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TOPICAL ISSUE

On Some Problems with Multivalued Mappings

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Abstract—We consider some problems with a set-valued mapping, which can be reduced to minimization of a homogeneous Lipschitz function on the unit sphere. Latter problem can be solved in some cases with a first order algorithm—the gradient projection method. As one of the examples, the case when set-valued mapping is the reachable set of a linear autonomous controlled system is considered. In several settings, the linear convergence is proven. The methods used in proofs follow those introduced by B.T. Polyak for the case where Lezanski–Polyak–Lojasiewicz condition holds. Unlike algorithms that use approximation of the reachable set, the proposed algorithms depend far less on dimension and other parameters of the problem. Efficient error estimation is possible. Numerical experiments confirm the effectiveness of the considered approach. This approach can also be applied to various set-theoretical problems with general set-valued mappings.

Keywords: gradient projection method, set-valued integral, strong convexity, supporting set, Lipschitz condition, nonsmooth analysis

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1. INTRODUCTION

Let $\mathbb{R}^n$ be a real Euclidean space with the inner product $(\cdot,\cdot)$ and norm $\|\cdot\| = \sqrt{(\cdot,\cdot)}$. Define the ball $\mathcal{B}_r(a) = \{x \in \mathbb{R}^n \ : \ \|x-a\| \leqslant r\}$, $(a \in \mathbb{R}^n,\ r > 0)$ and the unit sphere $\mathcal{S}_1 = \partial\mathcal{B}_1(0)$. Denote by $\operatorname{int}\mathcal{N}$ and $\partial\mathcal{N}$ the interior and the boundary of a set $\mathcal{N} \subset \mathbb{R}^n$, respectively. Recall that the supporting function for a closed convex set $\mathcal{N} \subset \mathbb{R}^n$ and vector $p \in \mathbb{R}^n$ is $s(p,\mathcal{N}) = \sup_{x\in\mathcal{N}}(p,x)$ and the supporting subset is $\mathcal{N}(p) = \{x \in \mathcal{N} \ : \ (p,x) = s(p,\mathcal{N})\}$. The set $\mathcal{N}(p)$ is called the supporting element if it is a singleton. For a convex compact set $\mathcal{N}$ the set $\mathcal{N}(p)$ is the subdifferential (in the sense of convex analysis) of the supporting function $s(p,\mathcal{N})$ at the point p. Let $P_{\mathcal{N}}x$ be the metric projection of a point $x \in \mathbb{R}^n$ onto a closed convex set $\mathcal{N}$.

Let $\mathcal{N} \subset \mathbb{R}^n\backslash\{0\}$ be a convex compact set and $f(p) = s(p,\mathcal{N})$. Consider the problem

$$\min_{\|p\|=1} f(p) = J. \tag{1}$$

It is obvious that the solution of problem (1) is a unit vector p_0 such that $p_0 = -z_0/\|z_0\|$, $P_{\mathcal{N}}0 = \{z_0\}$ and $J = (p_0, z_0)$. Also $z_0 \in \mathcal{N}(p_0)$. Thus finding the projection of zero $z_0 = P_{\mathcal{N}}0$ is equivalent to the problem (1). The general projection problem can be solved the same way as $P_{\mathcal{N}}x = x + P_{\mathcal{N}+(-x)}0$.

There are many ways to solve the problem of projecting a point onto a convex closed set $\mathcal{N}$, that depend on how the set $\mathcal{N}$ is defined. If the set $\mathcal{N}$ is a polyhedron, then it can be solved with the help of quadratic programming: $\min\|x\|^2$ under conditions $(p_i,x) \leqslant s(p_i,\mathcal{N})$, where $\{p_i\}$ is the set of unit normals to $\mathcal{N}$. Method of alternating projections under the transversality condition can be found in [1, Section 8.5]. In [2], the author considers properties of projector operators. They also consider convergence of an iterative projection/reflection algorithm for finding points that achieve a local minimum distance between two closed convex sets or one closed convex set and a closed prox-regular set. Usefulness of conditional gradient-like methods for determining projections onto convex sets was considered in [3]. In [4], the authors proposed an iterative algorithm for metric projection of a point onto a level set of a quadric function. Some algorithms for finding the Bregman projection of a point onto a closed convex set can be found in [5].

The best rate of convergence for the algorithms considered in the papers above is linear. Besides that, in many cases, the considered algorithms do not allow one to obtain an efficient computational procedure.

Further we shall assume that we know supporting function $s(p,\mathcal{N})$ and supporting subset $\mathcal{N}(p)$. "We know" means that we can efficiently compute $s(p,\mathcal{N})$ and $\mathcal{N}(p)$ for any vector $p \in \mathbb{R}^n \backslash \{0\}$.

Suppose that $\mathcal{M} \subset \mathbb{R}^n$ is a convex compact set and $\mathcal{R}(\cdot) : [0,T] \to 2^{\mathbb{R}^n}$, $\mathcal{R}(0) = \{0\}$, is a set-valued mapping with convex compact values that is continuous in Hausdorff metric. Consider a few problems that can be solved in the framework of statement (1).

Problem (P1). For given $t \geqslant 0$, find the distance between sets $\mathcal{R}(t)$ and $\mathcal{M}$, i.e. the value of $\rho(\mathcal{R}(t),\mathcal{M}) = \inf_{x\in\mathcal{R}(t),\ y\in\mathcal{M}} \|x-y\|$. Find minimal $t \geqslant 0$, so that $\rho(\mathcal{R}(t),\mathcal{M}) = 0$.

Problem (P2). For given $t \geqslant 0$, check whether the inclusion $\mathcal{R}(t) \subset \mathcal{M}$ holds. Find maximal $t \geqslant 0$, so that $\mathcal{R}(t) \subset \mathcal{M}$.

Problem (P3). For given $t \geqslant 0$, check whether the inclusion $\mathcal{R}(t) \supset \mathcal{M}$ holds. Find minimal $t \geqslant 0$, so that $\mathcal{R}(t) \supset \mathcal{M}$.

Problems (P1)–(P3) can be stated for an arbitrary set-valued continuous mapping with convex compact images $\mathcal{R}(t)$ and a convex compact set $\mathcal{M}$. Consider a particular case of a set-valued integral of the form

$$\mathcal{R}(t) = \int_0^t \mathcal{F}(s)\, ds, \tag{2}$$

where $\mathcal{F}$ is a set-valued mapping with convex compact values. By default we shall assume that $0 \in \mathcal{F}(s)$ for all $s \geqslant 0$. The last integral is treated as the Aumann integral [6]

$$\int_0^t \mathcal{F}(s)\, ds = \left\{ \int_0^t u(s)\, ds \ : \ u(s) \in \mathcal{F}(s)\text{—a measurable selector} \right\}.$$

By the Lyapunov theorem on vector measures [7] the value of the integral is convex and compact. From formula (2) and the inclusion $0 \in \mathcal{F}(s)$ for all $s \in [0,t]$ we conclude that $\{\mathcal{R}(t)\}_{t\geqslant 0}$ is increasing: $\mathcal{R}(t_1) \subset \mathcal{R}(t_2)$ for all $0 \leqslant t_1 \leqslant t_2$. It is also possible to consider a set $\mathcal{M}(t)$ depending on t.

The support function and supporting subset for integral (2) can be calculated easily: for a unit vector p and any $t \geqslant 0$ we get

$$s(p,\mathcal{R}(t)) = s\left(p, \int_0^t \mathcal{F}(s)\, ds\right) = \int_0^t s\left(p,\mathcal{F}(s)\right)\, ds, \qquad \mathcal{R}(t)(p) = \int_0^t \mathcal{F}(s)(p)\, ds. \tag{3}$$

Another class of sets for which we know the supporting function and the supporting element are finite sums of linear images of some fixed sets $\mathcal{M}$ with known $s(p,\mathcal{M})$ and $\mathcal{M}(p)$, e.g. ellipsoids. Suppose that $\mathcal{R}(t) = \sum_{k=1}^m A_k(t)B_1(0)$, where $A_k(t)$ are continuous nondegenerate matrices for all

$t \geqslant 0$. Then

$$s(p, \mathcal{R}(t)) = \sum_{k=1}^{m} s(p, A_k(t)B_1(0)) = \sum_{k=1}^{m} \|A_k^T(t)p\|, \qquad \mathcal{R}(t)(p) = \sum_{k=1}^{m} \frac{A_k(t)A_k^T(t)p}{\|A_k^T(t)p\|}. \tag{4}$$

Note that a finite sum of ellipsoids is, in general, not an ellipsoid.

Our most important example is the reachable set of an autonomous linear controlled system, which is described by a differential inclusion

$$x'(t) \in Ax(t) + \mathcal{U}, \ x(0) = 0, \qquad x \in \mathbb{R}^n, \quad A \in \mathbb{R}^{n\times n}, \tag{5}$$

where $\mathcal{U} \subset \mathbb{R}^n$ is a compact, $0 \in \mathcal{U}$. The reachable set (all points to which the system can arrive at the given moment of time) can be represented in the form

$$\mathcal{R}(t) = \int_0^t e^{As}\mathcal{U}\, ds. \tag{6}$$

The most important strengthening of the convexity condition is the concept of strong convexity with radius R. The set in $\mathbb{R}^n$ is strongly convex with radius R if it can be represented as an intersection of closed balls of radius R [8, 9]. This property can also be defined via the modulus of convexity [10]. In [8], the authors proved that the set-valued integral (2) is strongly convex if the multifunction $\mathcal{F}(s)$ has strongly convex values. In [11], the local strong convexity in certain sense was proved for integral (2) with $\mathcal{F}(s) = A(s)U$, where $A(s)$ is a certain class of smooth matrices and U is a polyhedron. In [12], the second order approximation in time of a Runge-Kutta type scheme for discretization of strongly convex differential inclusions was considered.

Various problems with set-valued integrals can be solved with the help of approximation of values of the integrals. In [13], the authors describe different methods to construct an approximation of the reachable set of a controlled system, see Table 1 therein. One of the most general and effective methods is based on the supporting function (it is also called hyperplane method), see, for example, [14]. We can consider an outer polyhedral approximation for $\mathcal{M}$ of the form

$$\{x \in \mathbb{R}^n \ : \ (p, x) \leqslant s(p, \mathcal{M}), \quad \forall p \in \mathbb{G}\}, \tag{7}$$

where $\mathbb{G} \subset \mathbb{R}^n$ is a finite grid of unit vectors and solve the problem for the approximation. The disadvantage of this approach is that a reasonable approximation can be obtained only in a space of low dimensions $2 \leqslant n < 5$, see [15].

There are also different approaches using special approximations, e.g. with zonotopes [16] or ellipsoidal technique [17]. The latter technique sometimes permits to describe the reachable set locally.

In the present paper we think $\mathcal{R}(t)$, $\mathcal{M}$, $\mathcal{N}$ to be either the value of a set-valued integral or a finite sum of ellipsoids. We shall show how to reduce different problems, e.g. (P1)–(P3), with such sets to the problem (1). The function $f(p)$ in (1) turns out to be the supporting function of some convex compact set $\mathcal{N}$, which depends on $\mathcal{R}(t)$ and $\mathcal{M}$. Lezanski-Polyak-Lojasiewicz (LPL) condition [18, formula (4.6)] is proven in problem (1), from which a linear convergence rate for gradient projection algorithm is obtained. The supporting function $f(p)$ and its gradient can be computed, e.g. using formula (3) for a set-valued integral or by (4) for sum of ellipsoids. With the supporting function and its gradient we obtain an efficient calculation scheme. We also consider a local condition of strong convexity: for some $R > 0$ for the solution p_0 of (1) the inclusion $\mathcal{N} \subset B_R(\mathcal{N}(p_0) - Rp_0)$ holds. Under this condition the problem can be solved with the help of the gradient projection method with a fixed step-size or with Armijo's step-size. We prove a linear rate of convergence for all algorithms and consider various examples.

There is another way to solve (1) using the conditional gradient (CG) method: take the function $g(x) = \frac{1}{2}\|x\|^2$, a starting point $x_1 \in \mathcal{N}$ and iterations $\overline{x}_k = \arg\max_{x\in\mathcal{N}}(-g'(x_k), x)$, $x_{k+1} \in \operatorname{Arg\,min}_{x\in[x_k,\overline{x}_k]} g(x)$. Note that, to ensure the linear convergence of this algorithm, strong convexity of $\mathcal{N}$ is usually required [18, Therorem 6.1, 5].

1.1. Notation and Auxiliary Results

Recall that for sets $\mathcal{M}$ and $\mathcal{N}$ from $\mathbb{R}^n$ we have $\mathcal{M}+\mathcal{N} = \{x+y \,:\, x\in\mathcal{M},\ y\in\mathcal{N}\}$ and $\mathcal{M}\stackrel{*}{-}\mathcal{N} = \{x \,:\, x+\mathcal{N}\subset\mathcal{M}\} = \bigcap_{x\in\mathcal{N}}(\mathcal{M}-x)$. These operations are called the Minkowski sum and difference of sets $\mathcal{M}$ and $\mathcal{N}$.

Denote by $\varrho(x,\mathcal{M}) = \inf_{y\in\mathcal{M}}\|x-y\|$ the distance from a point x to a set $\mathcal{M}$.

The *Hausdorff distance* on the space of convex compacts in $\mathbb{R}^n$ can be defined like this: for any convex compact sets $\mathcal{M},\mathcal{N}\subset\mathbb{R}^n$

$$h(\mathcal{M},\mathcal{N}) = \max_{\|p\|=1}|s(p,\mathcal{M}) - s(p,\mathcal{N})|.$$

Define $[a]_-$: $[a]_- = |a|$ for $a \leqslant 0$ and $[a]_- = 0$ for $a > 0$. Then $[\min_{\|p\|=1}(s(p,\mathcal{M}) - s(p,\mathcal{N}))]_-$ is the halfdistance from $\mathcal{N}$ to $\mathcal{M}$ and it is equal to $\max_{x\in\mathcal{N}}\varrho(x,\mathcal{M})$.

Suppose that the set $\mathcal{R}(t)$ (6) depends on parameter t. Then we shall denote the *supporting set* for a vector p by $\mathcal{R}(t)(p)$. From the Aumann's or Riemann's definition of the integral for any matrix $J\in\mathbb{R}^{m\times n}$ we have $J\mathcal{R}(t) = \int_0^t Je^{As}\mathcal{U}\,ds$. In particular, for any vector $p\in\mathbb{R}^n$

$$\mathcal{R}(t)(p) = \int_0^t (e^{As}\mathcal{U})(p)\,ds.$$

A set $\mathcal{M}\subset\mathbb{R}^n$ is *strongly convex with radius* $R>0$ if we can represent $\mathcal{M}$ as intersection of some collection of closed Euclidean balls with radius R. For any strongly convex set $\mathcal{M}$ with radius $R>0$ there exists another strongly convex set $\mathcal{N}$ with radius R such that $\mathcal{M}+\mathcal{N} = \mathcal{B}_R(0)$ [8, 19]. Strong convexity of a compact convex set $\mathcal{M}$ with radius R is equivalent to the Lipschitz condition for the supporting element $\mathcal{M}(p)$ on the unit sphere: for all $\|p\| = \|q\| = 1$ we have $\|\mathcal{M}(p) - \mathcal{M}(q)\| \leqslant R\|p-q\|$ [8].

We shall say that a convex set $\mathcal{M}\subset\mathbb{R}^n$ is *uniformly smooth with constant* $r>0$ if we have $\mathcal{M} = \mathcal{M}_0 + \mathcal{B}_r(0)$, where $\mathcal{M}_0\subset\mathbb{R}^n$ is a convex compact set. For more details see [20, Definition 2.1].

Let $\mathcal{S}_0\subset\mathbb{R}^n$ be a smooth manifold without boundary, $\overline{x}\in\mathcal{S}_0$, $\varepsilon>0$. For a differentiable function $f:\mathcal{S}_0+\operatorname{int}\mathcal{B}_\varepsilon(0)\to\mathbb{R}$ define $\mathcal{S} = \mathcal{S}(f,\overline{x}) = \{x\in\mathcal{S}_0 : f(x)\leqslant f(\overline{x})\}$. Assume $\mathcal{S}$ to be a smooth manifold with the boundary $\partial\mathcal{S}\subset\{x\in\mathcal{S}_0 : f(x) = f(\overline{x})\}$. We shall say that *the Lezanski–Polyak–Lojasiewicz (LPL)* condition holds on $\mathcal{S}$ [18; 21, Section 3.2] with a constant $\mu>0$ if $\Omega = \operatorname{Arg\,min}_{x\in\mathcal{S}} f(x)\neq\emptyset$ and for all $x\in\mathcal{S}$ the following inequality holds

$$\|P_{\mathcal{T}_x}f'(x)\|^2 \geqslant \mu(f(x) - f(\Omega)). \qquad (\star)$$

Here $\mathcal{T}_x$ is the tangent subspace to the manifold $\mathcal{S}$ at the point $x\in\mathcal{S}$, $P_{\mathcal{T}_x}$ is the orthogonal projector onto $\mathcal{T}_x$, $f'(x)$ is the Frechet gradient of the function f at the point $x\in\mathcal{S}$.

Lemma 1. *For any nonzero vectors* $p,q\in\mathbb{R}^n$ *we have* $\left\|\frac{p}{\|p\|} - \frac{q}{\|q\|}\right\| \leqslant \frac{\|p-q\|}{\sqrt{\|p\|\,\|q\|}}$,

Proposition 1 [8]. *Suppose that a set-valued mapping* $\mathcal{F}:[0,t]\to 2^{\mathbb{R}^n}$ *is continuous in the Hausdorff metric and has strongly convex images* $\mathcal{F}(s)$ *with radius* $R(s)$ *for all* $s\in[0,t]$, *that is integrable at* $[0,t]$. *Then the integral* $\mathcal{P} = \int_0^t F(s)\,ds$ *is strongly convex with radius* $R = \int_0^t R(s)\,ds$.

It should be mentioned that the set-valued integral can be strongly convex even when $\mathcal{F}(s)$ is not. itself. For example, this situation typically takes place for the reachable set $\mathcal{R}(t)$ of system (5) in dimension $n=2$ [22]. Nevertheless, the reachable set in dimensions $n\geqslant 3$ is often not strongly convex.

Let us look at an elementary example of a system mentioned in (5) (a similar system is considered in Example 1 below). Let the control set be a segment: $\mathcal{U} = \text{co}\ \{\pm v\}$. Define an analytic function $g_p(s) = (p, e^{As}v)$. The supporting set $\mathcal{R}(t)(p)$ is a singleton, provided that $g_p(s) \not\equiv 0$. This is guaranteed by full rank conditions

$$\text{span}\{A^i v\}_{i=0}^{n-1} = \mathbb{R}^n \quad \Leftrightarrow \quad \text{span}\,\mathcal{R}(t) = \mathbb{R}^n.$$

Since g_p is analytic, the equation $g_p = 0$ has a finite number of roots in $[0, t]$. The supporting element can be written down as

$$\mathcal{R}(t)(p) = \int_0^t e^{As} v \times \text{sign}\, g_p(s)\, ds = \sum_{i=0}^{k} \epsilon_i \int_{s_i(p)}^{s_{i+1}(p)} e^{As} v\, ds, \tag{8}$$

where $s_i(p)$, $i = \overline{1,k}$ are the roots of $g_p(s)$, $s_0 = 0$, $s_{k+1} = t$, $\epsilon_i = \pm 1$ is equal to sign of $g_p(s)$ when $s \in [s_i, s_{i+1}]$. Therefore, the behaviour of supporting element is defined by dependence of roots of analytic function $g_p(s)$ on parameter p. If all roots are simple and lie on interval $(0, t)$, then it follows from implicit function theorem that the support element depends smoothly on p in the neighbourhood. Therefore, the supporting element is locally Lipschitz. On the other hand, g_p can have roots with multiplicity greater than one belonging to $[0, t]$. In this case the supporting element is typically not locally Lipschitz, which means that the strong convexity fails. This is illustrated in the example below. However, it is easy to show the set of vectors p, such that g_p has non-simple zeros on $[0, t]$, has measure zero on the unit sphere. Some generalizations of this approach to set-valued integrals can be seen in [11].

Note that if all eigenvalues of A are real, then the number of switchings in optimal control $u(t) = \mathcal{U}(e^{A^T(T-t)}p) = v \times \text{sgn} g_p(T - t)$ is no greater than $n - 1$, it is a special case of Feldbaum theorem, see [23, Theorem 2.11]. In the examples below, we consider a dynamical system defined by $\dot{x} = Ax + Bu$, $u \in \mathcal{U}$, $t \in [0, T]$. The optimal control that guides the system to the support element $\mathcal{R}(t)(p)$ is [24]:

$$u(t) = \mathcal{U}(B^\top e^{A^\top (T-t)}), \quad t \in [0, T]. \tag{9}$$

Consider the system

$$\dot{x} = Ax + Bu, \quad x(0) = 0, \quad u \in \mathbb{R}: \quad |u| \leqslant 1, \quad A = \begin{bmatrix} -1 & 1 & 0 \\ 0 & -1 & 1 \\ 0 & 0 & -1 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{10}$$

Following what was said above, let $g_p(s) = (p, e^{As}B) = \frac{1}{2}e^{-s}\,(p_1 s^2 + 2p_2 s + 2p_3)$.

Let $p_0 = \frac{1}{3}(2, -2, 1)$, note that $g_{p_0}(s) = \frac{1}{3}e^{-s}(s-1)^2$ has a multiple root $s = 1$. We are interested in behaviour of supporting element near p_0. Remember that $f(s) \asymp g(s)$, $s \to 0$, if $f(s) = O(g(s))$ and $g(s) = O(f(s))$, $s \to 0$. Define for $\varepsilon \in (0, 1)$ a unit vector $q = q(\varepsilon) = \frac{(2,-2,1-\varepsilon)}{\sqrt{9-2\varepsilon+\varepsilon^2}}$. It is easy to see that $\|p - q(\varepsilon)\| \asymp \varepsilon$, $\varepsilon \to 0$, and to find the roots $g_{q(\varepsilon)} =: s_{1,2}(\varepsilon) = 1 \pm \sqrt{\varepsilon}$. Then for $t > 1 + \sqrt{\varepsilon}$ we can write down the supporting element in the following way:

$$\mathcal{R}(t)(p) - \mathcal{R}(t)(q) = \int_{1-\sqrt{\varepsilon}}^{1+\sqrt{\varepsilon}} e^{-s}(s^2,\ 2s,\ 2)^\top\, ds,$$

$$\|\mathcal{R}(t)(p) - \mathcal{R}(t)(q)\| \geqslant \int_{1-\sqrt{\varepsilon}}^{1+\sqrt{\varepsilon}} 2e^{-s}\, ds \asymp \sqrt{\varepsilon},\ \varepsilon \to 0.$$

Therefore, the supporting element fails to be Lipschitz in a neighbourhood of p_0, so the reachable set $\mathcal{R}(t)$ is not strongly convex.

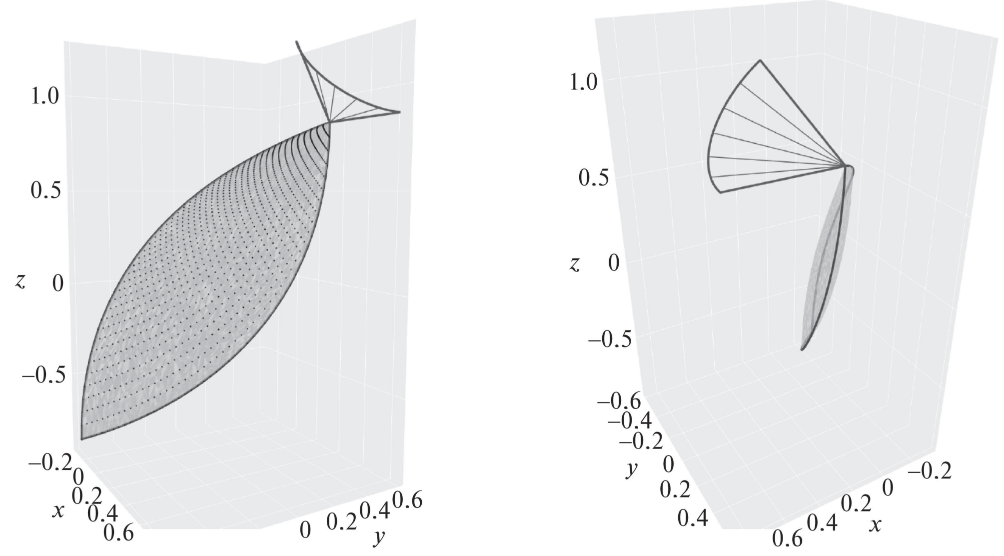

Fig. 1. Attainable set of (10) and normal vectors, where the supporting element is not locally Lipschitz, $t = 2$.

The reachable set for $t = 2$ can be seen at Fig. 1. Normal vectors, for which the supporting element is not locally Lipschitz, can be seen in the upper part of the figure. They lie on the boundary of the normal cone at the tip of the set. Moreover, it is evident that the reachable set is structured like a CW complex. This structure appears as a result of (8), since the supporting element can be determined by positions and multiplicities of the roots of $g_p(s)$ on $[0, t]$ and the sign of g_p around the left end of the segment. If the system has a matrix with real eigenvalues, then the overall multiplicity of roots of g_p is not greater that $n - 1$. It can be shown, that in this case an arbitrary configuration of roots substituted into (8) produces a point from $\partial Rs(t)$. Evaluating (8) on sets of roots with different overall multiplicities allows us to extract curvilinear edges and faces from the reachable set. Some generalization of the above arguments can be seen in [11].

Lemma 2. *Suppose that $A_1 = J^{-1}AJ$ is the Jordan form of the matrix A from system* (5), *$\mathcal{U}_1 = J^{-1}\mathcal{U}$, where $J \in \mathbb{R}^{n\times n}$ is the transfer matrix. If the set $\mathcal{R}_1(t) = \int_0^t e^{A_1 s}\mathcal{U}_1\,ds$ is strongly convex with radius r, then $\mathcal{R}(t) = \int_0^t e^{As}\mathcal{U}\,ds$ is also strongly convex with radius $R = r\alpha^2/\beta$, where $\alpha = \|J\| = \max_{\|h\|=1}\|Jh\|$, $\beta = \min_{\|h\|=1}\|Jh\|$.*

Note that by [25, Theorem 3] any ellipsoid

$$\mathcal{N} = \left\{ x \in \mathbb{R}^n \ :\ \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2} \leqslant 1 \right\}, \quad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0,$$

is strongly convex with radius $R = \frac{\lambda_1^2}{\lambda_n}$.

Lemma 3. *Suppose that in system* (5) *$\mathcal{U}$ is uniformly smooth with constant $r > 0$. Then $\mathcal{R}(t)$* (6) *is uniformly smooth with constant $r_0 = r\int\limits_0^t \frac{\lambda_n^2(s)}{\lambda_1(s)}\,ds$, where $\lambda_1(s) \geqslant \ldots \geqslant \lambda_n(s) > 0$ are the semiaxes of the ellipsoid $e^{As}\mathcal{B}_1(0)$.*

Note that by the proof of Lemma 3 any ellipsoid

$$\mathcal{N} = \left\{ x \in \mathbb{R}^n \ :\ \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2} \leqslant 1 \right\}, \quad \lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n > 0,$$

is uniformly smooth with constant $r = \frac{\lambda_n^2}{\lambda_1}$.

In particular, Lemmas 2 and 3 show that it is enough to consider system (5) with the Jordan form of the matrix A.

The next proposition estimates the rate of decrease for a Lipschitz differentiable function per step of the gradient projection method.

Proposition 2 [26, Lemma 2]. *Consider the problem* $\min_{\mathcal{M}} f(x)$ *in* $\mathbb{R}^n$. *Suppose that* $\mathcal{M}$ *is a closed set,* f' *is a Lipschitz function with constant* L_1. *Fix* $0 < \lambda \leqslant \frac{1}{L_1}$. *Assume that* $x_0 \in \mathcal{M}$ *and* $y_0 \in P_{\mathcal{M}}(x_0 - \lambda f'(x_0))$. *Then*

$$f(x_0) - f(y_0) \geqslant \frac{1}{2}\left(\frac{1}{\lambda} - L_1\right)\|x_0 - y_0\|^2.$$

For the validity of the previous formula the Lipschitz condition for f' with constant L_1 is essential on the segment $[x_0, y_0]$, see the proof of [27, Proposition 2.2].

1.2. Additional Assumptions on $\mathcal{R}(s)$

When solving problems (P1)–P(3) we will require some additional assumptions on sets we work with. Here we will enumerate all of them, we will only need some of them for each problem.

(1) $\mathcal{R}(s)$ is strongly convex with radius $R_T > 0$ for all $s \in [0, T]$.
(2) $\mathcal{M}$ is uniformly smooth with constant $r > 0$: $\mathcal{M} = \mathcal{M}_0 + \mathcal{B}_r(0)$, also
  (a) $\mathcal{M}_0$ us strongly convex with constant $R_0 > 0$.
  (b) $r > R_T$.
(3) $\mathcal{M}$ is strongly convex with constant $R_0 > 0$.
(4) $\mathcal{U}$ is uniformly smooth with constant $r_{\mathcal{U}} > 0$: $\mathcal{U} = \mathcal{U}_0 + \mathcal{B}_{r_{\mathcal{U}}}(0)$.
(5) $r(t) > R_0$, where $r(t) = r_{\mathcal{U}} \int_0^t \frac{\lambda_n^2(s)}{\lambda_1(s)}$ and $\lambda_1(s) \geqslant \ldots \geqslant \lambda_n(s)$ are the semiaxes of the ellipsoid $e^{As}\mathcal{B}_1(0)$.

The first assumption is fulfilled if, for example, the set $e^{As}\mathcal{U}$ is strongly convex with radius $R(s) > 0$. Then from proposition 1 and linearity of the integral it follows that

$$\mathcal{R}(T) = \int_0^T e^{As}\mathcal{U}\,ds = \int_0^t e^{As}\mathcal{U}\,ds + \int_t^T e^{As}\mathcal{U}\,ds = \mathcal{R}(t) + \int_t^T e^{As}\mathcal{U}\,ds,$$

then we obtain that the set

$$\mathcal{R}(t) = \bigcap \left\{ \mathcal{R}(T) - x \ : \ x \in \int_t^T e^{As}\mathcal{U}\,ds \right\}$$

is strongly convex with radius $R_T = \int_0^T R(s)\,ds$ for all $t \in [0, T]$.

1.3. Structure of the Paper

In Sections 2–4 we formulate sufficient conditions and prove results about linear convergence of the gradient projection method for a particular optimization problem with supporting functions to which problems (P1)–(P3) are reduced. This solves problems for a fixed $t \in [0, T]$.

In Section 5 we discuss how we can find the starting point p_1 for the iteration process. Estimates of the probability of finding p_1 using random search are given.

In Section 6 we discuss the results of numerical experiments. Here we also consider an algorithm for finding the optimal t in problems (P1)–(P3).

2. PROBLEM (P1)

Assumptions: 1, 2(a).

For all $t \in [0,T]$ consider the set $\mathcal{N}(t) = \mathcal{R}(t) + (-\mathcal{M}_0)$. The set $\mathcal{N}(t)$ is strongly convex with radius $R = R_T + R_0$ as a sum of strongly convex sets [19]. The equality $\mathcal{R}(t) \cap \mathcal{M} = \emptyset$ can be reformulated as follows: the distance from zero to $\mathcal{N}(t)$ is more than $r > 0$. If the last assertion is true, then $0 \notin \mathcal{R}(t) + (-\mathcal{M})$ and otherwise $0 \in \mathcal{R}(t) + (-\mathcal{M})$. Using the supporting function we can check the inclusion as follows: for the function $f(p) = s(p, \mathcal{N}(t)) = s(p, \mathcal{R}(t)) + s(p, -\mathcal{M}_0)$ find

$$\min_{\|p\|=1} f(p) = J. \tag{11}$$

If $J < -r$, then the distance from zero to the set $\mathcal{N}(t)$ is greater than r. If $J \geqslant -r$, then the distance from zero to the set $\mathcal{N}(t)$ is no greater than r and hence $0 \in \mathcal{R}(t) + (-\mathcal{M})$. Note that

$$f'(p) = \mathcal{R}(t)(p) + (-\mathcal{M}_0)(p) = \int_0^t (e^{As}\mathcal{U})(p)\,ds + (-\mathcal{M}_0)(p). \tag{12}$$

Theorem 1. *Fix $\varepsilon \in (0,1)$. Suppose that in (11) $J < 0$. Then under above mentioned assumptions the function f in (11) satisfies the LPL condition on the manifold $\mathcal{S} = \{p \in \mathcal{S}_1 : f(p) \leqslant 0\}$ with constant $\mu = |J|$. Also the function f has Lipschitz continuous gradient on the set $\{p \in \mathbb{R}^n : 1-\varepsilon \leqslant \|p\| \leqslant 1+\varepsilon\}$ with Lipschitz constant $L_1 = \frac{R}{1-\varepsilon} = \frac{R_T+R_0}{1-\varepsilon}$.*

Consider the following iteration process

$$p_1 \in \mathcal{S} \text{ (i.e. } f(p_1) \leqslant 0), \quad p_{k+1} = P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)), \qquad \lambda \in \left(0, \frac{1}{L_1}\right]. \tag{13}$$

If $p_k \in \mathcal{S}$, then $p_{k+1} \in \mathcal{S}$. Indeed, by Proposition 2

$$f(p_k) - f(p_{k+1}) \geqslant \frac{1}{2}\left(\frac{1}{\lambda} - L_1\right)\|p_k - p_{k+1}\|^2 \geqslant 0, \quad f(p_{k+1}) \leqslant f(p_k) \leqslant 0.$$

Consider the point $p_k - \lambda f'(p_k)$. We have

$$\|p_k - \lambda f'(p_k)\| \geqslant (p_k, p_k - \lambda f'(p_k)) = 1 - \lambda(p_k, f'(p_k)) = 1 - \lambda f(p_k) \geqslant 1.$$

Theorem 2. *Suppose that the function f is Lipschitz continuous with constant $L = \|\mathcal{N}(t)\|$, the function f' is Lipschitz continuous on S_1 with constant $R = R_T + R_0$. Suppose that $J < 0$. Put $L_1 = 2R$.*

Fix $\lambda \in (0, \min\{\frac{1}{L_1}, \frac{1}{2L}\})$. Then algorithm (13) converges to a point of minimum $p_0 \in \mathcal{S}_1$ at a linear rate:

$$f(p_{k+1}) - f(p_0) \leqslant q(f(p_k) - f(p_0)),$$
$$\|p_{k+1} - p_k\| \leqslant q^{k/2}\sqrt{2\lambda(f(p_1) - f(p_0))},$$
$$q = 1 - \frac{\lambda|J|}{2L\lambda + 2} \in (0,1).$$

The next example shows that the sharpness condition of the type $\exists \alpha > 0$ that $f(p) - f(p_0) \geqslant \alpha\|p - p_0\|$ for all $p \in \mathcal{S}$ does not hold.

Consider $L > r > 0$, $\|p_0\| = 1$ and the set $\mathcal{N} = \mathcal{B}_r(-Lp_0)$. Then for all $p \in \mathcal{S}_1$ we have

$$s(p, \mathcal{N}) - s(p_0, \mathcal{N}) = L(1 - (p, p_0)) = \frac{L}{2}\|p - p_0\|^2.$$

Remark 1. The above results can be proven under more local assumptions. Instead of the strong convexity assumption 1 of $\mathcal{R}(T)$ with radius R_T we can require the fulfillment for all $p \in \mathcal{S}$ the supporting principle for the set $\mathcal{R}(t)$: there exists $R_T > 0$ with

$$\mathcal{R}(t) \subset B_{R_T}(\mathcal{R}(t)(p) - R_T p), \quad \forall p \in \mathcal{S}. \tag{14}$$

Assumption 2(a) concerning $\mathcal{M}$ must be met.

In this situation the set $Z(t) = \mathcal{R}(t) + (-\mathcal{M}_0)$ satisfies the supporting principle for all $p \in \mathcal{S}$ with radius $R = R_T + R_0$:

$$Z(t) \subset B_R(\mathcal{N}(t)(p) - Rp), \quad \forall p \in \mathcal{S}.$$

For any $p, q \in \mathcal{S}$ we get $\|\mathcal{N}(t)(p) - Rp - \mathcal{N}(t)(q)\|^2 \leqslant R^2$, $\|\mathcal{N}(t)(q) - Rq - \mathcal{N}(t)(p)\|^2 \leqslant R^2$ and $\|\mathcal{N}(t)(p) - \mathcal{N}(t)(q)\|^2 \leqslant 2R(p, \mathcal{N}(t)(p) - \mathcal{N}(t)(q))$,

$$\|\mathcal{N}(t)(q) - \mathcal{N}(t)(p)\|^2 \leqslant 2R(q, \mathcal{N}(t)(q) - \mathcal{N}(t)(p)) = 2R(-q, \mathcal{N}(t)(p) - \mathcal{N}(t)(q)),$$

hence $\|\mathcal{N}(t)(p) - \mathcal{N}(t)(q)\| \leqslant R\|p - q\|$. Keeping in mind that for any $p, q \in \mathcal{S}$ the small arc of the circle of radius 1 with center 0 and endpoints p, q belongs to $\mathcal{S}$, we can repeat proofs of Theorems 1 and 2 for the considered case. In the generalization of Theorem 1 we should take $p, q \in \mathbb{R}^n$ with $\frac{p}{\|p\|}$, $\frac{q}{\|q\|} \in \mathcal{S}$, i.e. Lipschitz condition will be proved on the set $\left\{p \in \mathbb{R}^n \, : 1 - \varepsilon \leqslant \|p\| \leqslant 1 + \varepsilon, \ \frac{p}{\|p\|} \in \mathcal{S}\right\}$.

3. PROBLEM (P2)

Assumptions: 1, 2(b), 3.

Fix $\varepsilon \in (0, r - R_T)$. Consider ε-neighbourhood $\mathcal{R}_\varepsilon(t) = \mathcal{R}(t) + \mathcal{B}_\varepsilon(0)$ of the set $\mathcal{R}(t)$. Inclusion $\mathcal{R}(t) \subset \mathcal{M}$ means that

$$\max_{x \in \mathcal{R}_\varepsilon(t)} \varrho(x, \mathcal{M}) \leqslant \varepsilon$$

and otherwise, if $\max_{x \in \mathcal{R}_\varepsilon(t)} \varrho(x, \mathcal{M}) > \varepsilon$, then $\mathcal{R}(t) \not\subset \mathcal{M}$. Using supporting functions we can formulate an equivalent problem: for the function $f(p) = s(p, \mathcal{M}) - s(p, \mathcal{R}_\varepsilon(t))$ find minimum

$$\min_{\|p\|=1} f(p) = J. \tag{15}$$

If $J \geqslant -\varepsilon$ then $\mathcal{R}(t) \subset \mathcal{M}$ and if $J < -\varepsilon$ then $\mathcal{R}(t) \not\subset \mathcal{M}$.

Let $\mathcal{S} = \{p \in \mathcal{S}_1 \ : \ f(p) \leqslant 0\}$. Suppose that $p_0 \in \mathcal{S}_1$ is a solution of (15).

Assume that $\mathcal{S} \neq \emptyset$. Consider an iteration process

$$p_1 \in \mathcal{S}, \qquad p_{k+1} = P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)). \tag{16}$$

Theorem 3. *Suppose that under assumptions of Section 3 we have* $J < 0$ *in problem* (15). *Let* $r_0 = r - R_T - \varepsilon > 0$, $L = \|\mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t)\| > 0$. *Then for any* $p_1 \in \mathcal{S}$ *and* $0 < \lambda \leqslant \min\{r_0^2/R_0^3, 1/(2L), 1/(2R_0)\}$ *iterations* (16) *converge at a linear rate to the solution* p_0*:*

$$\|p_{k+1} - p_0\| \leqslant q\|p_k - p_0\|, \qquad q = \sqrt{1 - \frac{2r_0^2}{R_0}\lambda + R_0^2\lambda^2} \in (0, 1).$$

Remark 2. As in Section 2, we can prove the above results under more local assumptions. Instead of the Assumption 1 on strong convexity of $\mathcal{R}(s)$ for all $s \in [0, T]$ with radius R_T we can require the fulfillment for all $p \in \mathcal{S}$ of the supporting principle for the set $\mathcal{R}(t)$: there exists $R_T > 0$ such that for a number $\varepsilon \in (0, r - R_T)$ we have

$$\mathcal{M}(p) - \mathcal{R}(t)(p) + \mathcal{R}(t) \subset B_{R_T}(\mathcal{M}(p) - R_T p) \subset B_{r-\varepsilon}(\mathcal{M}(p) - (r - \varepsilon)p) \subset \mathcal{M}, \quad \forall p \in \mathcal{S}. \tag{17}$$

Assumptions 2(b), 3 concerning $\mathcal{M}$ must be met.

In the considered situation we have

$$\mathcal{M}(p) - \mathcal{R}(t)(p) - \varepsilon p + \mathcal{R}_\varepsilon(t) \subset \mathcal{M}, \quad \forall p \in \mathcal{S} \tag{18}$$

and hence $f'(p) = \mathcal{M}(p) - \mathcal{R}(t)(p) - \varepsilon \times p = \mathcal{M}(p) - \mathcal{R}_\varepsilon(t)(p) = \left(\mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t)\right)(p)$ for all $p \in \mathcal{S}$ because $f'(p) \in \mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t)$ and $(p, f'(p)) = s(p, \mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t))$ for all $p \in \mathcal{S}$. Indeed, fix $p \in \mathcal{S}$. From the inclusion $f'(p) + \mathcal{R}_\varepsilon(t) \subset \mathcal{M}$ we get $f'(p) \in \mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t)$. On the other hand $(p, f'(p)) + s(p, \mathcal{R}_\varepsilon(t)) = s(p, \mathcal{M})$ and thus $(p, f'(p)) = s(p, M) - s(p, \mathcal{R}_\varepsilon(t)) \geqslant \operatorname{co}(s(p, M) - s(p, \mathcal{R}_\varepsilon(t)) = s(p, \mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t))$.

The next steps repeat the proof of Theorem 3.

4. PROBLEM (P3)

Assumptions: 1, 3, 4, 5.

Note that by Lemma 3 the set $\mathcal{R}(t)$ is uniformly smooth with constant $r(t)$ and hence $R_T \geqslant r(t)$.

Fix $\varepsilon \in (0, r(t) - R_0)$. Consider ε-neighbourhood $\mathcal{M}_\varepsilon = \mathcal{M} + \mathcal{B}_\varepsilon(0)$ of the set $\mathcal{M}$. Inclusion $\mathcal{R}(t) \supset \mathcal{M}$ means that

$$\max_{x \in \mathcal{M}_\varepsilon} \varrho(x, \mathcal{R}(t)) \leqslant \varepsilon$$

and otherwise, if $\max_{x \in \mathcal{M}_\varepsilon} \varrho(x, \mathcal{R}(t)) > \varepsilon$, then $\mathcal{R}(t) \not\supset \mathcal{M}$. On the base of supporting functions we can formulate the next equivalent problem: for the function $f(p) = s(p, \mathcal{R}(t)) - s(p, \mathcal{M}_\varepsilon) = s(p, \mathcal{R}(t)) - s(p, \mathcal{M}) - \varepsilon\|p\|$ find minimum

$$\min_{\|p\|=1} f(p) = J. \tag{19}$$

If $J \geqslant -\varepsilon$ then $\mathcal{R}(t) \supset \mathcal{M}$ and if $J < -\varepsilon$ then $\mathcal{R}(t) \not\supset \mathcal{M}$.

As usual, $\mathcal{S} = \{p \in \mathcal{S}_1 \;:\; f(p) \leqslant 0\}$. Suppose that $p_0 \in \mathcal{S}_1$ is a solution of (19).

Assume that $\mathcal{S} \neq \emptyset$. Consider an iteration process

$$p_1 \in \mathcal{S}, \qquad p_{k+1} = \mathcal{P}_{\mathcal{S}_1}(p_k - \lambda f'(p_k)). \tag{20}$$

Theorem 4. *Suppose that under assumptions of Section 4 we have $J < 0$ in problem (19). Let $r = r(t) - R - \varepsilon > 0$, $L = \|\mathcal{R}(t) \overset{*}{-} \mathcal{M}_\varepsilon\|$. Then for any $p_1 \in \mathcal{S}$ and $0 < \lambda \leqslant \min\{r^2/R_T^3, 1/(2L), 1/(2R_T)\}$ iterations (20) converges at a linear rate to the solution p_0:*

$$\|p_{k+1} - p_0\| \leqslant q\|p_k - p_0\|, \qquad q = \sqrt{1 - \frac{2r^2}{R_T}\lambda + R_T^2\lambda^2} \in (0, 1).$$

Remark 3. As in Section 3, we can also prove the above results under more local assumptions. Instead of the strong convexity Assumption 3 of $\mathcal{M}$ with radius R_0 we can require the fulfillment for all $p \in \mathcal{S}$ of the supporting condition for the set $\mathcal{M}$: there exists $R_0 > 0$ such that

$$\mathcal{M} \subset B_{R_0}(\mathcal{M}(p) - R_0 p), \quad \forall p \in \mathcal{S}. \tag{21}$$

Assumptions 1, 4 and 5 must be met.

5. CHOOSING THE INITIAL POINT

We choose p_1 using random search: in problems (P1)–(P3) we sample a random vector $p_1 \in \mathcal{S}_1$ from a uniform distribution and check the inequality $f(p_1) \leqslant 0$. If it fails, we choose another random vector $p_1 \in \mathcal{S}_1$ and so on. In the present section we estimate the probability $\mathbb{P}(\{f(p_1) \leqslant 0\})$ to find an appropriate vector p_1. As an example, let us consider (P1) for fixed $t > 0$. Recall, that

$J < 0$ is the solution of problem (11). By assumptions for (P1), the set $\mathcal{N}(t)$ is strongly convex with radius $R > 0$. Denote $z_0 = P_{\mathcal{N}(t)}0$, $p_0 = -z_0/\|z_0\|$. For a set $\mathcal{M} \subset \mathbb{R}^n$ define cone $\mathcal{M}$ to be the (convex) conic hull of the set $\mathcal{M}$, i.e. $\operatorname{cone}\mathcal{M} = \{\sum_{i=1}^n \lambda_i x_i \ : \ x_i \in \mathcal{M},\ \lambda_i \geqslant 0\}$. For a pair of points $x, y \in \mathbb{R}^n$, $x \neq y$, define the ray $[x, y) = \{x + t(y - x) \ : \ t \geqslant 0\}$.

Let $D > 0$ and $H = \{x \in \mathbb{R}^n : (p_0, x - z_0) = 0\}$. Suppose that $\mathcal{K} = \operatorname{cone}\,(H \cap B_D(z_0)) \supset \operatorname{cone}\mathcal{N}(t)$. For example, D can be the diameter of the set $\mathcal{N}(t)$, i.e. $D = \sup_{x,y\in\mathcal{N}(t)} \|x - y\|$.

The set $\mathcal{K}$ is a cone of revolution with axis $[0, z_0)$. The angle between the axis and a generatrix is equal to α, $\tan\alpha = \frac{D}{|J|}$. The polar set $\mathcal{K}^- = \{p \in \mathbb{R}^n \ : \ (p, q) \leqslant 0 \quad \forall q \in \mathcal{K}\}$ is also a cone of revolution with axis $[0, -z_0)$ and the angle between the axis and a generatrix is equal to $\beta = \frac{1}{2}\pi - \alpha$, thus $\cos\beta = \frac{D}{\sqrt{D^2+J^2}}$.

By the definition of $\mathcal{K}$ we have for any $p_1 \in \mathcal{S}_1 \cap \mathcal{K}^-$ that $f(p_1) \leqslant 0$. Denote $\mathcal{S}_{\text{cap}} = \mathcal{S}_1 \cap \mathcal{K}^-$ and $\mathcal{S}_0 = \mathcal{K}^- \cap H_0$, here $H_0 = \{x \in \mathbb{R}^n \ : \ (p_0, x) = \cos\beta\}$. Note that $\mathcal{S}_0 = H_0 \cap B_{r_0}(\cos\beta \times p_0)$ with $r_0 = \sin\beta = \frac{|J|}{\sqrt{D^2+J^2}}$. $(n-1)$-Lebesgue's measure $\mu_{n-1}\mathcal{S}_0 \leqslant \mu_{n-1}\mathcal{S}_{\text{cap}}$ and thus

$$\mathbb{P}(\{f(p_1) \leqslant 0\}) \geqslant \frac{\mu_{n-1}\mathcal{S}_{\text{cap}}}{\mu_{n-1}\mathcal{S}_1} \geqslant \frac{\mu_{n-1}\mathcal{S}_0}{\mu_{n-1}\mathcal{S}_1} = \frac{r_0^{n-1}}{n}\frac{V_{n-1}}{V_n} = \frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{|J|}{\sqrt{D^2+J^2}}\right)^{n-1},$$

$V_n = \frac{\pi^{n/2}}{\Gamma\left(\frac{n}{2}+1\right)}$ is the volume of a unit ball in $\mathbb{R}^n$.

Suppose now that $B_r(z_0 - rp_0) \subset \mathcal{N}(t)$ for some $r > 0$. Then consider a cone of revolution $\mathcal{K} = \operatorname{cone} B_r(z_0 - rp_0) \subset \operatorname{cone}\mathcal{N}(t)$ with axis $[0, z_0)$. The angle between the axis and a generatrix of $\mathcal{K}$ is equal to α, $\sin\alpha = \frac{r}{r+|J|}$. Define a polar cone $\mathcal{K}^- \supset (\operatorname{cone}\mathcal{N}(t))^-$ with the angle β between the axis $[0, -z_0)$ and a generatrix, $\cos\beta = \frac{r}{r+|J|}$. We have for any $p_1 \in \mathcal{S}_1$ with $f(p_1) \leqslant 0$ that $p_1 \in \mathcal{S}_{\text{cap}}$, as previously $\mathcal{S}_{\text{cap}} = \mathcal{S}_1 \cap \mathcal{K}^-$. Define $\mathcal{S}_0^1 = \mathcal{K}^- \cap H_1$ with $H_1 = \{x \in \mathbb{R}^n \ : \ (p_0, x) = 1\}$. From the elementary planimetry it is easy to see that $\mathcal{S}_0^1 = H_1 \cap B_{r_1}(p_0)$, $r_1 = \tan\beta = \frac{\sqrt{2r|J|+|J|^2}}{r}$. Then $\mu_{n-1}\mathcal{S}_0^1 \geqslant \mu_{n-1}\mathcal{S}_{\text{cap}}$ and

$$\mathbb{P}(\{f(p_1) \leqslant 0\}) \leqslant \frac{\mu_{n-1}\mathcal{S}_{\text{cap}}}{\mu_{n-1}\mathcal{S}_1} \leqslant \frac{\mu_{n-1}\mathcal{S}_0^1}{\mu_{n-1}\mathcal{S}_1} = \frac{r_1^{n-1}}{n}\frac{V_{n-1}}{V_n} = \frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{\sqrt{2r|J|+|J|^2}}{r}\right)^{n-1}.$$

Finally for a set $\mathcal{N}(t)$ of diameter D that is also uniformly smooth with constant r we have

$$\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{|J|}{\sqrt{D^2+J^2}}\right)^{n-1} \leqslant \mathbb{P}(\{f(p_1) \leqslant 0\}) \leqslant \frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{\sqrt{2r|J|+|J|^2}}{r}\right)^{n-1}. \tag{22}$$

Similarly with the right estimate in (22) for an R-strongly convex set $\mathcal{N}(t)$ one can prove that

$$\frac{1}{n}\frac{V_{n-1}}{V_n}\left(\frac{\sqrt{2R|J|+|J|^2}}{R+|J|}\right)^{n-1} \leqslant \mathbb{P}(\{f(p_1) \leqslant 0\}).$$

This estimate shows that $\mathbb{P}(\{f(p_1) \leqslant 0\}) \asymp |J|^{n-1}$ when $J \to 0$. In our consideration $|J|$ is of the order $\varepsilon > 0$ and in this case the left inequality in (22) gives a more reasonable estimate because in most examples the value of D is much less than R.

The estimated probability can be very small and strongly influences calculations when either $|J|$ is close to zero or n is large. In our experiments in the examples below for n in range $3 \leqslant n \leqslant 12$ we found p_1 in a few dozens attempts at most (for problems (P1), (P2)). Sometimes we needed about 1000 attempts to find the vector p_1 in problem (P3). One of the reasons is that $D > 0$ in the above estimate can be chosen to be significantly smaller than the diameter $\mathcal{N}(t)$, since we only need the fulfillment of the inclusion $\operatorname{cone}(H \cap B_D(z_0)) \supset \operatorname{cone}\mathcal{N}(t)$.

Sometimes we can choose p_1 deterministically, see the Algorithm from Section 9.

The step size λ for solving problems (P1)–(P3) can be chosen using the Armijo rule. Its detailed description can be found in [29].

6. MODELING AND EXAMPLES

Some of the considered examples are low-dimensional ($n = 3$) for ease of interpretation by a reader. As shown in the following, convergence rates for such examples and for examples of higher dimension are the same.

6.1. Problem (P1). Example 1

In this example we calculate the point of time at which the reachable set $\mathcal{R}$ first intersects the target set $\mathcal{M}$.

Consider the system

$$\dot{x} = Ax + Bu, \quad x(0) = 0, \quad u \in \mathbb{R}: \quad |u| \leqslant 1, \quad A = \begin{bmatrix} -1.3 & 1 & 0 \\ 0 & -1.3 & 1 \\ 0 & 0 & -1.3 \end{bmatrix}, \quad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{23}$$

The target set is $\mathcal{M} = \mathcal{M}_0 + B_r(0)$, where $\mathcal{M}_0$ is the ball $B_{0.2}(0.7, -0.3, 0.35)$, $r = 0.5$. Recall that $f(p)$ in problem (11) depends on t, i.e. $f(p,t) = s(p, \mathcal{R}(t)) + s(p, -\mathcal{M}_0)$.

We first consider the auxiliary problem of finding the distance between sets $\mathcal{R}(t)$ and $\mathcal{M}$ for $t = 1$, with initial condition $p_1 = (0{,}03123620,\ -0{,}72453809,\ 0{,}68852659)$, $f(p_1, 1) = -0{,}05270947$.

Figure 2,a: Convergence of the gradient projection algorithm for the auxiliary problem $\min_{\|p\|=1} f(p,t)$ for $t = 1$. Approximation of the convergence rate is $f(p_k, 1) - f(p_0, 1) \approx 0.2486 \times 0.83043^k$. The found solution is $p_0 = (0.87540058,\ -0.46926876,\ 0.11602002)$ with $f(p_0, 1) = -\,0.573989$.

The reachable set and the point closest to the target set are depicted on Fig. 3.

When searching for the minimal time at which intersection occurs, we only know the search interval $[0, T]$, but not the starting point p_1 for arbitrary moment of time from the interval. There are two different strategies. The first one is to randomly find $p_1 \in \mathcal{S}_1$ with $f(p_1, t) < 0$ for a given t and increase t by a small amount. However, due to the time-related nature of (P1) there is a better algorithm. This algorithm involves keeping track of suitable p, $f(p,t) < 0$, while increasing the time.

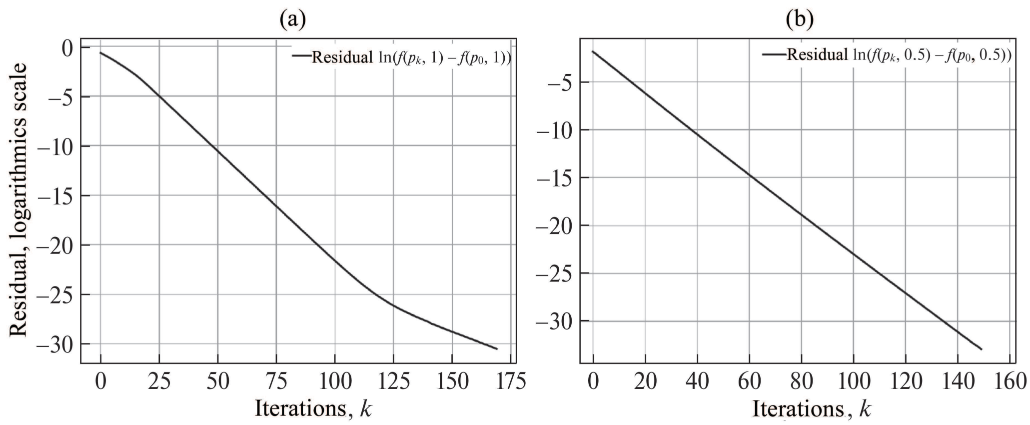

Fig. 2. Convergence of gradient projection algorithm with step size $\lambda = 0.1$. (a) Problem (P1), Example 1, (b) problem (P1), Example 2.

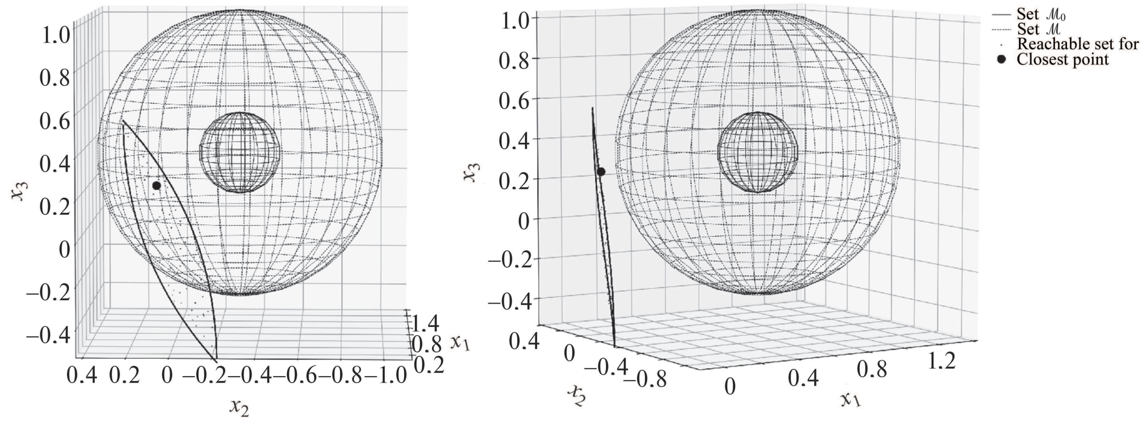

Fig. 3. The point of the reachable set $\mathcal{R}(t)$ $(t = 1)$ closest to the target set found by gradient projection algorithm (problem (P1), Example 1).

Algorithm for problem (P1) (finding minimal time)

Data: $T > 0$, $f(p,t)$, $r > 0$, tolerance $\varepsilon_{\text{tol}} > 0$, bounds $t_{\text{lower}} = 0, t_{\text{upper}} = T$, time step $\Delta_t > 0$.

(1) Put $t \leftarrow 0$ and find initial p_1 satisfying $f(p_1,0) < 0$ first. Then run the gradient projection method which gives $p(0) = \arg\min_{\|p\|=1} f(p,0) : f(p(0),0) < 0$.

(2) Put $t_{\text{test}} = \min\{t + \Delta_t, t_{\text{upper}}\}$.
If $f(p(t), t_{\text{test}}) \geqslant 0$, then set $\Delta_t \leftarrow \Delta_t/2$ and repeat this step.
If $f(p(t), t_{\text{test}}) < 0$, then proceed to Step (3).

(3) Run the gradient projection method (13) for function $f(p, t_{\text{test}})$ with initial point $p_1 = p(t)$. It results in p_0 and $J = f(p_0, t_{\text{test}}) = \min_{\|p\|=1} f(p, t_{\text{test}}) < 0$.

(4) If $J > -r + \varepsilon_{\text{tol}}$, then the reachable set intersects the set $\mathcal{M}$. Update $t_{\text{upper}} \leftarrow t_{\text{test}}$, $\Delta_t \leftarrow \frac{1}{2}\min\{\Delta_t, t_{\text{upper}} - t_{\text{lower}}\}$ and proceed to Step (2) with the same t and $p(t)$. Otherwise continue with Step (5).

(5) If $J < -r - \varepsilon_{\text{tol}}$, then the reachable set has yet to reach the set $\mathcal{M}$. Update $t_{\text{lower}} \leftarrow t_{\text{test}}$, $\Delta_t \leftarrow \min\{2\Delta_t, \frac{t_{\text{upper}} - t_{\text{lower}}}{2}\}$.
Also update $t \leftarrow t_{\text{test}}, p(t) \leftarrow p_0$ and continue with Step (2).
Otherwise finish with Step (6).

(6) A solution is found within given tolerance: $|J + r| \leqslant \varepsilon_{\text{tol}}$. Return $t_0 = t_{\text{test}}$ as the optimal time for problem (P1), and p_0.

Notes: the algorithm performs bisection-like search on the time interval $[0, T]$. Probability of finding suitable p_1 at Step (1) may be estimated using results from Section 5. However, it can be found non-randomly at Step (1) if we can somehow find a unit separation vector $p_1 \in \mathbb{R}^n$ such that $(p_1, x) \leqslant 0$ for all $x \in -\mathcal{M}_0$. Further at each Step (2), the initial value p_1 of the gradient projection algorithm is chosen non-randomly. At Step (5), the time step is doubled for faster search. The algorithm may also operate if the value T is unknown (i.e. $t_{\text{upper}} = \infty$), but for $t_{\text{upper}} > T$ convergence conditions for the gradient projection algorithm may be violated. Nevertheless the invariance $t_{\text{lower}} \leqslant t_{\text{test}} \leqslant t_{\text{upper}}$ is satisfied.

The algorithm stops when we obtain J with a given tolerance ε_{tol}, in all examples here and below $\varepsilon_{\text{tol}} = 10^{-7}$ and at the final stage $t_{\text{upper}} - t_{\text{lower}} \sim 10^{-6}$. We also can stop the algorithm with a given precision with respect to the time t: e.g. when $t_{\text{upper}} - t_{\text{lower}} \leqslant \varepsilon_{time}$ we finish calculations and take $t \in [t_{\text{lower}}, t_{\text{upper}}]$. Here $\varepsilon_{time} > 0$ is an admissible time error.

For system (23) Algorithm converges in 21 steps. The optimal time is 2.7383842,

$$p_0 = (0.77091811, -0.60777697, 0.19050571).$$

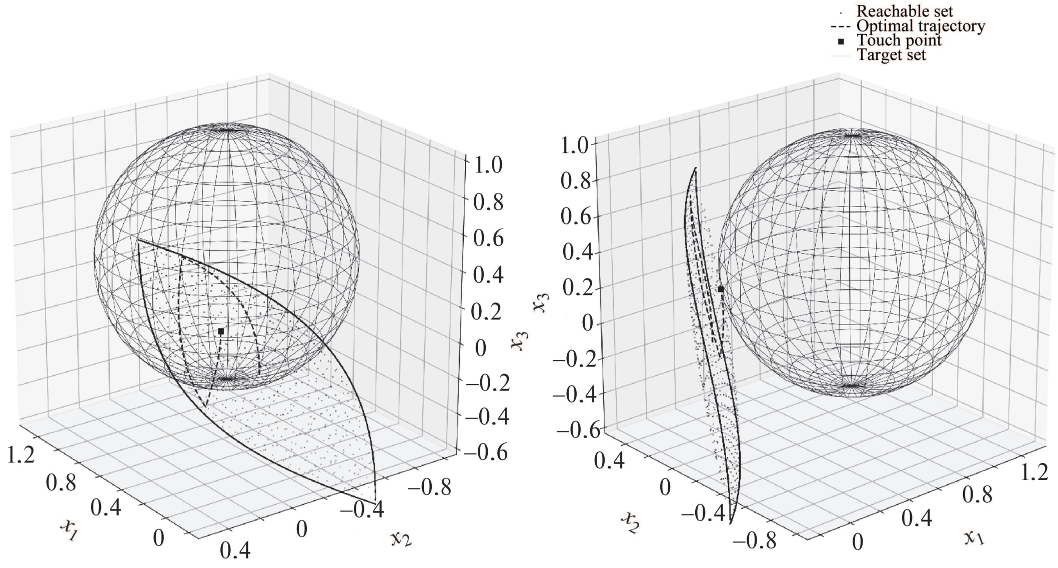

Fig. 4. Attainable set at the moment of intersection and the optimal trajectory (problem (P1), Example 1).

Figure 4 depicts the reachable set and the target set at the moment when they intersect. The optimal trajectory (as in 9) with two switches can also be seen. Also see [24].

As was shown in Introduction, the reachable set $\mathcal{R}(t)$ of system (23) is not strongly convex. For $\mathcal{U} = B\times[-1,1]$ and $t>0$ we have $s(p,\mathcal{R}(t)) = \int_0^t e^{-1.3s}|p_1\frac{s^2}{2} + p_2 s + p_3|\,ds$ for any $p = (p_1,p_2,p_3)\in\mathcal{S}_1$. For the solution $p_0 = (0.77091811, -0.60777697, 0.19050571)$ and $t = 2.73838\dots$ we have the roots $s_1(p_0) < s_2(p_0)$ of the equation $p_1\frac{s^2}{2} + p_2 s + p_3 = 0$ for $p = p_0$. By the inverse function theorem the roots $\mathcal{S}_1 \ni p \to s_i(p)$, $i = 1, 2$, of the equation $p_1\frac{s^2}{2} + p_2 s + p_3 = 0$ are analytic in some neighbourhood of the point $p_0 \in \mathcal{S}_1$. In other words, there exists a number $\gamma > 0$ such that the functions

$$\mathcal{S}_1 \cap B_\gamma(p_0) \ni p \to s_i(p), \quad i = 1, 2,$$

are Lipschitz continuous with some constant $L > 0$. Moreover, we can choose the number $\gamma > 0$ so that the first components of p and q are strictly positive and $\max\{s_1(p), s_1(q)\} \leqslant \min\{s_2(p), s_2(q)\}$ for all $p, q \in \mathcal{S}_1 \cap B_\gamma(p_0)$.

Fix a pair of points $p, q \in \mathcal{S}_1 \cap B_\gamma(p_0)$. Put $M = \max\limits_{s\in[0,t]} \|e^{As}\|$. Then $|s_i(p) - s_i(q)| \leqslant L\|p - q\|$ for $i = 1, 2$ and for the supporting elements, using the estimate $\|\mathcal{U}(e^{A^Ts}p) - \mathcal{U}(e^{A^Ts}q)\| \leqslant 2$, we have

$$\|\mathcal{R}(t)(p) - \mathcal{R}(t)(q)\| = \sum_{i=1}^{2} \left\| \int\limits_{s_i(p)}^{s_i(q)} e^{As}(\mathcal{U}(e^{A^Ts}p) - \mathcal{U}(e^{A^Ts}q))\,ds \right\| \leqslant 4ML\|p - q\|.$$

Thus the part of surface $\{\mathcal{R}(t)(p) \ : \ p \in \mathcal{S}_1 \cap B_\gamma(p_0)\}$ is a part of a strongly convex set with radius $R = 4ML$. In the present example it's enough for convergence of the gradient projection algorithm at time t. The same situation takes place for a time less than t.

6.2. Problem (P1), Example 2

Consider an example in $\mathbb{R}^{12}$.
$A = \mathrm{diag}(-0.3, -0.8, -1, -0.7, -0.71, -0.52, -0.37, -0.05, -0.25, -0.89, -0.99, -0.2)$, $\mathcal{U} = B_1(0)$. The target set is $\mathcal{M} = \mathcal{M}_0 + B_r(0)$, where $\mathcal{M}_0$ is the ball $B_{0.4}(0.3 \times \mathbf{1})$ ($\mathbf{1} = (1, 1, \dots, 1)$), $r = 0.2$, step-size $\lambda = 0.1$.

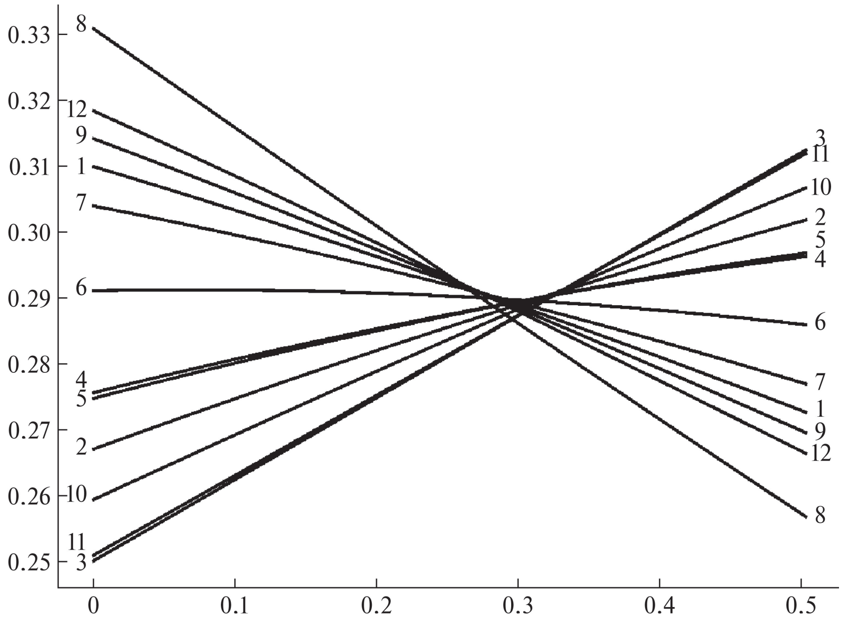

Fig. 5. The kth components u_k of the optimal control (problem (P1), Example 2).

Figure 2,b: convergence of the gradient projection algorithm for the auxiliary problem $\min_{\|p\|=1} f(p,t)$ for the time $t = 0.5$ and the initial condition

$$p_1 = (0.02046203,\ 0.24278712,\ 0.2199823,\ 0.33539534,\ 0.11750331,\ 0.07584814,\ 0.44196329,\ 0.14159412,\ 0.08314335,\ 0.32560626,\ 0.49401057,\ 0.43339861)$$

with $f(p_1, 0.5) = -0.047713028083805786$.

Approximation of convergence rate is $f(p_k, 0.5) - f(p_0, 0.5) \approx 0.1218 \times 0.8122^k$.

The optimal value is

$$p_0 = (0.2730037,\ 0.30197686,\ 0.3125336,\ 0.29647251,\ 0.29702965,\ 0.28619273,\ 0.27727228,\ 0.2572461,\ 0.26991497,\ 0.30680235,\ 0.31202019,\ 0.26679398)$$

with $f(p_0, 0.5) = -0.2023841828091369$.

Algorithm converges in 21 steps to the point

$$p_0 = (0.27281666,\ 0.3021221,\ 0.31280135,\ 0.29655398,\ 0.29711758,\ 0.28615572,\ 0.27713348,\ 0.25688441,\ 0.26969324,\ 0.30700357,\ 0.31228196,\ 0.26653741)$$

and the optimal time is 0.503150463104248.

Figure 5 illustrates the optimal control (per components, each line means one of 12 components).

6.3. Problem (P2). Example 3

The reachable set (as in (23)) is touching the target set from the inside.

The target set is the ellipsoid $\mathcal{M} = \{x : (x-c)^T Q(x-c) \leqslant R^2\}$, with

$$Q = \begin{bmatrix} 4.5 & -1.2 & -1.6 \\ -1.2 & 6.8 & -2.3 \\ -1.6 & -2.3 & 8 \end{bmatrix}, \quad c = \begin{bmatrix} -3.4 \\ -3.8 \\ 0.3 \end{bmatrix}, \quad R = 12.$$

Recall that $f(p,t) = s(p,\mathcal{M}) - s(p,\mathcal{R}_\varepsilon(t))$, here we take $\varepsilon = 0.05$, step-size $\lambda = 0.2$.

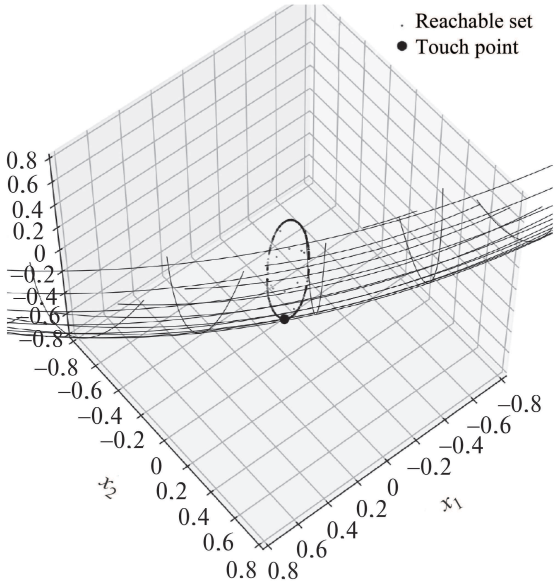

Fig. 6. Solution of problem (P2), Example 3.

Figure 6: For system (23) a similar bisection algorithm converges in 19 steps (i.e. $|J + \varepsilon| \leqslant \varepsilon_{\text{tol}} = 10^{-7}$). The optimal time is $t = 1{,}64610733$, $p_0 = (0{,}36800454,\ 0{,}72705740 - 0{,}57962073)$.

6.4. Problem (P2). Example 4. Homothete Inside the Target Set

We solve problem (P2) for a homothete, i.e. the problem is stated as

$$\max_{t \geqslant 0} t : \quad t\mathcal{R} \subset \mathcal{M}. \tag{24}$$

Define $\mathcal{M} = B_{10}(0)$, i.e. the ball centered at 0 of raduis 10. The set $\mathcal{R}$ is a strongly convex segment with endpoints $[-0.1, 3, 2.05884573]$, $[-1.9, 3, -1.05884573]$ and radius of strong convexity $R = 3$, i.e. $\mathcal{R}$ is the intersection of all closed balls of radius $R = 3$ containing the endpoints.

The supporting element for a unit vector $p = (p_1, \dots, p_n)$ for a strongly convex segment with endpoints $[-ae_1, ae_1]$ and radius of strong convexity $R > a$ is equal to $Rp - \frac{\sqrt{R^2-a^2}}{\sqrt{1-p_1^2}}(I - e_1 e_1^T)p$ if $\arctan(\frac{p_1}{\sqrt{1-p_1^2}}) < \arcsin(\frac{a}{R})$, otherwise it is equal to $\operatorname{sign}(p_1) a e_1$. We shall consider the homothety $t\mathcal{R}$, with parameter $\varepsilon = 0.1$ in the definition of f in (15), and step-size $\lambda = 0.2$.

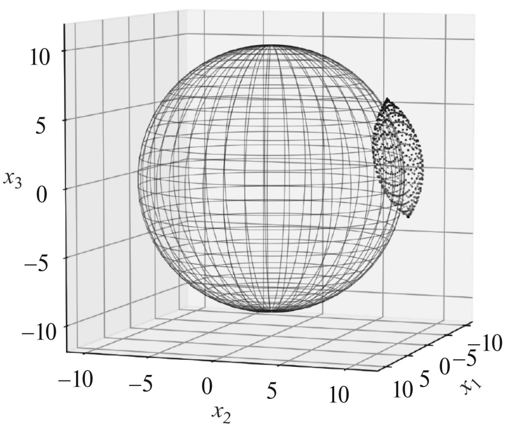

Fig. 7. Problem (P2), Example 4. The homothete is not contained inside $\mathcal{M}$ when $t = 3$.

For $t = 3$ the set $t\mathcal{R}$ is not contained in $\mathcal{M}$ (see Fig. 7). An algorithm, similar to one for problem (P2), in 21 steps gives the optimal value $t_0 = 2.62904820$ and $p_0 = (-0.3425777, 0.93398621, 0.10153957)$ (i.e. $|J + \varepsilon| \leqslant \varepsilon_{\text{tol}} = 10^{-7}$).

6.5. Problem (P3). Example 5

Consider an example in $\mathbb{R}^{10}$
$A = \text{diag}(0.1, 0.75, 0.8, 0.81, 0.82, 0.95, 1.0, 1.0, 1.05, 1.1)$, $\mathcal{U} = B_1(0)$. The target set is $\mathcal{M} = B_{0.1}(0.1 \times \mathbf{1})$, $(\mathbf{1} = (1, 1, \ldots, 1))$, $\varepsilon = 0.1$, step-size $\lambda = 0.1$.

We need 21 runs of the gradient projection algorithm to get the solution point

$$p_0 = (0.44643102,\ 0.32328081,\ 0.3153902,\ 0.3138356,\ 0.31228874,\\ 0.29286442,\ 0.28572048,\ 0.28572048,\ 0.27875066,\ 0.27195027)$$

and the optimal time is $t_0 = 0.35823087$.

7. CONCLUSION

In this paper we used a minimization Problem 1 to propose effective solution methods for several other problems (P1)—(P3) that involve distances and inclusions between sets. Linear convergence of proposed algorithms is proven. Several examples are given to prove the effectiveness of proposed solutions.

APPENDIX

A.1. PROOF OF LEMMA 1

Multiply both sides of the inequality by $\sqrt{\|p\|\,\|q\|}$ and take the square.

A.2. PROOF OF LEMMA 3

By the equality $e^{As} = Je^{A_1 s}J^{-1}$ we get

$$\mathcal{R}(t) = \int_0^t Je^{A_1 s}J^{-1}\mathcal{U}\,ds = \int_0^t Je^{A_1 s}\mathcal{U}_1\,ds = J\mathcal{R}_1(t).$$

The result follows from [25, Theorem 3].

A.3. PROOF OF LEMMA 3

We have $\mathcal{U} = \mathcal{U}_0 + \mathcal{B}_r(0)$. Then $\mathcal{R}(t) = \mathcal{R}_0(t) + r\int_0^t e^{As}\mathcal{B}_1(0)\,ds$,

$$\mathcal{R}_0(t) = \int_0^t e^{As}\mathcal{U}_0\,ds.$$

It is enough to prove that the ellipsoid $e^{As}\mathcal{B}_1(0)$ is uniformly smooth with constant $r(s) = \frac{\lambda_n^2(s)}{\lambda_1(s)}$. Consider orthonormal basis where the ellipsoid $e^{As}\mathcal{B}_1(0)$ has a canonical form

$$\mathcal{N} = \left\{x \in \mathbb{R}^n \ :\ \sum_{k=1}^{n} \frac{x_k^2}{\lambda_k^2} \leqslant 1\right\}, \quad \lambda_k = \lambda_k(s).$$

Then the matrix $L = \text{diag}\,\{\lambda_1, \ldots, \lambda_n\}$ gives $L\mathcal{B}_1(0) = \mathcal{N}$. The ellipsoid $\mathcal{V} = \{x\ :\ \sum_{k=1}^n \lambda_k^2 x_k^2 \leqslant 1\}$ is strongly convex with radius $\rho = \lambda_1/\lambda_n^2$. Hence there exists another compact convex set $\mathcal{P}$ with $\mathcal{V} + \mathcal{P} = \mathcal{B}_\rho(0)$ and, taking in mind that $L\mathcal{V} = \mathcal{B}_1(0)$, we have

$$L\mathcal{V} + L\mathcal{P} = L\mathcal{B}_\rho(0) = \rho L\mathcal{B}_1(0) = \rho\mathcal{N} \quad \Leftrightarrow \quad \frac{1}{\rho}\mathcal{B}_1(0) + \frac{1}{\rho}\mathcal{P} = \mathcal{N}.$$

Thus the set $\mathcal{N}$ is uniformly smooth with constant $\frac{1}{\rho} = \lambda_n^2/\lambda_1$.

A.4. PROOF OF THEOREM 1

Let I be the identity matrix. Assume that $p_0 \in \mathcal{S}_1$ is the solution of problem (1). From the necessary condition of extremum $f(p_0) = (p_0, f'(p_0)) = -\|f'(p_0)\|$. Then $P_{\mathcal{T}_p} = I - pp^T$ for any $p \in \mathcal{S}_1$ and $\|(I - pp^T)f'(p)\|^2 = \|f'(p)\|^2 - f^2(p)$. Hence for all $p \in \mathcal{S}$ we get

$$\|f'(p)\|^2 - f^2(p) = (\|f'(p)\| - f(p))(\|f'(p)\| + f(p_0) + f(p) - f(p_0)).$$

From the inequality $f(p) \leqslant 0$ and the fact that the supporting element $f'(p_0) = \mathcal{N}(t)(p_0)$ has minimal norm, we have $\|f'(p)\| - f(p) \geqslant \|f'(p)\| \geqslant \|f'(p_0)\| = |J|$. It remains to note that $\|f'(p)\| + f(p_0) = \|f'(p)\| - \|f'(p_0)\| \geqslant 0$.

For any vectors $p, q \in \mathbb{R}^n$, $1 - \varepsilon \leqslant \|p\|, \|q\| \leqslant 1 + \varepsilon$, by Lemma 1 we obtain that $\left\| \frac{p}{\|p\|} - \frac{q}{\|q\|} \right\| \leqslant \frac{\|p-q\|}{\sqrt{\|p\|\,\|q\|}}$. Fix such p, q. Then by Lipschitz continuity of the supporting element $f'(\xi) = \mathcal{N}(t)(\xi)$ on the unit sphere with Lipschitz constant R and by the equality $f'(\xi) = f'(\xi/\|\xi\|)$, for all $\xi \neq 0$, we get

$$\|f'(p) - f'(q)\| \leqslant R \left\| \frac{p}{\|p\|} - \frac{q}{\|q\|} \right\| \leqslant \frac{R\|p-q\|}{\sqrt{\|p\|\,\|q\|}} \leqslant \frac{R}{1-\varepsilon}\|p - q\|. \quad \square$$

A.5. PROOF OF THEOREM 2

Define $q_k = p_k - \lambda f'(p_k)$, $\|q_k\| \geqslant 1 - \lambda\|f'(p_k)\| \geqslant 1 - \lambda L \geqslant \frac{1}{2}$. By $\|p_k\| = \|p_{k+1}\| = 1$, Lemma 1 and from the inequality

$$\|p_{k+1} - p_k\| = \|P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)) - p_k\| \leqslant \frac{\|p_k - q_k\|}{\sqrt{\|p_k\|\,\|q_k\|}} \leqslant \lambda\sqrt{2}\|f'(p_k)\| \leqslant \lambda\sqrt{2}L \leqslant \frac{1}{\sqrt{2}}$$

we get $[p_k, p_{k+1}] \subset \{p \in \mathbb{R}^n : \frac{1}{2} \leqslant \|p\| \leqslant \frac{3}{2}\}$. By Theorem 1 f' is Lipschitz continuous on the segment $[p_k, p_{k+1}]$ with constant $L_1 = R/(1 - \frac{1}{2}) = 2R$.

We also have the LPL condition for the function f on the set $\mathcal{S}$ by Theorem 1 with $\mu = |J|$.

Fix λ from the proposition and $\ell = \frac{1}{\lambda} \geqslant L_1$. Put $z_k = \|\ell p_k - f'(p_k)\| - (p_k, p_k - f'(p_k)) \geqslant 0$,

$$z_k = \frac{\|(I - p_k p_k^T)f'(p_k)\|^2}{\|\ell p_k - f'(p_k)\| + (p_k, p_k - f'(p_k))} \geqslant \frac{\|(I - p_k p_k^T)f'(p_k)\|^2}{2\|\ell p_k - f'(p_k)\|}. \tag{A.1}$$

We have

$$\|p_{k+1} - p_k\|^2 = 2 - 2\frac{(p_k, \ell p_k - f'(p_k))}{\|\ell p_k - f'(p_k)\|} = \frac{2z_k}{\|\ell p_k - f'(p_k)\|}$$

and from the Lipschitz property of f' on the segment $[p_k, p_{k+1}]$ with constant L_1

$$\begin{aligned} f(p_{k+1}) - f(p_k) &\leqslant (f'(p_k), p_{k+1} - p_k) + \frac{L_1}{2}\|p_{k+1} - p_k\|^2 \\ &= (p_k, L_1 p_k - f'(p_k)) - \left(L_1 p_k - f'(p_k), \frac{\ell p_k - f'(p_k)}{\|\ell p_k - f'(p_k)\|}\right) \\ &= \left(\ell p_k - f'(p_k) + (L_1 - \ell)p_k, p_k - \frac{\ell p_k - f'(p_k)}{\|\ell p_k - f'(p_k)\|}\right), \end{aligned}$$

$$f(p_{k+1}) - f(p_k) \leqslant -z_k + (L_1 - \ell)\left(p_k, p_k - \frac{\ell p_k - f'(p_k)}{\|\ell p_k - f'(p_k)\|}\right) = -z_k + \frac{L_1 - \ell}{\|\ell p_k - f'(p_k)\|} z_k \leqslant -z_k.$$

From (A.1) and from the LPL condition with $\mu = |J|$ we obtain that

$$f(p_{k+1}) - f(p_k) \leqslant -\frac{\|(I - p_k p_k^T)f'(p_k)\|^2}{2\|\ell p_k - f'(p_k)\|} \leqslant -\frac{|J|}{2\|\ell p_k - f'(p_k)\|}(f(p_k) - f(p_0)).$$

Define $\varphi(p) = f(p) - f(p_0)$ for all $p \in \mathcal{S}_1$. From the estimate $\|\ell p_k - f'(p_k)\| \leqslant \ell + \|f'(p_k)\| \leqslant \ell + L$ we have

$$\varphi(p_{k+1}) \leqslant \left(1 - \frac{|J|}{2\ell + 2L}\right) \varphi(p_k) = q\varphi(p_k)$$

and $q \in (0, 1)$ because $|J| = \varrho(0, \mathcal{N}(t)) \leqslant \|\mathcal{N}(t)\| = L$.

For the points $\{p_k\}$ we have (note that $\|p_k - \lambda f'(p_k)\| \geqslant 1$)

$$\|p_{k+1} - p_k\|^2 \leqslant \frac{2z_k}{\|\ell p_k - f'(p_k)\|} \leqslant \frac{2\lambda(f(p_k) - f(p_{k+1}))}{\|p_k - \lambda f'(p_k)\|} \leqslant 2\lambda\varphi(p_k).$$

A.6. PROOF OF THEOREM 3

Consider $f(p)$:

$$f(p) = s(p, \mathcal{M}_0) + r\|p\| - s(p, \mathcal{R}_\varepsilon(t)).$$

The set $\mathcal{R}_\varepsilon(t)$ is strongly convex with radius $R_T + \varepsilon < r$. Hence there exists another convex compact set $\mathcal{N}(t)$ with $\mathcal{R}_\varepsilon(t) + \mathcal{N}(t) = \mathcal{B}_{R_T+\varepsilon}(0)$ and $r\|p\| - s(p, \mathcal{R}_\varepsilon(t)) = (r - R_T - \varepsilon)\|p\| + s(p, \mathcal{N}(t))$. Thus for all $p \in \mathbb{R}^n$

$$f(p) = s(p, \mathcal{M}_0) + (r - R_T - \varepsilon)\|p\| + s(p, \mathcal{N}(t)) = s(p, \mathcal{M}_0 + \mathcal{N}(t) + \mathcal{B}_{r-R_T-\varepsilon}(0))$$

and the function $f(p)$ is the supporting function of the set $\mathcal{N}(t) = \mathcal{M} \overset{*}{-} \mathcal{R}_\varepsilon(t) = \mathcal{M}_0 + \mathcal{N}(t) + \mathcal{B}_{r-R_T-\varepsilon}(0)$. The latter set is strongly convex with radius R_0 and uniformly smooth with constant $r_0 = r - R_T - \varepsilon > 0$. The function f' is Lipschitz on the set $\mathcal{S}_1$ with constant R_0 and as in the proof of Proposition 2 $[p_k, p_{k+1}] \subset \{p \in \mathbb{R}^n \,:\, \frac{1}{2} \leqslant \|p\| \leqslant \frac{3}{2}\}$. Thus for any point p from the segment $[p_k, p_{k+1}]$ we have $\|p\| \geqslant \frac{1}{2}$ and for any $p, q \in [p_k, p_{k+1}]$ by Lemma 1

$$\|f'(p) - f'(q)\| = \left\|f'\left(\frac{p}{\|p\|}\right) - f'\left(\frac{q}{\|q\|}\right)\right\| \leqslant R_0 \left\|\frac{p}{\|p\|} - \frac{q}{\|q\|}\right\| \leqslant R_0 \frac{\|p - q\|}{\sqrt{\|p\|\,\|q\|}} \leqslant 2R_0\|p - q\|,$$

i.e. f' is Lipschitz on any segment $[p_k, p_{k+1}]$ with constant $2R_0$. From the Lipschitz property of f' and Proposition 2 $f(p_k) \leqslant 0$ for all k.

$$\|p_{k+1} - p_0\|^2 = \|P_{\mathcal{S}_1}(p_k - \lambda f'(p_k)) - P_{\mathcal{S}_1}(p_0 - \lambda f'(p_0))\|^2,$$

$\|p_k - \lambda f'(p_k)\| \geqslant 1$, $\|p_0 - \lambda f'(p_0)\| \geqslant 1$, i.e. $p_k - \lambda f'(p_k) \notin \operatorname{int} B_1(0)$, $p_0 - \lambda f'(p_0) \notin \operatorname{int} B_1(0)$ and thence

$$\|p_{k+1} - p_0\|^2 \leqslant \|p_k - p_0 + \lambda(f'(p_k) - f'(p_0))\|^2$$
$$\leqslant \|p_k - p_0\|^2 - 2\lambda(p_k - p_0, f'(p_k) - f'(p_0)) + \lambda^2\|f'(p_k) - f'(p_0)\|.$$

From the strong convexity of the set $\mathcal{N}(t)$ with radius R_0 we have $\|f'(p_k) - f'(p_0)\| \leqslant R_0\|p_k - p_0\|$. Also by the strong convexity of the set $\mathcal{N}(t)$ with radius R_0 we have [28, Theorem 2.1 (h)] $(p_k - p_0, f'(p_k) - f'(p_0)) \geqslant \frac{1}{R_0}\|f'(p_k) - f'(p_0)\|^2$ and by the uniform smoothness of the set $\mathcal{N}(t)$ with constant r_0 [28, Definition 3.2, Theorem 3.6]

$$(p_k - p_0, f'(p_k) - f'(p_0)) \geqslant \frac{1}{R_0}\|f'(p_k) - f'(p_0)\|^2 \geqslant \frac{r_0^2}{R_0}\|p_k - p_0\|^2.$$

Thus $\|p_{k+1} - p_0\|^2 \leqslant q^2\|p_k - p_0\|^2$.

A.7. PROOF OF THEOREM 4

Repeat the proof of Theorem 3. In particular, the function $f(p)$ is the supporting function for the set $\mathcal{R}(t) \overset{*}{-} \mathcal{M}_\varepsilon = \mathcal{R}(t) \overset{*}{-} \mathcal{M} \overset{*}{-} \mathcal{B}_\varepsilon(0)$. The last set is strongly convex with constant R_T and uniformly smooth with constant r. □

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