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**NONLINEAR SYSTEMS**

# Stability of Solutions to Extremal Problems with Constraints Based on $\lambda$-Truncations

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**Abstract**—In this paper, we consider finite- and infinite-dimensional optimization problems with constraints of general type. We obtain sufficient conditions for stability of a strict solution and conditions for stability of a set of solutions with more than one point in it according to small perturbations of the problem parameters. For finite-dimensional extremal problems with equality-type constraints, we obtain stability conditions based on the construction of $\lambda$-truncations of mappings.

*Keywords*: abstract optimization problems, extremal problems with constraints, solution stability

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## 1. INTRODUCTION

In the theory and practice of extremal problems solving, one often has to deal with uncertain parameters involved in the optimized functional and/or the constraints in the problem. In this regard, a significant amount of research were aimed in recent decades at studying the dependence of solutions to extremal problems on these parameters. The key question is how much the solution to a given extremal problem will change (assuming that this solution exists) with a "small" change in parameters near some given values. In the case of the same "small" changes in the solution, we call this solution stable, and the magnitude of these changes is usually called an index of sensitivity of the solution with respect to small changes in parameters [1].

Abstract optimization problems with constraints and uncertain parameters and the stability of their solutions were considered earlier in [2, 3]. It is also the main topic in [4]. The fundamentals of sensitivity theory for finite-dimensional problems with constraints are described in detail in [1]. A significant number of results on stability of solutions to optimization problems with constraints were obtained under additional regularity conditions; see, for example, the latest results concerning the Robinson regularity condition [5, 6]. Stability analysis in the absence of such additional assumptions still remains a non-trivial task [7]. At the same time, the issue of estimating sensitivity of solutions is relevant for a wide class of finite-dimensional extremal problems and optimal control problems with uncertain and random parameters [8].

In this paper, we consider the following constrained minimization problem:

$$f(x,\sigma) \to \inf, \quad x \in \Phi(\sigma),$$

where $\sigma$ is a parameter. This statement naturally includes parameterized maximization problem of the same type. In Sections 2, 3 we assume that variable $x$ takes values in an arbitrary Banach space, the parameter $\sigma$ takes values in a topological space, and the function $f$ and the set-valued mapping $\Phi$ satisfy natural continuity assumptions only. Under these assumptions, we obtain sufficient conditions for stability of a strict solution $x_0$ to the problem for a certain fixed value of the parameter $\sigma = \sigma_0$ (Lemma 1). In Section 4 the obtained result is generalized to the case when the minimum of the problem for $\sigma = \sigma_0$ is attained not at a single point, but on an entire subset of the set $\Phi(\sigma_0)$ (Lemma 2). In Section 5, we use the construction of $\lambda$-truncations [9] to concretize these results to a finite-dimensional case with constraints of equality type.

## 2. PRELIMINARIES

Let a Banach space $X$ and a point $x_0 \in X$ be given. Denote by $B_r(x_0) \subset X$ a closed ball of radius $r > 0$ with center at point $x_0$.

Let $R > 0$ be some fixed radius and $\Phi(\sigma)$, $\sigma \in \Sigma$, be given non-empty closed sets, where $\Sigma$ is a given topological (in particular, metric) space. Moreover, let $\Phi(\sigma) \subset B_R(x_0)$ $\forall \sigma \in \Sigma$.

Consider the problem

$$f(x, \sigma) \to \inf, \quad x \in \Phi(\sigma) \tag{1}$$

for each fixed $\sigma \in \Sigma$, playing the role of a parameter. Here $f(\cdot, \sigma)$ are given continuous functions on $B_R(x_0)$.

The sets $\Phi(\sigma)$ have the following meaning. As a rule, they are given in the form $\Phi(\sigma) = \{x \in X : F(x, \sigma) \in C(\sigma)\} \cap B_R(x_0)$. Here $F : X \times \Sigma \to Y$ is a given continuous mapping, and $C(\sigma) \subset Y$ are given non-empty closed sets that define the constraints of the problem (1), $Y$ is a normed space. If we consider the problem without constraints, then $\Phi(\sigma) \equiv B_R(x_0)$.

Let a point $\sigma_0 \in \Sigma$ be given and assume that the first countability axiom holds at $\sigma_0$ (for example, $\sigma_0$ is an arbitrary point in a metric space $\Sigma$). From the set of the problems (1) we separately consider the problem

$$f(x) := f(x, \sigma_0) \to \inf, \quad x \in \Phi(\sigma_0). \tag{2}$$

Suppose that the functions $f(\cdot, \sigma)$ converge to $f = f(\cdot, \sigma_0)$ as $\sigma \to \sigma_0$ uniformly on $B_R(x_0)$. The latter means that for any $\varepsilon > 0$ there exists a neighborhood $O(\sigma_0) \subset \Sigma$ of the point $\sigma_0$ such that

$$|f(x, \sigma) - f(x)| < \varepsilon \quad \forall x \in B_R(x_0) \quad \forall \sigma \in O(\sigma_0). \tag{3}$$

We will assume that for each $\sigma$ the infimum in problem (1) is finite. Let also $x_0 \in \Phi(\sigma_0)$, and let $x_0$ be the strict solution to the problem (2), i.e.

$$f(x_0) < f(x) \quad \forall x \in \Phi(\sigma_0) : x \neq x_0. \tag{4}$$

Let us introduce two assumptions on the function $f = f(\cdot, \sigma_0)$. The first one is that there exists a non-decreasing function $\delta : \mathbb{R}_+ \to \mathbb{R}_+ \cup \{+\infty\}$ such that $\delta(0) = 0$, $\delta(r) > 0$ for $r > 0$ and

$$f(x) \geqslant f(x_0) + \delta(\|x - x_0\|) \quad \forall x \in \Phi(\sigma_0). \tag{5}$$

The second assumption is that

$$x \in \Phi(\sigma), \quad \widetilde{x} \in \Phi(\sigma_0), \quad \|x - \widetilde{x}\| \to 0, \quad \sigma \to \sigma_0 \quad \Rightarrow \quad f(x) \geqslant f(\widetilde{x}) - 1(\sigma), \tag{6}$$

where $1(\sigma) > 0$ for any $\sigma \neq \sigma_0$ and $1(\sigma) \to 0$ as $\sigma \to \sigma_0$. In other words, for any $\varepsilon > 0$ there exists a neighborhood $O(\sigma_0) \subset \Sigma$ of the point $\sigma_0$ and a real $\gamma > 0$ such that for all $\sigma \in O(\sigma_0)$ we have

$$x \in \Phi(\sigma), \quad \widetilde{x} \in \Phi(\sigma_0), \quad \|x - \widetilde{x}\| < \gamma \quad \Rightarrow \quad f(x) \geqslant f(\widetilde{x}) - \varepsilon.$$

The following assumption concerns the set-valued mapping $\Phi : \Sigma \rightrightarrows B_R(x_0)$. Namely, we will assume that it is upper semicontinuous at the point $\sigma_0$, i.e.

$$\Phi(\sigma) \subset \Phi(\sigma_0) + B_{1(\sigma)}(0) =: B_{1(\sigma)}(\Phi(\sigma_0)) \quad \text{as} \quad \sigma \to \sigma_0. \tag{7}$$

Here the function $1(\sigma)$ is the same as in (6), the symbol $+$ denotes the Minkowski addition (for more information about set-valued mappings, see, for example, [10, Chapter 1] or [11, §17]).

## 3. STABILITY OF THE SOLUTION TO AN EXTREMAL PROBLEM

Let by definition $x(\sigma_0) = x_0$. Let a non-negative scalar function $\psi$ be given on $\Sigma$ such that $\psi(\sigma) \to \psi(\sigma_0) = 0$ as $\sigma \to \sigma_0$ and for any $\sigma \in \Sigma$, $\sigma \neq \sigma_0$, there exists $x(\sigma) \in \Phi(\sigma)$ with the property

$$f(x(\sigma), \sigma) \leqslant f(x, \sigma) + \psi(\sigma) \quad \forall x \in \Phi(\sigma). \tag{8}$$

The specified function $\psi$ always exists. Indeed, if the space $X$ is finite-dimensional, then the solution $x(\sigma)$ to the problem (1) of minimizing a continuous function on a compact set $\Phi(\sigma)$ exists and we can put $\psi = 0$. Now let the Banach space $X$ be infinite-dimensional. Then for any $\psi(\sigma) > 0$, $\sigma \neq \sigma_0$, the infimum in the problem (1) is finite, so its solution exists, but only up to $\psi$. Therefore, there exists $x(\sigma)$ such that (8) is satisfied.

**Definition 1.** The solution $x_0$ to the extremal problem (2) is called stable if for any points $x(\sigma) \in \Phi(\sigma)$, $\sigma \neq \sigma_0$, satisfying the condition (8), occurs

$$x(\sigma) \to x_0 \quad \text{as} \quad \sigma \to \sigma_0.$$

The question is, when the solution $x_0$ to the problem (2) is stable? One of the answers to this question is given below.

**Lemma 1.** *Let $\varphi(\sigma_0) := x_0$, and for any $\sigma \neq \sigma_0$ there exists $\varphi(\sigma) \in \Phi(\sigma)$ such that*

$$\varphi(\sigma) \to x_0 \quad \textit{as} \quad \sigma \to \sigma_0. \tag{9}$$

*Let, in addition, the point $x_0$ be the strict solution to the problem* (2), *and let the conditions* (3), (5)–(7) *hold. Then the solution $x_0$ to the problem* (2) *is stable.*

**Proof of Lemma 1.**

Without loss of generality, we assume that $x_0 = 0$ and $f(0) = 0$.

Consider the contrary. Then, by virtue of the first countability axiom, this means that there exist sequences $\{\sigma_i\}$ and $\{x_i\}$ such that $\sigma_i \to \sigma_0$, $\forall i$ $x_i = x(\sigma_i) \in \Phi(\sigma_i)$ and (8) holds, but the sequence $\{x_i\}$ does not converge to zero, i.e. there exists a real $\varepsilon$ such that $0 < 2\varepsilon < R$ and, after passing to a subsequence, we have $\|x_i\| \geqslant 2\varepsilon$ for all $i$.

By (7) there exist points $\widetilde{x}_i \in \Phi(\sigma_0)$ such that

$$\|x_i - \widetilde{x}_i\| \to 0. \tag{10}$$

This means that for all sufficiently large $i$ we have $\|\widetilde{x}_i\| \geqslant \varepsilon$. Hence, by assumption (5), $f(\widetilde{x}_i) \geqslant \delta(\|\widetilde{x}_i\|) \geqslant \delta(\varepsilon) > 0$ holds, i.e. there exists $\delta_0 = \delta(\varepsilon)/2 > 0$ such that $f(\widetilde{x}_i) \geqslant 2\delta_0$ for all sufficiently large $i$. But then, due to (6) and (10), we have $f(x_i) \geqslant f(\widetilde{x}_i) - \delta_0/2 \geqslant \frac{3}{2}\delta_0$ for all sufficiently large $i$. Therefore, since by virtue of (3) the functions $f_i := f(\cdot, \sigma_i)$ converge uniformly to the function $f$ on $B_R(0)$, for all sufficiently large $i$ the inequality $f_i(x_i) \geqslant \delta_0$ holds.

Let us put $\varphi_i = \varphi(\sigma_i)$ and $\psi_i = \psi(\sigma_i)$. By (9) $\varphi_i \to 0$, and by virtue of the properties of the function $\psi$ we have $\psi_i \to 0,\ i \to \infty$. Therefore, by virtue of (3), there exists a positive integer $i_0$ such that for all $i \geqslant i_0$

$$|f_i(x) - f(x)| + \psi_i \leqslant \frac{\delta_0}{4} \quad \forall x \in B_R(0). \tag{11}$$

From the continuity of $f$, increasing the positive integer $i_0$ if necessary, we obtain that

$$f(\varphi_i) \leqslant \frac{\delta_0}{4}, \quad i \geqslant i_0. \tag{12}$$

For $i \geqslant i_0$, by virtue of (8) for an arbitrary $x \in \Phi(\sigma_i)$, $f_i(x_i) \leqslant f_i(x) + \psi_i$ holds. In particular, for $x = \varphi_i \in \Phi(\sigma_i)$ we obtain that for all sufficiently large $i$

$$f_i(x_i) \leqslant f_i(\varphi_i) + \psi_i = \Big(f_i(\varphi_i) - f(\varphi_i) + \psi_i\Big) + f(\varphi_i) \leqslant \frac{\delta_0}{4} + f(\varphi_i) \leqslant \frac{\delta_0}{2}.$$

Here the last inequality follows from (12), and the previous one follows from (11).

Thus, we have proved that $f_i(x_i) \leqslant \delta_0/2$ for all large $i$. But by construction we have $f_i(x_i) \geqslant \delta_0 > 0$, i.e. a contradiction is obtained. Lemma 1 is proved.

Let us discuss the stability conditions in Lemma 1. We start with the assumption (5). As the following example shows, in a finite-dimensional space $X$ it always holds when (4) is fulfilled.

*Example 1.* Let the Banach space $X$ be finite-dimensional, and $x_0$ be the strict minimum of the function $f$ on the set $\Phi(\sigma_0)$, i.e. the condition (4) holds. Then the function $\delta$ satisfying the estimate (5) is determined by the formula

$$\delta(r) = \min\Big\{f(x) - f(x_0) :\ x \in \Phi(\sigma_0),\ \|x - x_0\| \geqslant r\Big\}, \quad r \geqslant 0.$$

Here we consider the minimum over the empty set to be equal to $+\infty$.

Obviously, for a finite-dimensional space $X$, due to the compactness of each of the sets $\Phi(\sigma_0) \cap \{x \in X : \|x - x_0\| \geqslant r\}$, the defined function $\delta$ satisfies all the assumptions. The example is complete.

Let us consider the case when the space $X$ is infinite-dimensional, the function $f$ is smooth in a neighborhood of the point $x_0$, and the set $\Phi(\sigma_0)$ is determined by a finite number of smooth constraints of equality and inequality type. In this case, if the sufficient second-order condition is satisfied in the problem (2) (see, for example, [12, §8] or [13, Theorem 8.1]) and the radius $R > 0$ of the ball $B_R(x_0)$ is small enough, then there exists $\varepsilon > 0$ such that the function $\delta(r) = \varepsilon r^2$ satisfies the estimate (5). Generally speaking, for the existence of a function $\delta$ satisfying (5), it is sufficient that any sufficient conditions for a strict minimum of the second or higher orders are satisfied (see, for example, [13, Theorems 8.3, 15.2, 15.3]) and the real $R > 0$ is small enough.

At the same time, the following counterexample shows that in an infinite-dimensional Hilbert (especially Banach) space $X$, if the sufficient conditions for a strict minimum of the second or higher orders in the problem are not satisfied, then the appropriate function $\delta$ may not exist for any value of $R > 0$. In this case, the assertion of Lemma 1 is failed even in a problem without conatraints. Thus, the assumption (5) is essential.

*Example 2.* Let $X = \ell_2$ be the space of real square-summable sequences with the usual inner product. Let $\{e_i\}$ denote the standard basis in $\ell_2$, i.e. sequences whose $i$th real equals 1 and the rest are zeros. Let $A$ denote a positive self-adjoint compact linear operator $A : X \to X$, which is defined by an infinite diagonal matrix containing $i^{-1}$ on the main diagonal, $i = 1, 2, \ldots$, and zeros outside it (see, for example, [14, § 6.9]).

Let $\Sigma = \{i^{-1} : i \in \mathbb{N}\} \cup \{0\} \subset \mathbb{R}$ with the topology induced from $\mathbb{R}$, and let $\sigma_0 = 0$, $f(x) = \langle Ax, x\rangle$. For $i \in \mathbb{N}$ we put $f_i(x) := f(x, i^{-1}) = \langle A_i x, x\rangle$, where $A_i$ is obtained from $A$ by replacing $i^{-1}$ with $-i^{-1}$ at the $i$th place on the diagonal. Next, we put $x_0 = 0$ and $\psi(\sigma) \equiv 0$. Let us also fix an arbitrary $R > 0$ and put $\Phi(\sigma) = B_R(0)$ $\forall\sigma$ (i.e., there is no constraint $x \in \Phi(\sigma)$).

Here the condition (7) is satisfied trivially. The condition (6) is satisfied by the following:

$$\begin{aligned}|f(x) - f(\widetilde{x})| &= |\langle Ax, x\rangle - \langle A\widetilde{x}, \widetilde{x}\rangle|\\ &\leqslant |\langle Ax, x\rangle - \langle A\widetilde{x}, x\rangle| + |\langle A\widetilde{x}, x\rangle - \langle A\widetilde{x}, \widetilde{x}\rangle| \leqslant \|A\|\|x - \widetilde{x}\|(\|x\| + \|\widetilde{x}\|)\\ &\leqslant 2R\|A\|\|x - \widetilde{x}\| \to 0 \quad \text{as} \quad \|x - \widetilde{x}\| \to 0, \quad x, \widetilde{x} \in B_R(0).\end{aligned}$$

As $\{\varphi(i^{-1})\}$ in the condition (9) we can take any sequence of elements from $B_R(0)$ converging to zero, for example $\{i^{-1}Re_1\}$. Condition (3) is satisfied, since by construction $f_i \to f$ as $i \to \infty$ uniformly in $x \in B_R(0)$. Thus, in this example, the conditions (3), (6), (7) and (9) hold.

At the same time, condition (5) is failed, since $\forall r > 0$ $\inf\{f(x) : ||x|| = r\} = 0$, and there is no corresponding function $\delta$. Let us show that the assertion of Lemma 1 is also failed. Indeed, the values of $f(x)$ are positive for $x \neq 0$, $f(0) = 0$, however $\inf\{f_i(x) : x \in B_R(0)\} = -R^2 i^{-1}$ for all $i \in \mathbb{N}$, and this infimum is attained at $x_i := Re_i \in B_R(0)$. The points $x_i = x(i^{-1})$ satisfy the condition (8) for $\psi = 0$, but the sequence $\{x_i\}$ does not converge to zero, i.e. the point $x_0 = 0$ is not the stable solution to the problem (2). The example is complete.

Let us turn to the assumption (6). From the Cantor theorem on the uniform continuity of a continuous function on a compact set it follows that the assumption (6) is automatically satisfied in a finite-dimensional space $X$.

Indeed, let $X$ be finite-dimensional. Then for an arbitrary fixed $R > 0$ the closed ball $B_R(x_0)$ is compact. From the conditions $x \in \Phi(\sigma)$, $\widetilde{x} \in \Phi(\sigma_0)$ and the assumption $\Phi(\sigma) \subset B_R(x_0)$ $\forall\sigma$, it follows that $x, \widetilde{x} \in B_R(x_0)$. By virtue of the Cantor theorem applied to the function $f$ on the compact set $B_R(x_0)$, we obtain that $|f(x) - f(\widetilde{x})| \to 0$ as $\|x - \widetilde{x}\| \to 0$, which proves (6).

If the space $X$ is infinite-dimensional, then the assumption (6) is already essential, even if this space is Hilbert. Let us give a corresponding example.

*Example 3.* Let $X = \ell_2$ be the Hilbert space, as in Example 2, $\{e_i\} \subset \ell_2$ be the standard basis. Let us put $x_0 = e_1$, $R = 2$, $B := B_2(e_1)$, $\Sigma = \{i^{-1} : i \in \mathbb{N}\} \cup \{0\}$, with $\sigma_0 = 0$ and $\psi(\sigma) \equiv 0$. The function $f(x, \sigma)$ does not depend on $\sigma$ and is determined by the formula

$$f(x, \sigma) = f(x) = 1 - \sum_{i=1}^{\infty} \max\{0, 1 - 2i\|x - e_i\|\}, \quad x \in \ell_2, \quad \sigma \in \Sigma. \tag{13}$$

Since the supports of continuous functions $x \mapsto \max\{0, 1 - 2i\|x - e_i\|\}$ are pairwise disjoint balls $B_{1/(2i)}(e_i)$, then the function $f : X \to \mathbb{R}$ is well defined and continuous (including on $B$).

Further, we put

$$\Phi(0) = \{e_1\} \cup \{u_i : i = 2, 3, \ldots\}, \quad \text{where} \quad u_i = \left(1 - \frac{1}{2i}\right)e_i;$$

$$\Phi(i^{-1}) = \{a_i, b_i\}, \quad \text{where} \quad a_i = \left(1 - \frac{1}{2i}\right)e_1, \quad b_i = \left(1 - \frac{1}{2(i+1)i}\right)e_i, \quad i = 1, 2, \ldots;$$

$$\delta(r) = r/2, \quad r \geqslant 0.$$

To carry out the following constructions, let us calculate the values of the function $f$ at some points. Since $e_1, a_i \in B_{1/2}(e_1)$, then

$$f(e_1) = 1 - \max\{0, 1 - 2\|e_1 - e_1\|\} = 0, \tag{14}$$

$$f(a_i) = 1 - \max\{0, 1 - 2\|a_i - e_1\|\} = \frac{1}{i}. \tag{15}$$

Since $u_i, b_i \in B_{1/(2i)}(e_i)$, then

$$f(u_i) = 1 - \max\{0, 1 - 2i\|u_i - e_i\|\} = 1, \tag{16}$$

$$f(b_i) = 1 - \max\{0, 1 - 2i\|b_i - e_i\|\} = 1 - \max\left\{0, 1 - \frac{1}{i+1}\right\} = \frac{1}{i+1}. \tag{17}$$

Let us show that for given $X, f, \Phi, \Sigma, x_0, \sigma_0$ and $R$, all assumptions of Lemma 1 are satisfied, except the assumption (6).

The assumption (3) is satisfied since $f$ does not depend on $\sigma$. The infimum in the problem (1) is finite for any $\sigma \neq \sigma_0$, since the set $\Phi(\sigma)$ is finite, and the infimum in the problem (2) is finite due to (14) and (16). Moreover, $f(x_0) = f(e_1) = 0 < 1 = f(x)$ for any $x \in \Phi(0) : x \neq x_0$, which follows (4). The relation (8) holds for $x(i^{-1}) = b_i \in \Phi(i^{-1})$, $i = 1, 2, \ldots$, $x(0) = e_1 \in \Phi(0)$ and $\psi = 0$. The inequality (5) is satisfied because

$$f(e_1) + \delta(\|u_i - e_1\|) \overset{(14)}{=} \delta(\|u_i - e_1\|) = \frac{\|u_i - e_1\|}{2} = \frac{1}{2}\sqrt{1 + \left(1 - \frac{1}{2i}\right)^2} < 1 \overset{(16)}{=} f(u_i)$$

for any $i = 2, 3, \ldots$. The mapping $\Phi$ is upper semicontinuous at zero because $\|a_i - e_1\| \to 0$ and $\|b_i - u_i\| \to 0$ as $i \to \infty$, i.e. (7) holds. Finally, the relation (9) obviously holds for $\varphi(i^{-1}) = a_i$, $i = 1, 2, \ldots$.

Thus, all assumptions of Lemma 1 are satisfied except (6). And this assumption (6) is failed since $\|b_i - u_i\| \to 0$, $f(b_i) \to 0$ and $f(u_i) \to 1$ as $i \to \infty$. The assertion of Lemma 1 is also failed, since $x(i^{-1}) = b_i \not\to e_1 = x_0$ as $i \to \infty$. The example is complete.

Let us turn to the assumption (7). The following example shows that it is also essential.

*Example 4.* Let $X = \mathbb{R}$, $\Sigma = \mathbb{R}$, $f(x, \sigma) \equiv x$, $x_0 = \sigma_0 = 0$, $R = 1$, $\Phi(\sigma) = [-1; 1]$ for $\sigma \neq 0$, $\Phi(0) = \{0\}$ and $\psi(\sigma) \equiv 0$. Then, obviously, $x_0 = 0$ is not the stable solution to the problem (2). It is directly verified that in this example all assumptions of Lemma 1 are satisfied, except for the assumption (7).

Note that this essential assumption (7) is fulfilled for a wide class of problems, for example, for all finite-dimensional problems with the inclusion-type constraints under natural continuity assumptions. Let us demonstrate this with the following example.

*Example 5.* Let $X$ be a finite-dimensional Banach space, $Y$ be a normed space, and $\Sigma$ be a metric space. Let $F : X \times \Sigma \to Y$ be a continuous mapping and $C \subset Y$ be a non-empty closed set.

For $\sigma \in \Sigma$ we put

$$\Phi(\sigma) = \{x \in X : F(x, \sigma) \in C\} \cap B_R(x_0) \neq \varnothing. \tag{18}$$

Then the set-valued mapping $\Phi$ satisfies the assumption (7). Indeed, to verify (7) by [11, Theorem 17.15] it is sufficient to show that $\Phi$ is a closed mapping. Recall that a set-valued mapping $\Phi : \Sigma \rightrightarrows B_R(x_0)$ is called closed if its graph $\operatorname{gph}\Phi := \{(\sigma, x) \in \Sigma \times B_R(x_0) : x \in \Phi(\sigma)\}$ is closed.

So, let the sequence of points $(\sigma_i, x_i) \in \operatorname{gph}\Phi$ converge to the point $(\widetilde{\sigma}, \widetilde{x}) \in \Sigma \times B_R(x_0)$. Let us check that $(\widetilde{\sigma}, \widetilde{x}) \in \operatorname{gph}\Phi \Leftrightarrow \widetilde{x} \in \Phi(\widetilde{\sigma}) \Leftrightarrow F(\widetilde{x}, \widetilde{\sigma}) \in C$. By assumption we have $F(x_i, \sigma_i) \in C$,

$\sigma_i \to \widetilde{\sigma}$ and $x_i \to \widetilde{x}$ as $i \to \infty$. But then, due to the continuity of $F$, $F(x_i, \sigma_i) \to F(\widetilde{x}, \widetilde{\sigma})$, $i \to \infty$. Therefore $F(\widetilde{x}, \widetilde{\sigma})$ is the limit point for the sequence of points $F(x_i, \sigma_i) \in C$. Since the set $C$ is closed, we obtain $F(\widetilde{x}, \widetilde{\sigma}) \in C$, i.e. $(\widetilde{\sigma}, \widetilde{x}) \in \operatorname{gph} \Phi$. This means that the mapping $\Phi$ is closed and (7) holds. The example is complete.

Similar reasoning is valid for the more general case of constraint $F(x, \sigma) \in C(\sigma)$, i.e. $\Phi(\sigma) = \{x \in X : F(x, \sigma) \in C(\sigma)\} \cap B_R(x_0)$, where $C : \Sigma \to Y$ is an upper semicontinuous at the point $\sigma_0$ closed-set-valued mapping.

The main assumption that must be checked to apply Lemma 1 is (9). It is clear that this assumption is essential.

*Example 6.* Let $X = \mathbb{R}$, $\sigma = [0; 1]$, $f(x, \sigma) \equiv x^2$, $x_0 = \sigma_0 = 0$, $R = 1$, $\Phi(\sigma) = \{1\}$ for $\sigma > 0$, $\Phi(0) = [-1; 1]$ and $\psi(\sigma) \equiv 0$. In this example, all assumptions of Lemma 1 are obviously satisfied, except for (9). Also, the point $x_0 = 0$ is naturally not the stable solution to the problem (2).

The assumption (9) is closely related to the concept of the implicit function. Below, when considering finite-dimensional problems in section 5, we formulate one of the possible conditions under which (9) is satisfied. Here we note that in the case when the constraint set does not depend on the parameter $\sigma$, i.e. $\Phi(\sigma) \equiv \Phi(\sigma_0)$, condition (9) holds for $\varphi(\sigma) \equiv x_0$.

Examples similar to Examples 3, 4, 6 discussed in this section and the following Examples 7, 9 can be constructed for an arbitrary pre-fixed $R > 0$.

## 4. STABILITY OF A SET OF SOLUTIONS

Let us now consider the case when in the problem (2) the minimum of the continuous function $f$ is attained not necessarily at the unique point $x_0$, but on a given non-empty set $X_0 \subset \Phi(\sigma_0)$, i.e. occurs

$$f(x) = \min\{f(\xi, \sigma_0) : \ \xi \in \Phi(\sigma_0)\} \quad \Leftrightarrow \quad x \in X_0.$$

Note that such a set $X_0$ is closed due to the continuity of the function $f$ and the closedness of the set $\Phi(\sigma_0)$.

The following example shows that even in one-dimensional case in the problem without constraints $f(x) \to \min$, $x \in B_R(x_0)$, if the set of solutions $X_0$ to the problem (2) is not a singleton, then each point in $X_0$ may not be a stable solution in sense of Definition 1.

*Example 7.* Let $X = \mathbb{R}$, $\Sigma = \{i^{-1} : i \in \mathbb{N}\} \cup \{0\}$, $x_0 = 0$, $R = 1$, $\psi = 0$, the function $f(x)$ vanishes for $x \in [-1/2; 1/2]$, strictly monotonically decreases on the segment $[-1; -1/2]$, strictly monotonically increases on $[1/2; 1]$ and is continuous on $[-1; 1]$. There are no constraints in the considered problem (2) and the function $f$ reaches its minimum on the set $X_0 = [-1/2; 1/2]$.

We define the functions $f_i(x) = f(x, i^{-1})$ to be continuous on $[-1; 1]$ and reach a negative minimum at a unique point $x_i \in [-1/2; 1/2]$. Moreover, we assume that $f_i \to f$ as $i \to \infty$ uniformly on $[-1; 1]$. Finally, suppose that for each point $x \in [-1/2; 1/2]$ it is possible to select a subsequence from the sequence $\{x_i\}$ converging to $x$. It is easy to see that such functions $f_i$ exist.

For the functions $f_i$ that uniformly converge to $f$ on the segment $[-1; 1]$, consider the problem without constraints $f_i(x) \to \min$, $x \in [-1; 1]$. By construction, the minimum in this problem is attained at the unique point $x_i$, and the limit points of the set $\{x_i\}$ cover the whole segment $X_0$ of the unit length. Hence, any point from the set $X_0$ can be approached arbitrarily by the minima of the perturbed functions $f_i$. Thus, none of these points is a stable solution to the problem (2). The example is complete.

Despite this example, one can obtain sufficient conditions for convergence of minima $x(\sigma)$, up to $\psi$, of the perturbed functions $f(x, \sigma)$ on the sets $\Phi(\sigma)$ in terms of distance between the point $x(\sigma)$

and the set $X_0$. Recall that

$$\operatorname{dist}(\xi, X_0) := \inf\{\|\xi - x\| : \ x \in X_0\}, \quad \xi \in X.$$

Put

$$f_0 := \inf\{f(x, \sigma_0) : \ x \in \Phi(\sigma_0)\}.$$

Assume that, in addition to the existing assumptions and conditions (3), (6) and (7), the function $f$ has the following property:

$$f(x) \geqslant f_0 + \delta\big(\operatorname{dist}(x, X_0)\big) \quad \forall x \in \Phi(\sigma_0), \tag{19}$$

where the function $\delta$ is the same as in (5). At the same time, (4) and the stronger assumption (5) may not hold.

Let us also introduce an additional assumption on uniform continuity of the function $f$ on the set $X_0$:

$$f(x) \to f_0 \quad \text{as} \quad \operatorname{dist}(x, X_0) \to 0, \quad x \in B_R(x_0). \tag{20}$$

**Lemma 2.** *Let for any $\sigma \in \Sigma$ there exist $\varphi(\sigma) \in \Phi(\sigma)$ such that*

$$\operatorname{dist}(\varphi(\sigma), X_0) \to 0 \quad \text{as} \quad \sigma \to \sigma_0, \tag{21}$$

*and assumptions* (3), (6), (7), (19) *and* (20) *hold. Then for any $x(\sigma) \in \Phi(\sigma)$, $\sigma \neq \sigma_0$, satisfying the condition* (8), *we have*

$$\operatorname{dist}(x(\sigma), X_0) \to 0 \quad \text{as} \quad \sigma \to \sigma_0.$$

**Proof of Lemma 2.**

Without loss of generality, we assume that $f_0 = 0$, $x_0 = 0$ and, hence, $B_R(x_0) = B_R(0)$.

Consider the contrary. Then there exist sequences $\{\sigma_i\}$ and $\{x_i\}$ such that $\sigma_i \to \sigma_0$, $\forall i\ x_i = x(\sigma_i) \in \Phi(\sigma_i)$ and (8) holds, but the sequence $\{\operatorname{dist}(x_i, X_0)\}$ does not converge to zero. It follows that there exists a real $\varepsilon$ such that $0 < 2\varepsilon < R$ and, after passing to a subsequence, $\operatorname{dist}(x_i, X_0) \geqslant 2\varepsilon$ holds for all $i$.

Let us put $\psi_i = \psi(\sigma_i)$ in the same way as in the proof of Lemma 1. Due to (7) there exist points $\widetilde{x}_i \in \Phi(\sigma_0)$ such that $\|x_i - \widetilde{x}_i\| \to 0$. This means that for all sufficiently large $i$ we have $\operatorname{dist}(\widetilde{x}_i, X_0) \geqslant \varepsilon$. Therefore, by assumption (19) $f(\widetilde{x}_i) \geqslant \delta(\operatorname{dist}(\widetilde{x}_i, X_0)) \geqslant \delta(\varepsilon) > 0$. Thus, there exists $\delta_0 > 0$ such that $f(\widetilde{x}_i) \geqslant 2\delta_0$ for all large $i$. Hence, by (6) we have $f(x_i) \geqslant \frac{3}{2}\delta_0$ for all sufficiently large $i$. Consequently, since by virtue of (3) the functions $f_i = f(\cdot, \sigma_i)$ converge uniformly to $f$ on $B_R(0)$, then $f_i(x_i) \geqslant \delta_0$ for all large $i$.

Due to the properties of the function $\psi$ we have $\psi_i \to 0$, $i \to \infty$. Therefore, by (3) there exist a positive integer $i_0$ such that for all $i \geqslant i_0$ we have

$$|f_i(x) - f(x)| + \psi_i \leqslant \frac{\delta_0}{4} \quad \forall x \in B_R(0).$$

By (21) there exists a sequence $\varphi_i \in \Phi(\sigma_i)$ with the property $\operatorname{dist}(\varphi_i, X_0) \to 0$. Therefore, increasing the positive integer $i_0$, from (20) with $x = \varphi_i$ we obtain that $f(\varphi_i) \leqslant \frac{\delta_0}{4}$. It follows, just as in Lemma 1, that $f_i(x_i) \leqslant \delta_0/2$ for $i \geqslant i_0$. But according to the construction, $f_i(x_i) \geqslant \delta_0 > 0$, i.e. a contradiction is obtained. Lemma 2 is proved.

Thus, Example 7 and Lemma 2 show that, under natural assumptions, although the solutions $x(\sigma)$, up to $\psi$, of the perturbed problems may not converge to any point in the set $X_0$; nevertheless they converge in the sense of the distance to the entire set $X_0$.

*Remark 1.* If $X_0 = \{x_0\}$, then Lemmas 1 and 2 are equivalent. Indeed, in this case the assumption (5) coincides with (19), since $\operatorname{dist}(x, X_0) = \|x - x_0\|$; assumption (9) coincides with (21), since $\varphi(\sigma) \to x_0 \Leftrightarrow \operatorname{dist}(\varphi(\sigma), X_0) \to 0$; and the assumption (20) is equivalent to the continuity of $f$ at the point $x_0$. The remaining assumptions of these two lemmas coincide.

The assumption (19) has the same sense as the assumption (5). In particular, for a finite-dimensional space $X$ the following analogue of Example 1 holds.

*Example 8.* Let the Banach space $X$ be finite-dimensional and the condition

$$f(x) = f_0 \quad \forall x \in X_0, \qquad f(x) > f_0 \quad \forall x \in \Phi(\sigma_0) \setminus X_0 \tag{22}$$

holds. Then the function $\delta$ satisfying the estimate (19) is determined by the formula

$$\delta(r) = \min\Big\{f(x) - f_0 :\ x \in \Phi(\sigma_0),\ \operatorname{dist}(x, X_0) \geqslant r\Big\}, \quad r \geqslant 0.$$

Let us discuss the assumption (20). If $X_0$ is the singleton $\{x_0\}$, then (20) is fulfilled automatically, which follows from continuity of the function $f$.

If $X$ is finite-dimensional, then $X_0$ is compact, since it is closed and lies in $B_R(x_0)$. Then (20) is also satisfied automatically, due the Cantor theorem on the uniform continuity of a continuous function on a compact set.

Now let $X$ be infinite-dimensional, and let the set $X_0$ be closed and bounded (since it is a subset of $B_R(x_0)$), but not compact. Then the assumption (20) is essential. Let us show this with the following example.

*Example 9.* Suppose that $X$, $\Sigma$, $\sigma_0$, $\psi$ and $f(x, \sigma)$ are the same as in Example 3. Let $x_0 = 0$, $R = 1$ and $B := B_1(0)$. Put

$$a = \frac{3}{4}e_1; \quad \Phi(i^{-1}) = \left\{\left(1 - \frac{1}{2i}\right)e_i, a\right\}, \quad i \in \mathbb{N}; \quad \Phi(0) = B.$$

Here $a$ is chosen so that $f(a) = \frac{1}{2}$.

From the definition of the function $f$ it is obvious that $X_0 = \{e_1, e_2, \ldots\}$ and $f_0 = 0$. Firstly, we show that (6) holds.

Let us take arbitrary sequences $\{x_i \in \Phi(i^{-1})\}$ and $\{\widetilde{x}_i \in \Phi(0)\}$ with the property $\|x_i - \widetilde{x}_i\| \to 0$. Now we take all $i$ such that $\|x_i - \widetilde{x}_i\| < 1/4$ and consider two cases: $x_i = a$ and $x_i \neq a$. Let first $x_i = a$. Then, since $\|a - \widetilde{x}_i\| < 1/4$, we have $\widetilde{x}_i \in B_{1/2}(e_1)$. Hence,

$$f(\widetilde{x}_i) = 1 - \max\{0, 1 - 2\|\widetilde{x}_i - e_1\|\} = 2\|\widetilde{x}_i - e_1\| \leqslant 2\|\widetilde{x}_i - x_i\| + 2\|x_i - e_1\|.$$

Moreover, since $f(x_i) = f(a) = 2\|a - e_1\| = 2\|x_i - e_1\|$, then

$$f(x_i) = 2\|x_i - e_1\| \geqslant f(\widetilde{x}_i) - 2\|\widetilde{x}_i - x_i\|.$$

Let now $x_i \neq a$. Then $x_i = (1-(2i)^{-1})e_i$ and direct calculation (16) shows that $f(x_i) = 1$. Since the largest value of the function $f$ is 1, then in this case we also have $f(x_i) \geqslant f(\widetilde{x}_i) \geqslant f(\widetilde{x}_i) - 2\|\widetilde{x}_i - x_i\|$. Thus, for every sufficiently large $i$ we have

$$f(x_i) \geqslant f(\widetilde{x}_i) - 2\|\widetilde{x}_i - x_i\| = f(\widetilde{x}_i) - 1(i^{-1}).$$

This means that (6) is fulfilled.

The assumption (19) is satisfied for $\delta(r) = r$, $r \geqslant 0$. Indeed, let $x \in B = \Phi(0)$. One of two cases is possible: either for some $i \geqslant 1$ we have $x \in B_{1/(2i)}(e_i)$ holds, nor $x \notin B_{1/(2i)}(e_i)$ for all $i$. In the first case $\mathrm{dist}(x, X_0) = \|x - e_i\| \leqslant 2i\|x - e_i\| = f(x)$, and in the second case $\mathrm{dist}(x, X_0) \leqslant 1 = f(x)$ due to $X_0 \subset B$.

Obviously, the remaining assumptions of Lemma 2 are also satisfied, except for the assumption (20), which is failed. Let us show this.

Indeed, for the sequence of points $x_i = (1 - \frac{1}{2i})e_i$ we have $\mathrm{dist}(x_i, X_0) = \|x_i - e_i\| = (2i)^{-1} \to 0$ as $i \to \infty$, however, $f(x_i) = 1$ for all $i$. But by construction, $f_0 = 0$ and, therefore, $f(x_i) \not\to f_0$. Thus, (20) is failed.

Now we show that the assertion of Lemma 2 is also failed. Indeed, for all $i$ we have $f((1-\frac{1}{2i})e_i) = 1$, and by construction $f(a) = \frac{1}{2}$. Consequently, the minimum of the function $f$ on the set $\Phi(i^{-1})$ for all $i$ is attained at the unique point $a$. Since $a \notin X_0$, the assertion of Lemma 2 is failed. The example is complete.

We can make a quite similar remark on the assumption (21) as that given at the end of Section 3. Namely, if the set of constraints in the problem (1) does not depend on the parameter $\sigma$, then the condition (21) is satisfied for $\varphi(\sigma) \equiv x \in X_0$, where $x$ is an arbitrary point in the set $X_0$.

## 5. FINITE-DIMENSIONAL PROBLEMS AND λ-TRUNCATIONS

Let us turn to the theory of finite-dimensional extremal problems with the parameter $\sigma \in \Sigma$, where $\Sigma \subset \mathbb{R}^m$. In this theory, the space $X$ is defined as $X = \mathbb{R}^n$, and the sets $\Phi(\sigma)$ are given by

$$\Phi(\sigma) = \{x \in X : \ F(x, \sigma) = 0\} \cap B_R(x_0), \quad \sigma \in \Sigma. \tag{23}$$

Here $F : \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R}^m$ is a continuous mapping, $n, m$ are positive integers, $x_0 \in \mathbb{R}^n$. We will assume that all sets $\Phi(\sigma)$ are non-empty, $\sigma \in \Sigma$.

Consider the problem

$$f(x, \sigma) \to \min, \quad F(x, \sigma) = 0, \quad x \in B_R(x_0), \quad \sigma \in \Sigma. \tag{24}$$

As before, we separately consider the problem

$$f(x) = f(x, \sigma_0) \to \inf, \quad F(x, \sigma_0) = 0, \quad x \in B_R(x_0), \tag{25}$$

and we will assume that $x_0$ is its strict solution. Here we obtain stability conditions for the solution $x_0$ to the problem (25) assuming that $F(x, \sigma)$ has the form $F(x, \sigma) = F(x) - \sigma$. In this particular case, $F$ is a given continuous mapping $F = (F_1, \ldots, F_m) : \mathbb{R}^n \to \mathbb{R}^m$, and $\Sigma$ is a neighborhood of a point $\sigma_0 \in \mathbb{R}^m$ with the natural metric induced from $\mathbb{R}^m$.

Before formulating the corresponding statement, we write down some constructions from [9]. In this section, the coordinates of a vector $x \in \mathbb{R}^n$ are denoted by subscripts, i.e. we have $x = (x_1, \ldots, x_n)$. The symbols $\langle \cdot, \cdot \rangle$ mean the inner product in $\mathbb{R}^n$.

Let $D$ denote the set of nonzero $n$-dimensional vectors $d$ with non-negative coordinates $d_i$, and let $\widehat{D} \subset D$ denote the set of all nonzero integer vectors $z = (z_1, \ldots, z_n) \in D$. For $x \in \mathbb{R}^n$, $z \in \widehat{D}$ and $d \in D$ we use the notations

$$x^z := \prod_{k=1}^{n} x_k^{z_k}, \quad |x|^d := \prod_{k=1}^{n} |x_k|^{d_k}.$$

Here we put $x_k^l = |x_k|^l = 1$ for $l = 0$. Vector $z = (z_1, \ldots, z_n)$ is called a multi-index of the monomial $x^z$, and $z_1 + \ldots + z_n$ is called its degree.

Let us start with the case $m = 1$. Consider a continuous function $g : \mathbb{R}^n \to \mathbb{R}$ and a vector $\lambda \in \mathbb{R}^n$ such that $\lambda > 0$ (here and below this means the coordinate-wise inequality $\lambda_i > 0$, $i = \overline{1, n}$).

**Definition 2.** The function $p : \mathbb{R}^n \to \mathbb{R}$ is called a $\lambda$-truncation of the function $g$ in a neighborhood of the point $x_0 \in \mathbb{R}^n$ if there exists $\alpha > 0$ such that the following conditions hold:

1) there exists a non-empty finite set $\widehat{Z} \subset \widehat{D}$ such that

$$p(x) = \sum_{z \in \widehat{Z}} p_z x^z, \quad p_z \in \mathbb{R}, \quad p_z \neq 0, \quad \langle \lambda, z \rangle = \alpha \quad \forall z \in \widehat{Z};$$

2) there exist a real $K > 0$, a finite set $Z \subset D$ and a continuous function $\Delta : \mathbb{R}^n \to \mathbb{R}$ such that function $g$ in a neighborhood of the point $x_0$ has the form

$$g(x) = g(x_0) + p(x - x_0) + \Delta(x - x_0),$$

and

$$|\Delta(x - x_0)| \leqslant K \sum_{d \in Z} |x - x_0|^d, \quad \langle \lambda, d \rangle > \alpha \quad \forall d \in Z.$$

Let us give an illustrative example to Definition 2.

*Example 10.* Let $n = 2$, $x_0 = 0$ and

$$g(x) = \sin x_1^3 + 2x_1 \sin x_2 + x_2^3 + 3x_1 x_2^2 + 4x_1^2 x_2.$$

We have the following $\lambda$-truncations (unique for each $\lambda$):

$$\begin{array}{lll}
\lambda = (1,1) \;\Rightarrow\; p(x) = 2x_1x_2, & \alpha = 2; \\
\lambda = (1,2) \;\Rightarrow\; p(x) = x_1^3 + 2x_1x_2, & \alpha = 3; \\
\lambda = (2,1) \;\Rightarrow\; p(x) = 2x_1x_2 + x_2^3, & \alpha = 3; \\
\lambda = (1,3) \;\Rightarrow\; p(x) = x_1^3, & \alpha = 3; \\
\lambda = (3,1) \;\Rightarrow\; p(x) = x_2^3, & \alpha = 3.
\end{array}$$

This function $g$ has no other $\lambda$-truncations.

**Definition 3.** A mapping $P : \mathbb{R}^n \to \mathbb{R}^m$ is called a $\lambda$-truncation of the mapping $F : \mathbb{R}^n \to \mathbb{R}^m$ in a neighborhood of the point $x_0$, if each component $P_i$ is a $\lambda$-truncation of the component $F_i$ in a neighborhood of the same point, $i = \overline{1,m}$.

Let a vector $h \in \mathbb{R}^n$ be given. A $\lambda$-truncation $P$ is called regular in direction $h$ if

$$P(h) = 0, \quad \operatorname{im} P'(h) = \mathbb{R}^m.$$

The $\lambda$-truncations constructed in Example 10 for $\lambda = (1,1)$ and $\lambda = (1,2)$ are regular in the directions $h = (0, \pm 1)$, for $\lambda = (1,1)$ and $\lambda = (2,1)$ are regular in the directions $h = (\pm 1, 0)$, and the $\lambda$-truncations for $\lambda = (1,3)$ and $\lambda = (3,1)$ do not have regular directions.

From [9, Theorem 1], we have the following statement.

**Proposition 1.** *Let $P$ be a $\lambda$-truncation of the mapping $F$ in a neighborhood of the point $x_0$ for $\lambda > 0$, let $P$ be regular in a direction $h \in \mathbb{R}^n$, and let $F(x_0) = \sigma_0$. Then there exist a neighborhood $O(\sigma_0)$ of the point $\sigma_0$ and positive reals $c$ and $\beta$ such that for any $\sigma \in O(\sigma_0)$ there exists a solution $\varphi = \varphi(\sigma)$ to the equation $F(x) = \sigma$ with the following estimate:*

$$\|\varphi(\sigma) - x_0\| \leqslant c\|\sigma - \sigma_0\|^\beta.$$

**Theorem 1.** *Let $R > 0$ and a continuous mapping $F : B_R(x_0) \to \Sigma$ be given. Consider the parameter-dependent problem of minimizing continuous functions $f(x, \sigma)$ with a finite number of continuous constraints of the equality type:*

$$f(x, \sigma) \to \min, \quad F(x) = \sigma, \quad x \in B_R(x_0), \quad \sigma \in \Sigma. \tag{26}$$

*Let there exist $h \in \mathbb{R}^n$, a vector $\lambda > 0$ and a mapping $P : \mathbb{R}^n \to \mathbb{R}^m$ such that $P$ is a $\lambda$-truncation of the mapping $F$ in a neighborhood of the point $x_0$, and $P$ is regular in the direction $h$. Then for any function $f$ satisfying only conditions* (3) *and* (4), *the point $x_0$ is the stable solution to problem* (26) *for $\sigma = \sigma_0$.*

**Proof of Theorem 1.** Put $\Phi(\sigma) = \{x : F(x) = \sigma\}$. Then, for the considered problem, all the assumptions of Lemma 1 are satisfied. Indeed, (5) follows from the finite-dimensionality of the space $X$, the condition (4) and Example 1. The assumption (6) follows from the finite-dimensionality of $X$ and the Cantor theorem. Finally, (7) follows from the finite-dimensionality of $X$ and the fact that (26) is the problem with equality-type constraints, i.e. the conditions of Example 5 are fulfilled with $C = \{0\}$ in (18).

Now we check (9). Due to the assumption of the theorem on the $\lambda$-truncation $P$ of the mapping $F$ in a neighborhood of the point $x_0$, all conditions of Proposition 1 are satisfied. It follows that the equation $F(x) = \sigma$ has a solution $\varphi(\sigma)$ in some neighborhood of the point $\sigma_0$ such that $\varphi(\sigma_0) = x_0$ and the function $\varphi$ is continuous at the point $\sigma_0$. Using the continuity of $\varphi$ and reducing the indicated neighborhood if necessary, we get $\varphi(\sigma) \in B_R(x_0)$ for all $\sigma$ from this neighborhood. For the rest $\sigma \in \Sigma$ we select $\varphi(\sigma) \in \Phi(\sigma)$ arbitrarily. Then by construction $\varphi(\sigma) \in \Phi(\sigma)$ $\forall \sigma \in \Sigma$ and, in addition, $\varphi(\sigma) \to x_0$ as $\sigma \to \sigma_0$. Therefore, the assumption (9) is satisfied. To complete the proof, it remains to apply Lemma 1. Theorem 1 is proved.

Let us give an example of using Theorem 1.

*Example 11.* Let $n = 3$, $m = 2$, $x_0 = 0$. Let $\Sigma$ be a neighborhood of zero in $\mathbb{R}^2$, and put $\sigma_0 = 0$. It is a straightforward task to ensure that for $\lambda = (9, 6, 4)$ the mapping

$$F(x) = (x_1^2 + x_2^3, x_2^2 + x_3^3), \quad x \in \mathbb{R}^3,$$

is a $\lambda$-truncation of itself in a neighborhood of zero, and, moreover, $F$ is regular in the direction $h = (1, -1, -1)$.

Consider the problem (26) with arbitrary functions $f(x, \sigma)$, $\sigma \in \Sigma$, continuous in some ball of radius $R > 0$ with center at zero in $\mathbb{R}^3$ and convergent uniformly on this ball to some function $f$ as $\sigma \to 0$, while $f$ has a strict minimum at zero. Then, by virtue of Theorem 1, the solution $x_0 = 0$ to problem (26) for $\sigma = \sigma_0$ is stable. The example is complete.

In conclusion, we present a result using the constructions from Section 4, in particular, Lemma 2. To do this, let us assume that in problem (25) the minimum is attained not at the point $x_0$, but on a given non-empty set $X_0 \subset \Phi(\sigma_0)$, where $\Phi(\sigma)$ is given by (23). Here we also put $F(x, \sigma) = F(x) - \sigma$. In other words, we assume that

$$f(x) = f_0 = \min\{f(\xi, \sigma_0) : \ F(\xi) = \sigma_0, \ \xi \in B_R(x_0)\} \quad \Leftrightarrow \quad x \in X_0.$$

Note that the latter is equivalent to (22).

**Theorem 2.** *Let $R > 0$ and a continuous mapping $F : B_R(x_0) \to \Sigma$ be given. Consider the parameter-dependent problem of minimizing continuous functions $f(x, \sigma)$ with a finite number of continuous constraints of the equality type:*

$$f(x, \sigma) \to \min, \quad F(x) = \sigma, \quad x \in B_R(x_0), \quad \sigma \in \Sigma.$$

*Let $X_0 \subset B_R(x_0)$ with $F(x) = \sigma_0$ and $f(x, \sigma_0) = f_0$ $\forall x \in X_0$.*

*Let there exist $h \in \mathbb{R}^n$, $\widetilde{x} \in X_0$, a vector $\lambda > 0$ and a mapping $P : \mathbb{R}^n \to \mathbb{R}^m$ such that $P$ is a $\lambda$-truncation of the mapping $F$ in a neighborhood of the point $\widetilde{x}$, regular in the direction $h$, and let $\widetilde{x} \in \operatorname{int} B_R(x_0)$, i.e. $\widetilde{x}$ does not lie on the boundary of the ball $B_R(x_0)$. Then, for any function $f$*

*satisfying only conditions* (3) *and* (22)*, the result of Lemma 2 holds. Namely, for any points* $x(\sigma) \in \Phi(\sigma)$*,* $\sigma \neq \sigma_0$*, satisfying the condition* (8)*, we have*

$$\operatorname{dist}(x(\sigma), X_0) \to 0 \quad as \quad \sigma \to \sigma_0.$$

**Proof of Theorem 2.** Let us show that all assumptions of Lemma 2 are satisfied. In fact, (6) and (7) were checked in the proof of Theorem 1. The assumption (19) follows from the finite-dimensionality of the space $X$, the condition (22) and Example 8. The assumption (20) follows from the finite dimensionality of $X$ and the Cantor theorem.

Now we check (21) in the same way as we have checked (9) in the previous theorem. Namely, due to the assumption on $\lambda$-truncation, we obtain for the equation $F(x) = \sigma$ a solution $\varphi(\sigma)$ in some neighborhood of $\sigma_0$ such that $\varphi(\sigma_0) = \widetilde{x}$ and the function $\varphi$ is continuous at the point $\sigma_0$. Using again the continuity of the function $\varphi$ and reducing the indicated neighborhood if necessary, we get $\varphi(\sigma) \in B_R(x_0)$ for all $\sigma$ from this neighborhood. The latter is possible due to the assumption $\widetilde{x} \in \operatorname{int} B_R(x_0)$. For the rest $\sigma \in \Sigma$ select $\varphi(\sigma) \in \Phi(\sigma)$ arbitrarily. Then by construction we have $\varphi(\sigma) \in \Phi(\sigma)$ $\forall \sigma \in \Sigma$ and, in addition, $\varphi(\sigma) \to \widetilde{x} \in X_0$ as $\sigma \to \sigma_0$. Therefore, the condition (21) is satisfied. To complete the proof, it remains to apply Lemma 2. Theorem 2 is proved.

Note that Theorem 1 follows directly from Theorem 2 in the case when $X_0 = \{x_0\}$. In this case, we put $\widetilde{x} = x_0$ in Theorem 2.

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