═══ **OPTIMIZATION, SYSTEM ANALYSIS, AND OPERATIONS RESEARCH** ═══

Convex Isoquants in Dea Models
with Selective Convexity

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Abstract—Models with selective convexity are an important class of data envelopment analysis (DEA) models. This type of model allows managers to consider variables such as ratios, averages, percentages, etc. The paper proposes algorithms for constructing input and output isoquants using volume variables in models with selective convexity. These algorithms help investigate the relationship between any volume variables in the model. Computational experiments confirm the reliability and efficiency of the proposed methods.

Keywords: data envelopment analysis, production possibility set, selective convexity, efficient frontier, isoquant

1. INTRODUCTION

The data envelopment analysis (DEA) approach arose as a generalization of simple indicators of units behavior to a multidimensional case. Mathematically, this approach leads to solving a large family of optimization problems. The founders of this approach were famous American scientists A. Charnes, W. Cooper, E. Rhodes and R. Banker [1, 2]. The FDH (free disposal hull) models appeared almost simultaneously with VRS formulation of DEA in the works of D. Deprins, L. Simar and G. Tulkens [3] in the end of last century. Constraints sets of the DEA models are convex, so optimization methods are widely used for DEA models. Production possibility set of the FDH models are non-convex. For this reason, the development of visualization methods for FDH models slows down.

The notion of selective convexity was proposed in [4]. This notion considers a range of new DEA models, where DEA and FDH models are two extreme cases. Such models expands the possibilities of DEA and FDH models, since problems with selective convexity include such variables into models as ratios, percentages, averages, etc.

The DEA and FDH models aim to develop models and instruments for analyzing the behavior of complex socio-economic systems, such as regions, banks, universities, hospitals, industrial facilities, etc. For developing and applying these models it was necessary to develop new approaches.

Visualization techniques are utilized in various fields of human activity, including the study of the behavior of large-scale socio-economic systems. It enables managers to construct the trajectories

of units' development, to obtain unknown dependencies between model components, detect and correct incorrectness in models, to explore the problem of units' separation and merging, as noted in [5]. In general, visualization enhances a manager's intuition in making strategic decisions.

However, there exist a few works [5–7] in the scientific papers devoted to the visualization of multidimensional production possibility sets and dispositions of production units in such figures. In [7], the methods for multidimensional visualization of convex DEA models were presented. In [5] a review of visualization methods in DEA is presented. Visualization means the construction of intersections of multidimensional polyhedral production possibility set with two- or three-dimensional hyperplanes. This approach reduces the efficiency analysis of production units to the investigation of well-known functions in economics, such as production function, isoquant, isocost, isoprofit, etc. [8, 9].

In paper [10], visualization methods were proposed for models with selective convexity, in which some of the variables are ratios. For such models, solution and visualization methods were proposed and for any two ratio input or output variables. The new methods have shown their efficiency on real-life problems.

Moreover, it was shown in paper [10] that not taking into account specifics of the task leads to significant distortions of the result. In this paper, algorithms are considered for construction of input and output isoquants in models with selective convexity with the use of volume variables.

2. BACKGROUND

Consider a set of production units (X_j, Y_j), $j = 1, \ldots, n$, where the vector of outputs $Y_j = (y_{1j}, \ldots, y_{rj}) \geqslant 0$ is produced from the vector of inputs $X_j = (x_{1j}, \ldots, x_{mj}) \geqslant 0$. All data are assumed to be nonnegative, but at least one component of every input and output vector is positive.

Now consider the notion of selective convexity [4]. Let the input and output sets I and O have the following partition

$$I = I^C \cup I^{NC}, \quad O = O^C \cup O^{NC},$$

where the subsets I^C and I^{NC}, and O^C and O^{NC}, are mutually disjoint.

Subsets I^C and O^C are called the subsets of volume inputs and outputs (volume measures). The complementary subsets $I^{NC} = I \setminus I^C$ and $O^{NC} = O \setminus O^C$ are marked as ratio inputs and outputs (ratio measures).

Let us assume that the set I^C contains the inputs from 1 to m', at the same time the set I^{NC} contains the inputs from $(m' + 1)$ to m. Then it is evident that any vector of inputs can be written in the form $X = (X^C, X^{NC})$, where X^C is the vector of the first m' components of X, and X^{NC} is the vector of the last components of X.

In the same way, let us assume that the set O^C contains the output components from 1 to r', and the set O^{NC} contains the output components from $(r' + 1)$ to r. Hence any vector of outputs can be written in the form $Y = (Y^C, Y^{NC})$.

The production possibility set T of the technology with selective convexity is determined by the following postulates [4].

(A1) Feasibility of observed data. Unit $(X_j, Y_j) \in T$ for any $j = 1, \ldots, n$.

(A2) Free disposability. $(X, Y) \in T$, and $Y \geqslant Y' \geqslant 0$ and $X' \leqslant X$ implies $(X', Y') \in T$.

(A3) Selective convexity. Let $(X', Y') \in T$ and $(X'', Y'') \in T$. Assume that $(X')_i = (X'')_i$ for all $i \in I^{NC}$, and $(Y')_r = (Y'')_r$ for all $r \in O^{NC}$. Then, for any $\lambda \in [0, 1]$, the unit $\lambda(X', Y') + (1 - \lambda)(X'', Y'') \in T$.

The production possibility set T, which satisfies (A1)–(A3) can be written in the following form:

$$T = \left\{ (X^C, X^{NC}, Y^C, Y^{NC}) \geqslant 0 \,\Bigg|\, \sum_{j=1}^{n} X_j^C \lambda_j \leqslant X^C, \ \sum_{j=1}^{n} Y_j^C \lambda_j \geqslant Y^C, \ \text{if } \lambda_j \geqslant 0, \right.$$

$$\left. \text{then } X_j^{NC} \leqslant X^{NC} \text{ and } Y_j^{NC} \geqslant Y^{NC}, \ \sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geqslant 0, \ j = 1, \ldots, n \right\}. \quad (1)$$

The selective convexity model combines two well-known DEA models. So if $I^{NC} = O^{NC} = \emptyset$ (all variables are volume), then set (1) determines the BCC model. If set (1) contains only ratios, i.e., $I^C = O^C = \emptyset$, then the selective convexity model becomes the FDH model.

Podinovski [4] used binary variables δ_j to transform set T to a mixed integer linear constraints. However, for construction of isoquants for variables from I^C and O^C, ratio variables I^{NC} and O^{NC} do not change. So the mixed integer constraints in this case can be replaced by the equivalent constraints $(X_j^{NC} - X^{NC})\lambda_j \leqslant 0$ and $(Y_j^{NC} - Y^{NC})\lambda_j \geqslant 0$; see [11, 12] and Remark 3 in [4].

3. ALGORITHM FOR CONSTRUCTION OF THE INPUT ISOQUANT

Input two-dimensional section of set T for unit (X_o, Y_o) is determined by the following formula

$$I_1(X_o, Y_o) = \{(X, Y) \mid X = X_o + \alpha d_1 + \beta d_2, \ Y = Y_o, \ \alpha, \beta \in E^1\}, \quad (2)$$

where $d_1, d_2 \in E^m$, $(X_o, Y_o) \in T$, vectors d_1 and d_2 are directional vectors, and d_1 is perpendicular to d_2.

Next, define the input two-dimensional isoquant as the intersection of the frontier and two-dimensional plane I_1.

$$\mathrm{Sec}_I(X_o, Y_o) = \{(X, Y) \mid (X, Y) \in \mathrm{WEff}_P T \cap I_1\}, \quad (3)$$

where $\mathrm{WEff}_P T$ is a set of weakly Pareto efficient points of set T.

Output two-dimensional section of set T for unit $(X_o Y_o)$ is written as

$$I_2(X_o, Y_o) = \{(X, Y) \mid X = X_o, \ Y = Y_o + \alpha g_1 + \beta g_2, \ \alpha, \beta \in E^1\}, \quad (4)$$

where $g_1, g_2 \in E^r$, $(X_o, Y_o) \in T$, g_1 is perpendicular to g_2.

Now, define the output two-dimensional isoquant as the intersection of the frontier and two-dimensional plane I_2.

$$\mathrm{Sec}_O(X_o, Y_o) = \{(X, Y) \mid (X, Y) \in \mathrm{WEff}_P T \cap I_2\}. \quad (5)$$

Consider an optimization algorithm for construction of the input isoquant for unit $(X_o Y_o)$. The isoquant is determined by directions $e_p \in E^{m'}$ and $e_s \in E^{m'}$, where e_p and e_s are unity vectors with ones in positions p and s, correspondingly. In addition, the inputs p and s belong to the set I^C.

Algorithm 1 (construction of the input isoquant).

Step 1. Find a leftmost point on the input isoquant going through unit (X_o, Y_o) and associated with directions $e_p \in E^{m'}$ and $e_s \in E^{m'}$.

Step 1a. Solve the following optimization problem.

$$\max \theta_1$$

$$\sum_{j=1}^{n} x_{sj}^{C} \lambda_j + \theta_1 \leqslant x_{so},$$

$$\sum_{j=1}^{n} x_{pj}^{C} \lambda_j + \tau_1 \leqslant x_{po},$$

$$\sum_{j=1}^{n} x_{ij}^{C} \lambda_j \leqslant x_{io},\ i \neq p, s,$$

$$\sum_{j=1}^{n} Y_{j}^{C} \lambda_j \geqslant Y_o \tag{6}$$

$$(X_{j}^{NC} - X_{o}^{NC}) \lambda_j \leqslant 0,\ j = 1, \ldots, n,$$

$$(Y_{j}^{NC} - Y_{o}^{NC}) \lambda_j \geqslant 0,\ j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ j = 1, \ldots, n,$$

where τ_1 and θ_1 are free variables.

Step 1b. Let θ_1^* be optimal objective of (6). Solve the following problem.

$$\max \tau_1$$

$$\sum_{j=1}^{n} x_{sj}^{C} \lambda_j + \theta_1^* \leqslant x_{so},$$

$$\sum_{j=1}^{n} x_{pj}^{C} \lambda_j + \tau_1 \leqslant x_{po},$$

$$\sum_{j=1}^{n} x_{ij}^{C} \lambda_j \leqslant x_{io},\ i \neq p, s,$$

$$\sum_{j=1}^{n} Y_{j}^{C} \lambda_j \geqslant Y_o, \tag{7}$$

$$(X_{j}^{NC} - X_{o}^{NC}) \lambda_j \leqslant 0,\ j = 1, \ldots, n,$$

$$(Y_{j}^{NC} - Y_{o}^{NC}) \lambda_j \geqslant 0,\ j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ j = 1, \ldots, n,$$

where τ_1 is a free variable.

Let $\tilde{Z}_1^1 = (X_o^C - \theta_1^* e_s - \tau_1^* e_p, X_o^{NC}, Y_o^C, Y_o^{NC})$, where θ_1^* and τ_1^* are optimal objectives of problems (6) and (7), respectively.

Step 2. Find the second point on the input isoquant going through unit (X_o, Y_o) and determined by directions $e_p \in E^{m'}$ and $e_s \in E^{m'}$.

Step 2a. Solve the following optimization problem.

$$\max \tau_2$$

$$\sum_{j=1}^{n} x_{sj}^{C} \lambda_j + \theta_2 \leqslant x_{so},$$

$$\sum_{j=1}^{n} x_{pj}^{C} \lambda_j + \tau_2 \leqslant x_{po},$$

$$\sum_{j=1}^{n} x_{ij}^{C} \lambda_j \leqslant x_{io}, \ i \neq p, s,$$

$$\sum_{j=1}^{n} Y_{j}^{C} \lambda_j \geqslant Y_{o}, \tag{8}$$

$$(X_{j}^{NC} - X_{o}^{NC})\lambda_j \leqslant 0, \ j = 1, \ldots, n,$$

$$(Y_{j}^{NC} - Y_{o}^{NC})\lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

where τ_2 and θ_2 are free variables.

Step 2b. Let τ_2^* be optimal objective of (8). Solve the following problem.

$$\max \theta_2$$

$$\sum_{j=1}^{n} x_{sj}^{C} \lambda_j + \theta_2 \leqslant x_{so},$$

$$\sum_{j=1}^{n} x_{pj}^{C} \lambda_j + \tau_2^* \leqslant x_{po},$$

$$\sum_{j=1}^{n} x_{ij}^{C} \lambda_j \leqslant x_{io}, \ i \neq p, s,$$

$$\sum_{j=1}^{n} Y_{j}^{C} \lambda_j \geqslant Y_{o}, \tag{9}$$

$$(X_{j}^{NC} - X_{o}^{NC})\lambda_j \leqslant 0, \ j = 1, \ldots, n,$$

$$(Y_{j}^{NC} - Y_{o}^{NC})\lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

where τ_2 is a free variable.

Let $\tilde{Z}_2^1 = (X_o^C - \theta_2^* e_s - \tau_2^* e_p, X_o^{NC}, Y_o^C, Y_o^{NC})$, where θ_2^* and τ_2^* are optimal objectives of problems (8) and (9), respectively.

Step 3. Set $l := 1$, $k := 1$, $i_1 := 1$, $i_2 := 2$. Create flow F_k^l with points $Z_{i_1}^l = Z_1^1$, $Z_{i_2}^l = Z_2^1$ of production possibility set T. Define set $M = \{Z_1^1, Z_2^1\}$.

Step 4. Perform the following operations. Take any unprocessed flow F_k^l, solve optimization problem of the following type

$$\max \beta_1$$
$$(Z_{i_1}^l + Z_{i_2}^l)/2 + \beta_1 d_1 + \tau d_2 \in T, \tag{10}$$

where β_1 and τ are scalar variables, vector d_1 is perpendicular to the vector d_2, it lies in the plane of the section, and is directed to the low left corner of a two-dimensional section, vector $d_2 = Z_{i_1}^l - Z_{i_2}^l$.
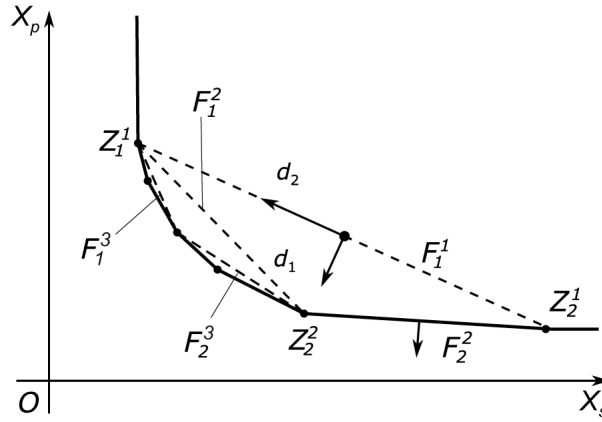
Fig. 1. Construction of input isoquant.

If optimal objective value of problem (10) $\beta_1^* > 0$, then start new flows $F_{k_1}^{l+1}$ and $F_{k_2}^{l+1}$ and solve optimization sub-problems.

Flow $F_{k_1}^{l+1}$ contains points

$$Z_{i_1}^{l+1} = Z_{i_1}^l, \quad Z_{i_2}^{l+1} = (Z_{i_1}^l + Z_{i_2}^l)/2 + \beta_1^* d_1 + \tau^* d_2,$$

where β_1^* and τ^* are optimal values of variables in (10).

Flow $F_{k_2}^{l+1}$ contains points

$$Z_{i_3}^{l+1} = Z_{i_2}^{l+1}, \quad Z_{i_4}^{l+1} = Z_{i_2}^l, \quad d_2 = Z_{i_4}^{l+1} - Z_{i_3}^{l+1},$$

where d_1 is perpendicular to the vector d_2.

If optimal solution value in (10) $\beta_1^* \leqslant 0$, then points $Z_{i_1}^l$ and $Z_{i_2}^l$ are angular points of the segment of input isoquant. Include these points to the set of corner points M. Flow F_k^l is deleted from the list of flow tasks.

Step 5. Set $l := l + 1$. If exist unprocessed flows F_k^l, then go to the Step 4, else go to the Step 6.

Step 6. Points of set M are angular points of input isoquant. Connect adjacent pair points by line segments. At last, add a vertical line after the first point, and a horizontal line starting from the last point. This completes construction of the input isoquant.

Figure 1 illustrates the construction of isoquant with the help of the algorithm. At first, the angular points Z_1^1 and Z_2^1 are found by solving models (6)–(9). Then the flow F_1^1 containing these two points is started. At the next steps, point Z_2^2 will be found with the help of solving problem (10); and the flow F_1^1 will be split into two flows F_1^2 and F_2^2, that will have vertices Z_1^1 and Z_2^2 for flow F_1^2 and vertices Z_2^2 and Z_2^2 for flow F_2^2.

After this, the computations are repeated until all segments of the isoquant are found.

For the algorithm presented above the following assertion is valid.

Assertion 1. *Algorithm constructs an input isoquant for production possibility set* (1) *in a finite number of steps.*

Proof. An input isoquant of the two-dimensional set (2) envelops this set or, in other words, it is a boundary of this set. At Steps 1 and 2 the algorithm founds two points Z_1^1 and Z_1^2 of the isoquant and determines segment $[Z_1^1, Z_1^2]$ belonging to the set (2). This initial approximation of the set (3) is found. After this, two optimization problems of the type (10) are solved. If $\beta_1^* > 0$ for at least one of this problem, then the algorithm starts new flows. The approximation of the set (3)

is expanded. If $\beta_1^* \leqslant 0$, then flow F_k^l is deleted from the list of flow tasks. Iterations continued if there exist unprocessed flows. However, all approximations of the set (3) belong to this set and they are expanded during the iterations. The last approximation coincides with set (3). Since the number of boundary segments is finite and the directions of the objective functions differ from each other at every iteration. This completes the proof.

4. ALGORITHM FOR CONSTRUCTION OF THE OUTPUT ISOQUANT

The algorithm for construction of the output isoquant can be written in a similar way. Next, we will focus only on the main differences. Let (X_o, Y_o) be a production unit for which the isoquant is being constructed, and let p and s be two outputs that determined that isoquant. At the first step, we find a rightmost vertex Z_1^1 of isoquant by solving the following optimization problems.

$$
\begin{aligned}
&\max \theta_1 \\
&\sum_{j=1}^{n} X_j^C \lambda_j \leqslant X_o, \\
&\sum_{j=1}^{n} y_{sj}^C \lambda_j - \theta_1 \geqslant y_{so}, \\
&\sum_{j=1}^{n} y_{pj}^C \lambda_j - \tau_1 \geqslant y_{po}, \\
&\sum_{j=1}^{n} y_{ij}^C \lambda_j \geqslant y_{io},\ i \neq p, s, \\
&(X_j^{NC} - X_o^{NC})\lambda_j \leqslant 0,\ j = 1, \ldots, n, \\
&(Y_j^{NC} - Y_o^{NC})\lambda_j \geqslant 0,\ j = 1, \ldots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ j = 1, \ldots, n,
\end{aligned}
\tag{11}
$$

where τ_1 and θ_1 are free variables.

$$
\begin{aligned}
&\max \tau_1 \\
&\sum_{j=1}^{n} X_j^C \lambda_j \leqslant X_o, \\
&\sum_{j=1}^{n} y_{sj}^C \lambda_j - \theta_1^* \geqslant y_{so}, \\
&\sum_{j=1}^{n} y_{pj}^C \lambda_j - \tau_1 \geqslant y_{po}, \\
&\sum_{j=1}^{n} y_{ij}^C \lambda_j \geqslant y_{io},\ i \neq p, s, \\
&(X_j^{NC} - X_o^{NC})\lambda_j \leqslant 0,\ j = 1, \ldots, n, \\
&(Y_j^{NC} - Y_o^{NC})\lambda_j \geqslant 0,\ j = 1, \ldots, n, \\
&\sum_{j=1}^{n} \lambda_j = 1,\ \lambda_j \geqslant 0,\ j = 1, \ldots, n,
\end{aligned}
\tag{12}
$$

where τ_1 is a free variable.

Point Z_1^1 is expressed as:

$$Z_1^1 = (X_o^C, X_o^{NC}, Y_o^C + \theta_1^* e_s + \tau_1^* e_p, Y_o^{NC}),$$

where $e_p \in E^{r'}$ and $e_s \in E^{r'}$ are direction vectors of isoquant, θ_1^* and τ_1^* are optimal objectives of problems (11) and (12), respectively.

Second vertex Z_2^1 of the output isoquant is determined using following problems.

$$\max \tau_2$$

$$\sum_{j=1}^{n} X_j^C \lambda_j \leqslant X_o,$$

$$\sum_{j=1}^{n} y_{sj}^C \lambda_j - \theta_2 \geqslant y_{so},$$

$$\sum_{j=1}^{n} y_{pj}^C \lambda_j - \tau_2 \geqslant y_{po},$$

$$\sum_{j=1}^{n} y_{ij}^C \lambda_j \geqslant y_{io}, \ i \neq p, s,$$

$$(X_j^{NC} - X_o^{NC})\lambda_j \leqslant 0, \ j = 1, \ldots, n,$$

$$(Y_j^{NC} - Y_o^{NC})\lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

(13)

where τ_2 and θ_2 are free variables.

$$\max \theta_2$$

$$\sum_{j=1}^{n} X_j^C \lambda_j \leqslant X_o,$$

$$\sum_{j=1}^{n} y_{sj}^C \lambda_j - \theta_2 \geqslant y_{so},$$

$$\sum_{j=1}^{n} y_{pj}^C \lambda_j - \tau_2^* \geqslant y_{po},$$

$$\sum_{j=1}^{n} y_{ij}^C \lambda_j \geqslant y_{io}, \ i \neq p, s,$$

$$(X_j^{NC} - X_o^{NC})\lambda_j \leqslant 0, \ j = 1, \ldots, n,$$

$$(Y_j^{NC} - Y_o^{NC})\lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

$$\sum_{j=1}^{n} \lambda_j = 1, \ \lambda_j \geqslant 0, \ j = 1, \ldots, n,$$

(14)

where τ_2 is a free variable.

Thus we have $Z_2^1 = (X_o^C, X_o^{NC}, Y_o^C + \theta_2^* e_s + \tau_2^* e_p, Y_o^{NC})$, where θ_2^* and τ_2^* are optimal objective values of problems (13) and (14), respectively.

Steps 3–6 of the algorithm for output isoquant coincide with the algorithm for the input isoquant. The only difference is that vector d_1 in model (10) must have positive p and s coordinates to secure the correct shape of the output isoquant.

Assertion 2. *Algorithm constructs an output isoquant for production possibility set* (1) *in a finite number of steps.*

The proof of this assertion is similar to the input isoquant case and hence omitted.

5. COMPUTATIONAL EXPERIMENTS

To perform the computational experiments we use a dataset with artificially generated DMUs. It contains 100 units with 6 variables (3 inputs and 3 outputs). The variables were generated randomly in a range from 5 to 95. Figure 2 shows three isoquants constructed for unit 78 (depicted by point Z_0) using three different models.

Curve *1* corresponds to the isoquant of BCC model, where all variables are from the set $I^C \cup O^C$, i.e., $I^{NC} \cup O^{NC} = \emptyset$. Curve *2* is associated with the model with selective convexity, where all variables are volume except two outputs y_2 and y_3 that are ratio variables. Third model differs from the previous only in inputs x_1 and x_2. In this model they belong to I^{NC}. The isoquant for this model is depicted as curve *3*. Input isoquant for FDH model looks exactly the same as curve *3*; it so happened that the two curves coincided. We see from Fig. 2 that BCC and FDH models are two extreme cases, and curve *2* lies between them. Points Z_1, Z_2, and Z_3 are radial projections of unit Z_0 onto the frontier of models 1, 2, and 3, respectively.

Figure 3 shows three output isoquants constructed for unit 78 (point Z_0 in the figure) using three different models. Curve *1* is associated with the output isoquant of the BCC model. Curve *2* corresponds to the model with one ratio output y_3. Curve *3* is obtained for the model with two ratio variables x_3 and y_3, and the rest are volume. Recall that the distances from the point Z_0 to the points Z_1, Z_2 and Z_3 in relative units are measures of efficiency in models 1, 2 and 3, respectively. This confirms the fact that the choice of the model significantly affects the accuracy of the analysis of the production units' behavior.
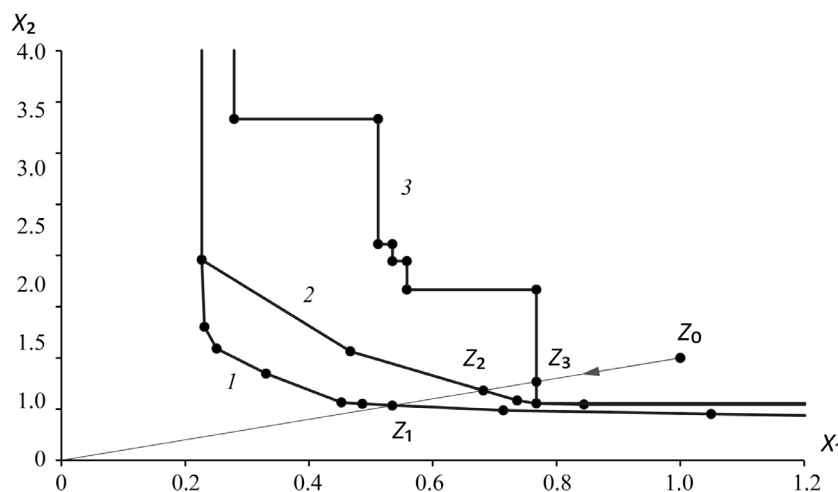
Fig. 2. Input isoquants for BCC model (curve *1*), model with selective convexity (curve *2*), and FDH model (curve *3*) for unit 78.
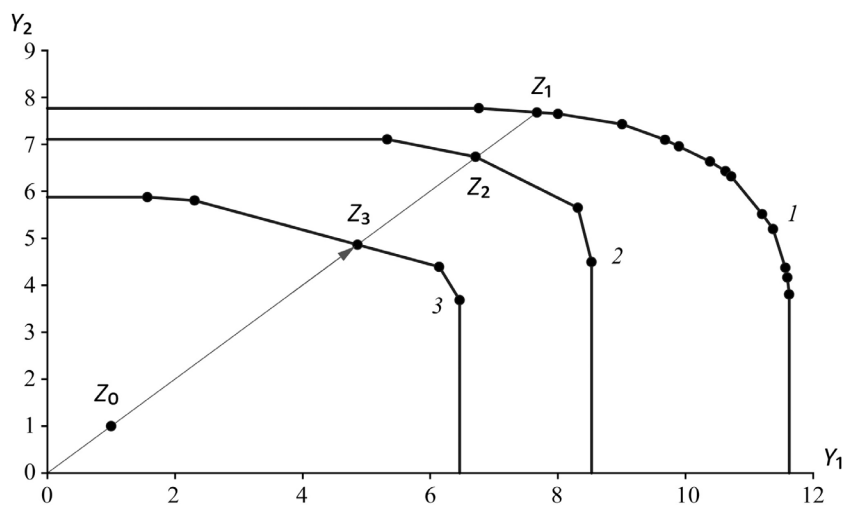
Fig. 3. Output isoquants for BCC model (curve *1*), model with one ratio variable (curve *2*), and model with two ratio variables (curve *3*) for unit 78.

6. CONCLUSION

Visualization plays a huge role in the science and practice of mankind. Indeed, the invention of the telescope by Giordano Bruno at the beginning of the 17th century allowed Newton at the end of this century to discover the laws of planetary motion and formulate as a result world-famous laws, without which it is impossible to create the modern development of science and technology. Visualization methods are used in many areas of human activity, no captain goes on a long trip without detailed maps, no doctor will start operation without a set of patient images, and no engineer will start construction without detailed drawings. However, the leaders of large-scale socio-economic systems often do not have all this instruments and rely on their intuition. However, the cost of an error may be quite huge.

The DEA and FDH technologies do not embrace all possible model cases for production units descriptions. In paper [4], the concept of selective convexity was proposed, which provides the development of a range of new DEA models [13–16], where FDH and DEA models are two extreme cases. Such modifications allow one to explain the class of model's variables and include the averages, percentages, ratios, etc. into DEA models.

In paper [10], algorithms were developed for the construction of input isoquants in DEA models with selective convexity with the use of ratio variables.

In this paper, algorithms are developed for construction of two-dimensional input and output isoquants with the use of volume input and output variables. The proposed algorithm requires considerably fewer computations than the algorithm [10] for ratio variables since it involves only linear problems, whereas the second uses mixed-integer programs.

Computational experiments documented that the proposed algorithms are reliable and efficient. The proposed algorithm allows parallel and distributed implementation similar to the approach proposed in [7]. The development of efficient parallel and distributed implementations [17–19] to speed up computations and conducting computational experiments with large-scale datasets we consider as a direction of our future research.

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