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A Coalitional Differential Game of Vaccine Producers

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Abstract—The paper proposes a game-theoretic model of competition and cooperation, including partial cooperation, of vaccine producers. Various versions of players' cooperation (partial and full) have been studied. The differential game has an infinite duration. For each possible coalition of players, the profits and production quantities of its members are determined. An stability analysis of possible coalition structures, as well as coalitions that are most attractive to customers has been made.

Keywords: differential game, Nash equilibrium, cooperative solution, coalition structure, stability

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1. INTRODUCTION

Modeling competition and cooperation between firms producing homogeneous goods has been an actual problem for many years and remains such a problem up to these days. In particular, during a disease epidemic, vaccine producing companies are actively developing new vaccines and putting them on the market, competing with each other on price. The feature of the competition of such firms is in their small number, therefore, when modeling their interaction on the market, it is possible to consider various cooperation scenarios, which is difficult to do with a large number of firms. Also, such a competition is characterized by a slow price change, i.e., the price has the so-called "memory effect," i.e., it responds with a delay to changes in demand. This paper attempts to propose an economic model of competition between vaccine manufacturers, which is an effective tool in the fighting against many diseases [1].

In the classical theory of cooperative games, it is assumed that one large or grand coalition is formed and the players maximize the payoff of this coalition [2–4], but in real life, firms can form several coalitions existing simultaneously, i.e., various coalition structures can be formed [5]. Players or firms can choose whether to be members of a coalition or to act independently. Coalition structures consisting of several coalitions are observed during the formation of political parties, cooperation in the market of goods and services, where large coalitions cannot be formed for different reasons [6–8], including the ban on monopoly associations in the market.

This paper proposes a differential game of competition between several firms when price dynamics are determined in a special way. Various coalition structures that can be formed in a three-person game are studied, as well as their stability [6–8]. A stable coalition structure is understood as one in which it is unprofitable for any player to deviate from her coalition, i.e., join another existing coalition or become an individual player [5–6]. The work deals with differential Nash equilibria for a given coalition structure, in which each coalition behaves as one player, maximizing its payoff. Necessary conditions for the Nash equilibria in open-loop strategies are obtained. The work does not assume the redistribution of payoffs between players within the coalition, i.e., a game with non-transferable utilities is considered. The existence of stable coalition structures in games with transferable utilities, when the Shapley vector or ES solution is considered as a solution, is studied in the papers [5–6]. The existence of stable coalition structures with respect to some cooperative solutions is considered in [9–11]. The paper [12] studies a coalition structure consisting of one large coalition and several singleton coalitions. Our work examines all possible coalition structures for a three-person game, using a numerical example to show theoretical results on the Nash equilibrium firms' production quantities, the existence of a stable structure, and which structure is preferable for consumers.

The work has the following structure. Section 2 presents a game-theoretic model. Section 3 formulates the main theoretical results on Nash equilibria in games under different coalition structures. The definition of a stable coalition structure is given in Section 4. A numerical example is presented in Section 5. Section 6 contains the conclusion to the work.

2. MODEL

We consider a model of a market consisting of firms manufacturing vaccine (or other production firms) that produce vaccines to fight the same disease, i.e., their vaccines can substitute each other. Denote by $N = \{1, 2, \ldots, n\}$ a set of firms, each of which has production of $q_i(\cdot)\colon [0, +\infty) \longrightarrow \mathbb{R}^+$ [13]. The total production Q_i of firm i over the entire time interval $[0, +\infty)$ is equal to

$$Q_i = \int_0^{+\infty} q_i(t)\,\mathrm{d}t.$$

Suppose that the initial vaccine price is given as $p(0) = p_0$, and at any time t the price satisfies a differential equation:

$$\dot{p}(t) = s\left(a - b\sum_{i=1}^{n} q_i(t) - p(t)\right), \quad p(0) = p_0, \tag{1}$$

where a and b are positive constants, while $a > c$ and $a - bQ \geqslant 0$, $s > 0$ is the parameter of price sensitivity to changes. Price dynamics (1) takes into account that a market price does not adapt immediately to market changes. A rate of change in a market price is determined by the difference between the current price and the price formed by a linear demand function, multiplied by the given constant s. Constant s shows market sensitivity to price changes. Firms are also assumed to have a quadratic function of production costs:

$$C(q_i) = c_i q_i + \frac{1}{2}{q_i}^2,$$

where c_i is a positive constant for any $i \in N$.

The profit of firm i is determined by the functional

$$J_i(q_1, \ldots, q_n) = \int_0^{+\infty} e^{-rt}\left(p(t)q_i(t) - c_i q_i(t) - \frac{1}{2}q_i^2(t)\right)\mathrm{d}t, \tag{2}$$

where $r > 0$ is a discount rate, the same for all players.

The differential game is defined by a set of players N, the players' payoff functions (2), and dynamics equation (1), while player $i \in N$ maximizes function (2) by choosing strategies $q_i(t)$. Denote the total output by $Q(t) = \sum_{i=1}^{n} q_i(t)$ [13].

As a solution, we will consider the Nash equilibrium in program strategies (open-loop strategies).

Definition 1. Nash equilibrium is a set of strategies $q = (q_1, q_2, \dots, q_n)$ such that

$$J_i(q_i^*, q_{-i}^*) \geqslant J_i(q_i, q_{-i}^*),$$

for any $q_i \geqslant 0$ and for any player $i \in N$.

In the next section, we consider a modification of a noncooperative differential game, assuming that players can cooperate, i.e., form coalitions of any sizes, thereby creating coalition partitions or structures of a set of players N. We make several assumptions about players' behavior in coalition structures:

1) If coalition structure $\pi = \{B_1, \dots, B_m\}$ is formed, consisting of m nonempty subsets of a set of players such that $B_i \cap B_j = \varnothing$ for any $i \neq j$, and $\cup_{k=1}^m B_k = N$, then players belonging to the same coalition maximize the total profit of this coalition.

2) Coalitions $B_1, \dots, B_m$ compete in the market, i.e., in the noncooperative setting, the optimality principle is the Nash equilibrium in a game of m players.

3) Players' payoffs are nontransferable, i.e., any player in a coalition receives its payoff according to the payoff function given by formula (2).

3. CASE OF THREE-PERSON GAME WITH DIFFERENT COALITION STRUCTURES

In this section, we formulate the necessary conditions of the Nash equilibria for the differential game described in the previous section and given coalition structures. The results are shown for a case of a three-person game, but if desired, they can be generalized to the case of a finite number of players. With a large number of players, the number of coalition structures determined by the Bell number in a recurrent way, is so large that it is not possible to provide conditions for the Nash equilibrium in a general case. For example, for five players the number of coalition structures is 52, for seven players it is equal to 877, and for ten players it equals 115 975.

3.1. Noncooperative game

Theorem 1. *In a three-person differential game defined by the players' payoff functions* (2), *dynamics equation* (1), *and coalition structure* $\{\{i\}, \{j\}, \{k\}\}$, *if the Nash equilibrium in admissible open-loop strategies exists, then it satisfies the system:*

$$q_i(t) = \frac{w_1 + w_3 + 3sbB_1B_2}{w_1 + w_3}\left(\frac{p_0(b(3r + 4s) + r + s)}{b(3r + 4s) + r + s} - \frac{((c_i + c_j + c_k + a)s + r(c_i + c_j + c_k))b}{b(3r + 4s) + r + s}\right.$$
$$\left. + \frac{a(r + s)}{b(3r + 4s) + r + s}\right) e^{-w_1 t} + \frac{sbB_2B_3}{w_3} + \frac{((c_i + c_j + c_k + a)s + r(c_i + c_j + c_k))b + a(r + s)}{b(3r + 4s) + r + s} - c_i,$$

$$q_j(t) = \frac{w_1 + w_3 + 3sbB_1B_2}{w_1 + w_3}\left(\frac{p_0(b(3r + 4s) + r + s)}{b(3r + 4s) + r + s} - \frac{((c_i + c_j + c_k + a)s + r(c_i + c_j + c_k))b}{b(3r + 4s) + r + s}\right.$$
$$\left. + \frac{a(r + s)}{b(3r + 4s) + r + s}\right) e^{-w_1 t} + \frac{sbB_2B_4}{w_3} + \frac{((c_i + c_j + c_k + a)s + r(c_i + c_j + c_k))b + a(r + s)}{b(3r + 4s) + r + s} - c_j,$$

$$q_k(t) = \frac{w_1 + w_3 + 3sbB_1B_2}{w_1 + w_3}\left(\frac{p_0(b(3r + 4s) + r + s)}{b(3r + 4s) + r + s}\right.$$
$$\left. - \frac{((c_i + c_j + c_k + a)s + r(c_i + c_j + c_k))b}{b(3r + 4s) + r + s} + \frac{a(r + s)}{b(3r + 4s) + r + s}\right) e^{-w_1 t}$$
$$+ \frac{sbB_2(B_3 - B_4)}{w_3} + \frac{((c_i + c_j + c_k + a)s + r(c_i + c_j + c_k))b + a(r + s)}{b(3r + 4s) + r + s} +$$
$$-sb\frac{c_i + c_j + c_k - 3a}{b(3r + 4s) + r + s} - c_k,$$

where

$$\begin{aligned}
w_1 &= \frac{-2sb + r - \sqrt{\Delta}}{2},\\
\Delta &= (4b^2 + 16b + 4)s^2 + (8rb + 4s)s + r^2,\\
w_3 &= bs + s + r,\\
B_1 &= b\left(r + \frac{4}{3}s\right) + \frac{r+s}{3},\\
B_2 &= -\frac{1}{b(3r+4s) + r + s},\\
B_3 &= ((-3c_i + c_j + c_k + a)s - r(2c_i - c_j - c_k))b + (r+s)(a - c_i),\\
B_4 &= ((c_i - 3c_j + c_k + a)s - r(c_i - 2c_j + c_k))b + (r+s)(a - c_j).
\end{aligned}$$

The expression of the Nash equilibrium price is given in the theorem proof.

Proof. Under coalition structure $\{\{i\},\{j\},\{k\}\}$, each firm individually maximizes its profit by choosing production. To find the Nash equilibrium in open-loop strategies, we use the Pontryagin maximum principle. The Hamiltonian of player $i \in N$ has the form:

$$H_i(q, p, \lambda_i) = pq_i - c_iq_i - \frac{1}{2}{q_i}^2 + s(a - bQ - p)\lambda_i, \tag{3}$$

where λ_i is an adjoint variable.

Maximizing Hamiltonian H_i by production q_i, we obtain the equation:

$$p - c_i - q_i - sb\lambda_i = 0,$$

from where we express q_i:

$$q_i = p - sb\lambda_i - c_i. \tag{4}$$

The system of differential equations with respect to λ_i, $i \in N$, and p is written as follows:

$$\begin{aligned}
\dot{\lambda}_i &= r\lambda_i - \frac{\partial H_i}{\partial p} = -p + (r + s + sb)\lambda_i + c_i,\ i \in N,\\
\dot{p} &= -(snb + s)p + s^2b^2\sum_{i\in N}\lambda_i + sb\sum_{i\in N}c_i + sa.
\end{aligned}$$

We write down the system of $n + 1 = 4$ differential equations:

$$\begin{cases}
\dot{p} = -(3sb + s)p + s^2b^2(\lambda_i + \lambda_j + \lambda_k) + sb(c_i + c_j + c_k) + sa,\\
\dot{\lambda}_i = -p + (sb + r + s)\lambda_i + c_i,\\
\dot{\lambda}_j = -p + (sb + r + s)\lambda_j + c_j,\\
\dot{\lambda}_k = -p + (sb + r + s)\lambda_k + c_k.
\end{cases}$$

Then rewrite this system of differential equations in a matrix form:

$$\begin{pmatrix}\dot{p}\\ \dot{\lambda}_i\\ \dot{\lambda}_j\\ \dot{\lambda}_k\end{pmatrix} = \begin{pmatrix} -3sb - s & s^2b^2 & s^2b^2 & s^2b^2\\ -1 & r+s+sb & 0 & 0\\ -1 & 0 & r+s+sb & 0\\ -1 & 0 & 0 & r+s+sb\end{pmatrix}\begin{pmatrix}p\\ \lambda_i\\ \lambda_j\\ \lambda_k\end{pmatrix} + \begin{pmatrix} sa + sb(c_i + c_j + c_k)\\ c_i\\ c_j\\ c_k\end{pmatrix}.$$

We find a solution to this system, i.e., λ_i, λ_j, λ_k and p, and to do this, we write a characteristic equation corresponding to the matrix of the system of differential equations. Characteristic equation

$$\begin{vmatrix} -3sb-s-w & s^2b^2 & s^2b^2 & s^2b^2 \\ -1 & r+s+sb-w & 0 & 0 \\ -1 & 0 & r+s+sb-w & 0 \\ -1 & 0 & 0 & r+s+sb-w \end{vmatrix} = 0$$

has three roots w_1, w_2 and w_3, which are written as

$$w_{1,2} = \frac{-2sb+r\pm\sqrt{\Delta}}{2},$$
$$w_3 = bs+s+r,$$

where

$$\Delta = (4b^2+16b+4)s^2+(8rb+4s)s+r^2.$$

Solutions λ_i, λ_j, λ_k and p can be written as follows:

$$\lambda_i(t) = -\frac{3B_1B_2}{w_1+w_3}A_3e^{-w_1t} - \frac{3B_1B_2}{w_3-w_2}A_4e^{w_2t} + A_2e^{w_3t} - \frac{B_2B_3}{w_3},$$
$$\lambda_j(t) = -\frac{3B_1B_2}{w_1+w_3}A_3e^{-w_1t} - \frac{3B_1B_2}{w_3-w_2}A_4e^{w_2t} + A_1e^{w_3t} - \frac{B_2B_4}{w_3},$$
$$\lambda_k(t) = B_5B_6\left(\lambda_i+\lambda_j-A_1e^{w_3t}+A_2e^{w_3t}\right)$$
$$+ B_5\left(B_7+(B_8+4bB_9+B_9)s-(3b+1)r^2+(3bB_9-B_9)r\right)A_3e^{-w_1t}$$
$$+ B_5\left(B_7+(B_8-4bB_9-B_9)s-(3b+1)r^2-(3bB_9+B_9)r\right)A_4e^{w_2t} - \frac{c_i+c_j+c_k-3a}{(s(4b+1)+r(3b+1))},$$

$$p(t) = A_4e^{w_2t} + A_3e^{-w_1t} + \frac{((c_i+c_j+c_k+a)s+r(c_i+c_j+c_k))b+a(r+s)}{b(3r+4s)+r+s},$$

where

$$B_1 = b(r+\frac{4}{3}s)+\frac{r+s}{3},$$
$$B_2 = -\frac{1}{b(3r+4s)+r+s},$$
$$B_3 = ((-3c_i+c_j+c_k+a)s-r(2c_i-c_j-c_k))b+(r+s)(a-c_i),$$
$$B_4 = ((c_i-3c_j+c_k+a)s-r(c_i-2c_j+c_k))b+(r+s)(a-c_j),$$
$$B_5 = -\frac{1}{2s^2b^2(s(4b+1)+r(3b+1))},$$
$$B_6 = s^3(8b^3+2b^2)+rs^2(6b^3+2b^2),$$
$$B_7 = -s^2(16b^2+12b+2),$$
$$B_8 = -r(12b^2+14b+3),$$
$$B_9 = \sqrt{4s^2b^2+8brs+16bs^2+r^2+4rs+4s^2},$$

and A_1, A_2, A_3, and A_4 are constants determined from the initial and limit conditions: $p(0)=p_0$ and $\lim_{t\to\infty} e^{-rt}\lambda_i(t)=0$, $\lim_{t\to\infty} e^{-rt}\lambda_j(t)=0$, and $\lim_{t\to\infty} e^{-rt}\lambda_k(t)=0$.

From condition $\lim_{t\to\infty} e^{-rt}\lambda_i(t) = 0$ it follows that $A_2 = 0$ and $A_4 = 0$, and from condition $\lim_{t\ to\infty} e^{-rt}\lambda_j(t) = 0$ it follows that $A_1 = 0$ and $A_4 = 0$, then

$$\lambda_i(t) = -\frac{3B_1B_2}{w_1+w_3}A_3e^{-w_1t} - \frac{B_2B_3}{w_3},$$
$$\lambda_j(t) = -\frac{3B_1B_2}{w_1+w_3}A_3e^{-w_1t} - \frac{B_2B_4}{w_3},$$
$$\lambda_k(t) = B_{10}A_3e^{-w_1t} - \frac{B_2B_5B_6(B_3+B_4)}{w_3} - \frac{c_i+c_j+c_k-3a}{(s(4b+1)+r(3b+1))},$$

where

$$B_{10} = -\frac{6B_1B_2B_5B_6}{w_1+w_3} + B_5B_7 + B_5B_8s + B_5B_9s(4b+1) - B_5r^2(3b+1) + B_5B_9r(3b+1),$$

then p can be written as:

$$p(t) = A_3e^{-w_1t} + \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s}.$$

Given the initial condition $p(0) = p_0$, we find the constant

$$A_3 = \frac{p_0(b(3r+4s)+r+s) - ((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s}.$$

Thus, after transformations, we get:

$$\begin{aligned}
p(t) = &-\frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b}{b(3r+4s)+r+s}e^{-w_1t} \\
&+ \frac{p_0(b(3r+4s)+r+s) + a(r+s)}{b(3r+4s)+r+s}e^{-w_1t} \\
&+ \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s},
\end{aligned}$$

$$\begin{aligned}
\lambda_i(t) = &-\frac{3B_1B_2}{w_1+w_3}\frac{p_0(b(3r+4s)+r+s)}{b(3r+4s)+r+s}e^{-w_1t} \\
&+ \frac{3B_1B_2}{w_1+w_3}\frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b}{b(3r+4s)+r+s}e^{-w_1t} \\
&+ \frac{a(r+s)}{b(3r+4s)+r+s}e^{-w_1t} - \frac{B_2B_3}{w_3},
\end{aligned}$$

$$\begin{aligned}
\lambda_j(t) = &-\frac{3B_1B_2}{w_1+w_3}\frac{p_0(b(3r+4s)+r+s)}{b(3r+4s)+r+s}e^{-w_1t} \\
&+ \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b}{b(3r+4s)+r+s}e^{-w_1t} \\
&+ \frac{a(r+s)}{b(3r+4s)+r+s}e^{-w_1t} - \frac{B_2B_4}{w_3},
\end{aligned}$$

$$\begin{aligned}
\lambda_k(t) = &B_{10}\frac{p_0(b(3r+4s)+r+s)}{b(3r+4s)+r+s}e^{-w_1t} \\
&- B_{10}\frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s}e^{-w_1t} \\
&- \frac{B_2B_5B_6(B_3-B_4)}{w_3} - \frac{c_i+c_j+c_k-3a}{(s(4b+1)+r(3b+1))}.
\end{aligned}$$

By replacing λ_i and p with their expressions in equation (4), we get:

$$q_i(t) = \frac{w_1 + w_3 + 3sbB_1B_2}{w_1 + w_3}\left(\frac{p_0(b(3r+4s)+r+s)}{b(3r+4s)+r+s}e^{-w_1t} + \frac{a(r+s)}{b(3r+4s)+r+s}e^{-w_1t} - \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b}{b(3r+4s)+r+s}e^{-w_1t}\right) + \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s} + \frac{sbB_2B_3}{w_3} - c_i,$$

$$q_j(t) = \frac{w_1 + w_3 + 3sbB_1B_2}{w_1 + w_3}\left(\frac{p_0(b(3r+4s)+r+s)}{b(3r+4s)+r+s}e^{-w_1t} - \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b}{b(3r+4s)+r+s}e^{-w_1t} + \frac{a(r+s)}{b(3r+4s)+r+s}e^{-w_1t}\right) + \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s} + \frac{sbB_2B_4}{w_3} - c_j,$$

$$q_k(t) = (1 - sbB_{10})\left(\frac{p_0(b(3r+4s)+r+s)}{b(3r+4s)+r+s}e^{-w_1t} - \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b}{b(3r+4s)+r+s}e^{-w_1t} + \frac{a(r+s)}{b(3r+4s)+r+s}e^{-w_1t}\right) + \frac{sbB_2(B_3 - B_4)}{w_3} + \frac{((c_i+c_j+c_k+a)s + r(c_i+c_j+c_k))b + a(r+s)}{b(3r+4s)+r+s} + {} - sb\frac{c_i + c_j + c_k - 3a}{b(3r+4s)+r+s} - c_k,$$

where

$$1 - sbB_{10} = \frac{w_1 + w_3 + 3sbB_1B_2}{w_1 + w_3}.$$

The proof is completed.

3.2. Cooperative Game Version

In this section, we consider the differential game described above, when all players from set N are united into one coalition, i.e., in a three-person game, the coalition structure is $\{\{i, j, k\}\}$. Thus, this version of the game corresponds to full cooperation.

Theorem 2. *In a differential game with players' payoff functions* (2) *with given price dynamics* (1), *when coalition structure* $\{\{1, 2, 3\}\}$ *is formed, if the Nash equilibrium in open-loop strategies exists, then the player's equilibrium strategy* $i = 1, 2, 3$ *satisfies the conditions:*

$$q_i = \frac{s + r - w_2}{3}A_1e^{w_2t} + \frac{\left(3a - \sum\limits_{j=1}^{3} c_j\right)(6bs + s + r) + \left(\sum\limits_{\substack{j=1\\ j\neq i}}^{3} c_j - 2c_i\right)(3br + 6sb + r + s)}{3(3br + 6bs + r + s)},$$

where

$$w_2 = \frac{r - \sqrt{12brs + 24bs^2 + r^2 + 4rs + 4s^2}}{2},$$

$$A_1 = \frac{(3br + 6sb + r + s)(3p_0 - c_i - c_j - c_k)}{(3br + 6sb + r + s)(3bs + s + r - w_2)} - \frac{(3a - c_i - c_j - c_k)(3bs + s + r)}{(3br + 6sb + r + s)(3bs + s + r - w_2)}.$$

Proof. We use the Pontryagin maximum principle. The Hamiltonian of coalition $\{i, j, k\}$, acting as a single player and maximizing the sum of players' profits, has the form:

$$H_{i,j,k} = p(q_i + q_j + q_k) - (c_i q_i + c_j q_j + c_k q_k) - \left(\frac{1}{2}{q_i}^2 + \frac{1}{2}{q_j}^2 + \frac{1}{2}{q_j}^2\right)$$
$$+ \lambda_{i,j,k} s(a - b(q_i + q_j + q_k) - p),$$

where $\lambda_{i,j,k}$ is an adjoint variable defined for coalition $\{i, j, k\}$. For simplicity, we introduce notation: $\lambda_{i,j,k} = \lambda$.

Maximizing Hamiltonian $H_{i,j,k}$ with respect to productions q_i, q_j, and q_k, we obtain the following system of equations:

$$p - c_i - q_i - sb\lambda = 0,$$
$$p - c_j - q_j - sb\lambda = 0,$$
$$p - c_k - q_k - sb\lambda = 0,$$

from where we find q_i, q_j, and q_k:

$$q_i = p - sb\lambda - c_i, \ i = 1, 2, 3.$$

The system of differential equations for λ and p is written as follows:

$$\dot{\lambda} = r\lambda - \frac{\partial H_{i,j,k}}{\partial p} = -3p + (r + s + 3sb)\lambda + c_i + c_j + c_k,$$
$$\dot{p} = -(3sb + s)p + 3s^2b^2\lambda + sb(c_i + c_j + c_k) + sa.$$

We rewrite the last system and get:

$$\begin{cases} \dot{p} = -(3sb + s)p + 3s^2b^2\lambda + sb(c_i + c_j + c_k) + sa, \\ \dot{\lambda} = -3p + (r + s + 3sb)\lambda + c_i + c_j + c_k. \end{cases}$$

We write this system of differential equations in a matrix form as

$$\begin{pmatrix} \dot{p} \\ \dot{\lambda} \end{pmatrix} = \begin{pmatrix} -3sb - s & 3s^2b^2n \\ -3 & r + s + 3sb \end{pmatrix} \begin{pmatrix} p \\ \lambda \end{pmatrix} + \begin{pmatrix} sa + sb(c_i + c_j + c_k) \\ c_i + c_j + c_k \end{pmatrix}.$$

We find a solution of this system, i.e., λ and p, and to do this, we write the characteristic equation corresponding to the matrix of the system of differential equations, which has the form:

$$\begin{vmatrix} -3sb - s - w & 3s^2b^2n \\ -3 & r + s + 3sb - w \end{vmatrix} = -w^2 + rw + (3sb + s)(s + r) + 3s^2b = 0.$$

The characteristic equation has two roots:

$$w_{1,2} = \frac{r \pm \sqrt{\Delta}}{2},$$

where

$$\Delta = r^2 + 12sbr + 24bs^2 + 4s^2 + 4sr.$$

Obviously, $w_1 > w_2$, and moreover, $w_1 > 0$, $w_2 < 0$.

We obtain the solution of the system of differential equations:

$$\lambda(t) = A_2 e^{w_1 t} + A_1 e^{w_2 t} + \frac{3a - c_i - c_j - c_k}{3br + 6bs + r + s},$$
$$p(t) = \Big(bs + \frac{s + r - w_1}{3}\Big) A_2 e^{w_1 t} + \Big(bs + \frac{s + r - w_1}{3}\Big) A_1 e^{w_2 t} + \frac{(3a - c_i - c_j - c_k)(3bs + s + r)}{9br + 18bs + 3r + 3s} + \frac{c_i + c_j + c_k}{3},$$

where A_1 and A_2 are constants determined from the initial and limit conditions: $p(0) = p_0$ and $\lim_{x\to\infty} e^{-rt}\lambda(t) = 0$. From condition $\lim_{x\to\infty} e^{-rt}\lambda(t) = 0$ it follows that $A_2 = 0$, then the solution can be written as:

$$\lambda(t) = A_1 e^{w_2 t} + \frac{3a - c_i - c_j - c_k}{3br + 6bs + r + s},$$
$$p(t) = \frac{3bs + s + r - w_2}{3} A_1 e^{w_2 t} + \frac{(3a - c_i - c_j - c_k)(3bs + s + r)}{9br + 18bs + 3r + 3s} + \frac{c_i + c_j + c_k}{3}.$$

Given the initial condition $p(0) = p_0$, we find constant A_1:

$$A_1 = \frac{3p_0(3br + 6bs + s + r) - (3a - c_i - c_j - c_k)(3bs + s + r)}{(3br + 6bs + s + r)(3bs + s + r - w_2)} - \frac{(3br + 6bs + s + r)(c_i + c_j + c_k)}{(3br + 6bs + s + r)(3bs + s + r - w_2)}.$$

Substituting A_1 into the solution, we get:

$$p(t) = \frac{3bs + s + r - w_2}{3} A_1 e^{w_2 t} + \frac{(3a - c_i - c_j - c_k)(3bs + s + r)}{9br + 18bs + 3r + 3s} + \frac{c_i + c_j + c_k}{3},$$
$$\lambda(t) = A_1 e^{w_2 t} + \frac{3a - c_i - c_j - c_k}{3br + 6bs + r + s}.$$

Substituting λ into the expressions for equilibrium productions, we obtain

$$q_i = \frac{s + r - w_2}{3} A_1 e^{w_2 t} + \frac{(3a - \sum_{\ell=1}^{3} c_\ell)(6bs + s + r)}{3(3br + 6bs + r + s)} + \frac{(c_j + c_k - 2c_i)(3br + 6sb + r + s)}{3(3br + 6bs + r + s)},$$

where $i \neq j$, $i \neq k$, $j \neq k$, $i, j, k \in \{1, 2, 3\}$. The proof is complete.

3.3. Case of Partial Cooperation

In this section, we consider the case when the formed coalition partition consists of two coalitions, there exist three such structures: $\{\{i, j\}, \{k\}\}$, $\{\{i, k\}, \{j\}\}$, and $\{\{j, k\}, \{i\}\}$. Taking into account the complexity of formulating the theorem defining the Nash equilibrium explicitly in such a game of two coalitions competing with each other, we present the formulation of the theorem for coalition structure $\{\{i, j\}, \{k\}\}$ and obtain the Nash equilibrium conditions with this structure.

Theorem 3. *For coalition structure $\{\{i, j\}, \{k\}\}$, in a differential game with players' payoff functions* (2) *and given price dynamics* (1), *if there exists the Nash equilibrium in admissible open-loop strategies, then it is defined as follows:*

$$q_i = p - c_i - sb\lambda_{ij},$$
$$q_j = p - c_j - sb\lambda_{ij},$$
$$q_k = p - c_k - sb\lambda_k,$$

where p, λ_{ij}, and λ_k are the solutions to the system of differential equations:

$$\begin{aligned} \dot{p} &= -(3bs+s)p + 2s^2b^2\lambda_{ij} + s^2b^2\lambda_k + sb(c_i+c_j+c_k) + sa, \\ \dot{\lambda}_{ij} &= -2p + (r+s+2sb)\lambda_{ij} + c_i + c_j, \\ \dot{\lambda}_k &= -p + (r+s+sb)\lambda_k + c_k \end{aligned}$$

with initial and limit conditions: $p(0)=p_0$, $\lim_{t\to\infty} e^{-rt}\lambda_{ij}(t)=0$ *and* $\lim_{t\to\infty} e^{-rt}\lambda_k(t)=0$.

Proof. In coalition $\{i,j\}$, firms maximize the coalition's total profit by choosing productions q_i and q_j. We use the Pontryagin maximum principle. The Hamiltonians for coalitions $\{i,j\}$ and $\{k\}$ are

$$\begin{aligned} H_{ij}(q_i,q_j,q_k,\lambda_{ij},p) &= p(q_i+q_j) - c_iq_i - c_jq_j - \frac{1}{2}(q_i^2+q_j^2) + s(a-bQ-p)\lambda_{ij}, \\ H_k(q_i,q_j,q_k,\lambda_k,p) &= pq_k - c_kq_k - \frac{1}{2}q_k^2 + s(a-bQ-p)\lambda_k, \end{aligned}$$

where λ_{ij} and λ_k are the adjoint variables for coalitions $\{i,j\}$ and $\{k\}$ respectively.

Maximizing Hamiltonian H_{ij} with respect to productions q_i and q_j, as well as H_k with respect to production q_k, we obtain the following system of equations:

$$\begin{aligned} p - c_i - q_i - sb\lambda_{ij} &= 0, \\ p - c_j - q_j - sb\lambda_{ij} &= 0, \\ p - c_k - q_k - sb\lambda_k &= 0, \end{aligned}$$

whose solution is

$$q_i = p - c_i - sb\lambda_{ij}, \tag{5}$$

$$q_j = p - c_j - sb\lambda_{ij}, \tag{6}$$

$$q_k = p - c_k - sb\lambda_k. \tag{7}$$

We write down the system of differential equations with respect to λ_{ij}, λ_k, and p:

$$\begin{aligned} \dot{p} &= -(3bs+s)p + 2s^2b^2\lambda_{ij} + s^2b^2\lambda_k + sb(c_i+c_j+c_k) + sa, \\ \dot{\lambda}_{ij} &= -2p + (r+s+2sb)\lambda_{ij} + c_i + c_j, \\ \dot{\lambda}_k &= -p + (r+s+sb)\lambda_k + c_k. \end{aligned}$$

Let us rewrite this system in a matrix form as

$$\begin{pmatrix} \dot{p} \\ \dot{\lambda}_{ij} \\ \dot{\lambda}_k \end{pmatrix} = \begin{pmatrix} -3sb-s & 2s^2b^2 & s^2b^2 \\ -2 & r+s+2sb & 0 \\ -1 & 0 & sb+r+s \end{pmatrix} \begin{pmatrix} p \\ \lambda_{ij} \\ \lambda_k \end{pmatrix} + \begin{pmatrix} sa + sb(c_i+c_j+c_k) \\ c_i+c_j \\ c_k \end{pmatrix}.$$

We find λ_{ij}, λ_k, and p, for this we write the characteristic equation, which looks like:

$$\begin{aligned} &\begin{vmatrix} -3sb-s-w & 2s^2b^2 & s^2b^2 \\ -2 & r+s+2sb-w & 0 \\ -1 & 0 & sb+r+s-w \end{vmatrix} \\ &= -(3sb+s+w)(w^2 - (2r+3sb+2s)w + (r+2sb+s)(r+sb+s)) \\ &+ s^2b^2(5r+6sb+5s-5w) = 0. \end{aligned}$$

It is difficult to write down the solution of this characteristic equation in a general case in an explicit form. Using, for example, the *Matlab* software, one can find the unique admissible solution of this equation and express the solutions of a system of differential equations through this and other parameters of the system. When solving a system of differential equations, we use the initial and limit conditions: $p(0) = p_0$, $\lim_{t\to\infty} e^{-rt}\lambda_{ij}(t) = 0$ and $\lim_{t\to\infty} e^{-rt}\lambda_k(t) = 0$. After finding solutions, we substitute λ_{ij}, λ_k, and p into expressions (5), (6), and (7), and obtain the Nash equilibrium strategies.

Remark 1. Conditions for Nash equilibrium in the case of coalition structures $\{\{i,k\},\{j\}\}$ and $\{\{j,k\},\{i\}\}$ can be found similar to Theorem 3. Solution of a system of differential equations from the proof of Theorem 3 was found using the Matlab program that is presented in Section 5, which considers a numerical example.

4. STABILITY OF COALITION STRUCTURES

In our model, the coalition structure is given exogenously, i.e., the players do not participate "actively" in forming a coalition structure. But even if the structure is given, the problem of its stability arises. By stability, it is natural to understand a stable structure at which no firm would prefer to leave its coalition in order to join another one, or become an individual player. Although firms in a coalition choose strategies that maximize the total coalition's profits, they can compare their own profits in these coalitions to decide which coalition is more preferable for them [5].

We give a definition of a stable coalition structure based on the Nash equilibrium principle, i.e., on the fact that it is nonprofitable for any player to individually deviate from a stable structure, i.e., to move to other coalitions or become an individual player.

Definition 2. Coalition structure $\pi = \{B_1, \ldots, B_m\}$ is said to be stable in a game with nontransferable payoffs if the following inequality holds for any player $i \in N$:

$$J_i^{\pi} \geqslant J_i^{\pi'} \text{ for all } B_j \in \pi \cup \emptyset,\ B_j \neq B(i).$$

Here J_i^{π} and $J_i^{\pi'}$ are the payoffs of player i in a game with given coalition structure π and π' respectively, where $\pi' = \{B(i)\setminus\{i\}, B_j \cup \{i\}, \pi_{-B(i)\cup B_j}\}$, $B(i)$ is a coalition from structure π which player i belongs to.

5. NUMERICAL EXAMPLE

To illustrate the theoretical results obtained in the previous sections, we consider a differential game described above between three firms from set $N = \{i,j,k\}$. It is assumed that firms can form any coalition structure: $\{\{i,j,k\}\}$ (cooperative version), $\{\{i\},\{j\},\{k\}\}$ (noncooperative version), $\{\{i,j\},\{k\}\}$, $\{\{i,k\},\{j\}\}$ and $\{\{j,k\},\{i\}\}$ (partially cooperative version).

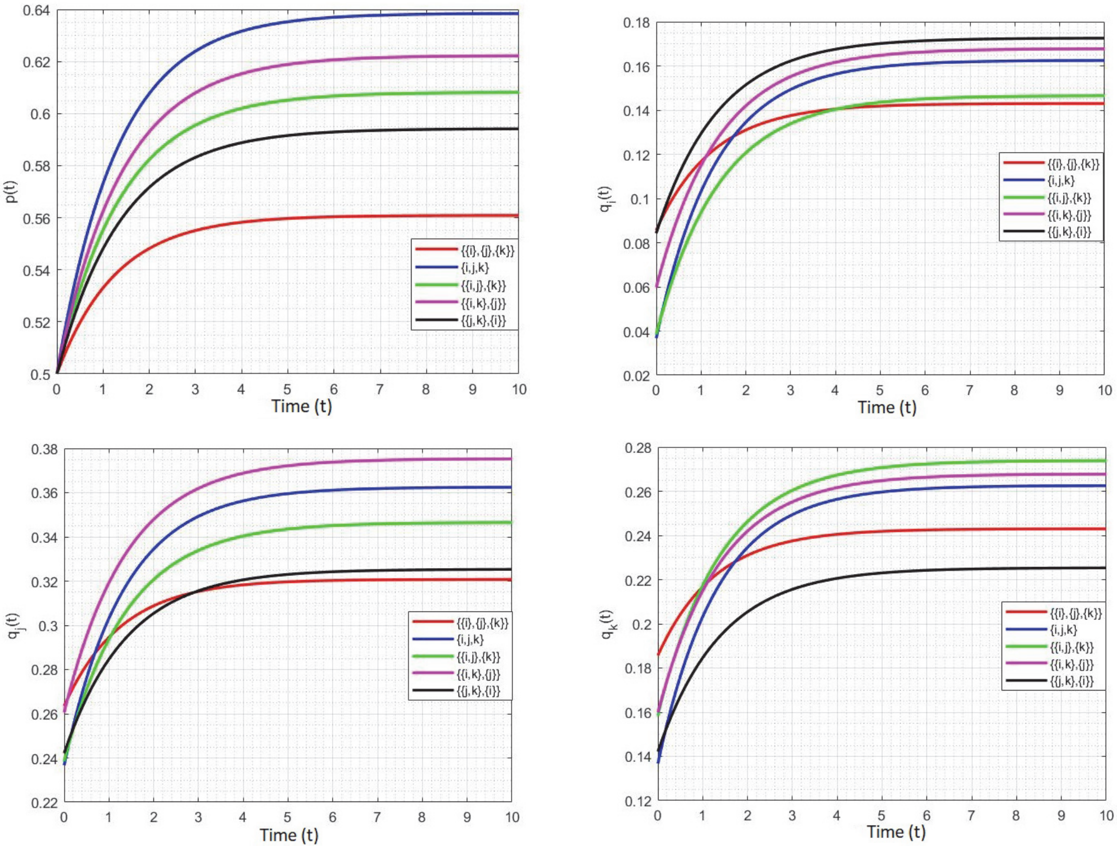
We use the following parameters for numerical simulations:

p_0	c_i	c_j	c_k	b	r	s	a	n
0.5	0.4	0.2	0.3	0.2	0.3	0.5	0.7	3

Applying Theorems 4.3.1, 4.3.2, and 4.3.3, we find equilibrium price $p(t)$ and strategies $q_i(t)$, $q_j(t)$, and $q_k(t)$ for all coalition structures. The result in graph form is shown in figure.

The profits of firms i, j, and k are calculated by substituting equilibrium $p(t)$, $q_i(t)$, $q_j(t)$, and $q_k(t)$ from Theorems 4.3.1, 4.3.2, and 4.3.3 into the firms' payoff functions (2). The firms' payoffs are given in Table 1.

The analysis of the obtained results shows (see Table 1) that for all firms coalition structure $\{\{i,j,k\}\}$ is preferable, since the payoffs of all firms are greater with this coalition structure than



Equilibrium prices and firms' productions for various coalition structures.

with any other one. Obviously, this coalition structure is stable in accordance with Definition 2, since it is unprofitable for any player to deviate from this coalition structure, i.e., to become an individual player.

The analysis of equilibrium prices with various coalition structures shows that the lowest price is formed under full competition, i.e., with coalition structure $\{\{i\},\{j\},\{k\}\}$. It means that this structure is the most preferable for consumers, followed by structures $\{\{j,k\},\{i\}\}$, $\{\{i,j\},\{k\}\}$, $\{\{i,k\},\{j\}\}$, $\{\{i,j,k\}\}$ in order of increasing price. As expected, $\{\{i,j,k\}\}$ coalition structure or full cooperation is the least preferred scenario for consumers. Table 2 shows price limits with different coalition structures. Of course, it is more profitable for firms to have coalition structure $\{\{i,j,k\}\}$, if we talk about firm profits, and consumers prefer competition, i.e., coalition structure $\{\{i\},\{j\},\{k\}\}$, when the lowest price is formed for them on the market.

Table 1. Firms' payoffs under various coalition structures

	J_i	J_j	J_k
$\{\{i\},\{j\},\{k\}\}$	0.0346	0.1952	0.0993
$\{\{i,j,k\}\}$	0.0600	0.2593	0.1430
$\{\{i,j\},\{k\}\}$	0.0460	0.2273	0.1237
$\{\{i,k\},\{j\}\}$	0.0557	0.2473	0.1344
$\{\{j,k\},\{i\}\}$	0.0467	0.2178	0.1123

Table 2. Equilibrium price limits $\bar{p} = \lim\limits_{t\to\infty} p(t)$ for various coalition structures

	$\{\{i\},\{j\},\{k\}\}$	$\{\{i,j,k\}\}$	$\{\{i,j\},\{k\}\}$	$\{\{i,k\},\{j\}\}$	$\{\{j,k\},\{i\}\}$
$\bar{p}$	0.5609	0.6384	0.6081	0.6222	0.5941

6. CONCLUSION

This paper proposes a model of competition in the market of manufacturers of vaccines or other products, when the product price has the property of the so-called “memory,” i.e., it is dynamically formed not only by demand, but also by the previous price value. The model is represented by a differential game of infinite duration, where the players’ strategies are production volumes. It is assumed that players-firms can form any coalitions, i.e., not only a grand coalition, but also coalitions of smaller sizes. In this paper, the Nash equilibrium is found in a game with a given coalition structure. The chapter considers the case of nontransferable utilities, i.e., players cannot redistribute payoffs in cooperation. A numerical example demonstrates theoretical results and analyzes the stability of coalition structures. Finally, we draw conclusions about which structures are preferable for consumers and firms.

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