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Signal Recognition without State Space Expansion Based on Observations Containing a Singular Interference: The Case of Nonlinear Parameters of Basis Functions

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Abstract—This paper proposes a novel method for recognizing a set of signals with linearly and nonlinearly included parameters from a given ensemble of signals under essential a priori uncertainty. Due to this uncertainty, well-known statistical methods become inapplicable. Signals may be present in an additive mixture containing an observation noise and a singular interference; the distribution law of the noise is unknown, and only its correlation matrix is specified. The novel method is invariant to this interference, does not require traditional state-space expansion, and ensures the decomposition and parallelization of the computational procedure. The signals and interference are represented using conventional linear spectral decompositions with unknown coefficients and given basis functions. Random and methodological errors, as well as the resulting computational effect, are analyzed. An illustrative example is provided.

Keywords: essential a priori uncertainty, singular interference, basis functions with nonlinear parameters, spectral coefficients, state space expansion, optimal estimation, unbiased estimation, invariance to interference, recognition algorithm, decomposition, parallel computing

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1. INTRODUCTION

In the publications [1–3], a method was developed for solving several applied problems under essential a priori uncertainty (e.g., masking and estimation of information processes for stationary and dynamic objects) based on observations containing a useful signal, some noise with an unknown distribution law (but with a given correlation matrix), and a parametric singular interference. By assumption, both the signal and the interference can be represented as a finite-dimensional linear combination of given basis functions with unknown spectral coefficients. The algorithms proposed in [1–3] are based on the idea of generalized invariant-unbiased estimation (GIUE) without state space expansion. This idea allows estimating any linear functionals (over the signal) without determining spectral coefficients in linear combinations of both the signal and the interference.

In this paper, GIUE is adopted for a more complex challenge, namely, recognizing a set of signals from a given ensemble using various information-measuring systems. Recognition means solving several problems related to the estimation, detection, discrimination, and resolution of signals (with linear or nonlinear parameters) under a priori uncertainty of various levels; for example, see [4–11]. Such problems arise in location, navigation, communication, radio reconnaissance, radio astronomy, telemetry, technical and medical diagnosis, information security, and many other fields. When probabilistic models are legitimate, these problems can be solved within known statistical methods (minimax, Bayesian, maximum likelihood, etc.).

There exists a wide range of signal recognition problems for which the traditional probabilistic approach [4–11] is difficult to implement, especially in the class of real-time information-measuring systems operating under essential a priori uncertainty, with minimum available statistical data. For estimation problems with such restrictions, the least squares (LS) method is often used; it involves the known correlation matrix of observation errors only and yields, by the well-known Gauss–Markov theorem, the best linear estimate [12, 13]. A classical approach to recognition for the above systems can be the extended least squares (XLS) method. This method is based on state space expansion and jointly estimates all parameters of signals and singular interference present in an input observation. As is known [12–15], for problems with nonlinearity and state-space expansion, XLS leads in practice to iterative procedures (i.e., requires sufficiently good initial conditions) and the well-known effect of "smearing accuracy." This effect is most pronounced in multidimensional recognition problems with ill-conditioned matrices. In addition, such an extension causes a significant increase in computing cost (and consequently, a decrease in computational speed).

In [1–3], a self-compensating parallel procedure was developed for the GIUE of linear functionals. Being an alternative to XLS, it gives optimal estimation algorithms for the parameters of a single signal under essential a priori uncertainty. This decomposition-based procedure requires no state space expansion, ensures the self-compensation of a singular interference without determining its spectral coefficients, and allows parallel computing. A significant computational effect is shown.

In this paper, GIUE is further developed to solve much more complex problems of recognizing a set of signals with minimum available statistical data on their observation noise. Here, the term "self-compensation" is given a broader interpretation since each signal in the observation equation is treated as an interference for another signal of the same equation. A novel signal recognition method is proposed for the nonlinear case. This method retains all the advantages of GIUE, does not require iterative procedures, and implements the principle of parallel computing with multichannel data processing [16–19]. In the statistical sense, the method involves no distribution laws but only information about the mean and correlation matrix of the observation noise. This feature is characteristic of the algorithms based on LS and XLS.

This paper uses no stochastic signals (e.g., Markovian ones), which are typical for the theory of linear and nonlinear filtering with its large amount of reliable statistical information. In many cases, stochastic filtering leads to current estimation algorithms with poor convergence and (or) long transients, therefore being inappropriate for the class of information-measuring systems considered below.

2. MODELS, CONSTRAINTS, AND CRITERIA. STATEMENT OF THE RECOGNITION PROBLEM WITH ESSENTIAL A PRIORI UNCERTAINTY

Consider the problem of recognizing signals from a given ensemble based on their observations in the form of the additive mixture

$$\begin{gathered} h(t) = \sum_{i=1}^{K} q_i s_i(t, \mathbf{A}_i, \boldsymbol{\zeta}_i) + \theta(t, \mathbf{B}) + \xi(t), \\ K \geqslant 1, \quad \mathbf{A}_i \in \wp_i \subset \mathbb{R}^{M_i}, \quad \boldsymbol{\zeta}_i \in \mathcal{J}_i \subset \mathbb{R}^{L_i}, \end{gathered} \tag{1}$$

with the following notations: $t \in [0, T]$ is continuous time; K specifies the number of possible signals in the ensemble; $\theta(t, \mathbf{B})$ is a singular interference; $\xi(t)$ is an observation noise; q_i is the parameter characterizing the presence ($q_i = 0$) or absence ($q_i = 1$) of the signal $s_i(t, \mathbf{A}_i, \boldsymbol{\zeta}_i)$ in the mixture; $\mathbf{A}_i$ and $\mathbf{B}$ are the vector spectral coefficients of the linear decomposition of the signal and interference, respectively, in given systems of basis functions; finally, $\boldsymbol{\zeta}_i$ is the signal's vector parameter that nonlinearly enters its basis functions. (Generally speaking, this parameter is unknown and will be called the basis parameter as well.)

For $K = 1$, we have the recognition problem for a single signal; if the mixture (1) with $K > 1$ contains only one signal, then the discrimination problem arises naturally; under an arbitrary number of signals in (1), the matter concerns the resolution problem. This model includes the following particular cases: $q_i = 0$ $\forall i = \overline{1, K}$ (all signals of the ensemble are absent) and $q_i = 1$ $\forall i = \overline{1, K}$ (all signals of the ensemble are present). Also, model (1) covers other signal recognition problems; for example, see [7, pp. 10–12]).

Assume that the sets $\wp_i$ and $\mathcal{J}_i$ are bounded and convex. The coordinates of the basis vector $\boldsymbol{\zeta}_i$ can be physical quantities, such as pulse duration, signal period, time delay, carrier frequency or Doppler frequency, some parameters of the modulating function, etc.

For example, equation (1) can be given the following physical interpretation. Let K targets be possibly located in the field of view of a radar station. For the radiated sounding signal $s(t)$, at the receiving point we have the sum of reflected signals $q_1 a_1 s(t - \tau_1) + q_2 a_2 s(t - \tau_2) + \ldots + q_K a_K s(t - \tau_K)$, where $\tau_i = 2R_i/c$, $i \in \{1, \ldots, K\}$, is the delay of the received signal with respect to the sounding one, R_i is the range to target i, and c is the speed of light. Hence, in this example, $\mathbf{A}_i = [a_i]$ and $\boldsymbol{\zeta}_i = [\tau_i]$ are the amplitude and delay of the signal, respectively (the linear and nonlinear parameters, respectively). Note that $\theta(t, \mathbf{B})$ can be the resulting interference, e.g., due to the side lobes of the receiving antenna pattern, the influence of the underlying surface, and natural or artificial spatially distributed interferences.

Suppose that an appropriate numerical grid of multidimensional (vector) nodes $\boldsymbol{\zeta}_{(d_i)}$ can be defined on the set $\mathcal{J}_i$. (Here, $d_i \in \{1, \ldots, D_i\}$ indicates the node number, D_i is the grid volume for the parameter $\boldsymbol{\zeta}_i$, and $\boldsymbol{\zeta}_{(d_i)}$ is the value of the parameter $\boldsymbol{\zeta}_i$ at node d_i.) Appropriateness means that the grid is sufficient for the discrete representation of $\boldsymbol{\zeta}_i$ with required accuracy on the entire set $\mathcal{J}_i$ (in the sense of resolution). To simplify the mathematical constructs and expressions below, let us introduce the vectors $\boldsymbol{q} = \left[q_i, i = \overline{1, K}\right]^{\mathrm{T}}$, $\boldsymbol{d} = \left[d_i, i = \overline{1, K}\right]^{\mathrm{T}}$ and $\boldsymbol{\zeta} = \left[\boldsymbol{\zeta}_i^{\mathrm{T}}, i = \overline{1, K}\right]^{\mathrm{T}}$ as well as the single multidimensional node $\boldsymbol{\zeta}_{(d)} = \left[\boldsymbol{\zeta}_{(d_i)}^{\mathrm{T}}, i = \overline{1, K}\right]^{\mathrm{T}}$.

For arbitrary values of $\mathbf{A}_i$, $\boldsymbol{\zeta}_i$, and $\mathbf{B}$, we use the linear finite-dimensional combinations

$$s_i(t, \mathbf{A}_i, \boldsymbol{\zeta}_i) = \mathbf{A}_i^{\mathrm{T}} \boldsymbol{\Psi}_i(t, \boldsymbol{\zeta}_i) \tag{2}$$

(the signal model) and

$$\theta(t, \mathbf{B}) = \mathbf{B}^{\mathrm{T}} \boldsymbol{\Omega}(t) \tag{3}$$

(the interference model), where $\mathbf{A}_i = \left[a_{im}, m = \overline{1, M_i}\right]^{\mathrm{T}}$ and $\mathbf{B} = \left[b_j, j = \overline{1, J}\right]^{\mathrm{T}}$ are the unknown vector spectral coefficients of the linear decompositions of the signal and interference, respectively, and $\boldsymbol{\Psi}_i(t, \boldsymbol{\zeta}_i) = \left[\psi_{im}(t, \boldsymbol{\zeta}_i), m = \overline{1, M_i}\right]^{\mathrm{T}}$ and $\boldsymbol{\Omega}(t) = \left[\omega_j(t), j = \overline{1, J}\right]^{\mathrm{T}}$ are given basis functions of the signal and interference, respectively.

Consider the discrete-time vector observation equation (on the grid of nodes $\{t_1, \ldots, t_N\}$)

$$\mathbf{H} = \sum_{i=1}^{K} q_i \mathbf{S}_i + \boldsymbol{\Theta} + \boldsymbol{\Xi}, \tag{4}$$

where

$$\mathbf{H} = \left[h_n, n = \overline{1, N}\right]^{\mathrm{T}}, \quad \mathbf{S}_i = \left[s_{in}(\mathbf{A}_i, \boldsymbol{\zeta}_i), n = \overline{1, N}\right]^{\mathrm{T}},$$

$$\boldsymbol{\Theta} = \left[\theta_n(\mathbf{B}), n = \overline{1, N}\right]^{\mathrm{T}}, \quad \boldsymbol{\Xi} = \left[\xi_n, n = \overline{1, N}\right]^{\mathrm{T}},$$

$$h_n = h(t_n), \quad s_{in}(\mathbf{A}_i, \boldsymbol{\zeta}_i) = s_i(t_n, \mathbf{A}_i, \boldsymbol{\zeta}_i), \quad \theta_n(\mathbf{B}) = \theta(t_n, \mathbf{B}), \quad \xi_n = \xi(t_n).$$

This equation is widespread in practice.

Assume that the noise $\mathbf{\Xi}$ is described by zero mean and the corresponding correlation matrix $\mathbf{K}^{\mathbf{\Xi}}$. No other probabilistic characteristics of $\mathbf{\Xi}$ are required.

For arbitrary values of $\mathbf{A}_i$, $\boldsymbol{\zeta}_i$, and $\mathbf{B}$, we have

$$\mathbf{S}_i = \mathbf{\Psi}_i \boldsymbol{A}_i, \tag{5}$$

$$\mathbf{\Theta} = \mathbf{\Omega}\mathbf{B}, \tag{6}$$

where $\mathbf{\Psi}_i = \left[\psi_{imn}(\boldsymbol{\zeta}_i), n = \overline{1,N},\, m = \overline{1,M_i}\right]$ and $\mathbf{\Omega} = \left[\omega_{jn}, n = \overline{1,N},\, j = \overline{1,J}\right]$ are the basis matrices of the signal $\mathbf{S}_i$ and interference $\mathbf{\Theta}$, respectively, $\psi_{imn}(\boldsymbol{\zeta}_i) = \psi_{im}(t_n, \boldsymbol{\zeta}_i)$, and $\omega_{jn} = \omega_j(t_n)$.

Suppose also that the extended functional basis $\{\mathbf{\Psi}_1(t,\boldsymbol{\zeta}_1), \ldots, \mathbf{\Psi}_K(t,\boldsymbol{\zeta}_K), \mathbf{\Omega}(t)\}$ is linearly independent on the grid $\{t_1, \ldots, t_N\}$ for any $\boldsymbol{\zeta}_1 \in \mathcal{J}_1, \boldsymbol{\zeta}_2 \in \mathcal{J}_2, \ldots, \boldsymbol{\zeta}_K \in \mathcal{J}_K$ (by analogy with [1–3]).

In the most general case, the problem of signal recognition involves detecting optimally each signal $s_i(t, \mathbf{A}_i, \boldsymbol{\zeta}_i)$ from a given ensemble (i.e., obtaining an estimate q_i^* for the parameter q_i) and finding estimates $\mathbf{A}_i^*$ and $\boldsymbol{\zeta}_i^*$ for $\mathbf{A}_i$ and $\boldsymbol{\zeta}_i$, respectively. In such conditions (multi-alternative solutions), a family of hypotheses Γ_l, $l \in \{1, \ldots, L\}$, where $L = 2^K \geqslant 2$, is introduced to characterize all possible alternatives (the presence or absence of ensemble signals in the observation (1)). In the sequel, $\Gamma^0 \in \{\Gamma_1, \ldots, \Gamma_L\}$ is a true hypothesis.

Let the case $q_i = 0\ \forall i = \overline{1,K}$ correspond to the hypothesis Γ_1 whereas the case $q_i = 1\ \forall i = \overline{1,K}$ to the hypothesis Γ_L.

To each hypothesis Γ_l we assign the model observation

$$\mathbf{H}_l = \sum_{i=1}^{K_l} \mathbf{S}_{il} + \mathbf{\Theta} + \mathbf{\Xi}, \quad l \in \{1, \ldots, L\}, \quad \mathbf{S}_{il} \in \{\mathbf{S}_1, \ldots, \mathbf{S}_K\}, \tag{7}$$

where $\mathbf{S}_{il} = \mathbf{\Psi}_{il}\mathbf{A}_{il}$ (see (5)) and K_l is the number of ensemble signals in the mixture by the hypothesis Γ_l.

In view of (7), the recognition problem is solved by minimizing the quadratic criterion $\chi^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l)$ for the extended vector of spectral coefficients $\mathbf{W}_l = \left[\mathbf{A}_{1l}^{\mathrm{T}}, \mathbf{A}_{2l}^{\mathrm{T}}, \ldots, \mathbf{A}_{K_l l}^{\mathrm{T}}, \mathbf{B}^{\mathrm{T}}\right]^{\mathrm{T}}$ using XLS:

$$(l^*, \boldsymbol{\zeta}_{l^*}, \mathbf{W}_{l^*}) = \arg\min_{l,\boldsymbol{\zeta}_l,\mathbf{W}_l} \chi^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l) = \arg\min_{l,\boldsymbol{\zeta}_l,\mathbf{W}_l} \left[\Delta^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l)\right]^{\mathrm{T}} \left(\mathbf{K}^{\mathbf{\Xi}}\right)^{-1} \Delta^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l), \tag{8}$$

where $\Delta^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l) = \mathbf{H} - \mathbf{H}^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l)$ and minimization with respect to the nonlinear parameter $\boldsymbol{\zeta}_l$ is performed at the nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$ of the numerical grid.

The criterion (8) ensures minimization considering the residuals $\Delta^{\mathrm{xls}}(\boldsymbol{\zeta}_l, \mathbf{W}_l)$ and the weight matrix $\left(\mathbf{K}^{\mathbf{\Xi}}\right)^{-1}$. Note that $\boldsymbol{\zeta}_{l^*}$ and $\mathbf{W}_{l^*}$ mean the estimates of $\boldsymbol{\zeta}_l$ and $\mathbf{W}_l$, respectively, within the optimal hypothesis Γ_{l^*}, $l^* \in \{1, \ldots, L\}$.

Obviously, this problem has a rather high dimension; in the case of ill-conditioned matrices, the estimation errors may significantly exceed the methodological error and devalue the optimal processing results of observations (see an illustrative example in Section 6 below). In addition, the criterion (8) provides no possibility of parallelizing the computational procedure.

The drawbacks of XLS can be appreciably overcome by using a modified GIUE procedure oriented to signal recognition. For this purpose, for a fixed number k, we write the observation (7) in two forms:

$$\mathbf{H}_l = \begin{cases} \mathbf{S}_{kl} + \mathbf{X}_{kl} + \mathbf{\Xi}, & k \in \{1, \ldots, K_l\} \\ \mathbf{\Theta} + \mathbf{X}_l + \mathbf{\Xi}, & \end{cases} \tag{9}$$

where $\mathbf{S}_{kl} = \mathbf{S}_{kl}(\mathbf{A}_{kl}, \boldsymbol{\zeta}_{kl})$, $\mathbf{X}_l = \mathbf{X}_l(\mathbf{A}_l, \boldsymbol{\zeta}_l)$, $\mathbf{A}_l = \left[\mathbf{A}_{1l}^{\mathrm{T}}, \mathbf{A}_{2l}^{\mathrm{T}}, \ldots, \mathbf{A}_{K_l l}^{\mathrm{T}}\right]^{\mathrm{T}}$, $\boldsymbol{\zeta}_l = \left[\boldsymbol{\zeta}_{il}^{\mathrm{T}}, i = \overline{1, K_l}\right]^{\mathrm{T}}$, $\mathbf{X}_{kl} = \sum\limits_{\substack{i=1\\ i\neq k}}^{K_l} \mathbf{S}_{il} + \boldsymbol{\Theta}$, and $\mathbf{X}_l = \sum\limits_{i=1}^{K_l} \mathbf{S}_{il}$.

The first form allows treating $\mathbf{X}_{kl}$ as a component interfering with the estimation of the useful signal $\mathbf{S}_{kl}$. The second form allows treating $\mathbf{X}_l$ as a component interfering with the estimation of the interference $\boldsymbol{\Theta}$.

Next, we need the optimal linear estimation matrices $\mathbf{P}_{kl}^{\mathbf{S}} = \left[p_{krnl}^{\mathbf{S}}, r = \overline{1, N}, n = \overline{1, N}\right]$, $\mathbf{P}_{kl}^{\mathbf{A}} = \left[p_{kmnl}^{\mathbf{A}}, m = \overline{1, M_{kl}}, n = \overline{1, N}\right]$, $\mathbf{P}_l^{\boldsymbol{\Theta}} = \left[p_{rnl}^{\boldsymbol{\Theta}}, r = \overline{1, N}, n = \overline{1, N}\right]$, and $\mathbf{P}_l^{\mathbf{B}} = \left[p_{jnl}^{\mathbf{B}}, j = \overline{1, J}, n = \overline{1, N}\right]$ to obtain optimal estimates (based on GIUE for fixed numbers l and $\boldsymbol{\zeta}_k$) for the signal $\mathbf{S}_{kl}$ and its spectral coefficient vector $\mathbf{A}_{kl}$ as well as for the interference $\boldsymbol{\Theta}$ and its spectral coefficient vector $\mathbf{B}$:

$$\mathbf{S}_{kl}^* = \mathbf{P}_{kl}^{\mathbf{S}} \mathbf{H}_l, \quad \mathbf{A}_{kl}^* = \mathbf{P}_{kl}^{\mathbf{A}} \mathbf{H}_l, \quad k = \overline{1, K_l}, \quad \boldsymbol{\Theta}_l^* = \mathbf{P}_l^{\boldsymbol{\Theta}} \mathbf{H}_l, \quad \mathbf{B}_l^* = \mathbf{P}_l^{\mathbf{B}} \mathbf{H}_l.$$

Within the hypothesis Γ_l, the matrices $\mathbf{P}_{kl}^{\mathbf{S}}$, $\mathbf{P}_{kl}^{\mathbf{A}}$, $\mathbf{P}_l^{\boldsymbol{\Theta}}$, and $\mathbf{P}_l^{\mathbf{B}}$ must ensure the following equalities:

$$\begin{cases} \mathbf{H}_{kl}^{\mathbf{S}} = \mathbf{P}_{kl}^{\mathbf{S}} \mathbf{H}_l = \mathbf{P}_{kl}^{\mathbf{S}} \mathbf{S}_{kl} + \mathbf{P}_{kl}^{\mathbf{S}} \mathbf{X}_{kl} + \mathbf{P}_{kl}^{\mathbf{S}} \boldsymbol{\Xi} = \mathbf{S}_{kl} + \boldsymbol{\Xi}_{kl}^{\mathbf{S}}, \quad k = \overline{1, K_l} \\ \mathbf{H}_{kl}^{\mathbf{A}} = \mathbf{P}_{kl}^{\mathbf{A}} \mathbf{H}_l = \mathbf{P}_{kl}^{\mathbf{A}} \mathbf{S}_{kl} + \mathbf{P}_{kl}^{\mathbf{A}} \mathbf{X}_{kl} + \mathbf{P}_{kl}^{\mathbf{A}} \boldsymbol{\Xi} = \mathbf{A}_{kl} + \boldsymbol{\Xi}_{kl}^{\mathbf{A}}, \quad k = \overline{1, K_l} \\ \mathbf{H}_l^{\boldsymbol{\Theta}} = \mathbf{P}_l^{\boldsymbol{\Theta}} \mathbf{H}_l = \mathbf{P}_l^{\boldsymbol{\Theta}} \boldsymbol{\Theta} + \mathbf{P}_l^{\boldsymbol{\Theta}} \mathbf{X}_l + \mathbf{P}_l^{\boldsymbol{\Theta}} \boldsymbol{\Xi} = \boldsymbol{\Theta} + \boldsymbol{\Xi}_l^{\boldsymbol{\Theta}} \\ \mathbf{H}_l^{\mathbf{B}} = \mathbf{P}_l^{\mathbf{B}} \mathbf{H}_l = \mathbf{P}_l^{\mathbf{B}} \boldsymbol{\Theta} + \mathbf{P}_l^{\mathbf{B}} \mathbf{X}_l + \mathbf{P}_l^{\mathbf{B}} \boldsymbol{\Xi} = \mathbf{B} + \boldsymbol{\Xi}_l^{\mathbf{B}}, \end{cases} \tag{10}$$

where $\boldsymbol{\Xi}_{kl}^{\mathbf{S}} = \mathbf{P}_{kl}^{\mathbf{S}} \boldsymbol{\Xi}$ and $\boldsymbol{\Xi}_l^{\Theta} = \mathbf{P}_l^{\Theta} \boldsymbol{\Xi}$ are the noises with zero means $M\{\boldsymbol{\Xi}_{kl}^{\mathbf{S}}\} = M\{\mathbf{P}_{kl}^{\mathbf{S}} \boldsymbol{\Xi}\} = \mathbf{P}_{kl}^{\mathbf{S}} M\{\boldsymbol{\Xi}\} = [\mathbf{0}]_{N\times 1}$, $M\{\boldsymbol{\Xi}_{kl}^{\mathbf{A}}\} = M\{\mathbf{P}_{kl}^{\mathbf{A}} \boldsymbol{\Xi}\} = \mathbf{P}_{kl}^{\mathbf{A}} M\{\boldsymbol{\Xi}\} = [\mathbf{0}]_{M_{kl}\times 1}$, $M\{\boldsymbol{\Xi}_l^{\Theta}\} = M\{\mathbf{P}_l^{\Theta} \boldsymbol{\Xi}\} = \mathbf{P}_l^{\Theta} M\{\boldsymbol{\Xi}\} = [\mathbf{0}]_{N\times 1}$, $M\{\boldsymbol{\Xi}_l^{\mathbf{B}}\} = M\{\mathbf{P}_l^{\mathbf{B}} \boldsymbol{\Xi}\} = \mathbf{P}_l^{\mathbf{B}} M\{\boldsymbol{\Xi}\} = [\mathbf{0}]_{J\times 1}$. (In these expressions, $M\{\cdot\}$ stands for the mathematical expectation and $\mathbf{0}$ is a zero column vector of an appropriate dimension specified by subscripts in square brackets.)

In addition, the correlation matrices $\mathbf{K}_{kl}^{\boldsymbol{\Xi}\mathbf{S}}$, $\mathbf{K}_{kl}^{\boldsymbol{\Xi}\mathbf{A}}$, $\mathbf{K}_l^{\boldsymbol{\Xi}\boldsymbol{\Theta}}$, and $\mathbf{K}_l^{\boldsymbol{\Xi}\mathbf{B}}$ of the random vectors $\boldsymbol{\Xi}_{kl}^{\mathbf{S}}$, $\boldsymbol{\Xi}_{kl}^{\mathbf{A}}$, $\boldsymbol{\Xi}_l^{\boldsymbol{\Theta}}$, and $\boldsymbol{\Xi}_l^{\mathbf{B}}$, respectively, must ensure the minimum conditions

$$\begin{cases} Sp\mathbf{K}_{kl}^{\boldsymbol{\Xi}\mathbf{S}} \longrightarrow \min\limits_{\mathbf{P}_{kl}^{\mathbf{S}}}, \quad k = \overline{1, K_l} \\ Sp\mathbf{K}_{kl}^{\boldsymbol{\Xi}\mathbf{A}} \longrightarrow \min\limits_{\mathbf{P}_{kl}^{\mathbf{A}}}, \quad k = \overline{1, K_l} \\ Sp\mathbf{K}_l^{\boldsymbol{\Xi}\boldsymbol{\Theta}} \longrightarrow \min\limits_{\mathbf{P}_l^{\boldsymbol{\Theta}}} \\ Sp\mathbf{K}_l^{\boldsymbol{\Xi}\mathbf{B}} \longrightarrow \min\limits_{\mathbf{P}_l^{\mathbf{B}}}, \end{cases} \tag{11}$$

where Sp is the matrix trace operator.

Formula (10) reflects two properties: unbiasedness with respect to the parameters of useful signals and singular interference, i.e.,

$$\begin{cases} \mathbf{P}_{kl}^{\mathbf{S}} \mathbf{S}_{kl} = \mathbf{S}_{kl}, \quad k = \overline{1, K_l} \\ \mathbf{P}_{kl}^{\mathbf{A}} \mathbf{S}_{kl} = \mathbf{A}_{kl}, \quad k = \overline{1, K_l} \\ \mathbf{P}_l^{\boldsymbol{\Theta}} \boldsymbol{\Theta} = \boldsymbol{\Theta} \\ \mathbf{P}_l^{\mathbf{B}} \boldsymbol{\Theta} = \mathbf{B}, \end{cases} \tag{12}$$

and invariance with respect to the interfering components $\mathbf{X}_{kl}$ and $\mathbf{X}_l$, i.e.,

$$\begin{cases} \mathbf{P}^{\mathbf{S}}_{kl}\mathbf{X}_{kl} = [\mathbf{0}]_{N\times 1}, & k = \overline{1, K_l} \\ \mathbf{P}^{\mathbf{A}}_{kl}\mathbf{X}_{kl} = [\mathbf{0}]_{M_{kl}\times 1}, & k = \overline{1, K_l} \\ \mathbf{P}^{\mathbf{\Theta}}_{l}\mathbf{X}_{l} = [\mathbf{0}]_{N\times 1} \\ \mathbf{P}^{\mathbf{B}}_{l}\mathbf{X}_{l} = [\mathbf{0}]_{J\times 1}. \end{cases} \tag{13}$$

Applying the matrices $\mathbf{P}^{\mathbf{S}}_{kl}$, $\mathbf{P}^{\mathbf{A}}_{kl}$, $\mathbf{P}^{\mathbf{\Theta}}_{kl}$, and $\mathbf{P}^{\mathbf{B}}_{kl}$ directly to the observation (1) yields the set of optimal estimates for all parameters of the signal recognition problem under fixed numbers l and $\boldsymbol{\zeta}_l$:

$$\begin{cases} \mathbf{S}^{*}_{kl} = \mathbf{P}^{\mathbf{S}}_{kl}\mathbf{H} = \sum\limits_{i=1}^{K_l} q_i\mathbf{P}^{\mathbf{S}}_{kl}\mathbf{S}_i + \mathbf{P}^{\mathbf{S}}_{kl}\mathbf{\Theta} + \mathbf{\Xi}^{\mathbf{S}}_{kl}, \quad k = \overline{1, K_l} \\ \mathbf{A}^{*}_{kl} = \mathbf{P}^{\mathbf{A}}_{kl}\mathbf{H} = \sum\limits_{i=1}^{K_l} q_i\mathbf{P}^{\mathbf{A}}_{kl}\mathbf{S}_i + \mathbf{P}^{\mathbf{A}}_{kl}\mathbf{\Theta} + \mathbf{\Xi}^{\mathbf{A}}_{kl}, \quad k = \overline{1, K_l} \\ \mathbf{\Theta}^{*}_{l} = \mathbf{P}^{\mathbf{\Theta}}_{l}\mathbf{H} = \sum\limits_{i=1}^{K_l} q_i\mathbf{P}^{\mathbf{\Theta}}_{l}\mathbf{S}_i + \mathbf{P}^{\mathbf{\Theta}}_{l}\mathbf{\Theta} + \mathbf{\Xi}^{\mathbf{\Theta}}_{l} \\ \mathbf{B}^{*}_{l} = \mathbf{P}^{\mathbf{B}}_{l}\mathbf{H} = \sum\limits_{i=1}^{K_l} q_i\mathbf{P}^{\mathbf{B}}_{l}\mathbf{S}_i + \mathbf{P}^{\mathbf{B}}_{l}\mathbf{\Theta} + \mathbf{\Xi}^{\mathbf{B}}_{l}. \end{cases} \tag{14}$$

In contrast to XLS, the GIUE-based estimates (14) of the signal and interference parameters are formed separately, which allows for parallel computing; moreover, due to the invariance conditions, the dimensions of the inverted matrices can be appreciably reduced (by analogy with [1–3]). This feature is well demonstrated in the illustrative example below.

In view of (12)–(14), for the true hypothesis Γ^0 with number $l^0 \in \{1, \ldots, L\}$, we obtain

$$\begin{cases} \mathbf{S}^{*}_{kl^0} = \mathbf{P}^{\mathbf{S}}_{kl^0}\mathbf{H} = q_k\mathbf{S}_k + \mathbf{\Xi}^{\mathbf{S}}_{kl^0}, \quad k = \overline{1, K_{l^0}} \\ \mathbf{A}^{*}_{kl^0} = \mathbf{P}^{\mathbf{A}}_{kl^0}\mathbf{H} = q_k\mathbf{A}_k + \mathbf{\Xi}^{\mathbf{A}}_{kl^0}, \quad k = \overline{1, K_{l^0}} \\ \mathbf{\Theta}^{*}_{l^0} = \mathbf{P}^{\mathbf{\Theta}}_{l^0}\mathbf{H} = \mathbf{\Theta} + \mathbf{\Xi}^{\mathbf{\Theta}}_{l^0} \\ \mathbf{B}^{*}_{l^0} = \mathbf{P}^{\mathbf{B}}_{l^0}\mathbf{H} = \mathbf{B} + \mathbf{\Xi}^{\mathbf{B}}_{l^0}. \end{cases} \tag{15}$$

If Γ_l does not match the true hypothesis Γ^0, then conditions (12) and (13) are violated, causing the residual

$$\Delta^{\text{giue}}(l, \boldsymbol{\zeta}_l) = \mathbf{H} - \mathbf{H}^{\text{giue}}(\mathbf{S}^{*}_{l}, \mathbf{\Theta}^{*}_{l}) = \mathbf{H} - \sum_{i=1}^{K_l} \mathbf{S}^{*}_{il} - \mathbf{\Theta}^{*}_{l},$$

where $\mathbf{S}^{*}_{l} = \left[(\mathbf{S}^{*}_{il})^{\mathrm{T}}, i = \overline{1, K_l}\right]^{\mathrm{T}}$, $\mathbf{S}^{*}_{il} = \mathbf{S}^{*}_{il}(\boldsymbol{\zeta}_l)$, and $\mathbf{\Theta}^{*}_{l} = \mathbf{\Theta}^{*}_{l}(\boldsymbol{\zeta}_l)$. In this case, the recognition problem within GIUE is solved by minimizing the quadratic criterion

$$(l^*, \boldsymbol{\zeta}_{l^*}) = \arg\min_{l, \boldsymbol{\zeta}_l} \chi^{\text{giue}}(l, \boldsymbol{\zeta}_l) = \arg\min_{l, \boldsymbol{\zeta}_l} \left[\Delta^{\text{giue}}(l, \boldsymbol{\zeta}_l)\right]^{\mathrm{T}} \left(\mathbf{K}^{\mathbf{\Xi}}\right)^{-1} (\Delta^{\text{giue}})(l, \boldsymbol{\zeta}_l); \tag{16}$$

the resulting estimates of the signal and singular interference parameters are given by $\mathbf{S}^{*}_{l^*} = \mathbf{S}^{*}_{l=l^*}$, $\mathbf{A}^{*}_{l^*} = \mathbf{A}^{*}_{l=l^*}$, $\mathbf{\Theta}^{*}_{l^*} = \mathbf{\Theta}^{*}_{l=l^*}$, and $\mathbf{B}^{*}_{l^*} = \mathbf{B}^{*}_{l=l^*}$, where $\mathbf{A}^{*}_{l} = \left[(\mathbf{A}^{*}_{kl})^{\mathrm{T}}, k = \overline{1, K_l}\right]^{\mathrm{T}}$.

In (16), minimization with respect to the nonlinear parameter is performed simultaneously at all nodes $\boldsymbol{\zeta}_l$ of the numerical grid by parallel computing.

Formulas (1)–(16) fully specify all the models, constraints, and criteria necessary to develop the novel signal resolution method under essential a priori uncertainty and compare this method with XLS. It is required to do the following: construct the matrices $\mathbf{P}_{kl}^{\mathbf{S}}$, $\mathbf{P}_{kl}^{\mathbf{A}}$, $\mathbf{P}_{kl}^{\boldsymbol{\Theta}}$, and $\mathbf{P}_{kl}^{\mathbf{B}}$ for fixed l and $\mathbf{d}$; using these matrices and the adopted optimality criterion, solve the optimal recognition problem of signals in decomposed form (with linear and nonlinear parameters) without traditional state space expansion under minimum a priori statistical information (the matrix $\mathbf{K}^{\boldsymbol{\Xi}}$ only), ensure the self-compensation of the singular interference and the parallel processing of observations; derive formulas for the random and methodological errors of the resulting estimates; compare the novel method with XLS in terms of computational efficiency; demonstrate the possibility of its comparison with known statistical signal recognition methods; finally, give an illustrative example confirming the advantages of the novel method over XLS.

In the general case, it is unknown which ensemble signals appear in the observation (1), and the value of the nonlinear parameter $\boldsymbol{\zeta}_i$ is also unknown for each of these signals $s_i(t, \mathbf{A}_i, \boldsymbol{\zeta}_i)$. In this case, solving the recognition problem leads to complex iterative procedures with the need to choose sufficiently good initial approximations. In practice, the complexity described above can be circumvented using a parallel algorithm that tests all possible hypotheses for the presence of signals and interference in (4) and examines all nodes for each hypothesis. This approach will be applied below to eliminate the nonlinearity and related difficulties. Such an idea was successfully employed in [20, 21] to solve several applied control problems.

3. CONSTRUCTING THE OPTIMAL LINEAR SELF-COMPENSATION–DECOMPOSITION ESTIMATION MATRICES

First, let the exact value of the parameter $\boldsymbol{\zeta}$ be known and the mixture (1) contain all K signals of the ensemble, i.e., $q_1 = q_2 = \ldots = q_K = 1$. (In this case, the subscript l can be omitted.) Then the recognition problem can be solved in the class of linear estimates in the form of K parallel computation algorithms

$$\mathbf{A}_k^* = \mathbf{P}_k^{\mathbf{A}}\mathbf{H}, \quad k = \overline{1, K}, \tag{17}$$

where $\mathbf{A}_k^*$ denotes the estimate of the vector $\mathbf{A}_k$ and $\mathbf{P}_k^{\mathbf{A}} = [p_{kmn}^A, m = \overline{1, M_k}, n = \overline{1, N}]$ is the matrix of unknown optimal estimation weights.

The correlation matrix of estimation errors based on (17) is given by

$$\mathbf{K}_k^{\mathbf{A}} = \mathbf{P}_k^{\mathbf{A}}\mathbf{K}^{\boldsymbol{\Xi}}\left(\mathbf{P}_k^{\mathbf{A}}\right)^{\mathrm{T}}, \quad k = \overline{1, K}. \tag{18}$$

The optimality criterion to find $\mathbf{P}_k^{\mathbf{A}}$ is minimizing the trace of the matrix $Sp\mathbf{K}_k^{\mathbf{A}}$ (see (11)). In addition, the supplementary conditions (12) (unbiasedness) and (13) (invariance) must hold.

To obtain the matrix $\mathbf{P}_k^{\mathbf{A}}$, we transform (9) to

$$\mathbf{H} = \mathbf{S}_k + \mathbf{X}_k + \boldsymbol{\Xi}, \tag{19}$$

where $\mathbf{X}_k = \left[x_{kn}, n = \overline{1, N}\right]^{\mathrm{T}}$, $\mathbf{X}_k = \mathbf{Y}_k\mathbf{C}_k$,

$\mathbf{Y}_k = [\boldsymbol{\Psi}_1 \vdots \ldots \vdots \boldsymbol{\Psi}_{k-1} \vdots \boldsymbol{\Psi}_{k+1} \vdots \ldots \vdots \boldsymbol{\Psi}_K \vdots \boldsymbol{\Omega}]$ is a matrix of dimensions $N \times (\overline{M}_k + J)$ and

$\mathbf{C}_k = \left[\mathbf{A}_1^{\mathrm{T}} \vdots \ldots \vdots \mathbf{A}_{k-1}^{\mathrm{T}} \vdots \mathbf{A}_{k+1}^{\mathrm{T}} \vdots \ldots \vdots \mathbf{A}_K^{\mathrm{T}} \vdots \mathbf{B}^{\mathrm{T}}\right]^{\mathrm{T}}$ is a vector of dimensions $(\overline{M}_k + J) \times 1$, $\overline{M}_k = M_1 + \ldots + M_{k-1} + M_{k+1} + \ldots + M_K$.

For the case under consideration, due to (12), (13), and (19), the unbiasedness and invariance conditions can be written as

$$\mathbf{P}_k^{\mathbf{A}}\boldsymbol{\Psi}_k - [\mathbf{E}]_{M_k\times M_k} = [\mathbf{0}]_{M_k\times M_k}, \tag{20}$$

$$\mathbf{P}_k^{\mathbf{A}}\mathbf{Y}_k = [\mathbf{0}]_{M_k\times(\overline{M}_k+J)}, \tag{21}$$

where $[\mathbf{0}]_{M_k\times M_k}$ and $[\mathbf{E}]_{M_k\times M_k}$ mean zero and identity matrices, respectively, and $[\cdot]_{M_k\times M_k}$ indicates the dimensions of a matrix in square brackets. (This notation is used throughout the paper.)

For the further presentation, we need the vector $\mathbf{P}_{km}^{\mathbf{A}} = [p_{pmn}^{\mathbf{A}}, n = \overline{1,N}]^{\mathrm{T}}$, containing all elements of the mth row of the matrix $\mathbf{P}_k^{\mathbf{A}}$. It allows finding the scalar estimate a_{km}^* of the coefficient a_{km} under fixed numbers k and m. Obviously, $(\mathbf{P}_{km}^{\mathbf{A}})^{\mathrm{T}}\mathbf{S}_k = a_{km}$ (unbiasedness) and $(\mathbf{P}_{km}^{\mathbf{A}})^{\mathrm{T}}\mathbf{X}_k = 0$ (invariance). By analogy with (20) and (21), we therefore have

$$(\mathbf{P}_{km}^{\mathbf{A}})^{\mathrm{T}}\boldsymbol{\Psi}_k - \mathbf{E}_{km}^{\mathrm{T}} = [\mathbf{0}]_{1\times M_k}, \tag{22}$$

$$\mathbf{Y}_k^{\mathrm{T}}\mathbf{P}_{km}^{\mathbf{A}} = [\mathbf{0}]_{(\overline{M}_k+J)\times 1}, \tag{23}$$

where $\mathbf{E}_{km}$ denotes a column vector whose components are all zero except the mth one.

Theorem. *The optimal estimation vector $\mathbf{P}_{km}^{\mathbf{A}}$ of the spectral coefficient a_{km} that minimizes the quadratic criterion $(\mathbf{P}_{km}^{\mathbf{A}})^{\mathrm{T}}\mathbf{K}^{\boldsymbol{\Xi}}\mathbf{P}_{km}^{\mathbf{A}}$ subject to the unbiasedness* (22) *and invariance* (23) *conditions is given by*

$$\mathbf{P}_{km}^{\mathbf{A}} = \boldsymbol{\Lambda}_k\mathbf{V}_k(\boldsymbol{\Psi}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{V}_k)^{-1}\mathbf{E}_{km}. \tag{24}$$

This result is proved in the Appendix.

The scalar estimate a_{km}^* of the coefficient is calculated as

$$a_{km}^* = \mathbf{H}^{\mathrm{T}}\mathbf{P}_{km}^{\mathbf{A}} = \mathbf{H}^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{V}_k(\boldsymbol{\Psi}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{V}_k)^{-1}\mathbf{E}_{km}. \tag{25}$$

In view of (24), passing from the scalar coefficient a_{km} to the vector $\mathbf{A}_k$ yields the optimal estimation matrix

$$\mathbf{P}_k^{\mathbf{A}} = \left[\boldsymbol{\Lambda}_k\mathbf{V}_k(\boldsymbol{\Psi}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{V}_k)^{-1}\right]^{\mathrm{T}}, \quad k = \overline{1,K}. \tag{26}$$

Substituting (26) into (17), we find the desired estimates

$$\mathbf{A}_k^* = \mathbf{P}_k^{\mathbf{A}}\mathbf{H} = \left[\boldsymbol{\Lambda}_k\mathbf{V}_k(\boldsymbol{\Psi}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{V}_k)^{-1}\right]^{\mathrm{T}}\mathbf{H}, \quad k = \overline{1,K}. \tag{27}$$

In turn, the optimal estimation matrix of the signals $\mathbf{S}_{kl}$ is given by

$$\mathbf{P}_k^{\mathbf{S}} = \boldsymbol{\Psi}_k\mathbf{P}_k^{\mathbf{A}} \tag{28}$$

whereas the optimal estimate by

$$\mathbf{S}_k^* = \boldsymbol{\Psi}_k\mathbf{A}_k^* = \boldsymbol{\Psi}_k\left[\boldsymbol{\Lambda}_k\mathbf{V}_k(\boldsymbol{\Psi}_k^{\mathrm{T}}\boldsymbol{\Lambda}_k\mathbf{V}_k)^{-1}\right]^{\mathrm{T}}\mathbf{H}, \quad k = \overline{1,K}. \tag{29}$$

The estimates (27) and (29) are optimal in the sense of their unbiasedness, efficiency (minimum variance), and invariance with respect to the resulting singular interference.

In addition to the matrices $\mathbf{P}_k^{\mathbf{A}}$ and $\mathbf{P}_k^{\mathbf{S}}$, in several signal recognition problems it is necessary to construct the optimal estimation matrices $\mathbf{P}^{\mathbf{B}}$ and $\mathbf{P}^{\boldsymbol{\Theta}}$ for the interference parameters under the unbiasedness and invariance conditions (by analogy with (22) and (23)). In this case, the matrices

$\mathbf{Y}_k^{\mathbf{\Theta}}$ and $\mathbf{C}_k^{\mathbf{\Theta}}$ are constructed instead of $\mathbf{Y}_{kl}$ and $\mathbf{C}_{kl}$ to obtain the estimates $\mathbf{B}^*$ and $\mathbf{\Theta}^*$ while treating all useful signals as interference components. The formulas for the matrices $\mathbf{P}^{\mathbf{B}}$ and $\mathbf{P}^{\mathbf{\Theta}}$, as well as those for the estimates $\mathbf{B}^*$ and $\mathbf{\Theta}^*$, are written by analogy with (26)–(29).

For signal recognition problems with possible hypotheses Γ_l and the nonlinear parameter γ, we construct a family of optimal estimation matrices for all values l and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$: $\mathbf{P}_{k(\mathbf{d})}^{\mathbf{A}}$, $\mathbf{P}_{k(\mathbf{d})}^{\mathbf{S}}$, $\mathbf{P}_{(\mathbf{d})}^{\mathbf{B}}$, and $\mathbf{P}_{(\mathbf{d})}^{\mathbf{\Theta}}$.

The estimation matrices presented in this section are necessary and sufficient for solving a wide range of problems related to signal recognition under essential a priori uncertainty. The next section considers the most common recognition problems: estimation (the signals present in the observation are known, and it is required to estimate their parameters only); detection (it is required to determine whether a useful signal is present in the observation); discrimination (only one signal from a given ensemble is present in the observation, and it is required to identify this signal); resolution (some signals from the ensemble may be present in the observation, and it is required to identify them). In addition, the problems of signal detection, discrimination, and resolution can be accompanied by the estimation of the parameters of signals and interference.

4. ALGORITHMS FOR SOLVING BASIC SIGNAL RECOGNITION PROBLEMS UNDER UNCERTAINTY

Algorithm for estimating signals with known nonlinear parameters (1). Let the mixture (1) contain all ensemble signals, i.e., $q_1 = q_2 = \ldots = q_K = 1$. It is required to estimate all coefficients $\mathbf{A}_k$ and the signals $\mathbf{S}_k$ without determining the coefficient $\mathbf{B}$ of the interference $\mathbf{\Theta}$, i.e., without state space expansion. The hypotheses are not used in this problem, so we omit the subscript l.

Step 1.1. Construct the matrices $\mathbf{P}_k^{\mathbf{A}}$ of the weight coefficients.

Step 1.2. Find the estimates $\mathbf{A}_k^* = \mathbf{P}_k^{\mathbf{A}}\mathbf{H}$ of the vector coefficients $\mathbf{A}_k$, $k = \overline{1, K}$.

Step 1.3. Construct the matrices $\mathbf{P}_k^{\mathbf{S}} = \mathbf{\Psi}_k \mathbf{P}_k^{\mathbf{A}}$, $k = \overline{1, K}$.

Step 1.4. Find the estimates $\mathbf{S}_k^* = \mathbf{P}_k^{\mathbf{S}} H$ of the signals $\mathbf{S}_k$, $k = \overline{1, K}$.

Algorithm for estimating signals and interference with unknown nonlinear parameters (2). Let the mixture (1) contain all ensemble signals, i.e., $q_1 = q_2 = \ldots = q_K = 1$. It is required to estimate all coefficients $\mathbf{A}_k$ and $\boldsymbol{\zeta}_k$, the signals $\mathbf{S}_k$, as well as the coefficient $\mathbf{B}$ and the interference $\mathbf{\Theta}$. The hypotheses are not used in this problem, so we omit the subscript l.

Step 2.1. Construct the matrices $\mathbf{P}_{k(\mathbf{d})}^{\mathbf{A}}$, $\mathbf{P}_{k(\mathbf{d})}^{\mathbf{S}}$, $\mathbf{P}_{(\mathbf{d})}^{\mathbf{B}}$, and $\mathbf{P}_{(\mathbf{d})}^{\mathbf{\Theta}}$ for all nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 2.2. Find the partial estimates $\mathbf{A}_{k(\mathbf{d})}^*$, $\mathbf{S}_{k(\mathbf{d})}^*$, $\mathbf{B}_{(\mathbf{d})}^*$, and $\mathbf{\Theta}_{(\mathbf{d})}^*$ for all nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 2.3. Calculate the residuals

$$\Delta^{\text{giue}}(\mathbf{d}) = \mathbf{H} - \sum_{i=1}^{K} \mathbf{S}_{i(\mathbf{d})}^* - \mathbf{\Theta}_{(d)}^*$$

for all nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 2.4. Find the estimate $\mathbf{d}^*$ of $\mathbf{d}$ in the form

$$\mathbf{d}^* = \arg\min_{\mathbf{d}} \chi^{\text{giue}}(\mathbf{d}) = \arg\min_{\mathbf{d}} \left[\Delta^{\text{giue}}(\mathbf{d})\right]^{\mathrm{T}} \left(\mathbf{K}^{\mathbf{\Xi}}\right)^{-1} \Delta^{\text{giue}}(\mathbf{d}).$$

Step 2.5. Find the resulting estimates $\mathbf{A}_{k\mathbf{d}^*}^*$, $\mathbf{S}_{k(\mathbf{d}^*)}^*$, $\boldsymbol{\zeta}_{k(\mathbf{d}^*)}^*$, $\mathbf{B}_{(\mathbf{d}^*)}^*$, and $\mathbf{\Theta}_{(\mathbf{d}^*)}^*$.

Algorithm for jointly detecting and estimating with known nonlinear parameters (3). In this case, $K = 1$, $L = 2$, and consequently, $\mathbf{H} = q\mathbf{S} + \mathbf{\Theta} + \mathbf{\Xi}$. Depending on the value of the coefficient q, two hypotheses are therefore possible: Γ_1 if $q = 0$ and Γ_2 if $q = 1$. It is required to obtain an estimate

$l^* \in \{0,1\}$ for the parameter $l \in \{0,1\}$ and construct the estimates $\mathbf{A}^*_{l^*}$, $\mathbf{S}^*_{l^*}$, $\mathbf{B}^*_{l^*}$, and $\mathbf{\Theta}^*_{l^*}$. In this case, the subscript k in formulas (26)–(29) can be omitted.

Step 3.1. Construct the matrices $\mathbf{P}^{\mathbf{A}}_{l=1}$, $\mathbf{P}^{\mathbf{A}}_{l=2}$ and $\mathbf{P}^{\mathbf{S}}_{l=1}$, $\mathbf{P}^{\mathbf{S}}_{l=2}$ for Γ_1 and Γ_2, respectively.

Step 3.2. Find the estimates $\mathbf{A}^*_{l=1}$, $\mathbf{S}^*_{l=1}$ (for the signal $\mathbf{S} = [\mathbf{0}]_{N\times 1}$ and the hypothesis Γ_1) and $\mathbf{A}^*_{l=2}$, $\mathbf{S}^*_{l=2}$ (for the signal $\mathbf{S} \neq [\mathbf{0}]_{N\times 1}$ and the hypothesis Γ_2).

Step 3.3. Construct the matrices $\mathbf{P}^{\mathbf{B}}_{l=1}$, $\mathbf{P}^{\mathbf{\Theta}}_{l=1}$ and $\mathbf{P}^{\mathbf{B}}_{l=2}$, $P^{\Theta}_{l=2}$ for Γ_1 and Γ_2, respectively.

Step 3.4. Find the estimates $\mathbf{B}^*_{l=1}$, $\mathbf{\Theta}^*_{l=1}$, $\mathbf{B}^*_{l=2}$, and $\mathbf{\Theta}^*_{l=2}$.

Step 3.5. Calculate the residuals

$$\Delta^{\text{giue}}(l) = \mathbf{H} - \mathbf{S}^*_l - \mathbf{\Theta}^*_l$$

and select the best hypothesis in terms of the adopted optimality criterion:

$$l^* = \arg\min_l \chi^{\text{giue}}(l) = \arg\min_l \left[\Delta^{\text{giue}}(l)\right]^{\mathrm{T}} (\mathbf{K}^{\mathbf{\Xi}})^{-1} \Delta^{\text{giue}}(l), \quad l^* \in \{0,1\}.$$

Step 3.6. Find the resulting estimates $\mathbf{A}^*_{l^*}$, $\mathbf{S}^*_{l^*}$, $\mathbf{B}^*_{l^*}$, and $\mathbf{\Theta}^*_{l^*}$.

Algorithm for jointly discriminating and estimating the parameters of signals and interference with unknown nonlinear parameters (4). In this case, K is arbitrary, $L = K$, and $\mathbf{H} = \mathbf{S}_l + \mathbf{\Theta} + \mathbf{\Xi}$, $l \in \{1,\ldots,L\}$. It is required to establish which signal from the given ensemble $\{\mathbf{S}_1,\ldots,\mathbf{S}_L\}$ appears in the observation, i.e., obtain the estimate $l^* \in \{1,\ldots,L\}$ of the parameter l, as well as construct the estimates $\mathbf{A}^*_{l^*}$, $\mathbf{S}^*_{l^*}$, $\mathbf{B}^*_{l^*}$, and $\mathbf{\Theta}^*_{l^*}$. In this case, the subscript k can be omitted.

Step 4.1. Construct the matrices $\mathbf{P}^{\mathbf{A}}_{l(\mathbf{d})}$, $\mathbf{P}^{\mathbf{S}}_{l(\mathbf{d})}$, $\mathbf{P}^{\mathbf{B}}_{l(\mathbf{d})}$, and $\mathbf{P}^{\mathbf{\Theta}}_{l(\mathbf{d})}$ for all Γ_l and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 4.2. Find the partial estimates $\mathbf{A}^*_{l(\mathbf{d})}$, $\mathbf{S}^*_{l(\mathbf{d})}$, $\mathbf{B}^*_{l(\mathbf{d})}$, and $\mathbf{\Theta}^*_{l(\mathbf{d})}$ for all Γ_l and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 4.3. Calculate the residuals

$$\Delta^{\text{giue}}(l,\mathbf{d}) = \mathbf{H} - \mathbf{S}^*_{l(d)} - \mathbf{\Theta}^*_{l(d)}$$

for all Γ_l and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 4.4. Select the best hypothesis in terms of the adopted optimality criterion:

$$(l^*,\mathbf{d}^*) = \arg\min_{l,\mathbf{d}} \chi^{\text{giue}}(l,\mathbf{d}) = \arg\min_{l,\mathbf{d}} \left[\Delta^{\text{giue}}(l,\mathbf{d})\right]^{\mathrm{T}} \left(\mathbf{K}^{\mathbf{\Xi}}\right)^{-1} \Delta^{\text{giue}}(l,\mathbf{d}).$$

Step 4.5. Find the resulting estimates $\mathbf{A}^*_{l^*(\mathbf{d}^*)}$, $\mathbf{S}^*_{l^*(\mathbf{d}^*)}$, $\mathbf{B}^*_{l^*(\mathbf{d}^*)}$, and $\mathbf{\Theta}^*_{l^*(\mathbf{d}^*)}$ for the hypothesis Γ_{l^*} and node $\boldsymbol{\zeta}_{(\mathbf{d}^*)}$.

Algorithm for jointly recognizing and estimating the parameters of signals and interference with unknown nonlinear parameters (5). Here, we have the general case (4). In this case, K is arbitrary and $L = 2^K \geqslant 2$. It is required to establish which signals from the given ensemble $\{\mathbf{S}_1,\ldots,\mathbf{S}_L\}$ appear in the observation, estimate their linear and nonlinear parameters as well as the interference parameters. In this case, the hypotheses Γ_l are used, where $l \in \{1,\ldots,2^K\}$.

Step 5.1. Construct the matrices $\mathbf{P}^{\mathbf{A}}_{kl(\mathbf{d})}$, $\mathbf{P}^{\mathbf{S}}_{kl(\mathbf{d})}$, $\mathbf{P}^{\mathbf{B}}_{l(\mathbf{d})}$, and $\mathbf{P}^{\mathbf{\Theta}}_{l(\mathbf{d})}$ for all Γ_l, $k = \overline{1,K_l}$, and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 5.2. Find the partial estimates $\mathbf{A}^*_{kl(\mathbf{d})}$, $\mathbf{S}^*_{kl(\mathbf{d})}$, $\mathbf{B}^*_{l(\mathbf{d})}$, and $\mathbf{\Theta}^*_{l(\mathbf{d})}$ for all Γ_l, $k = \overline{1,K_l}$, and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 5.3. Calculate the residuals

$$\Delta^{\text{giue}}(l,\mathbf{d}) = \mathbf{H} - \sum_{k=l}^{K_l} \mathbf{S}^*_{kl(\mathbf{d})} - \mathbf{\Theta}^*_{(\mathbf{d})}$$

for all Γ_l and nodes $\boldsymbol{\zeta}_{(\mathbf{d})}$.

Step 5.4. Select the best hypothesis in terms of the adopted optimality criterion:

$$(l^*, \mathbf{d}^*) = \arg\min_{l,\mathbf{d}} \chi^{\text{giue}}(l, \mathbf{d}) = \arg\min_{l,\mathbf{d}} \left[\Delta^{\text{giue}}(l, \mathbf{d})\right]^{\mathrm{T}} \left(\mathbf{K}^{\boldsymbol{\Xi}}\right)^{-1} \Delta^{\text{giue}}(l, \mathbf{d}).$$

Step 5.5. Find the resulting estimates $\mathbf{A}^*_{kl^*(\mathbf{d}^*)}$, $\mathbf{S}^*_{kl^*(\mathbf{d}^*)}$ (where $k = \overline{1, K_l}$), $\mathbf{B}^*_{l^*(\mathbf{d}^*)}$, and $\boldsymbol{\Theta}^*_{l^*(\mathbf{d}^*)}$ for the hypothesis Γ_{l^*} and node $\boldsymbol{\zeta}_{(\mathbf{d}^*)}$.

Note. The optimality criteria used in these algorithms ensure the minimum mutual influence of neighbor ensemble signals, the self-compensation of the singular interference, and noise smoothing (potentially comparable to the capabilities of XLS).

The algorithms only illustrate some possibilities of the developed method. Signal recognition under uncertainty may have other problem statements considering the peculiarities of the purpose and application of information-measuring systems at hand. Obviously, the proposed method can be combined with traditional probabilistic approaches depending on the conditions of system operation.

The need to process observations for multiple hypotheses naturally leads to organizing 2^L channels of parallel computations while considering all nodes of the numerical grid for the nonlinear parameters of useful signals. The proposed method seems very promising for modern real-time information-measuring systems.

5. SIGNAL RESOLUTION ANALYSIS

Due to the linearity of the proposed method, the correlation matrices of estimation errors are significantly simpler to find. Considering (18) and (26), for the hypothesis Γ_{l^*} and node $\boldsymbol{\zeta}_{(d^*)}$, we express the correlation matrix of the estimate as follows:

$$\mathbf{K}^{\mathbf{A}}_{kl^*(\mathbf{d}^*)} = \mathbf{P}^{\mathbf{A}}_{kl^*(\mathbf{d}^*)} \mathbf{K}^{\boldsymbol{\Xi}} \left(\mathbf{P}^{\mathbf{A}}_{kl^*(\mathbf{d}^*)}\right)^{\mathrm{T}}, \quad k = \overline{1, K_{i^*}}, \tag{30}$$

where $\mathbf{P}^{\mathbf{A}}_{kl^*(\mathbf{d}^*)} = \left[\Lambda_{kl^*(\mathbf{d}^*)} \mathbf{V}_{k^*(\mathbf{d}^*)} \left(\boldsymbol{\Psi}^{\mathrm{T}}_{k^*(\mathbf{d}^*)} \boldsymbol{\Lambda}_{k^*(\mathbf{d}^*)} \mathbf{V}_{k^*(\mathbf{d}^*)}\right)^{-1}\right]^{\mathrm{T}}$.

In each particular case, the expression (30) allows assessing the potential capabilities of the method considering system requirements. By analogy with (30), we write mathematical formulas for the correlation matrices $\mathbf{K}^{\mathbf{S}}_{kl^*(\mathbf{d}^*)}$, $\mathbf{K}^{\mathbf{B}}_{l^*(\mathbf{d}^*)}$, and $\mathbf{K}^{\boldsymbol{\Theta}}_{l^*(\mathbf{d}^*)}$, characterizing estimation accuracies for signal samples, their spectral coefficients, and even the singular interference. For example, when estimating the scalar coordinates a_{km} of the vector $\mathbf{A}_k$ and the samples s_{kn} of the signal $\mathbf{S}_k$, the variances of the errors are given by

$$\left(\sigma^{\mathbf{A}}_{kml^*(\mathbf{d}^*)}\right)^2 = \left(\mathbf{P}^{\mathbf{A}}_{kml^*(\mathbf{d}^*)}\right)^{\mathrm{T}} \mathbf{K}^{\boldsymbol{\Xi}} \mathbf{P}^{\mathbf{A}}_{kml^*(\mathbf{d}^*)}, \quad m = \overline{1, M_{kl^*}},$$
$$\left(\sigma^{\mathbf{S}}_{kml^*(\mathbf{d}^*)}\right)^2 = \left(\mathbf{P}^{\mathbf{S}}_{kml^*(\mathbf{d}^*)}\right)^{\mathrm{T}} \mathbf{K}^{\boldsymbol{\Xi}} \mathbf{P}^{\mathbf{S}}_{kml^*(\mathbf{d}^*)}, \quad n = \overline{1, N}.$$

To assess the characteristics of signal detection, discrimination, and resolution, as well as the possibility of comparing the novel and well-known methods, it suffices to specify the most appropriate distribution law of observation noise (in current observation conditions; as a rule, the Gaussian law) and, if necessary, some a priori probabilities (e.g., the appearance of useful signals in the observation) and the values of risks from incorrect decisions. As a result, the characteristics of the method can be investigated within known statistical approaches [4–10]: the corresponding likelihood functions are constructed and used to find the most important linear functionals for a particular problem (e.g., the probabilities of false alarm, correct detection, etc.).

Applying the constructed matrices $\mathbf{P}^{\mathbf{A}}_{kl(\mathbf{d})}$, $\mathbf{P}^{\mathbf{S}}_{kl(\mathbf{d})}$, $\mathbf{P}^{\mathbf{B}}_{l(\mathbf{d})}$, and $\mathbf{P}^{\boldsymbol{\Theta}}_{l(\mathbf{d})}$ directly to (4), we obtain a set of transformed observation equations with the new noises $\boldsymbol{\Xi}^{\mathbf{A}}_{kl(\mathbf{d})} = \mathbf{P}^{\mathbf{A}}_{kl(\mathbf{d})}\boldsymbol{\Xi}$, $\boldsymbol{\Xi}^{\mathbf{S}}_{kl(\mathbf{d})} = \mathbf{P}^{\mathbf{S}}_{kl(\mathbf{d})}\boldsymbol{\Xi}$, $\boldsymbol{\Xi}^{\mathbf{B}}_{kl(\mathbf{d})} = \mathbf{P}^{\mathbf{B}}_{kl(\mathbf{d})}\boldsymbol{\Xi}$, and $\boldsymbol{\Xi}^{\boldsymbol{\Theta}}_{kl(\mathbf{d})} = \mathbf{P}^{\boldsymbol{\Theta}}_{kl(\mathbf{d})}\boldsymbol{\Xi}$. Note that these equations preserve additivity.

Consequently, given the distribution law of the original noise $\boldsymbol{\Xi}$, it is easy to write the distribution laws of the new noises and calculate the values of the functionals of interest. This approach serves to compare the novel method with XLS and well-known statistical approaches to signal recognition.

According to the expressions (27) and (29), the novel method requires inverting the matrix $\overline{\boldsymbol{\Phi}}_{kl}$ of dimensions $(\overline{M}_{kl} + J) \times (\overline{M}_{kl} + J)$ and the matrix $\boldsymbol{\Psi}^{\mathrm{T}}_{kl}\boldsymbol{\Lambda}_{kl}\mathbf{V}_{kl}$ of dimensions $M_{kl} \times M_{kl}$.

For a fixed number k, XLS requires inverting a matrix of the higher dimensions $(M_{1l} + M_{2l} + \ldots + M_{K_l} + J) \times (M_{1l} + M_{2l} + \ldots + M_{K_l} + J)$.

Obviously, if the inverted matrices are ill-conditioned, the novel method may turn out rather efficient in the computational sense.

Let the model observation equation have the form

$$\mathbf{H}_l = (\mathbf{S}_{kl} + \Delta\mathbf{S}_{kl}) + (\mathbf{X}_{kl} + \Delta\mathbf{X}_{kl}) + \boldsymbol{\Xi},$$

where $\Delta\mathbf{S}_{kl}$ and $\Delta\mathbf{X}_{kl}$ are addends for the signal and interference describing the tails of the functional series.

In this case, the estimate a^*_{kml} (calculated without the additional terms) is given by

$$a^*_{kml} = (\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}\mathbf{H}_l = (\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}(\mathbf{S}_{kl} + \boldsymbol{\Delta}\mathbf{S}_{kl}) + (\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}(\mathbf{X}_{kl} + \boldsymbol{\Delta}\mathbf{X}_{kl}) + (\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}\boldsymbol{\Xi}.$$

Accordingly, the true value a_{kml} can be represented as

$$a_{kml} = (\mathbf{P}^{\mathbf{A}}_{kml} + \boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}(\mathbf{S}_{kl} + \boldsymbol{\Delta}\mathbf{S}_{kl}) + (\mathbf{P}^{\mathbf{A}}_{kml} + \boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}(\mathbf{X}_{kl} + \boldsymbol{\Delta}\mathbf{X}_{kl}),$$

where $\boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{kml}$ is an addend for the weight column describing the tails.

Denoting by $M\{\cdot\}$ mathematical expectation, we define the mean of the methodological error as

$$\overline{\Delta a}_{kml} = M\{a_{kml} - a^*_{kml}\} = (\boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}(\mathbf{S}_{kl} + \boldsymbol{\Delta}\mathbf{S}_{kl}) + (\boldsymbol{\Delta}\mathbf{P}^{\mathbf{A}}_{kml})^{\mathrm{T}}(\mathbf{X}_{kl} + \boldsymbol{\Delta}\mathbf{X}_{kl}), \tag{31}$$

where $M\{\boldsymbol{\Xi}\} = [\mathbf{0}]_{N\times 1}$.

Formulas (31) and (30) can be used together to select the necessary parameters of the novel method that minimize the resulting estimation error in each particular case.

By analogy with [1–3], necessary and sufficient conditions for the existence of a unique solution of the estimation problem require nondegeneracy and some restrictions on the ranks of several matrices. In practice, these conditions are satisfied by rationally choosing the functional bases and the number of degrees of freedom in the signal and singular interference models as well as by setting appropriate observation conditions. All these issues are related to the planning of computational experiments and will not be considered below since they require separate studies in each particular case.

To assess the computational efficiency of the novel method, it suffices to use the results of [2], where the possibility of implementing the GIUE procedure based on distributed data processing was demonstrated. The time to obtain the desired estimates can be an indicator of the computational efficiency of the method. This time is determined by the performance of the distributed environment, the total number of operations required to implement the method, and the programming technique. According to [2], since the GIUE procedure requires no state space expansion, the estimation methods implemented on its basis can provide a significant gain in computational efficiency. In [2], the potential gain was quantified for a particular example.

6. AN ILLUSTRATIVE EXAMPLE

This artificially chosen example is simple enough and, at the same time, clearly demonstrates the achievable computational effect of the novel method in comparison with XLS. (Note that only *the estimation algorithm* (1) will be considered below.) For this purpose, we use appropriate initial data leading to the parameter estimation problem of signals with ill-conditioned matrices.

Let $K = 2$ and $\mathbf{H} = \mathbf{S}_1 + \mathbf{S}_2 + \boldsymbol{\Theta} + \boldsymbol{\Xi}$, where $\mathbf{A}_1 = [a_{11}, a_{12}, a_{13}, a_{14}]^{\mathrm{T}}$, $\boldsymbol{\Theta}_1(t) = [1, t^2, t^3, t^5]^{\mathrm{T}}$, $M_1 = 4$, $\mathbf{S}_1 = \mathbf{A}_1^{\mathrm{T}}\boldsymbol{\Theta}_1$, $\mathbf{A}_2 = [a_{21}, a_{22}, a_{23}]^{\mathrm{T}}$, $\boldsymbol{\Theta}_2(t) = [t, t^4, t^6]^{\mathrm{T}}$, $M_2 = 3$, $\mathbf{S}_2 = \mathbf{A}_2^{\mathrm{T}}\boldsymbol{\Theta}_2$, $\mathbf{B} = [b_1, b_2]^{\mathrm{T}}$,

$$\boldsymbol{\Omega}(t) = [\omega_1(t), \omega_2(t)]^{\mathrm{T}},\ J = 2,\ \boldsymbol{\Psi}_1 = \begin{bmatrix} 1 & t_1^2 & t_1^3 & t_1^5 \\ 1 & t_2^2 & t_2^3 & t_2^5 \\ \vdots & \vdots & \vdots & \vdots \\ 1 & t_N^2 & t_N^3 & t_N^5 \end{bmatrix},\ \boldsymbol{\Psi}_2 = \begin{bmatrix} 1 & t_1^4 & t_1^6 \\ 1 & t_2^4 & t_2^6 \\ \vdots & \vdots & \vdots \\ 1 & t_N^4 & t_N^6 \end{bmatrix},\ \boldsymbol{\Omega} = \begin{bmatrix} \omega_1(t_1) & \omega_2(t_1) \\ \omega_1(t_2) & \omega_1(t_2) \\ \vdots & \vdots \\ \omega_1(t_N) & \omega_2(t_N) \end{bmatrix},$$

$\mathbf{X}_1 = \mathbf{Y}_1\mathbf{C}_1$, $\mathbf{X}_2 = \mathbf{Y}_2\mathbf{C}_2$, $\mathbf{Y}_1 = [\boldsymbol{\Psi}_2 \vdots \boldsymbol{\Omega}]$ is a matrix of dimensions $N \times 5$, $\mathbf{Y}_2 = [\boldsymbol{\Psi}_1 \vdots \boldsymbol{\Omega}]$ is a matrix of dimensions $N \times 6$, $\mathbf{C}_1 = [a_{21}, a_{22}, a_{23}, b_1, b_2]^{\mathrm{T}}$, and $\mathbf{C}_2 = [a_{11}, a_{12}, a_{13}, a_{14}, b_1, b_2]^{\mathrm{T}}$.

In the observation equation, $a_{11} = 10^3$, $a_{12} = 50$, $a_{13} = 10$, $a_{14} = 3$, $a_{21} = -2 \times 10^3$, $a_{22} = -2$, $a_{23} = 1$, $\mathbf{K}^{\boldsymbol{\Xi}} = \mathrm{diag}\left[\sigma_n^2, n = \overline{1, N}\right]$, $\sigma_n^2 = \sigma^2 = 4 \times 10^{-2}$, $t_{t+1} - t_n = 0.5$, and $N = 300$.

The quality of estimation is measured by the relative value $\delta a_{km} = 10^2\,|a_{km}^* - a_{km}| \,/\, |\bar{a}_{km}^*|$ (in percentage), where $\bar{a}_{km}^* = \max\{a_{km}^*, |a_{km}|\}$. Let the singular interference be a linear combination of two basis functions:

$$\theta(t, b_1, b_2) = b_1\omega_1(t) + b_2\omega_2(t) = b_1 \sin(\alpha_1 t) + b_2 \exp(\alpha_2 t),$$

where α_1 and α_2 are arbitrary numbers.

The interference parameters are chosen randomly here: their values are not essential since this method ensures the full self-compensation of the interference under any parameter values.

All calculations were carried out to an accuracy of 15 digits by averaging the results of one thousand experiments. The novel method yielded the following estimates for the coordinates of the vectors $\boldsymbol{\delta}\mathbf{A}_1 = \left[\delta a_{1m}, m = \overline{1, 4}\right]^{\mathrm{T}}$ and $\boldsymbol{\delta}\mathbf{A}_2 = \left[\delta a_{2m}, m = \overline{1, 3}\right]^{\mathrm{T}}$:

$$\delta a_{11} = 1.585, \quad \delta a_{12} = 0.621, \quad \delta a_{13} = 0.082, \quad \delta a_{14} = 1.967 \times 10^{-5},$$
$$\delta a_{21} = 1.643 \times 10^{-4}, \quad \delta a_{22} = 1.409 \times 10^{-7}, \quad \delta a_{23} = 2.756 \times 10^{-11}.$$

In turn, the corresponding XLS-based estimates were

$$\delta a_{11} = 43.053, \quad \delta a_{12} = 24.902, \quad \delta a_{13} = 2.686, \quad \delta a_{14} = 0.138,$$
$$\delta a_{21} = 2.968 \times 10^{-3}, \quad \delta a_{22} = 2.906 \times 10^{-6}, \quad \delta a_{23} = 3.846 \times 10^{-10}.$$

Obviously, there is a significant gain in accuracy and the XLS-based estimates (under the current data) turn out to be of little use.

According to the comparative analysis, the novel method also significantly reduces the computational error of estimation, which is due to decomposition and the reduced dimension of the computational procedure. In the course of calculations using the Euclidean norm, we determined the condition numbers (ν_1 and ν_2) of the inverted matrices $\boldsymbol{\Psi}_\mathbf{1}^{\mathrm{T}}\boldsymbol{\Lambda}_\mathbf{1}\mathbf{V}_\mathbf{1}$ and $\boldsymbol{\Psi}_\mathbf{2}^{\mathrm{T}}\boldsymbol{\Lambda}_\mathbf{2}\mathbf{V}_\mathbf{2}$ as well as the condition number ν of the corresponding combined matrix in XLS. The results were $\nu_1 = 5.283 \times 10^{15}$, $\nu_2 = 1.276 \times 10^{14}$, and $\nu = 4.537 \times 10^{26}$.

Besides, the number of addition and multiplication operations required to implement XLS and the novel method was calculated. The relative computational gain was 1.35 times, which also confirms the efficiency of the novel method.

During the numerical experiments, the correlation matrices $\mathbf{K}_1^{\mathbf{A}}$ and $\mathbf{K}_2^{\mathbf{A}}$ were also calculated (by analogy with (30)). To shorten the expressions, we provide only the diagonal elements of these matrices:

$$\left(\sigma_{11}^{\mathbf{A}}\right)^2 = 0.031, \quad \left(\sigma_{12}^{\mathbf{A}}\right)^2 = 3.631 \times 10^{-6}, \quad \left(\sigma_{13}^{\mathbf{A}}\right)^2 = 2.306 \times 10^{-9}, \quad \left(\sigma_{14}^{\mathbf{A}}\right)^2 = 0,$$
$$\left(\sigma_{21}^{\mathbf{A}}\right)^2 = 2.287 \times 10^{-5}, \quad \left(\sigma_{22}^{\mathbf{A}}\right)^2 = 0, \quad \left(\sigma_{23}^{\mathbf{A}}\right)^2 = 0.$$

For XLS, they were as follows:

$$\left(\sigma_{11}^{\mathbf{A}}\right)^2 = 0.755, \quad \left(\sigma_{12}^{\mathbf{A}}\right)^2 = 7.002 \times 10^{-4},$$
$$\left(\sigma_{13}^{\mathbf{A}}\right)^2 = 1.236 \times 10^{-6}, \quad \left(\sigma_{14}^{\mathbf{A}}\right)^2 = 4.781 \times 10^{-14},$$
$$\left(\sigma_{21}^{\mathbf{A}}\right)^2 = 0.073 \times 10^{-6}, \quad \left(\sigma_{22}^{\mathbf{A}}\right)^2 = 5.096 \times 10^{-10}, \quad \left(\sigma_{23}^{\mathbf{A}}\right)^2 = 0.$$

Since both methods under comparison are linear and optimal, the variance of estimation errors should not differ (when neglecting computational errors): the methods inherit the same potential estimation accuracy. However, according to the calculation results, the variance estimates for the novel method are considerably smaller than those for XLS. The reasons have been discussed above; moreover, the computational errors for the given formulas significantly depend on the dimensions of the inverted ill-conditioned matrices.

7. CONCLUSIONS

The novel method can be effectively combined with orthogonal decomposition algorithms [13] and algorithms for solving ill-posed problems [22, 23]. With the possibility of decomposition and parallelization of computational procedures, this method gives a more efficient solution for an entire range of applied problems related to parallel processing of measurements in various fields. The signal resolution algorithms under singular interferences developed above are easy to implement in special computers of real-time information-measuring systems.

Compact analytical formulas of the novel method have been derived. For a particular applied problem, they can be used to preselect the appropriate models of signals and interferences as well as their parameter values to achieve the method's potential capabilities. This method belongs to the linear class, so all procedures reduce to elementary mathematical operations over vectors and matrices. Moreover, it can be combined with traditional approaches to solving applied problems related to optimal and quasi-optimal processing of measurements.

The results of this paper can also be applied to the class of dynamical systems with measurable output. For this purpose, it is necessary to use the well-known combined method of reference integral curves and generalized invariant-unbiased estimation [3, 24, 25]. With a pre-built family of reference curves or surfaces of a required volume, the state and output of such systems can also be represented as a finite linear envelope of a given functional basis.

In future publications on this topic, it is reasonable to investigate in detail the efficiency of the method under constraints imposed on the signal and singular interference models.

APPENDIX

We optimally estimate the coefficient a_{km} using the method of Lagrange multipliers by optimizing the function

$$F(\mathbf{P}_{km}^{\mathbf{A}}, \gamma_{km}, \boldsymbol{\eta}_{km}) = \left(\mathbf{P}_{km}^{\mathbf{A}}\right)^{\mathrm{T}} \mathbf{K}^{\boldsymbol{\Xi}} \mathbf{P}_{km}^{\mathbf{A}} + \boldsymbol{\gamma}_{km}^{\mathrm{T}} \mathbf{Y}_k^{\mathrm{T}} \mathbf{P}_{km}^{\mathbf{A}} + \left[\left(\mathbf{P}_{km}^{\mathbf{A}}\right)^{\mathrm{T}} \boldsymbol{\Psi}_k - \mathbf{E}_{km}^{\mathrm{T}}\right] \boldsymbol{\eta_{km}}, \tag{A.1}$$

where $\boldsymbol{\gamma}_{km} = \left[\gamma_{kmn}, n = \overline{1, \overline{M}_k + J}\right]^{\mathrm{T}}$ and $\boldsymbol{\eta}_{km} = \left[\eta_{kmn}, n = \overline{1, M_k}\right]^{\mathrm{T}}$ are the vector Lagrange multipliers corresponding to the unbiasedness (22) and invariance (23) conditions.

Differentiating the function (A.1) with respect to all arguments yields the system of linear algebraic equations

$$\partial F/\partial \mathbf{P}^{\mathbf{A}}_{km} = 2\mathbf{K}^{\boldsymbol{\Xi}}\mathbf{P}^{\mathbf{A}}_{km} + \mathbf{Y}_k\boldsymbol{\gamma}_{km} + \boldsymbol{\Psi}_k\boldsymbol{\eta}_{km} = [\mathbf{0}]_{n\times 1},$$
$$\partial F/\partial \boldsymbol{\gamma}_{km} = \mathbf{Y}^{\mathrm{T}}_k\mathbf{P}^{\mathbf{A}}_{km} = [\mathbf{0}]_{\left(\overline{M}_k+J\right)\times 1},$$
$$\partial F/\partial \boldsymbol{\eta}_{km} = \boldsymbol{\Psi}^{\mathrm{T}}_k\mathbf{P}^{\mathbf{A}}_{km} - \mathbf{E}_{km} = [\mathbf{0}]_{M_k\times 1}.$$

Let us introduce the matrices

$$\mathbf{V}_k = \left(\mathbf{K}^{\boldsymbol{\Xi}}\right)^{-1}\boldsymbol{\Psi}_k, \quad \overline{\mathbf{V}}_k = \left(\mathbf{K}^{\boldsymbol{\Xi}}\right)^{-1}\mathbf{Y}_k,$$
$$\boldsymbol{\Phi}_k = \boldsymbol{\Psi}^{\mathrm{T}}_k\mathbf{V}_k, \quad \overline{\boldsymbol{\Phi}}_k = \mathbf{Y}^{\mathrm{T}}_k\overline{\mathbf{V}}_k,$$
$$\mathbf{Z}_k = \mathbf{Y}^{\mathrm{T}}_k\mathbf{V}_k, \quad \overline{\mathbf{Z}}_k = \boldsymbol{\Psi}^{\mathrm{T}}_k\overline{\mathbf{V}}_k.$$

With these notations, the weight column vector—the solution of the system of equations above—is given by

$$\mathbf{P}^{\mathbf{A}}_{km} = 2^{-1}\left(\mathbf{V}_k\boldsymbol{\eta}_{km} - \overline{\mathbf{V}}_k\boldsymbol{\gamma}_{km}\right). \tag{A.2}$$

We multiply the left- and right-hand sides of (A.2) on the left by the matrix $\mathbf{Y}^{\mathrm{T}}_k$. In view of the invariance condition (23), after trivial transformations, it follows that

$$\boldsymbol{\gamma}_{km} = \overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Z}_k\boldsymbol{\zeta}_{km}. \tag{A.3}$$

Similarly, multiplying (A.2) by the matrix $\boldsymbol{\Psi}^{\mathrm{T}}_k$ and considering the unbiasedness condition (22) yield

$$\boldsymbol{\eta}_{km} = 2\Phi_k\left(\mathbf{E}_{km} + 2^{-1}\overline{\mathbf{Z}_k}\boldsymbol{\gamma}_{km}\right). \tag{A.4}$$

Resolving (A.3) and (A.4) for $\boldsymbol{\gamma}_{km}$ and $\boldsymbol{\eta}_{km}$, we obtain

$$\boldsymbol{\gamma}_{km} = 2\overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Z}_k\left([\mathbf{E}]_{M_k\times M_k} - \boldsymbol{\Phi}^{-1}_k\overline{\mathbf{Z}}_k\overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Z}_k\right)^{-1}\boldsymbol{\Phi}^{-1}_k\mathbf{E}_{km}, \tag{A.5}$$
$$\boldsymbol{\eta}_{km} = 2\left([\mathbf{E}]_{M_k\times M_k} - \boldsymbol{\Phi}^{-1}_k\overline{\mathbf{Z}}_k\overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Z}_k\right)^{-1}\boldsymbol{\Phi}^{-1}_k\mathbf{E}_{km}. \tag{A.6}$$

Substituting (A.5) and (A.6) into (A.2) gives

$$\mathbf{P}^{\mathbf{A}}_{km} = \left(\mathbf{V}_k - \overline{\mathbf{V}}_k\overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Z}_k\right)\left([\mathbf{E}]_{M_k\times M_k} - \boldsymbol{\Phi}^{-1}_k\overline{\mathbf{Z}}_k\overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Z}_k\right)^{-1}\boldsymbol{\Phi}^{-1}_k\mathbf{E}_{km}. \tag{A.7}$$

Finally, denoting $\boldsymbol{\Lambda}_k = [\mathbf{E}]_{N\times N} - \overline{\mathbf{V}}_k\overline{\boldsymbol{\Phi}}^{-1}_k\mathbf{Y}^{\mathrm{T}}_k$, we write the expression (A.7) in the compact form (24). The proof of this theorem is complete.

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