time instants.

Here we briefly explain the purpose of the procedures in use. Having the measurable signals $\Phi(t)$ at hand, the application of the filter (4.1) allows one to obtain a regression equation with measurable regressor and regressand with respect to parameters $\overline{\vartheta}(t) = \begin{bmatrix} \Theta_i{}^{\mathrm{T}} & \dot{\Theta}_i^{\mathrm{T}} & e_n^{\mathrm{T}} x\left(t_i^+\right) \end{bmatrix}^{\mathrm{T}}$, where $\Theta\left(t_i^+\right) = \Theta_i$, $\dot{\Theta}\left(t_i^+\right) = \dot{\Theta}_i$ are the values of the system parameters $\Theta(t)$ and the rate of their change at the time instant t_i^+. The normalisation (4.2) ensures that all signals used in further procedures belong to L_∞ space. The extension and mixing procedures (4.3), (4.4) allow one to transform the vector regressor $\overline{\varphi}_n(t) \in \mathbb{R}^{2n+3}$ into a scalar one $\Delta(t) \in \mathbb{R}$. Owing to $\Delta(t) \in \mathbb{R}$, the elimination (4.5) separates the regression equation under consideration into two ones with respect to the parameters of the piecewise-constant approximation of $a(t)$ and $b(t)$. By substitution (4.6) of (4.5) into the matching conditions (see assumption 2), we transform the equations with respect to approximation of the system parameters into the ones with respect to approximation $\theta(t)$ of the perturbation parameters. Smoothing (4.7a), (4.7b) allows one to ensure sufficient smoothness of the signals $\Upsilon(t)$ and $\Omega(t)$.

Having at hand the regression equation (4.8) that regressor and regressand are based only on measurarble signals $\Phi(t)$, we can indirectly implement the law (3.5) and guarantee the achievement of the goal (2.3).

Theorem 1. *Let* $\Phi \in \mathrm{PE} \Rightarrow \overline{\varphi}_n \in \mathrm{PE}$ *with the period* $T_s < T$*, Assumptions 1–2 be met, then there exists* $T_{\min} > 0$ *such that for all* $0 < T < T_{\min}$ *the control law* (3.5a) *with the adaptive law*

$$\begin{gathered}\dot{\hat{\theta}}(t) = -\gamma(t)\Omega(t)\left(\Omega(t)\hat{\theta}(t) - \Upsilon(t)\right)\\ = -\gamma(t)\Omega^2(t)\tilde{\theta}(t) + \gamma(t)\Omega(t)w(t), \quad \hat{\theta}\left(t_0^+\right) = \hat{\theta}_0,\\ \gamma(t) = \begin{cases} 0, & \text{if } \Omega(t) < \rho \in (0, \Omega_{\mathrm{LB}}], \\ \dfrac{\gamma_1}{\Omega^2(t)} & \text{otherwise,}\end{cases}\end{gathered} \tag{4.9}$$

in case $i \leqslant i_{\max} < \infty$ *for* $\xi(t) = \begin{bmatrix} e_{ref}^{\mathrm{T}}(t) & vec^{\mathrm{T}}\left(\tilde{\theta}(t)\right)\end{bmatrix}^{\mathrm{T}}$*, ensures that:*

1) $\forall t \geqslant t_0^+ \quad \xi(t) \in L_\infty$,
2) $\lim\limits_{t\to\infty} \|\xi(t)\| \leqslant \Delta_\xi(T)$ (exp), $\lim\limits_{T\to 0} \Delta_\xi(T) = 0.$

Proof of theorem is presented in Appendix.

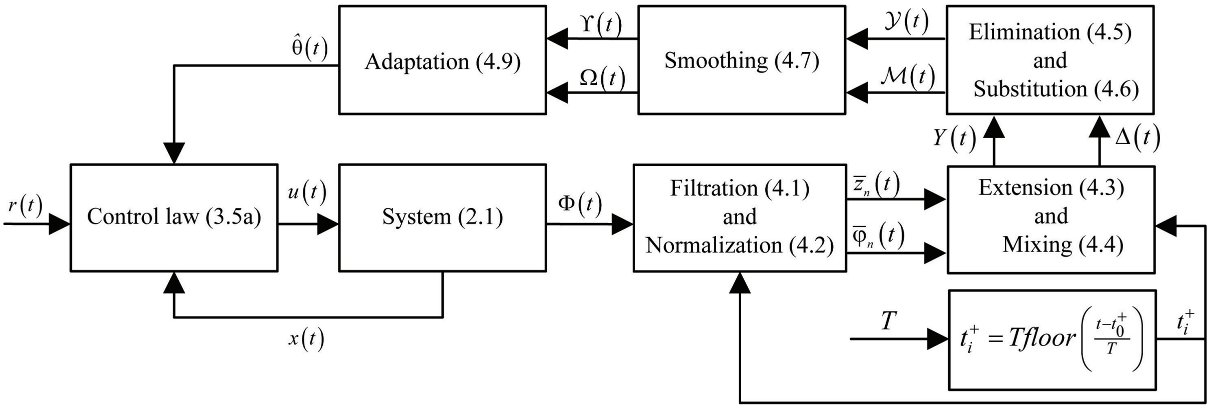

Fig. 2. Block diagram of proposed adaptive control system.

The block diagram of the obtained algorithm for adaptive control of systems with time-varying unknown parameters (2.1) is presented in Fig. 2.

Thus, the developed control system consists of a control law (3.5a), an adaptive law (4.9), a set of procedures (4.1)–(4.7) to process the measurable signals. In contrast to existing adaptive control methods [4–14], the proposed approach does not require any *a priori* information about the system parameters $a(t)$ and $b(t)$, does not use high-gain in control or adaptive laws, guarantees global exponential convergence of the error $\xi(t)$ to the bounded neighbourhood of the equilibrium, which can be adjusted by the parameter T.

Remark 2. The feature of the proposed solution is the relationship between the steady-state error $\Delta_\xi(T)$, the length of the Taylor series expansion interval T and the period of the regressor persistent excitation T_s. The problem is that the parameter T cannot be made smaller than the value of the regressor excitation period T_s. However, for a fixed period T_s and a minimum possible $T < T_{\min}$ such that $T - T_s > 0$, the error $\xi(t)$ may be bounded in an unacceptably large neighbourhood of the equilibrium point $\Delta_\xi(T)$. Therefore, in order to reduce the steady-state error, it is necessary, first of all, to ensure a persistent excitation of the regressor with a sufficiently small period T_s, which in practice can be achieved by addition of a high-frequency or random test signal to the reference $r(t)$.

5. NUMERICAL EXPERIMENTS

In Matlab/Simulink numerical experiments have been conducted for the proposed adaptive system using the explicit Euler solver with a constant step time of $\tau_s = 10^{-3}$ s.

The system (2.1) was considered with $n = 2$. The initial conditions, the parameters of the system and reference model (2.2) were chosen as

$$\begin{gathered} x_0 = \begin{bmatrix} -1 & 1 \end{bmatrix}^{\mathrm{T}}, \quad b(t) = 3 + \cos(0.4t)\sin(0.1t), \quad a_{ref}^{\mathrm{T}}(t) = \begin{bmatrix} -8 & -4 \end{bmatrix}, \\ a^{\mathrm{T}}(t) = \begin{bmatrix} 2 + \sin(0.1t) & 1 + 5\left(1 - e^{\frac{-1}{25}t}\right) \end{bmatrix}, \quad b_{ref} = 8. \end{gathered} \tag{5.1}$$

First, we verified the preliminary conclusions made in Proposition 1. We picked $\gamma_1 = 50$ as the filter constant (3.5b), and defined the reference as $r(t) = 10$. Figure 3 presents the comparison of the error $e_{1ref}(t)$ for different T.

The obtained results validated the conclusions made in Proposition 1. Indeed, a decrease of T resulted in a decrease of the steady-state value of the tracking error $e_{ref}(t)$ when the control law (3.5a) with (3.5b) was applied. Having checked proposition 1, we proceeded to verify the main result.

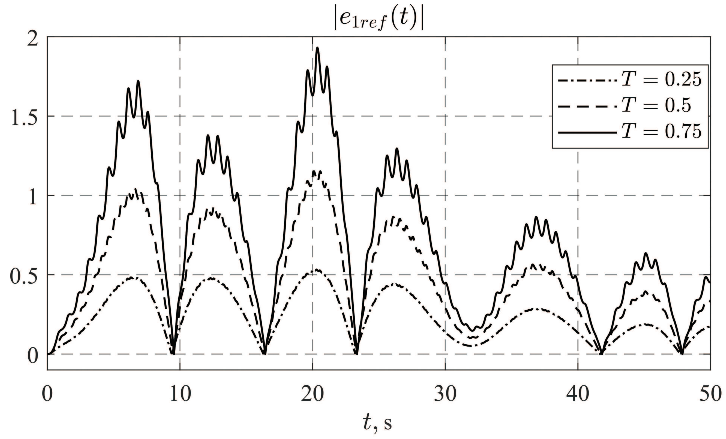

Fig. 3. Behavior of $|e_{1ref}(t)|$ for different T.

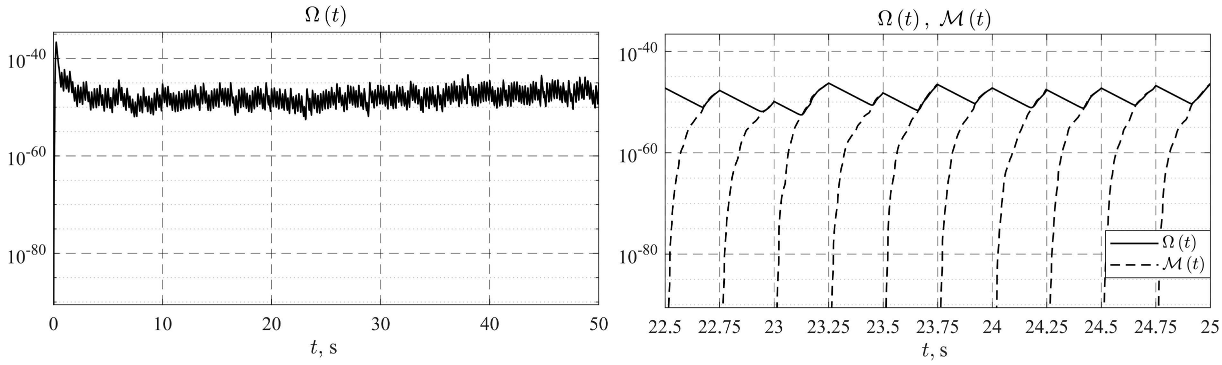

Fig. 4. Behavior of regressors $\mathcal{M}(t)$ and $\Omega(t)$.

The parameters of the filters (4.1), (4.3), (4.7) and the adaptive law (4.9) were chosen as

$$l = 10, \quad \sigma = \frac{0.05}{T}, \quad k = 50, \quad \rho = 10^{-72}, \quad \gamma_1 = 100, \quad T = 0.25,$$

the reference was picked as $r(t) = 1 + r_d(t)$ for $r_d(t) \sim \mathcal{N}(0,\ 10^{-2})$. A random signal $r_d(t)$ was added to a unity reference signal to ensure that $\overline{\varphi}_n \in \mathrm{PE}$ for the closed-loop system (3.1).

Figure 4 depicts the behavior of the regressors $\mathcal{M}(t)$ and $\Omega(t)$ on the logarithmic scale.

It follows from the obtained results that despite the fact that the filters (4.1) and (4.3) were reset every T seconds, the regressor $\Omega(t)$ (unlike $\mathcal{M}(t)$) was globally bounded away from zero starting from some time instant, which confirms the theoretical conclusions made in statement (a) of Proposition 2. Figure 4 demonstrates the importance of the smoothing procedure (4.7), which, as can be seen, allows one to (i) average the values of the regressor $\mathcal{M}(t)$ over the period T, and (ii) avoid discontinuities caused by the reinitialisation of the filters (4.1) and (4.3).

Figure 5 shows the behavior of (a) the state $x(t)$ when the control law (3.5a) with (3.5b) and with (4.9) is used, (b) the estimates of $\hat{\theta}_{\mathrm{i}}(t)$ and the true parameters $\theta_i(t) + 1$ shifted by one for clarity of illustration, (c) the control signal (3.5a) with (4.9).

Figure 6 compares the values of the integral control quality index of tracking $e_{ref}(t)$ and parametric $\tilde{\theta}(t)$ errors for different values of T.

The simulation results illustrate the conclusions of Propositions 1, 2 and theorem. The goal (2.3) is achieved, and the steady-state values of the errors $e_{ref}(t)$ and $\tilde{\theta}(t)$ are directly proportional to the parameter T.

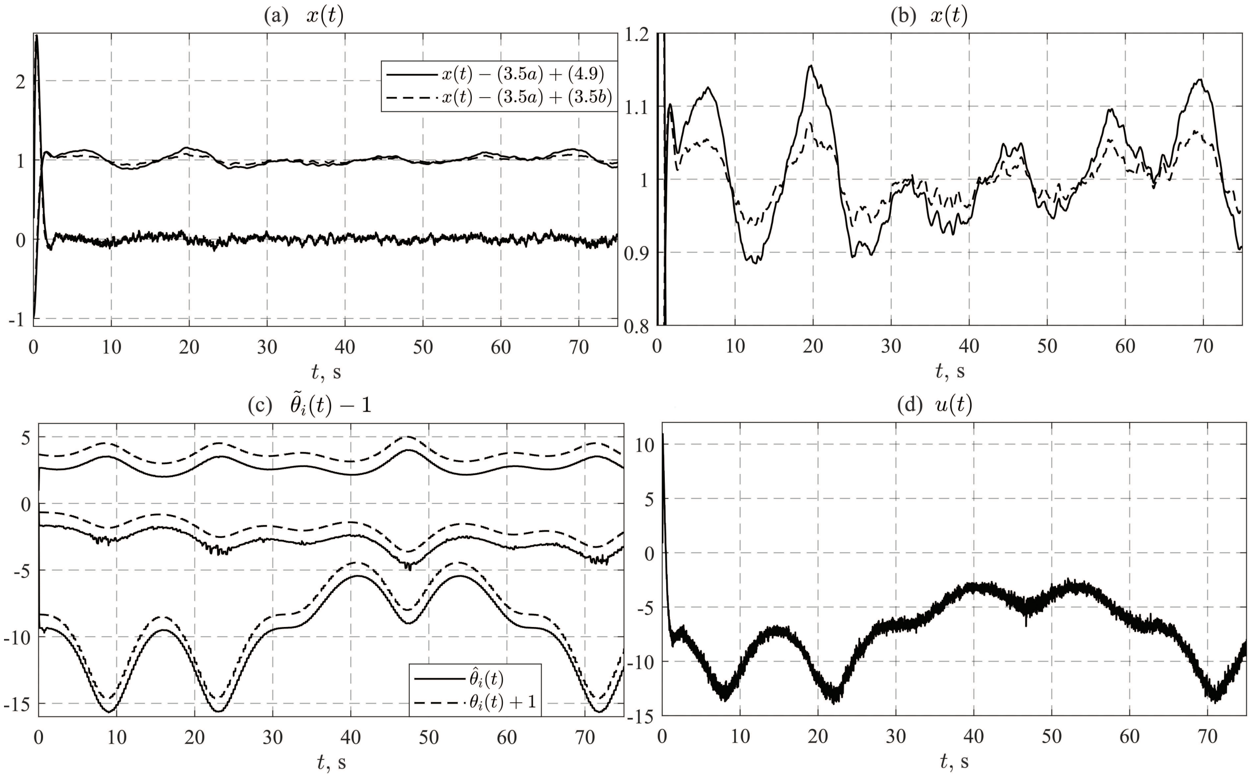

Fig. 5. Behavior of (a)–(b) state $x(t)$ when control law (3.5a) used with (3.5b) and with (4.9), (c) estimates $\hat{\theta}_i(t)$ and ideal parameters $\theta_i(t) + 1$ shifted by one for clarity of illustration, (d) control signal (3.5a) with (4.9).

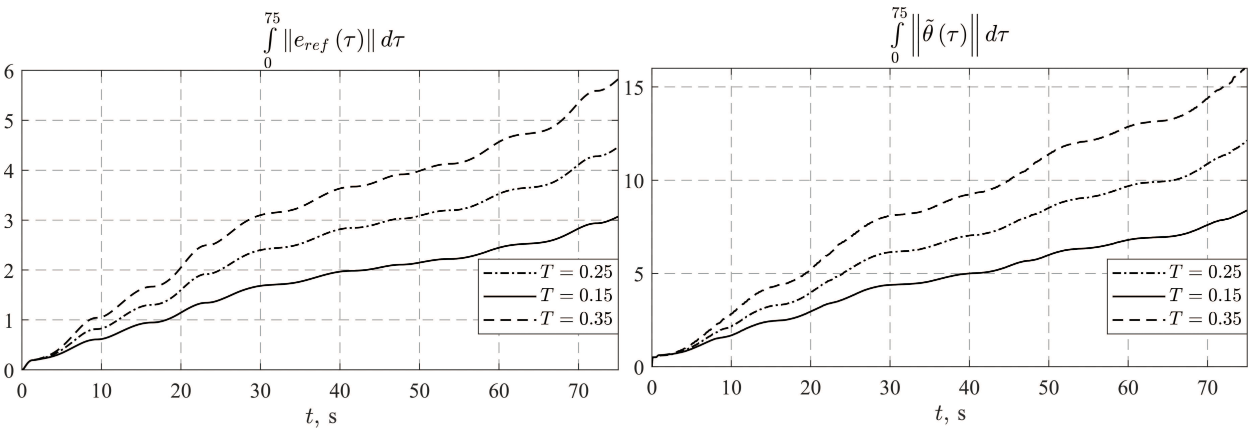

Fig. 6. Comparison of integral control quality indexes.

6. CONCLUSION

The problem of tracking of a linear time-invariant reference model by a linear time-varying system is solved. It is proposed to approximate the unknown time-varying parameters of the ideal control law by piecewise-constant parameters. Parametric identification methods proposed in [16, 19] are combined to identify these piecewise-constant parameters. The resulting adaptive control system requires persistent excitation of the regressor with a sufficiently small period to achieve the control goal, but it does not require *a priori* information about the unknown parameters of the system.

FUNDING

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APPENDIX

Proof of Proposition 1. The proof of the proposition is divided into two steps. At the first one we analyse the properties of the parametric error $\tilde{\theta}(t)$, at the second one — the properties of the tracking error $e_{ref}(t)$.

Step 1. Owing to proposition 1 from [19], if $i \leqslant i_{\max} < \infty$, then for the differential equation

$$\dot{\tilde{\theta}}(t) = -\gamma_1 \tilde{\theta}(t) - \dot{\theta}(t), \quad \tilde{\theta}\left(t_0^+\right) = \hat{\theta}_0 - \theta\left(t_0^+\right),$$

the following upper bound holds

$$\left\|\tilde{\theta}(t)\right\| \leqslant \beta_{\max} e^{-\gamma_1\left(t-t_0^+\right)}, \quad \beta_{\max} > 0, \tag{A.1}$$

where $\dot{\theta}(t) = \sum\limits_{q=1}^{i} \Delta_q^\theta \delta\left(t - t_q^+\right)$, and $\delta : [t_0^+; \infty) \to \{0, \infty\}$ is the Dirac function.

Step 2. The following quadratic form is introduced:

$$\begin{gathered} V_{e_{ref}} = e_{ref}^{\mathrm{T}} P e_{ref} + \frac{a_0^2}{\gamma_1} e^{-2\gamma_1\left(t-t_0^+\right)}, \quad H = \operatorname{blockdiag}\left\{P, \ \frac{a_0^2}{\gamma_1}\right\}, \\ \underbrace{\lambda_{\min}(H)}_{\lambda_{\mathrm{m}}} \|\overline{e}_{ref}\|^2 \leqslant V\left(\|\overline{e}_{ref}\|\right) \leqslant \underbrace{\lambda_{\max}(H)}_{\lambda_M} \|\overline{e}_{ref}\|^2, \end{gathered} \tag{A.2}$$

where $\overline{e}_{ref}(t) = \left[e_{ref}^{\mathrm{T}}(t) \ \ e^{-\gamma_1\left(t-t_0^+\right)}\right]^{\mathrm{T}}$, $P = P^{\mathrm{T}} > 0$ is the solution of the below-given Lyapunov equation in case $\lambda_{\min}(Q) > 2$:

$$A_{ref}^{\mathrm{T}} P + P A_{ref} = -Q, \quad Q = Q^{\mathrm{T}} > 0.$$

The derivative of the quadratic form (A.2) is written as:

$$\begin{gathered} \dot{V}_{e_{ref}} = e_{ref}^{\mathrm{T}}\left(A_{ref}^{\mathrm{T}} P + P A_{ref}\right) e_{ref} - 2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} + 2e_{ref}^{\mathrm{T}} P e_n b \tilde{\theta}^{\mathrm{T}} \omega + 2 e_{ref}^{\mathrm{T}} P e_n b \delta_{\theta_0}^{\mathrm{T}} \omega \\ = -e_{ref}^{\mathrm{T}} Q e_{ref} - 2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} + 2e_{ref}^{\mathrm{T}} P e_n b \tilde{\theta}^{\mathrm{T}}\left(\omega_{e_{ref}} + \omega_r\right) + 2e_{ref}^{\mathrm{T}} P e_n b \delta_{\theta_0}^{\mathrm{T}}\left(\omega_{e_{ref}} + \omega_r\right) \\ \leqslant -\lambda_{\min}(Q)\|e_{ref}\|^2 - 2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} \\ + 2\lambda_{\max}(P) b_{\max} \|e_{ref}\|^2 \left\|\tilde{\theta}\right\| + 2\lambda_{\max}(P) \overline{\omega}_r b_{\max}\|e_{ref}\| \left\|\tilde{\theta}\right\| \\ + 2\lambda_{\max}(P) b_{\max} \dot{\mathcal{K}}_{\max} T \|e_{ref}\|^2 + 2\lambda_{\max}(P) b_{\max} \overline{\omega}_r \dot{\mathcal{K}}_{\max} T \|e_{ref}\|, \end{gathered} \tag{A.3}$$

where

$$\|\omega(t)\| \leqslant \underbrace{\left\|\left[e_{ref}(t) \ \ 0\right]\right\|}_{\left\|\omega_{e_{ref}}(t)\right\| = \left\|e_{ref}(t)\right\|} + \underbrace{\left\|\left[x_{ref}(t) \ \ r(t)\right]\right\|}_{\|\omega_r(t)\| \leqslant \overline{\omega}_r} \leqslant \|e_{ref}(t)\| + \overline{\omega}_r.$$

Having applied Young's inequality twice:

$$\begin{gathered} 2\lambda_{\max}(P) \overline{\omega}_r b_{\max}\|e_{ref}\| \left\|\tilde{\theta}\right\| \leqslant \|e_{ref}\|^2 + \lambda_{\max}^2(P) \overline{\omega}_r^2 b_{\max}^2 \left\|\tilde{\theta}\right\|^2, \\ 2\lambda_{\max}(P) b_{\max} \overline{\omega}_r \dot{\mathcal{K}}_{\max} T \|e_{ref}\| \leqslant \lambda_{\max}^2(P) b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2 + \|e_{ref}\|^2, \end{gathered} \tag{A.4}$$

equation (A.3) is rewritten as:

$$\begin{gathered}\dot{V}_{e_{ref}} \leqslant \left[-\lambda_{\min}(Q) + 2\lambda_{\max}(P)\,b_{\max}\left(\left\|\tilde{\theta}\right\| + \dot{\mathcal{K}}_{\max}T\right) + 2\right]\left\|e_{ref}\right\|^2 \\ -2a_0^2 e^{-2\gamma_1\left(t-t_0^+\right)} + \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2\left\|\tilde{\theta}\right\|^2 + \lambda_{\max}^2(P)\,b_{\max}^2\overline{\omega}_r^2\dot{\mathcal{K}}_{\max}^2T^2.\end{gathered} \tag{A.5}$$

As the parametric error $\tilde{\theta}(t)$ converges to zero exponentially (A.2), then, if $\lambda_{\min}(Q) > 2$, then there definitely exists a time instant $t_{e_{ref}} \geqslant t_0^+$ and constants $T_{\min} > 0,\ a_0 > \lambda_{\max}(P)\,\overline{\omega}_r b_{\max}\beta_{\max}$ such that for all $t \geqslant t_{e_{ref}}$ and $0 < T < T_{\min}$ it holds that

$$\begin{gathered}-\lambda_{\min}(Q) + 2\lambda_{\max}(P)\,b_{\max}\left(\beta_{\max}e^{-\gamma_1\left(t_{e_{ref}}-t_0^+\right)} + \dot{\mathcal{K}}_{\max}T\right) + 2 = -c_1 < 0, \\ \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2\beta_{\max}^2 - 2a_0^2 = -c_2 < 0.\end{gathered} \tag{A.6}$$

Then the upper bound of the derivative (A.5) for all $t \geqslant t_{e_{ref}}$ is written as

$$\dot{V}_{e_{ref}} \leqslant -\eta_{\overline{e}_{ref}}V_{e_{ref}} + \lambda_{\max}^2(P)\,b_{\max}^2\overline{\omega}_r^2\dot{\mathcal{K}}_{\max}^2T^2, \tag{A.7}$$

where $\eta_{\overline{e}_{ref}} = \min\left\{\frac{c_1}{\lambda_{\max}(P)},\ \frac{c_2\gamma_1}{a_0^2}\right\}$.

The solution of the differential inequality (A.7) for all $t \geqslant t_{e_{ref}}$ is obtained as

$$V_{e_{ref}}(t) \leqslant e^{-\eta_{\overline{e}_{ref}}\left(t-t_{e_{ref}}\right)}V_{e_{ref}}\left(t_{e_{ref}}\right) + \frac{\lambda_{\max}^2(P)\,b_{\max}^2\overline{\omega}_r^2\dot{\mathcal{K}}_{\max}^2T^2}{\eta_{\overline{e}_{ref}}}. \tag{A.8}$$

Tending time to infinity for (A.8) and considering expression for $V_{e_{ref}}$, it is concluded that (2.3) holds, which completes the proof.

Proof of Proposition 2. Owing to assumption 2 and following (3.2)–(3.3), we apply the Taylor formula (1.3) to the parameters $\Theta(t)$ to obtain:

$$\Theta(t) = \Theta\left(t_i^+\right) + \overbrace{\dot{\Theta}\left(t_i^+\right)\left(t-t_i^+\right) + \underbrace{\int_{t_i}^{t}(t-\zeta)\ddot{\Theta}(\zeta)\,d\zeta}_{\delta_1(t)}}^{\delta_0(t)}, \tag{A.9}$$

where $\Theta\left(t_i^+\right) = \Theta_i,\ \dot{\Theta}\left(t_i^+\right) = \dot{\Theta}_i$ are the values of the system parameters $\Theta(t)$ and the rate of their change at the time instant t_i^+, $\|\delta_1(t)\| \leqslant 0.5\ddot{\Theta}_{\max}T^2$ denotes the bounded reminder of the first order $(p = 1)$, $\|\delta_0(t)\| \leqslant \dot{\Theta}_{\max}T$ is the bounded reminder of the zeroth order $(p = 0)$.

Equation (A.9) is rewritten in the matrix form

$$\Theta(t) = \Lambda\left(t,\ t_i^+\right)\vartheta(t) + \delta_1(t), \tag{A.10}$$

where $\vartheta(t) = \begin{bmatrix}\Theta_i{}^{\mathrm{T}} & \dot{\Theta}_i^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}} \in \mathbb{R}^{2(n+1)}$.

The substitution of (A.10) into (2.1) yields

$$\dot{x}(t) = A_0x + e_n\left(\Phi^{\mathrm{T}}(t)\,\Lambda\left(t, t_i^+\right)\vartheta(t) + \Phi^{\mathrm{T}}(t)\,\delta_1(t)\right). \tag{A.11}$$

The expression $x(t) - l\overline{x}(t)$ is differentiated to obtain

$$\dot{x}(t) - l\dot{\overline{x}}(t) = -l\left(x(t) - l\overline{x}(t)\right) + A_0x + e_n\left(\Phi^{\mathrm{T}}(t)\,\Lambda\left(t, t_i^+\right)\vartheta(t) + \Phi^{\mathrm{T}}(t)\,\delta_1(t)\right). \tag{A.12}$$

The solution of (A.12) is written as

$$\begin{gathered} x(t) - l\overline{x}(t) = e^{-l\left(t-t_i^+\right)}x(t_i) + A_0\overline{x}(t) + \int\limits_{t_i^+}^{t} e^{-l(t-\tau)}e_n\Phi^{\mathrm{T}}(\tau)\Lambda\left(\tau, t_i^+\right)\vartheta(\tau)\,d\tau \\ + \int\limits_{t_i^+}^{t} e^{-l(t-\tau)}e_n\Phi^{\mathrm{T}}(\tau)\delta_1(\tau)\,d\tau = A_0\overline{x}(t) + e_n\overline{\varphi}(t)\overline{\vartheta}(t) + e_n\underbrace{\int\limits_{t_i^+}^{t} e^{-l(t-\tau)}\Phi^{\mathrm{T}}(\tau)\delta_1(\tau)\,d\tau}_{\varepsilon_0(t)}, \end{gathered} \tag{A.13}$$

where $\overline{\vartheta}(t) = \left[\vartheta^{\mathrm{T}}(t) \;\; e_n^{\mathrm{T}}x\left(t_i^+\right)\right]^{\mathrm{T}} \in \mathbb{R}^{2n+3}$, and the third equality is not violated since the reset of the filter states (4.1) and the change of parameters occur synchronously at a known time instant t_i^+, i.e. $\overline{\vartheta}(t) = \text{const}$ for all $t \in \left[t_i^+, t_i^+ + T\right)$.

Equation (A.13) is substituted into (4.2) to obtain

$$\overline{z}_n(t) = n_s(t)\,e_n^{\mathrm{T}}\left[x(t) - l\overline{x}(t) - A_0\overline{x}(t)\right] = \overline{\varphi}_n^{\mathrm{T}}(t)\,\overline{\vartheta}(t) + \overline{\varepsilon}_0(t), \tag{A.14}$$

where $\overline{z}_n(t) \in \mathbb{R}$, $\overline{\varphi}_n(t) \in \mathbb{R}^{2n+3}$ and the perturbation $\overline{\varepsilon}_0(t) \in \mathbb{R}$ is bounded as follows (see definitions of $\Phi(t)$ and $\overline{\varphi}_n(t)$):

$$\|\overline{\varepsilon}_0(t)\| = \left\| n_s(t)\int\limits_{t_i^+}^{t} e^{-l(t-\tau)}\Phi^{\mathrm{T}}(\tau)\delta_1(\tau)\,d\tau \right\| \leqslant \left\|\overline{\varphi}_n^{\mathrm{T}}(t)\right\| 0.5\ddot{\Theta}_{\max}T^2. \tag{A.15}$$

Owing to the multiplication of the regression equation (A.14) by $n_s(t)$, the regressor $\overline{\varphi}_n^{\mathrm{T}}(t)$, the regressand $\overline{z}_n(t)$ and the perturbation $\overline{\varepsilon}_0(t)$ are bounded. In addition, according to the upper bound (A.15), the perturbation $\overline{\varepsilon}_0(t)$ can be reduced by decreasing the parameter T. Therefore, further on we will use the definition $\overline{\varepsilon}_0(t) := \overline{\varepsilon}_0(t, T)$ and imply that any perturbation obtained by transformation of $\overline{\varepsilon}_0(t, T)$ can also be reduced by a reduction of T.

Having applied (4.3) and multiplicated $z(t)$ by $\operatorname{adj}\{\varphi(t)\}$, we have (commutativity of the filter (4.3a) is not violated as its reinitialization and parameters change happen synchronously at a known time instant t_i^+, *i.e.* $\overline{\vartheta}(t) = \text{const}$ for all $t \in [t_i^+, t_i^+ + T)$)

$$\begin{gathered} Y(t) := \operatorname{adj}\{\varphi(t)\}\,z(t) = \Delta(t)\,\overline{\vartheta}(t) + \overline{\varepsilon}_1(t, T), \\ \operatorname{adj}\{\varphi(t)\}\,\varphi(t) = \det\{\varphi(t)\}\,I_{2(n+1)+1} = \Delta(t)\,I_{2(n+1)+1}, \\ \overline{\varepsilon}_1(t, T) = \operatorname{adj}\{\varphi(t)\}\int\limits_{t_i^+}^{t} e^{-\sigma\left(\tau-t_i^+\right)}\overline{\varphi}_n(\tau)\,\overline{\varepsilon}_0(\tau, T)\,d\tau, \end{gathered} \tag{A.16}$$

where $Y(t) \in \mathbb{R}^{2n+3}$, $\Delta(t) \in \mathbb{R}$, $\overline{\varepsilon}_1(t, T) \in \mathbb{R}^{2n+3}$.

Owing to $\Delta(t) \in \mathbb{R}$, the elimination (4.5) allows one to obtain the following from (A.16)

$$\begin{gathered} z_a(t) = Y^{\mathrm{T}}(t)\,\mathfrak{L}_a = \Delta(t)\,\vartheta_a^{\mathrm{T}}(t) + \overline{\varepsilon}_1^{\mathrm{T}}(t, T)\,\mathfrak{L}_a, \\ z_b(t) = Y^{\mathrm{T}}(t)\,\mathfrak{L}_b = \Delta(t)\,\vartheta_b(t) + \overline{\varepsilon}_1^{\mathrm{T}}(t, T)\,\mathfrak{L}_b, \end{gathered} \tag{A.17}$$

where $z_a(t) \in \mathbb{R}^{1\times n}$, $z_b(t) \in \mathbb{R}$, and $\vartheta_a(t)$, $\vartheta_b(t)$ are the first order approximations of the parameters $a(t)$ and $b(t)$, respectively (components of the vector Θ_i).

In case Assumption 2 is met, following the definition of the signal $\mathcal{K}(t)$, the first order approximations $\theta_x(t)$ and $\theta_r(t)$ of the parameters $k_x(t)$ and $k_r(t)$, respectively, satisfy the equations

$$a_{ref}^{\mathrm{T}} - \vartheta_a^{\mathrm{T}}(t) = \vartheta_b(t)\,\theta_x(t)\,,\;\; b_{ref} = \vartheta_b(t)\,\theta_r(t)\,. \tag{A.18}$$

where $\theta(t) = \begin{bmatrix}\theta_x(t) & \theta_r(t)\end{bmatrix}^{\mathrm{T}}$.

Each equation from (A.18) is multiplied by $\Delta(t)$. Equations (A.17) are substituted into the obtained result to have equation (4.6):

$$\begin{gathered}\mathcal{Y}(t) = \mathcal{M}(t)\,\theta(t) + d(t, T)\,,\\ \mathcal{Y}(t) := \begin{bmatrix}\Delta(t)\,a_{ref}^{\mathrm{T}} - z_a(t) & \Delta(t)\,b_{ref}\end{bmatrix}^{\mathrm{T}},\\ \mathcal{M}(t) := z_b(t)\,,\\ d(t, T) := -\begin{bmatrix}\overline{\varepsilon}_1^{\mathrm{T}}(t, T)\,\mathfrak{L}_a + \overline{\varepsilon}_1^{\mathrm{T}}(t, T)\,\mathfrak{L}_b\theta_r(t) & \overline{\varepsilon}_1^{\mathrm{T}}(t, T)\,\mathfrak{L}_b\theta_r(t)\end{bmatrix}^{\mathrm{T}},\end{gathered} \tag{A.19}$$

where $\mathcal{Y}(t) \in \mathbb{R}^{n+1}$, $\mathcal{M}(t) \in \mathbb{R}$, $d(t,\ T) \in \mathbb{R}^{n+1}$.

Owing to (A.19), the solution of (4.7a) is written as

$$\Upsilon(t) = \int\limits_{t_0^+}^{t} e^{\int\limits_t^\tau k d\tau}\mathcal{M}(\tau)\,\theta(\tau)\,d\tau + \int\limits_{t_0^+}^{t} e^{\int\limits_t^\tau k d\tau} d(\tau,\ T)\,d\tau \pm \Omega(t)\,\theta(t) = \Omega(t)\,\theta(t) + w(t)\,, \tag{A.20}$$

where

$$w(t) = \Upsilon(t) - \Omega(t)\,\theta(t)\,.$$

Equation (A.20) completes the proof of the fact that equation (4.8) can be obtained via procedure (4.1)–(4.7).

In order to prove statement (a), the regressor $\Omega(t)$ is represented as:

$$\begin{aligned}&\Omega(t) = \Omega_1(t) + \Omega_2(t)\,,\\ &\dot{\Omega}_1(t) = -k\left(\Omega_1(t) - \Delta(t)\,\vartheta_b(t)\right),\;\; \Omega_1\left(t_0^+\right) = 0,\\ &\dot{\Omega}_2(t) = -k\left(\Omega_2(t) - \overline{\varepsilon}_1^{\mathrm{T}}(t, T)\,\mathfrak{L}_b\right),\;\; \Omega_2\left(t_0^+\right) = 0.\end{aligned} \tag{A.21}$$

As $k > 0$ and the perturbation $\overline{\varepsilon}_1(t, T)$ is bounded, then $\Omega_2(t)$ is bounded, moreover, for all $t \geqslant t_0^+$ the following holds

$$|\Omega_2(t)| \leqslant \Omega_{2\max}(T)\,, \tag{A.22}$$

and there exists a limit $\lim_{T\to 0}\Omega_{2\max}(T) = 0$ for the upper bound as, following (A.15)–(A.19), the value of $\overline{\varepsilon}_1(t, T)$ can be arbitrarily reduced by reduction of T.

The next aim is to analyze $\Omega_1(t)$. The solution of the first differential equation from (A.21) is written for all $t \in \left[t_i^+ + T_s,\, t_{i+1}^+\right)$ as

$$\Omega_1(t) = \phi\left(t, t_i^+ + T_s\right)\Omega_1\left(t_i^+ + T_s\right) + \int\limits_{t_i^+ + T_s}^{t} \phi(t, \tau)\,\Delta(\tau)\,\vartheta_b(\tau) d\tau, \tag{A.23}$$

where $\phi(t, \tau) = e^{-\int\limits_\tau^t k d\tau}$.

The upper bound is required for the signal $\Omega_1(t)$ over the time range under consideration. To this end, we need bounds for $\Delta(t)$, and, in its turn, the ones for $\varphi(t)$.

As, according to the premises of the proposition, $\overline{\varphi}_n \in \mathrm{PE}$ for $T_s < T$, then $\overline{\varphi}_n \in \mathrm{FE}$ over $\left[t_i^+, t_i^+ + T_s\right]$ (this fact can be validated by substitution of $t = t_i^+$ into (1.2)). Then for all $t \in \left[t_i^+ + T_s, t_{i+1}^+\right)$ the following lower bound holds for the regressor $\varphi(t)$

$$\begin{aligned}\varphi(t) &= \int\limits_{t_i^+}^{t} e^{-\sigma\left(\tau - t_i^+\right)} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\,d\tau \\ &\geqslant \int\limits_{t_i^+}^{t_i^+ + T_s} e^{-\sigma\left(\tau - t_i^+\right)} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\,d\tau \\ \geqslant e^{-\sigma\left(t_{i+1}^+ - t_i^+\right)} &\int\limits_{t_i^+}^{t_i^+ + T_s} \overline{\varphi}_n(\tau)\overline{\varphi}_n^{\mathrm{T}}(\tau)\,d\tau \geqslant \alpha e^{-\sigma\left(t_{i+1}^+ - t_i^+\right)} I_{n+1}.\end{aligned} \tag{A.24}$$

On the other hand, as $\left\|\overline{\varphi}_n(t)\right\|^2 \leqslant \overline{\varphi}_n^{\max}$, then there exists an upper bound

$$\varphi(t) \leqslant \overline{\varphi}_n^{\max} \int\limits_{t_i^+}^{t} e^{-\sigma\left(\tau - t_i^+\right)} d\tau \leqslant \overline{\varphi}_n^{\max} \frac{1 - e^{-\sigma\left(t - t_i^+\right)}}{\sigma} \leqslant \sigma^{-1}\overline{\varphi}_n^{\max}, \tag{A.25}$$

and, therefore, for all $t \in \left[t_i^+ + T_s, t_{i+1}^+\right)$ it holds that $\Delta_{UB} \geqslant \Delta(t) \geqslant \Delta_{LB} > 0$.

Taking into consideration that, following Assumptions 1 and 2, $b_{\max} \geqslant |b(t)| \geqslant b_{\min} > 0$, and $\vartheta_b(t)$ is the approximation of first order of $b(t)$, then the following holds for the multiplication $\Delta(t)\vartheta_b(t)$

$$\forall t \in \left[t_i^+ + T_s, t_{i+1}^+\right) \;\; \Delta_{UB} b_{\max} \geqslant \left|\Delta(t)\vartheta_b(t)\right| \geqslant \Delta_{LB} b_{\min} > 0. \tag{A.26}$$

Having applied (A.21) and (A.26) and considered that $0 \leqslant \phi(t,\tau) \leqslant 1$, the following estimates hold for $\Omega_1(t)$

$$\begin{gathered}\forall t \in \left[t_0^+, t_0^+ + T_s\right] \;\; \Omega_1(t) \equiv 0, \\ \forall i \geqslant 1 \;\; \forall t \in \left[t_i^+ + T_s, t_{i+1}^+\right] \;\; \Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right)\Delta_{UB} b_{\max} \geqslant \Omega_1(t) \\ \geqslant \phi\left(t_{i+1}^+, t_i^+ + T_s\right)\left(\Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right)\Delta_{LB} b_{\min}\right) > 0,\end{gathered} \tag{A.27}$$

from which we have

$$\begin{gathered}\forall t \geqslant t_0 + T_s \quad \Omega_{1\max} \geqslant \Omega_1(t) \geqslant \Omega_{1\min} > 0, \\ \Omega_{1\max} = \min_{\forall i \geqslant 1}\left\{\phi\left(t_{i+1}^+, t_i^+ + T_s\right)\left(\Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right)\Delta_{LB} b_{\min}\right)\right\}, \\ \Omega_{1\min} = \max_{\forall i \geqslant 1}\left\{\Omega_1\left(t_i^+ + T_s\right) + \left(t_{i+1}^+ - t_i^+ - T_s\right)\Delta_{UB} b_{\max}\right\}.\end{gathered} \tag{A.28}$$

Then, using (A.28) and (A.23), the bounds for the regressor $\Omega(t)$ are written

$$\forall t \geqslant t_0 + T_s \;\; \Omega_{1\max} + \Omega_{2\max}(T) \geqslant |\Omega(t)| \geqslant \Omega_{1\min} - \Omega_{2\max}(T), \tag{A.29}$$

and, therefore, considering $\lim_{T\to 0}\Omega_{2\max}(T)=0$, there exists $T_{\min}>0$ such that for all $0<T<T_{\min}$ and $t\geqslant t_0+T_s$ the following inequality holds

$$\Omega_{\rm UB}\geqslant\Omega(t)\geqslant\Omega_{\rm LB}>0, \tag{A.30}$$

which was to be proved in statement (a).

In order to prove the statement (b), the disturbance $w(t)$ is differentiated with (A.20) and (4.7) at hand

$$\begin{aligned}\dot w(t)&=\dot\Upsilon(t)-\dot\Omega(t)\theta(t)-\Omega(t)\dot\theta(t)\\&=-k(\Upsilon(t)-\mathcal{Y}(t))+k(\Omega(t)-\mathcal{M}(t))\theta(t)-\Omega(t)\dot\theta(t)\\&=-k(\Upsilon(t)-\mathcal{M}(t)\theta(t)-d(t,T))+k(\Omega(t)-\mathcal{M}(t))\theta(t)-\Omega(t)\dot\theta(t)\\&=-k(\Upsilon(t)-\Omega(t)\theta(t))-\Omega(t)\dot\theta(t)+kd(t,T)\\&=-kw(t)-\Omega(t)\dot\theta(t)+kd(t,T),\quad w\left(t_0^+\right)=0_{n+1}.\end{aligned} \tag{A.31}$$

The solution of (A.31) is represented as:

$$\begin{aligned}&w(t)=w_1(t)+w_2(t),\\&\dot w_1(t)=-kw_1(t)-\Omega(t)\dot\theta(t),\quad w_1\left(t_0^+\right)=0_{n+1},\\&\dot w_2(t)=-kw_2(t)+kd(t,T),\quad w_2\left(t_0^+\right)=0_{n+1}.\end{aligned} \tag{A.32}$$

As for the first differential equation from (A.32), in Proposition 2 from [19] it is proved (up to notation) that the following inequality holds

$$\|w_1(t)\|\leqslant w_{1\max}\phi\left(t,t_0^++T_s\right), \tag{A.33}$$

when $i\leqslant i_{\max}<\infty$.

As $k>0$ and the disturbance $d(t,T)$ is bounded, then $w_2(t)$ is also bounded, and consequently, the following inequality holds

$$\|w_2(t)\|\leqslant w_{2\max}(T), \tag{A.34}$$

where the limit $\lim_{T\to 0}w_{2\max}(T)=0$ holds, as the input of the second differential equation from (A.32) depends only from the value of $d(t,T)$, which, in its turn, according to (A.15)–(A.19), can be reduced arbitrarily by reduction of T. The combination of the inequalities (A.33) and (A.34) in accordance with (A.32) completes the proof of proposition.

Proof of Theorem 1. Proof of theorem is similar to the above-given proof of Proposition 1.

Step 1. For all $t\geqslant t_0^++T_s$ the solution of the differential equation (4.9) is written as

$$\begin{aligned}\tilde\theta(t)=\phi\left(t,t_0^++T_s\right)\tilde\theta\left(t_0^++T_s\right)+\int\limits_{t_0^++T_s}^{t}\phi(t,\tau)\frac{\gamma_1 w(\tau)}{\Omega(\tau)}d\tau\\-\int\limits_{t_0^++T_s}^{t}\phi(t,\tau)\sum_{q=1}^{i}\Delta_q^\theta\delta\left(\tau-t_q^+\right)d\tau,\end{aligned} \tag{A.35}$$

where $\phi(t,\tau)=e^{-\int_\tau^t\gamma_1 d\tau}$.

Then, following the proof of Theorem 1 from [19], if $i \leqslant i_{\max} < \infty$, then the boundedness of the parametric error (A.35) can be shown:

$$\begin{gathered} \left\|\tilde{\theta}(t)\right\| \leqslant \beta_{\max} e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_0\right)}+\frac{\gamma_1 w_{1\max}}{\Omega_{LB}} \int\limits_{t_0^{+}+T_s}^{t} \phi(t, \tau) \phi\left(\tau, t_0^{+}+T_s\right) d\tau \\ +\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}} \int\limits_{t_0^{+}+T_s}^{t} \phi(t, \tau) d\tau \leqslant\left(\beta_{\max}+\frac{2 w_{1\max}}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_0\right)}+\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}. \end{gathered} \tag{A.36}$$

Step 2. The following quadratic form is introduced for all $t \geqslant t_0^{+}+T_s$:

$$\begin{gathered} V_{e_{ref}}=e_{ref}^{\mathrm{T}} P e_{ref}+\frac{4 a_0^2}{\gamma_1} e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_s\right)}, \quad H=\operatorname{blockdiag}\left\{P, \ \frac{4 a_0^2}{\gamma_1}\right\}, \\ \underbrace{\lambda_{\min}(H)}_{\lambda_{\mathrm{m}}}\left\|\bar{e}_{ref}\right\|^2 \leqslant V\left(\left\|\bar{e}_{ref}\right\|\right) \leqslant \underbrace{\lambda_{\max}(H)}_{\lambda_M}\left\|\bar{e}_{ref}\right\|^2, \\ \bar{e}_{ref}(t)=\left[e_{ref}^{\mathrm{T}}(t) \quad e^{-\frac{\gamma_1}{4}\left(t-t_0^{+}-T_s\right)}\right]^{\mathrm{T}}. \end{gathered} \tag{A.37}$$

Similar to proof of Proposition 1, the derivative of (A.37) is written as

$$\begin{gathered} \dot{V}_{e_{ref}} \leqslant\left[-\lambda_{\min}(Q)+2 \lambda_{\max}(P) b_{\max}\left(\left\|\tilde{\theta}\right\|+\dot{\mathcal{K}}_{\max} T\right)+2\right]\left\|e_{ref}\right\|^2 \\ +\lambda_{\max}^2(P) \bar{\varpi}_r^2 b_{\max}^2\left\|\tilde{\theta}\right\|^2+\lambda_{\max}^2(P) b_{\max}^2 \bar{\varpi}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2-2 a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_s\right)}. \end{gathered} \tag{A.38}$$

As for all $t \geqslant t_0^{+}+T_s$ the parametric error $\tilde{\theta}(t)$ meets the inequality (A.36), then, considering

$$\begin{gathered} \left\|\tilde{\theta}(t)\right\|^2 \leqslant\left(\beta_{\max}+\frac{2 w_{1\max}}{\Omega_{LB}}\right)^2 e^{-\gamma_1\left(t-t_0^{+}-T_0\right)}+\left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 \\ +2\left(\beta_{\max}+\frac{2 w_{1\max}}{\Omega_{LB}}\right) \frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}} e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_0\right)} \\ \leqslant\left(\beta_{\max}+\frac{2 w_{1\max}}{\Omega_{LB}}\right)\left(\beta_{\max}+\frac{2\left(w_{1\max}+\gamma_1 w_{2\max}(T)\right)}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_0\right)}+\left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 \\ =\bar{\beta}_{\max} e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_0\right)}+\left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 \end{gathered}$$

the upper bound of (A.38) is written as follows:

$$\begin{gathered} \dot{V}_{e_{ref}} \leqslant\left[-\lambda_{\min}(Q)+2+2 \lambda_{\max}(P) b_{\max}\right. \\ \left.\times\left(\left(\beta_{\max}+\frac{2 w_{1\max}}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_s\right)}+\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}+\dot{\mathcal{K}}_{\max} T\right)\right]\left\|e_{ref}\right\|^2 \\ +\lambda_{\max}^2(P) \bar{\varpi}_r^2 b_{\max}^2 \bar{\beta}_{\max} e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_s\right)}+\lambda_{\max}^2(P) \bar{\varpi}_r^2 b_{\max}^2\left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 \\ +\lambda_{\max}^2(P) \bar{\varpi}_r^2 b_{\max}^2 \dot{\mathcal{K}}_{\max}^2 T^2-2 a_0^2 e^{-\frac{\gamma_1}{2}\left(t-t_0^{+}-T_s\right)}. \end{gathered} \tag{A.39}$$

There definitely exists a time instant $t_{e_{ref}} \geqslant t_0^+ + T_s$ and constants $T \to 0$, $a_0 > \lambda_{\max}(P)\overline{\omega}_r b_{\max}\overline{\beta}_{\max}^{\frac{1}{2}}$ such that for all $t \geqslant t_{e_{ref}}$ it holds that

$$\begin{gathered}
-\lambda_{\min}(Q) + 2 + 2\lambda_{\max}(P)\, b_{\max}\left(\left(\beta_{\max} + \frac{2w_{1\max}}{\Omega_{LB}}\right) e^{-\frac{\gamma_1}{2}\left(t_{e_{ref}} - t_0^+ - T_s\right)} \right. \\
\left. + \frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}} + \dot{\mathcal{K}}_{\max} T\right) = -c_1 < 0, \\
\lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \overline{\beta}_{\max} - 2a_0^2 = -c_2 < 0.
\end{gathered} \tag{A.40}$$

Then the upper bound for the derivative (A.39) for all $t \geqslant t_{e_{ref}}$ is obtained as

$$\dot{V}_{e_{ref}} \leqslant -\eta_{\overline{e}_{ref}} V_{e_{ref}} + \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 + \lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \dot{\mathcal{K}}_{\max}^2 T^2, \tag{A.41}$$

where $\eta_{\overline{e}_{ref}} = \min\left\{\frac{c_1}{\lambda_{\max}(P)},\ \frac{c_2\gamma_1}{4a_0^2}\right\}$.

The solution of the differential inequality (A.41) for all $t \geqslant t_{e_{ref}}$ is written as

$$\begin{gathered}
V_{e_{ref}}(t) \leqslant e^{-\eta_{\overline{e}_{ref}}\left(t - t_{e_{ref}}\right)} V_{e_{ref}}\left(t_{e_{ref}}\right) \\
+ \frac{1}{\eta_{\overline{e}_{ref}}}\left(\lambda_{\max}^2(P)\,\overline{\omega}_r^2 b_{\max}^2 \left(\frac{\gamma_1 w_{2\max}(T)}{\Omega_{LB}}\right)^2 + \lambda_{\max}^2(P)\, b_{\max}^2 \overline{\omega}_r^2 \dot{\mathcal{K}}_{\max}^2 T^2\right),
\end{gathered} \tag{A.42}$$

which completes the proof of statement (ii) of theorem.

Step 3. Owing to (A.36) and (A.42), the error $\tilde{\theta}(t)$ is bounded for all $t \geqslant t_0^+ + T_s$, and the error $e_{ref}(t)$ — for all $t \geqslant t_{e_{ref}}$. Then, to prove the statement (i), we need to show that $\tilde{\theta}(t)$ is bounded over $\left[t_0^+, t_0^+ + T_s\right)$, and $e_{ref}(t)$ is bounded over $\left[t_0^+, t_{e_{ref}}\right)$.

In the conservative case, the inequality $\Omega(t) \leqslant \Omega_{LB}$ is satisfied over $\left[t_0^+, t_0^+ + T_s\right)$, whence, owing to $\dot{\hat{\theta}}(t) = 0_{n+1}$, if Assumption 1 is met, it follows that the parametric error $\tilde{\theta}(t) = \hat{\theta}\left(t_0^+\right) - \theta(t)$ is bounded over $\left[t_0^+, t_0^+ + T_s\right)$ and, as a consequence, for all $t \geqslant t_0^+$.

Considering the time range $\left[t_0^+, t_{e_{ref}}\right)$ and taking into account the notation from (A.3), (A.18), the error equation (3.1) is written in the following form:

$$\dot{e}_{ref}(t) = \left(A_{ref} + e_n b(t)\left(\hat{\theta}_x(t) - k_x(t)\right)\right) e_{ref}(t) + e_n b(t)\left(\hat{\theta}^{\mathrm{T}}(t) - \mathcal{K}^{\mathrm{T}}(t)\right)\omega_r(t),$$

which, as it has been proved that $\tilde{\theta}(t)$ is bounded for all $t \geqslant t_0^+$ and Assumptions 1 and 2 are met, allows one, using Theorem 3.2 from [20], to make the conclusion that 1) $e_{ref}(t)$ is bounded over $\left[t_0^+, t_{e_{ref}}\right)$, 2) $\xi(t) \in L_\infty$ for all $t \geqslant t_0^+$.

REFERENCES

1. Polyak, B.T. and Tsypkin, Ya.Z., Optimal Pseudogradient Adaptation Algorithms, *Autom. Remote Control*, 1981, vol. 41, pp. 1101–1110.

2. Polyak, B.T. and Tsypkin, Ya.Z., Robust Pseudogradient Adaptation Algorithms, *Autom. Remote Control*, 1981, vol. 41, no. 10, pp. 1404–1409.

3. Ioannou, P. and Sun, J., *Robust Adaptive Control*, New York: Dover, 2013.

4. Fradkov, A.L., Lyapunov-Bregman functions for speed-gradient adaptive control of nonlinear time-varying systems, *IFAC-PapersOnLine*, 2022, vol. 55, no. 12, pp. 544–548.

5. Goel, R. and Roy, S.B., Composite adaptive control for time-varying systems with dual adaptation, *arXiv preprint arXiv:2206.01700*, 2022, pp. 1–6.

6. Na, J., Xing, Y., and Costa-Castelló, R., Adaptive estimation of time-varying parameters with application to roto-magnet plant, *IEEE Transactions on Systems, Man, and Cybernetics: Systems*, 2018, vol. 51, no. 2, pp. 731–741.

7. Chen, K. and Astolfi, A., Adaptive control for systems with time-varying parameters, *IEEE Transactions on Automatic Control*, 2020, vol. 66, no. 5, pp. 1986–2001.

8. Patil, O.S., Sun, R., Bhasin, S., and Dixon, W.E., Adaptive control of time-varying parameter systems with asymptotic tracking, *IEEE Transactions on Automatic Control*, 2022, vol. 67, no. 9, pp. 4809–4815.

9. Putov, V.V., Methods of Adaptive Control Systems Design for Nonlinear Time-Varying Dynamic Plants with Functional-Parametric Uncertainty, *Thesis . . . Dr. of Technical Sciences*, SPbGETU 'LETI'. SPb., 1993, 590 p. In Russian.

10. Putov, V.V., Polushin, I.G., Lebedev, V.V., and Putov, A.V., Generalisation of the majorizing functions method for the problems of adaptive control of nonlinear dynamic plants, *Izvestiya SPbGETU LETI*, 2013, no. 8, pp. 32–37. In Russian.

11. Glushchenko, A. and Lastochkin, K., Exponentially Stable Adaptive Control. Part III. Time-Varying Plants, *Autom. Remote Control*, 2023, vol. 84, no. 11, pp. 1232–1247.

12. Pagilla, P.R. and Zhu, Y., Adaptive control of mechanical systems with time-varying parameters and disturbances, *J. Dyn. Sys., Meas., Control*, 2004, vol. 126, no. 3, pp. 520–530.

13. Quoc, D.V., Bobtsov, A.A., Nikolaev, N.A., and Pyrkin, A.A., Stabilization of a linear non-stationary system under conditions of delay and additive sinusoidal perturbation of the output, *Journal of Instrument Engineering*, 2021, vol. 64, no. 2, pp. 97–103. In Russian.

14. Dat, V.Q. and Bobtsov, A.A., Output Control by Linear Time-Varying Systems using Parametric Identification Methods, *Mekhatronika, Avtomatizatsiya, Upravlenie*, 2020, vol. 21, no. 7, pp. 387–393. In Russian.

15. Grigoryev, V.V., Design of control equations for variable parameter systems, *Autom. Remote Control*, 1983, vol. 44, no. 2, pp. 189–194.

16. Glushchenko, A. and Lastochkin, K., Robust Time-Varying Parameters Estimation Based on I-DREM Procedure, *IFAC-PapersOnLine*, 2022, vol. 55, no. 12, pp. 91–96.

17. Dieudonne, J., *Foundations of Modern Analysis*, New York, Academic Press, 1960.

18. Leiva, H. and Siegmund, S., A necessary algebraic condition for controllability and observability of linear time-varying systems, *IEEE Transactions on Automatic Control*, 2003, vol. 48, no. 12, pp. 2229–2232.

19. Glushchenko, A.I. and Lastochkin, K.A., Exponentially Stable Adaptive Control. Part II. Switched Systems, *Autom. Remote Control*, 2023, vol. 84, no. 3, pp. 260–291.

20. Khalil, H., *Nonlinear Systems*, 3rd ed., Upper Saddle River, NJ, USA: Prentice-Hall, 2002.

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