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NONLINEAR SYSTEMS

Output Stabilization of Lurie-Type Nonlinear Systems in a Given Set

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Abstract—This paper considers the problem of stabilizing the output variables of a Lurie-type nonlinear system in a given set at any time instant. A special output transformation is used to reduce the original constrained problem to that of analyzing the input-to-state stability of a new extended system without constraints. For this system, nonlinear control laws are obtained using the technique of linear matrix inequalities. Examples are given to illustrate the effectiveness of the method proposed and confirm the theoretical conclusions.

Keywords: Lurie-type nonlinear system, stabilization, nonlinear control, coordinate transformation, stability, linear matrix inequalities

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1. INTRODUCTION

Guaranteeing the desired quality of transients is a key criterion in the design of automatic control systems. Classical control methods, such as modal control [1], adaptive robust control [2, 3], etc., ensure control performance only in the steady-state mode. The transient mode remains uncontrollable.

Control problems for linear plants with a guarantee for the controlled variable to stay in a given set at any time instant were presented in [4–6]. Within this approach, control performance is ensured not only in the steady-state mode but also in the transient mode. Such problems often arise in practice, e.g., when controlling electric power systems to maintain the frequency and voltage of electric generators in specified ranges [7, 8], when stabilizing the formation pressure of oil production, where the pressure at the wellhead must strictly belong to a given band [9], etc. To solve such problems, the authors [4, 5] proposed a method based on a special output transformation that reduces the original control problem with output constraints to a new control problem without constraints on the auxiliary variable. For the class of linear plants, the corresponding control problems were well studied and solved in [4]. However, they remain open for Lurie-type nonlinear systems.

Below we consider Lurie-type systems with an unstable linear part and unknown bounded disturbances and pose the problem of stabilizing such systems in a given set of output variables. The remainder of this paper is organized as follows. Section 2 formulates the problem of stabilizing the controlled variables of Lurie-type nonlinear systems in given sets. In section 3, control design methods are proposed. Finally, section 4 provides some numerical examples in MATLAB to illustrate the theoretical results.

The following notations are used in the presentation: $\mathbb{R}^n$ is the n-dimensional Euclidean space with the Euclidean norm $|\cdot|$; $\mathbb{R}^{n\times m}$ is the set of all real matrices of dimensions $n\times m$ with the Euclidean norm $\|\cdot\|$; for $A\in\mathbb{R}^{n\times n}$, the relation $A\succ 0$ ($A\prec 0$) means that A is a positive (negative, respectively) definite matrix, whereas the relation $A\succeq 0$ ($A\preceq 0$) means that A is a nonnegative (nonpositive, respectively) definite matrix; $I, 0$, and $diag\{\cdot\}$ are identity, zero, and diagonal, respectively, matrices of appropriate dimensions; $\mathbf{1_m}\in\mathbb{R}^m$ is an m dimensional vector composed of unit elements; $col\{\cdot\}\in\mathbb{R}^m$ is a column vector in the space $\mathbb{R}^m$; finally, the symbol "$\star$" indicates a symmetric block in a symmetric matrix.

2. PROBLEM STATEMENT

Consider a Lurie-type nonlinear system of the form

$$\begin{aligned}\dot{x}(t) &= Ax(t)+Bu(t)+G\phi(z(t))+Df(t),\\ y(t) &= Lx(t),\quad z(t)=Cx(t),\end{aligned}\tag{1}$$

where $t\geqslant 0$; $x(t)\in\mathbb{R}^n$ is the vector of measured states; $u(t)\in\mathbb{R}^m$ is the control variable (input); $y(t)=col\{y_1(t),\ldots,y_m(t)\}\in\mathbb{R}^m$ is the controlled output; $f(t)\in\mathbb{R}^l$ is an unknown disturbance such that $|f(t)|\leqslant\bar{f}$; $z(t)\in\mathbb{R}^q$ is the argument of the nonlinearity ϕ; the matrices $A\in\mathbb{R}^{n\times n}$, $B\in\mathbb{R}^{n\times m}$, $G\in\mathbb{R}^{n\times q}$, $D\in\mathbb{R}^{n\times l}$, $L\in\mathbb{R}^{m\times n}$, and $C\in\mathbb{R}^{q\times n}$ are known. The pair of matrices (A,B) is controllable, and the pair of matrices (A,L) is observable. System (1) has a relative degree of $\mathbf{1_m}$, i.e., $det(LB)\neq 0$ [10, 11]. The unknown nonlinearity $\phi(\cdot):\mathbb{R}^q\to\mathbb{R}^q$ satisfies sector constraints: for all $z, \phi(z)=col\{\phi_1(z_1),\ldots,\phi_q(z_q)\}\in\mathbb{R}^q$,

$$k_{1i}\leqslant\frac{\phi_i(z_i)}{z_i}\leqslant k_{2i},\quad\forall z_i\neq 0,\ i=1,\ldots,q,\tag{2}$$

where k_{1i} and k_{2i} are some known constants.

This paper aims at designing a control law that stabilizes the output $y(t)$ of the plant (1) in a given set at any time instant:

$$\mathcal{Y}=\left\{y(t)\in\mathbb{R}^m:\underline{g}_i(t)<y_i(t)<\overline{g}_i(t),\ i=1,\ldots,m\right\},\quad\forall t\geqslant 0,\tag{3}$$

where $\underline{g}_i(t)$ and $\overline{g}_i(t)$ are bounded differentiable functions with bounded first derivatives. These functions can be selected by the designers according to system performance requirements. To illustrate the objective of control, Fig. 1 shows a given pipe where the output must be at any time instant.

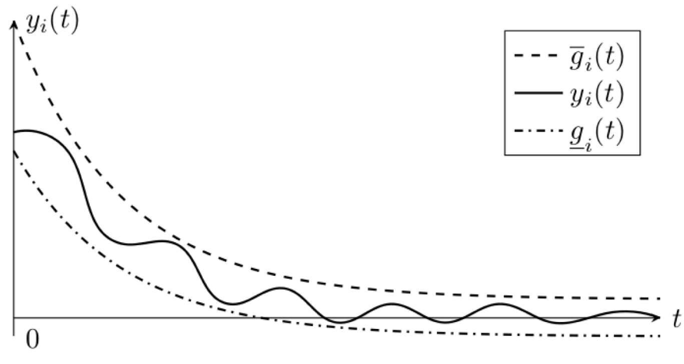

Fig. 1. The objective of control: one illustration.

3. SOLUTION METHOD

Following [4, 5], we introduce the output transformation

$$\varepsilon(t) = \Phi(y(t), t), \tag{4}$$

where $\varepsilon(t) = col\{\varepsilon_i(t), i = 1, \ldots, m\} \in \mathbb{R}^m$ and $\Phi : \mathcal{Y} \times [0, \infty) \to \mathbb{R}^m$ is a differentiable function (with respect to all arguments) in the diagonal form that satisfies several conditions:

(a) There exists the inverse mapping

$$y = \Phi^{-1}(\varepsilon, t), \forall \varepsilon \in \mathbb{R}^m, \quad t \geqslant 0. \tag{5}$$

(b) The function $\Phi^{-1}(\varepsilon, t)$ is differentiable with respect to ε and t, and $\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial \varepsilon} \succ 0 \forall \varepsilon \in \mathbb{R}^m$ and $t \geqslant 0$.

(c) $\underline{g}_i(t) < \Phi_i^{-1}(\varepsilon_i, t) < \overline{g}_i(t),\ i = 1, \ldots, m,\ \forall \varepsilon_i \in \mathbb{R}$ and $t \geqslant 0$.

(d) $\left| \frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial t} \right| < \gamma$ for ε and $t \geqslant 0$, where $\gamma > 0$ is some constant defined by the transformation (4).

In this paper, the functions $\Phi_i^{-1}(\varepsilon_i, t)$ depend on $\varepsilon_i \in \mathbb{R}$ and t; therefore, the matrix $\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial \varepsilon}$ has the diagonal form. Some knowledge regarding the dynamics of the variable $\varepsilon(t)$ is needed to construct a control law. For this purpose, we take the total time derivative of the function $y(t)$ considering (5):

$$\dot{y} = \frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon} \dot{\varepsilon} + \frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial t}. \tag{6}$$

Due to (1) and $det \left(\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial \varepsilon} \right) \neq 0$, the expression (6) can be written as

$$\dot{\varepsilon} = \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon} \right)^{-1} \left[LAx + LBu + LG\phi + LDf - \frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial t} \right]. \tag{7}$$

In (7), $LDf(t)$ and $\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial t}$ are bounded values. Applying the change $\psi(t) = LDf(t) - \frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial t}$ yields $|\psi(t)| \leqslant \kappa$, where $\kappa = \|LD\| \bar{f} + \gamma$. In view of this change, (7) reduces to

$$\dot{\varepsilon} = \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon} \right)^{-1} \left[LAx + LBu + LG\phi + \psi \right]. \tag{8}$$

We recall the main result of [4] to solve the problem.

Theorem 1. *Let conditions* (a)–(d) *hold for the transformation* (4). *If there exists a control law* $u(t)$ *under which the solutions of* (8) *and* (1) *are bounded, then* $y(t) \in \mathcal{Y}$.

Remark 1. Conditions (a)–(d) affect the choice of the transformation function (4) only: they do not guarantee condition (3). For example, if the trajectories $\varepsilon(t)$ tend to infinity in a finite time, $y(t)$ will converge to a boundary of the pipe defined by (3). Therefore, after choosing the output transformation function based on conditions (a)–(d), it is required to obtain a control law that would ensure the boundedness of $\varepsilon(t)$. Theorem 1 reduces the original control problem (1) with the constraints (3) on the output $y(t)$ to an auxiliary control problem without any constraints on the variable $\varepsilon(t)$.

Now we find a control law $u(t)$ ensuring the boundedness of $\varepsilon(t)$. Consider a Lyapunov function of the form $V = \frac{1}{2}\varepsilon^{\mathrm{T}}\varepsilon$. According to (8),

$$\dot{V} = \varepsilon^{\mathrm{T}}\dot{\varepsilon} = \varepsilon^{\mathrm{T}} \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \Big[LAx + LBu + LG\phi + \psi\Big]. \tag{9}$$

Let the set Ω be an open Euclidean ball in the space $\mathbb{R}^m$, i.e.,

$$\Omega = \left\{\varepsilon \in \mathbb{R}^m : |\varepsilon| < \sqrt{2c},\ c > 0\right\}, \tag{10}$$

where c is a given positive number. The idea is to stabilize the trajectory $\varepsilon(t)$ in the set Ω. For $\varepsilon(t)$ to stay in Ω, it suffices to guarantee the negative derivative of the Lyapunov function for all ε outside the set Ω, i.e., $\dot{V} < 0\ \forall \varepsilon \notin \Omega$. (See the concept of input-to-state stability in the book [12].) The derivative of the Lyapunov function in (9) contains the matrix $\left(\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial \varepsilon}\right)^{-1}$, which is positive definite. In particular, if the plant (1) is one-dimensional, then $\left(\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial \varepsilon}\right)^{-1}$ is a positive scalar not affecting the sign of $\dot{V} < 0$. The control law can be constructed using the technique of linear matrix inequalities (LMIs) as described in [6]. Below, we present control design procedures for a particular case of one-dimensional systems and extend them to the general case of multidimensional ones.

Remark 2. The sector nonlinearity condition (2) can be written as the norm constraint $|\frac{\phi_i(z_i)}{z_i}| \leqslant \overline{k}_i = \max\{|k_{1i}|,\ |k_{2i}|\}$. Hence, it follows that $|\phi(z)| \leqslant \mu|z|$, where $\mu = \sqrt{q}\max_i\{\overline{k}_i\},\ i = 1, \ldots, q$. When passing from the original nonlinearity sector to the new one, the nonlinearity range will be expanded: $[k_{1i}, k_{2i}] \subset [-\mu, \mu]$. Then a control law will be designed for any nonlinearity in the new sector. In other words, this law can handle any nonlinearity in the original sector as well (i.e., it has higher "robustness" to the nonlinearity).

3.1. One-Dimensional Systems

We define a piecewise continuous control law of the form

$$u = -(LB)^{-1}[K\varepsilon + LAx + \mu \operatorname{sgn}(\varepsilon)\|LG\|\|C\||x|], \tag{11}$$

where $K \in \mathbb{R}$ is the desired gain and

$$\operatorname{sgn}(\varepsilon) = \begin{cases} 1, & \varepsilon \geqslant 0, \\ -1, & \varepsilon < 0. \end{cases}$$

The following result is true for the one-dimensional system.

Theorem 2. *Let conditions* (a)–(d) *hold for the transformation* (4), *and let* $\frac{\partial \Phi^{-1}(\varepsilon,t)}{\partial \varepsilon} > 0$ *for any* $\varepsilon \in \mathbb{R}$ *and* $t \geqslant 0$. *For given numbers* $c,\ \alpha > 0$, *assume the existence of a positive number* K *and positive coefficients* $\tau_i,\ i = 1, 2$, *such that*

$$\begin{aligned} &\begin{bmatrix} -K + \alpha + 0.5\tau_1 & 0.5 \\ \star & -\tau_2 \end{bmatrix} \leqslant 0, \\ &-c\tau_1 + \kappa^2\tau_2 \leqslant 0. \end{aligned} \tag{12}$$

Then the control law (11) *ensures the target condition* (3).

The proof of Theorem 2 is postponed to the Appendix.

3.2. Multidimensional Systems

Proposition 1. *Consider given block matrices*

$$M = \begin{bmatrix} Q & 0 \\ \star & Q \end{bmatrix} \succ 0, \quad N = \begin{bmatrix} N_{11} & N_{12} \\ \star & N_{22} \end{bmatrix} \prec 0,$$

where $Q, N_{11}, N_{12}, N_{21}, N_{22} \in \mathbb{R}^{n\times n}$ *are diagonal matrices. Then the matrix*

$$MN = \begin{bmatrix} QN_{11} & QN_{12} \\ \star & QN_{22} \end{bmatrix}$$

is negative definite.

Proposition 1 is used to prove the main result of this subsection, see below. The proof of Proposition 1 is provided in the Appendix.

We define a piecewise continuous control law of the form

$$u = -(LB)^{-1}\left[K\varepsilon + LAx + \bar{\sigma}\mu \mathrm{Sign}(\varepsilon)\left\|\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\right\| \|LG\|\|C\||x|\right], \tag{13}$$

where $K \in \mathbb{R}^{m\times m}$ is the gain matrix and $\bar{\sigma}$ is a constant determined by the transformation (4). In addition, $\mathrm{Sign}(\varepsilon) = col\{\mathrm{sgn}(\varepsilon_i), i = 1, \ldots, m\}$.

Substituting the control law (13) into (8) gives the closed loop system

$$\dot{\varepsilon} = \left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\left[-K\varepsilon - \bar{\sigma}\mu \mathrm{Sign}(\varepsilon)\left\|\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\right\| \|LG\|\|C\||x| + LG\phi + \psi\right]. \tag{14}$$

We arrive at the following result.

Theorem 3. *Let conditions* (a)–(d) *hold for the transformation* (4)*, and let* $0 \prec \frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon} \preceq \overline{\sigma}I$ *for any* $\varepsilon \in \mathbb{R}^m$ *and* $t \geqslant 0$*. For a given number* $c > 0$*, assume the existence of a diagonal matrix* $K \in \mathbb{R}^{m\times m}$ *and positive coefficients* τ_i*,* $i = 1, 2$*, such that*

$$\begin{aligned} &\begin{bmatrix} -K + [(0.5\tau_1 - \alpha)\sigma + \beta]I & 0.5I \\ \star & -\tau_2\sigma I \end{bmatrix} \preceq 0, \\ &-c\tau_1 + \kappa^2\tau_2 \leqslant 0 \end{aligned} \tag{15}$$

for any $\sigma \in (0, \overline{\sigma}]$ *and* $\alpha > 0$*,* $\beta > 0$*.*

Then the control law (13) *ensures the target condition* (3)*.*

The proof of Theorem 3 can be found in the Appendix.

Remark 3. The technique of LMIs and the S-procedure allow analyzing the input-to-state stability of the closed loop system under unknown bounded disturbances. Moreover, the gain for ε in (11), (13) can be obtained by finding an admissible solution of (12), (15), which is easy to do using widespread solvers for semidefinite programming problems (e.g., SEDUMI [14], SDPT3 [15], CSDP [16], and others.)

Remark 4. Obviously, the parameter c in (12), (15) is related to the radius of the open balls Ω attracting the system trajectories $\varepsilon(t)$: this radius equals $\sqrt{2c}$. Decreasing the value of c will reduce the radius of the ball and, in turn, the limit value of $\varepsilon(t)$. Therefore, a decrease in the limit value of $\varepsilon(t)$ will also reduce the fluctuation of the variable $y(t)$ in the set $\mathcal{Y}$ due to the exogenous disturbance $f(t)$.

4. NUMERICAL EXAMPLES

4.1. Example 1. One-Dimensional System

Consider an unstable plant of the form (1) with the following parameters:

$$A=\begin{bmatrix}0 & 1\\ 2 & -3\end{bmatrix},\ B=\begin{bmatrix}0\\ 1\end{bmatrix},\ G=\begin{bmatrix}0\\ 0.1\end{bmatrix},\ D=\begin{bmatrix}1\\ 1\end{bmatrix},\ L=\begin{bmatrix}2 & 1\end{bmatrix},\ C=\begin{bmatrix}1 & 2\end{bmatrix},$$
$$f(t)=0.1+\sin(3t)+0.5sat(d(t)),\ \ \phi(z)=\sin(z).$$

where $sat\{\cdot\}$ is the saturation function and $d(t)$ is a white noise with an intensity and sampling time of 0.1. Then $\bar{f}=1.6$ and $\mu=1$.

Let the function $\varepsilon(t)$ be specified as

$$\varepsilon(t)=\ln\left(\frac{y(t)-\underline{g}(t)}{\overline{g}(t)-y(t)}\right).$$

Consequently, the inverse function $\Phi^{-1}(\varepsilon(t),t)$ is given by

$$\Phi^{-1}(\varepsilon,t)=\frac{\overline{g}(t)e^{\varepsilon}+\underline{g}(t)}{e^{\varepsilon}+1}.$$

For all $\varepsilon\in\mathbb{R}$ and $t\geqslant 0$, we have

$$\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}=\frac{e^{\varepsilon}(\overline{g}(t)-\underline{g}(t))}{(e^{\varepsilon}+1)^2}>0$$

and

$$\left|\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial t}\right|=\left|\frac{\dot{\overline{g}}(t)e^{\varepsilon}+\dot{\underline{g}}(t)}{e^{\varepsilon}+1}\right|\leqslant\max\left\{\sup_{t\geqslant 0}|\dot{\overline{g}}(t)|,\sup_{t\geqslant 0}|\dot{\underline{g}}(t)|\right\}=\gamma. \tag{16}$$

Let the functions $\underline{g}(t)$ and $\overline{g}(t)$ be specified as

$$\overline{g}(t)=\begin{cases}-3\cos(t)+0.2, & t<2\pi,\\ \cos(t)+2.2, & t\geqslant 2\pi,\end{cases}$$
$$\underline{g}(t)=\begin{cases}3\cos(t)-0.2, & t<2\pi,\\ \cos(t)+1.8, & t\geqslant 2\pi.\end{cases}$$

The control law (11) can be written as

$$u=-(LB)^{-1}\left[K\ln\left(\frac{y-\underline{g}}{\overline{g}-y}\right)+LAx+\mu\mathrm{sgn}\left(\ln\left(\frac{y-\underline{g}}{\overline{g}-y}\right)\right)\|LG\|\|C\||x|\right].$$

In view of (16), we find $\gamma=3$ and $\kappa=8.6$. Inequality (12) was solved using YALMIP [17] with SEDUMI. For $c=100$ and $\alpha=2$, the result is $\tau_1=3.10$, $\tau_2=4.14$, and $K=6.54$; for $c=0.1$ and $\alpha=2$, the result is $\tau_1=45.47$, $\tau_2=0.04$, and $K=35.36$.

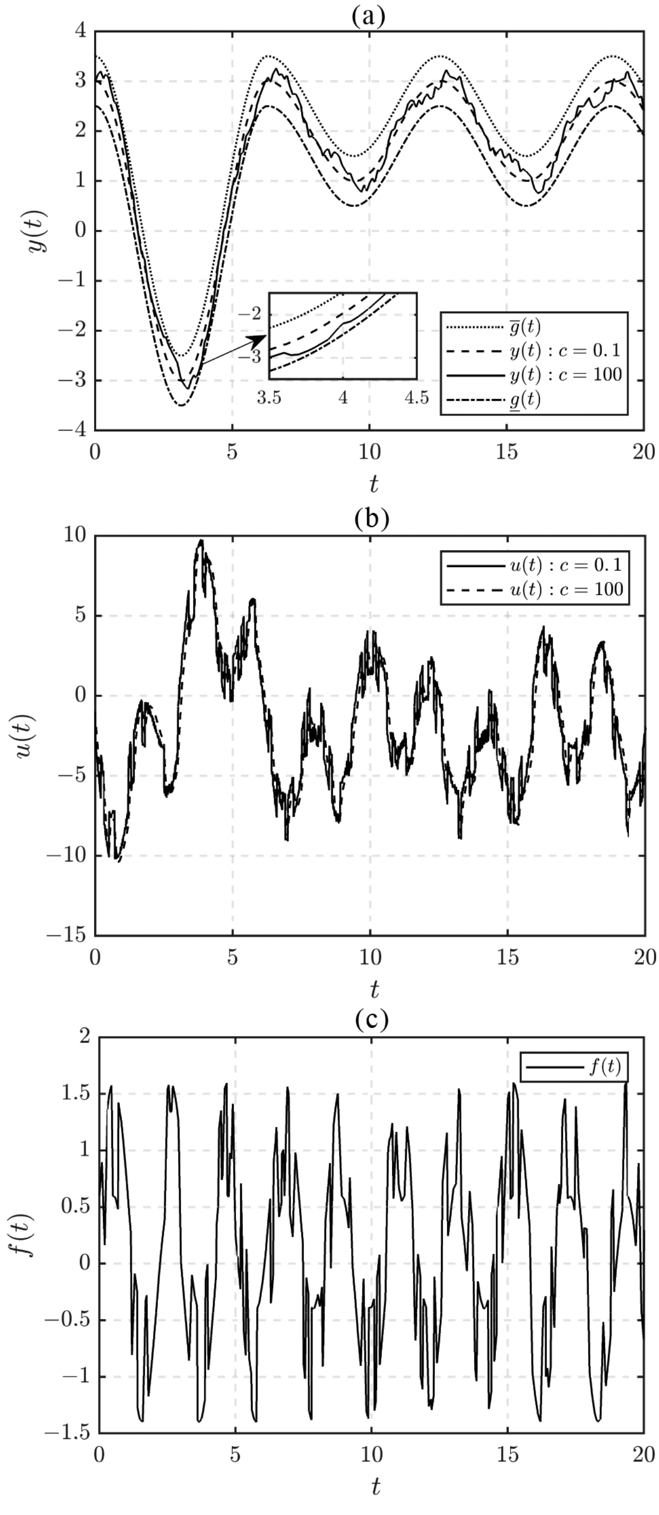

Fig. 2. Transients in the closed loop system for $c = 0.01$ and $c = 100$: (a) output $y(t)$, (b) control variable $u(t)$, and (c) disturbance $f(t)$.

The transients in $y(t)$, $u(t)$, and $f(t)$ for $x(0) = col\{1, 1\}$ are shown in Fig. 2. According to Fig. 2a, the output $y(t)$ never reaches the boundaries of the given set. Note also that the smaller the parameter c is, the better the effect of exogenous disturbances will be suppressed. The fluctuations of the control variable in Fig. 2b are explained by the disturbance $f(t)$ present in the system.

4.2. Example 2. Multidimensional System

We demonstrate control performance for an unstable double-input double-output plant with the following parameters:

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0.1 & 2 & -3 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 1 \\ 1 & 3 \end{bmatrix}, \quad G = \begin{bmatrix} 0.1 & 0 \\ 0 & 0.1 \\ 0,1 & 0.1 \end{bmatrix},$$

$$D = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}, \quad L = \begin{bmatrix} 2 & 1 & 1 \\ 1 & 2 & 1 \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 0 & 1 \\ 1 & 1 & 1 \end{bmatrix},$$

$$\phi(z) = col\{z_1 + \sin(z_1), \sin(z_2)\},$$

where $f(t)$ is the same as in Example 1. Then $\bar{f} = 1.6$ and $\mu = 2$.

Let $\Phi(y(t), t) = diag\{\Phi_1(y_1(t), t), \Phi_2(y_2(t), t)\}$, where Φ_i, $i = 1, 2$, are the same as in Example 1, i.e., $\Phi(y_i(t), t) = \ln\left(\frac{y_i(t) - \underline{g}_i(t)}{\overline{g}_i(t) - y_i(t)}\right)$. Consequently, $\Phi^{-1}(\varepsilon_i, t) = \frac{\overline{g}_i(t)e^{\varepsilon_i} + \underline{g}_i(t)}{e^{\varepsilon_i} + 1}$. For $\varepsilon \notin \Omega$, we have $0 \prec \frac{\partial\Phi^{-1}(\varepsilon, t)}{\partial\varepsilon} \preceq \overline{\sigma}I$, where $\overline{\sigma} = \frac{1}{4}\max_i\left[\sup_{t\geqslant 0}(\overline{g}_i(t) - \underline{g}_i(t))\right]$, $i = 1, 2$.

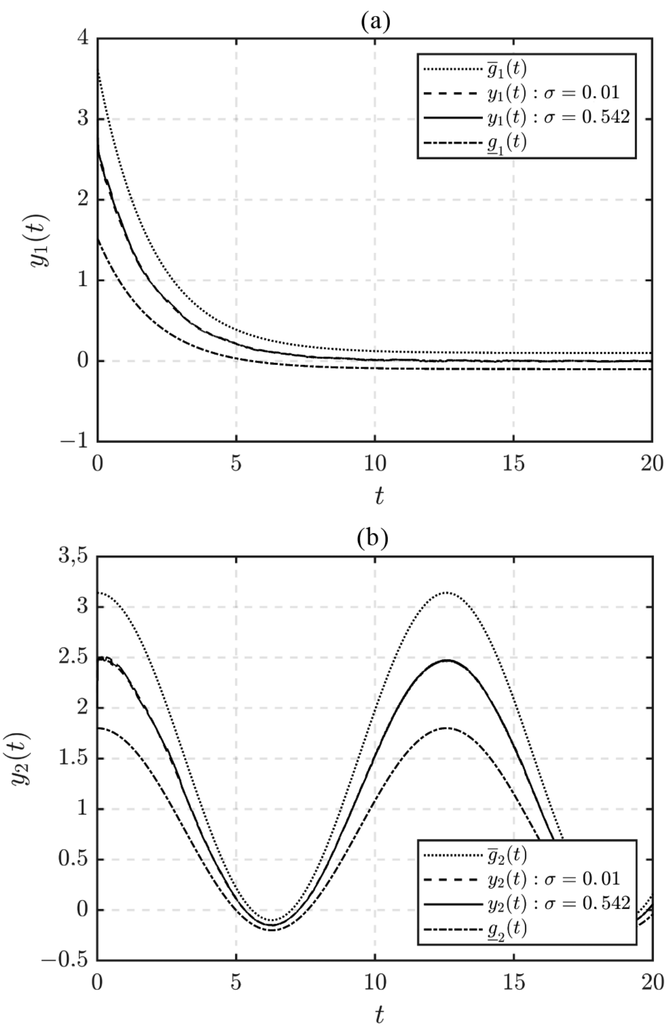

Fig. 3. Transients in the closed loop system for $c = 0.1$, $\sigma = 0.01$, and $\sigma = 0.542$: (a) $y_1(t)$ and (b) $y_2(t)$.

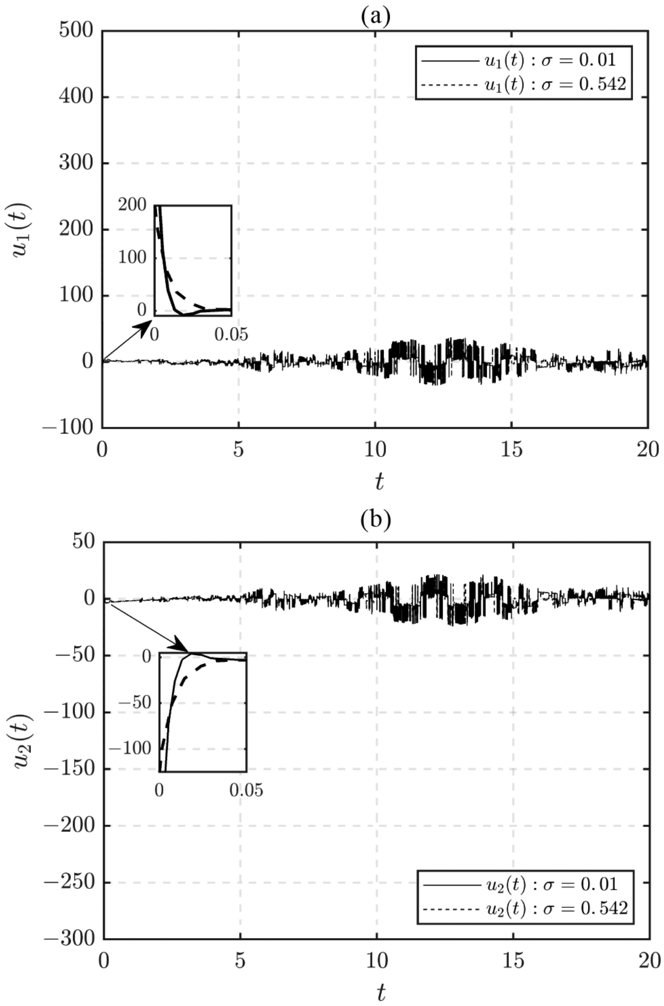

Fig. 4. Control variables in the closed loop system for $c = 0.1$, $\sigma = 0.01$, and $\sigma = 0.542$: (a) $u_1(t)$ and (b) $u_2(t)$.

Let the parameters of the constraint functions $\underline{g}(t)$ and $\overline{g}(t)$ be specified as

$$\begin{aligned}
\overline{g}_1(t) &= 3.52e^{-0.5t} + 0.1,\\
\underline{g}_1(t) &= 1.62e^{-0.5t} - 0.1,\\
\overline{g}_2(t) &= 1.62\cos(0.5t) + 1.52,\\
\underline{g}_2(t) &= \cos(0.5t) + 0.8.
\end{aligned}$$

In this case,

$$\gamma = \sqrt{2}\max_i \left\{ \sup_{t\geqslant 0} |\dot{\overline{g}}_i(t)|,\ \sup_{t\geqslant 0} |\dot{\underline{g}}_i(t)| \right\} = 2.49, \quad i = 1, 2$$

and

$$\kappa = 11.54, \quad \overline{\sigma} = 0.542.$$

For $c = 0.1$, we solve inequality (15) under some values of $\sigma \in (0,\ 0.542]$:

$\tau_1 = 527.72$, $\tau_2 = 0.3$, and $K = diag\{108.42,\ 108.42\}$ (for $\sigma = 0.01$) and

$\tau_1 = 92.33$, $\tau_2 = 0.05$, and $K = diag\{39.26,\ 39.26\}$ (for $\sigma = 0.542$).

Figure 3 presents the transients of $y_1(t)$ and $y_2(t)$ for $x(0) = col\left\{\frac{5}{3}, \frac{2}{3}, -1\right\}$ whereas Fig. 4 the control variables $u_1(t)$ and $u_2(t)$. According to Fig. 3, the outputs always belong to the given pipes.

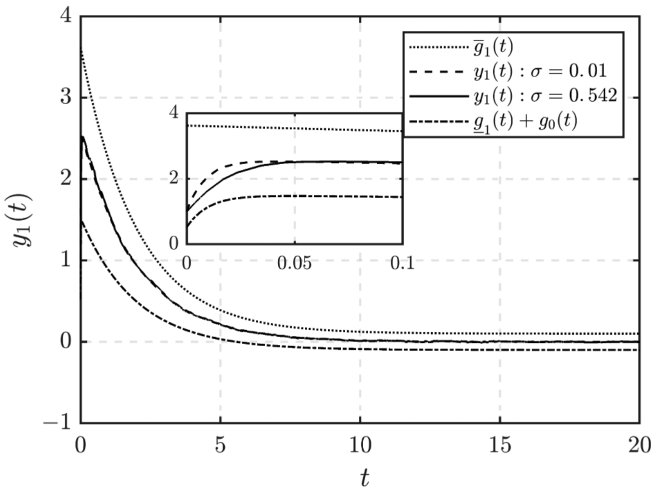

Fig. 5. Transients of $y_1(t)$ for $x(0) = col\{-\frac{1}{3}, \frac{2}{3}, 1\}$.

Remark 5. In the above examples, the initial values of the outputs are supposed to belong to a given set. However, if they are outside it, the control design method will fail: by the transformation (3), the outputs must be specified inside this set. This drawback can be eliminated by adding a fast exponentially decaying function to the limit functions, so the new limits will cover the initial conditions. Figure 5 shows the transients of $y_1(t)$ for $x(0) = col\left\{-\frac{1}{3}, \frac{2}{3}, 1\right\}$, i.e., $y_1(0) = 1$ falls beyond the initial set $\mathcal{Y}$. The function $g_0(t) = -e^{-100t}$ is added to the function $\underline{g}_1(t)$ so that the initial condition $y_1(0)$ is bounded from below by the new constraint function.

5. CONCLUSIONS

This paper has proposed a new method for stabilizing the output variables of nonlinear Lurie-type systems in given sets at any time instant. The method is based on a special output transformation and the technique of LMIs. With this transformation, the original problem with a constraint on the output variables is reduced to a problem without any constraints on the auxiliary variable. The control law for the new perturbing closed-loop system is designed using the Lyapunov function method in combination with the technique of LMIs. Simulation results in MATLAB/Simulink have illustrated the effectiveness of the method and confirmed the theoretical conclusions.

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APPENDIX

Proof of Theorem 2. Substituting (11) into (8) yields the closed loop system

$$\dot{\varepsilon} = \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \Big[- K\varepsilon - \mu \,\mathrm{sgn}(\varepsilon)\|LG\|\|C\||x| + LG\phi + \psi \Big]. \tag{A.1}$$

We choose a Lyapunov function of the form $V = \frac{1}{2}\varepsilon^2$. Its total time derivative along the solutions of (A.1) is given by

$$\dot{V} = \varepsilon\dot{\varepsilon} = \varepsilon\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\Big[-K\varepsilon - \mu\,\mathrm{sgn}(\varepsilon)\|LG\|\|C\||x| + LG\phi + \psi\Big]. \tag{A.2}$$

For $V \geqslant c$, we require the condition $\dot{V} \leqslant -2\alpha V$, where α is any known positive number, i.e., $\dot{V} < 0\ \forall\varepsilon \notin \Omega$. Due to $LG\phi \leqslant |LG\phi| \leqslant \mu\|LG\|\|C\||x|$ and the constraint $|\psi| \leqslant \kappa$, the above conditions can be written as

$$\begin{aligned} &(-K+\alpha)\varepsilon^2 + \varepsilon\psi \leqslant 0\ \forall\,(\varepsilon,\psi): \\ &0.5\varepsilon^2 \geqslant c,\ \ \psi^2 \leqslant \kappa^2. \end{aligned} \tag{A.3}$$

Denoting $z = col\{\varepsilon,\psi\}$, we represent (A.3) in the matrix form

$$\begin{gathered} z^{\mathrm{T}}\begin{bmatrix} -K+\alpha & 0.5 \\ \star & 0 \end{bmatrix} z \leqslant 0, \\ z^{\mathrm{T}}\begin{bmatrix} -0.5 & 0 \\ \star & 0 \end{bmatrix} z \leqslant -c,\ \ z^{\mathrm{T}}\begin{bmatrix} 0 & 0 \\ \star & 1 \end{bmatrix} z \leqslant \kappa^2. \end{gathered} \tag{A.4}$$

By the S-procedure [13], inequalities (A.4) hold under conditions (12). Hence, system (A.1) is input-to-state stable, and the variable $\varepsilon(t)$ is bounded. Owing to the transformation (4), the output $y(t)$ is also bounded, and the state vector $x(t)$ of system (1) possesses the same property accordingly. Therefore, the control variable $u(t)$ in (11) is bounded as well. Due to Theorem 1, the target condition (3) holds.

The proof of Theorem 2 is complete.

Proof of Proposition 1. Obviously, the matrix MN is symmetric. Let $\lambda_i, x_i,\ i = 1,\dots,2n$, be the eigenvalues and eigenvectors of the matrix MN, respectively. Then

$$x_i^{\mathrm{T}} NMNx_i = \lambda_i x_i^{\mathrm{T}} Nx_i.$$

Hence, the values λ_i can be expressed as

$$\lambda_i = \frac{x_i^{\mathrm{T}} NMNx_i}{x_i^{\mathrm{T}} Nx_i}.$$

Since $M \succ 0$ and $N = N^{\mathrm{T}} \prec 0$, we obtain $NMN \succ 0$, i.e., $x^{\mathrm{T}} NMNx > 0\ \forall x \neq 0$. In view of $x^{\mathrm{T}} Nx < 0\ \forall x \neq 0$, it follows that $\lambda_i < 0,\ i = 1,\dots,2n$. All eigenvalues of the symmetric matrix MN are negative, so the matrix MN is negative definite.

The proof of Proposition 1 is complete.

Proof of Theorem 3. We choose a Lyapunov function of the form $V = \frac{1}{2}\varepsilon^{\mathrm{T}}\varepsilon$. Its total time derivative along the solutions of (14) is given by

$$\begin{aligned} \dot{V} = \varepsilon^{\mathrm{T}}\dot{\varepsilon} = \varepsilon^{\mathrm{T}}\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\Big[&-K\varepsilon \\ &-\bar{\sigma}\mu\,\mathrm{Sign}(\varepsilon)\left\|\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1}\right\|\|LG\|\|C\||x| + LG\phi + \psi\Big]. \end{aligned} \tag{A.5}$$

Formula (A.5) can be written as

$$\dot{V} = \dot{V}_1 + \dot{V}_2, \tag{A.6}$$

where

$$\dot{V}_1 = -\varepsilon^{\mathrm{T}} \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} K\varepsilon + \varepsilon^{\mathrm{T}} \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \psi,$$

$$\dot{V}_2 = -\varepsilon^{\mathrm{T}} \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \mathrm{Sign}(\varepsilon)\bar{\sigma}\mu \left\| \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \right\| \|LG\| \|C\| |x| + \varepsilon^{\mathrm{T}} \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} LG\phi.$$

Considering $\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right) \leqslant \bar{\sigma} I$, we estimate $\dot{V}_2$ as

$$\dot{V}_2 \leqslant -\left(\sum_{i=1}^{v} |\varepsilon_i|\right) \bar{\sigma}^{-1}\bar{\sigma}\mu \left\| \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \right\| \|LG\| \|C\| |x| + \mu|\varepsilon| \left\| \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \right\| \|LG\| \|C\| |x| \leqslant 0.$$

Based on this inequality, the condition $\dot{V} \leqslant 0$ is equivalent to $\dot{V}_1 \leqslant 0$. In the case under study, $\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1}$ is a matrix and cannot be neglected when analyzing the sign definiteness of $\dot{V}$, in contrast to the previous section. For $V \geqslant c$, we require the condition $\dot{V} \leqslant -2\alpha V$, where α is any known positive number. Due to the constraints $|\psi| \leqslant \kappa$, the above conditions can be written as

$$\begin{gathered} -\varepsilon^{\mathrm{T}} \left[\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} K + \alpha I\right] \varepsilon + \varepsilon^{\mathrm{T}} \left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \psi \leqslant 0 \\ \forall\, (\varepsilon, \psi) : 0.5\varepsilon^{\mathrm{T}}\varepsilon \geqslant c, \psi^{\mathrm{T}}\psi \leqslant \kappa^2. \end{gathered} \tag{A.7}$$

Denoting $z = col\{\varepsilon, \psi\}$, $z \in \mathbb{R}^{2m}$, we represent (A.7) in the matrix form

$$\begin{gathered} z^{\mathrm{T}} \begin{bmatrix} -\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} K - \alpha I & 0.5\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \\ \star & 0 \end{bmatrix} z \leqslant 0, \\ z^{\mathrm{T}} \begin{bmatrix} -0.5I & 0 \\ \star & 0 \end{bmatrix} z \leqslant -c, \quad z^{\mathrm{T}} \begin{bmatrix} 0 & 0 \\ \star & I \end{bmatrix} z \leqslant \kappa^2. \end{gathered} \tag{A.8}$$

By the S-procedure, inequalities (A.8) hold under the conditions

$$\begin{gathered} \begin{bmatrix} -\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} K - \alpha I + 0.5\tau_1 I & 0.5\left(\frac{\partial \Phi^{-1}(\varepsilon, t)}{\partial \varepsilon}\right)^{-1} \\ \star & -\tau_2 I \end{bmatrix} \prec 0, \\ -c\tau_1 + \kappa^2\tau_2 \leqslant 0. \end{gathered} \tag{A.9}$$

The first inequality in (A.9) is equivalent to

$$\begin{bmatrix} \left(\dfrac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} & 0 \\ \star & \left(\dfrac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} \end{bmatrix} \times \begin{bmatrix} -K+(0.5\tau_1-\alpha)\dfrac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon} & 0.5I \\ \star & -\tau_2\dfrac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon} \end{bmatrix} \prec 0. \tag{A.10}$$

Since $\left(\frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon}\right)^{-1} \succ 0$, by Proposition 1, the latter inequality holds if, for any $\beta > 0$,

$$\begin{bmatrix} -K+(0.5\tau_1-\alpha)\dfrac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon} & 0.5I \\ \star & -\tau_2\dfrac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon} \end{bmatrix} \preceq -\beta I \prec 0. \tag{A.11}$$

Due to condition (A.7), it is required to ensure $\dot V < 0$ for all ε from the set $\{\varepsilon \in \mathbb{R}^m : |\varepsilon| \geqslant \sqrt{2c}, c > 0\}$. In addition, for all ε from this set, we have an interval uncertainty in (A.11) with $0 \prec \frac{\partial\Phi^{-1}(\varepsilon,t)}{\partial\varepsilon} \preceq \overline{\sigma} I$. Conditions (A.7) will be valid if the LMIs (15) are feasible for any $\sigma \in (0,\ \overline{\sigma}]$. Moreover, obviously, there always exist a matrix K and τ_1, $\tau_2 > 0$ such that (15) are feasible. Indeed, using Schur's complement lemma [13], we write (15) as

$$\begin{gathered} -\tau_2\sigma+\beta<0, \\ -K+[(0.5\tau_1-\alpha)\sigma+\beta]I+\frac{1}{\tau_2\sigma-\beta}I \preceq 0, \\ -c\tau_1+\kappa^2\tau_2 \leqslant 0, \\ \tau_1>0,\ \tau_2>0, \\ \alpha>0,\ \beta>0,\ 0<\sigma\leqslant\overline{\sigma}. \end{gathered} \tag{A.12}$$

For a given number $c > 0$ and fixed numbers σ, α, and β, inequalities (A.12) always have finite solutions (K, τ_1, τ_2). Thus, according to Theorem 1, the control law (13) with the gain matrix K satisfying (15) ensures the target condition (3).

The proof of Theorem 3 is complete.

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