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Inserting a Maximum-Mass Spacecraft into a Target Orbit Using a Limited-Thrust Engine with Releasing the Separable Part of Its Launch Vehicle into the Earth's Atmosphere

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Abstract—Space debris is an urgent problem of our time. This paper considers the idea of reducing near-Earth space debris by releasing the spent additional fuel tank (AFT) and the booster's central block (CB) into the Earth's atmosphere. The spacecraft transfer from a reference circular orbit of an artificial Earth satellite to a target elliptical orbit is optimized. The transition maneuvers are carried out using a booster with a high limited-thrust engine and the AFT. The second zonal harmonic of the Earth's gravitational field is taken into account. The optimal control problem is solved based on Pontryagin's maximum principle. Bulky derivatives are calculated using a specially developed numerical-analytical differentiation technique. The Pontryagin extremals obtained below are the next step in implementing the problem hierarchy methodology.

Keywords: spacecraft, space debris, additional fuel tank, booster, limited-thrust problem, transfer optimization, release into the Earth's atmosphere, numerical-analytical differentiation

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1. INTRODUCTION

Space debris is an urgent problem of our time. The approaches to solving this problem can be divided into two large groups: prevention and cleaning. A detailed literature review on the topic was presented in [1]. In addition, we mention the works [2, 3], covering the monitoring issues of man-made space debris, and [4–7], describing different means of capture and removal of large-size space debris.

This paper considers the idea of reducing near-Earth space debris (an approach from the prevention group) by releasing the spent parts of spacecraft launch vehicles in orbits touching the conditional boundary of the atmosphere at the transition maneuvers stage of insertion into the target orbit. The problem under study is to optimize the spacecraft transfer from a reference circular orbit of an artificial Earth satellite of a given radius and inclination to a target elliptical orbit using a booster with a high limited-thrust engine and an additional fuel tank (AFT), with releasing the AFT and the booster's central block (CB) into the Earth's atmosphere. The final ascent maneuver from the target orbit to the geostationary orbit (GEO) is considered within the simplified apsidal pulse scheme and performed using the satellite engine.

This work is the next step in implementing the problem hierarchy methodology, which consists in the sequential formalization and solution of a series of problems, where each previously obtained solution is used as an initial approximation in the next one. Initially, the simplest problem was solved in the apsidal pulse statement [8]: according to the results, for the optimal transfer trajectory with separation of the first impulse action and a limit of 1.5 km/s on the characteristic velocity of the final ascent maneuver, the cost of releasing the CB's spent parts (stages) turn out to be small. If the characteristic velocity of the final ascent maneuver is less than 1.47 km/s, the trajectory structure changes and the cost of releasing the AFT and CB into the atmosphere becomes significant. In the next step [9, 10], the problem was solved without assuming the apsidality of impulse actions. It was established that in the problem with a phase constraint on the maximum possible distance between the spacecraft and the Earth and an unlimited transfer time, the solution is apsidal and coincides with that obtained in the previous step. The need to solve the problem in a modified pulse statement (considering the release of the AFT and CB) [1], representing the third step of the problem hierarchy methodology, was due to the difficulty of a direct transition to the problem with a high limited thrust: the modified Newton's method did not converge when using the solution of the second-step problem as an initial approximation.

Well, this paper aims at constructing Pontryagin extremals in the problem with high limited thrust. The structure of this extremal (the sequence and approximate location of active segments on the trajectory) is known from the previous studies conducted in the pulse statement. The first series of transition maneuvers of a spacecraft to the target orbit is performed using fuel from the AFT. After exhausting this fuel, the spacecraft is in an orbit touching the conditional boundary of the Earth's atmosphere (with a perigee altitude of 100 km). On the passive flight segment, lasting 120 s, the AFT is released. The spacecraft returns to a safe orbit (with a perigee altitude of 200 km) by an additional activation of the spacecraft engine. This activation, as well as the subsequent ones, are performed using fuel from the CB's main tank.

After performing the second series of maneuvers, the spacecraft is in a target orbit from which the characteristic velocity of final ascent maneuvers to the GEO is bounded by a given value. According to the earlier studies [1, 8–10], the cost of releasing is small for the bi-elliptical final ascent scheme. In the target orbit, the satellite is separated from the CB. Due to the last engine activation on the residual fuel from the main tank in the neighborhood of the target orbit apogee, the CB is transferred to an orbit touching the conditional boundary of the Earth's atmosphere, and the satellite is transferred to the GEO using its engines.

In this paper, we consider two different but similar problem statements. Within the first one, by assumption, the tanks contain exactly as much fuel as is necessary to perform the corresponding maneuvers, the dry mass of the AFT and the mass of the CB's main tank are proportional to the mass of their fuel with a coefficient α, and the engine mass is proportional to the thrust-to-weight ratio with a coefficient β [11]. Within the second statement, the following mass characteristics of the booster are given: the dry masses of the AFT and the CB's main tank as well as limits on the masses of fuel in the AFT and the CB's main tank.

The objective functional in the problems below is the payload mass, i.e., the mass of the spacecraft remaining in the target orbit after undocking the CB.

The problems under consideration are formalized as optimal control problems for a set of dynamic systems. Based on the corresponding Pontryagin's maximum principle [12], they are reduced to multipoint boundary-value problems. The boundary-value problems of the maximum principle are solved numerically by the shooting method [13, 14]. Using the previous studies, we choose the computational schemes of the shooting method and good initial approximations of the required shooting parameters. The Cauchy problem is solved by the 8(7)th order Dorman–Prince method with automatic step selection [15]; the system of nonlinear equations, by Newton's method in the Isaev–Sonin modification [16] with the Fedorenko normalization [17] used in convergence conditions; the system of linear equations arising therein, by the Gaussian elimination technique with selection of the leading element by column and recalculation [18]. The bulky derivatives in the transversality conditions are considered through numerical-analytical differentiation [19].

2. PROBLEM STATEMENT

The transfer is considered in the rectangular Cartesian frame related to the Earth's center. The axis z of this frame is perpendicular to the equatorial plane and has south-to-north direction; the axis x lies in the equatorial plane and is directed along the node line of the initial circular orbit from the descending node to the ascending one; the axis y completes the frame to the right-hand triple.

The motion of the spacecraft's center of mass in the central Newtonian gravitational field in a vacuum is described by the system of differential equations

$$\begin{gathered}\dot{x}(t) = v_x(t), \quad \dot{y}(t) = v_y(t), \quad \dot{z}(t) = v_z(t), \\ \dot{v}_x(t) = -\frac{\mu x(t)}{r^3(t)} + \frac{P_x(t)}{m(t)}, \quad \dot{v}_y(t) = -\frac{\mu y(t)}{r^3(t)} + \frac{P_y(t)}{m(t)}, \\ \dot{v}_z(t) = -\frac{\mu z(t)}{r^3(t)} + \frac{P_z(t)}{m(t)}, \quad \dot{m}(t) = -\frac{P(t)}{c}\end{gathered} \tag{1}$$

with the following notations: $x(t)$, $y(t)$, and $z(t)$ are the coordinates of the spacecraft's center of mass at a time instant t; $r = \sqrt{x^2(t) + y^2(t) + z^2(t)}$ is the distance between the spacecraft and the Earth's center at a time instant t; $v_x(t)$, $v_y(t)$, and $v_z(t)$ are the velocity vector components of the spacecraft's center of mass at a time instant t; $M(0)$ is the spacecraft mass at the initial time instant; $M(t)$ is the spacecraft mass at a time instant t; $m(t) = M(t)/M(0)$ is the dimensionless mass of the spacecraft (used in calculations); $\vec{F}(t) = (F_x(t), F_y(t), F_z(t))$ is the jet thrust vector at a time instant t; $F(t) = |\vec{F}(t)| = \sqrt{F_x^2(t) + F_y^2(t) + F_z^2(t)}$ is the magnitude of the jet thrust vector; $\vec{P}(t) = (P_x(t), P_y(t), P_z(t)) = (F_x(t)/M(0), F_y(t)/M(0), F_z(t)/M(0))$ is the dimensionless jet thrust vector; $n = F_{\max}/(M(0)g_{\text{Ear}})$ is the initial thrust-to-weight ratio; $P(t) = \sqrt{P_x^2(t) + P_y^2(t) + P_z^2(t)}$ is the magnitude of the dimensionless jet thrust vector at a time instant t; $\mu = 398\,601.19\ \text{km}^3/\text{s}^2$ is the gravitational parameter of the Earth; $c = P_{\text{spe}}g_{\text{Ear}}$ is the jet velocity; P_{spe} is the specific thrust; finally, $g_{\text{Ear}} = 9.80665\ \text{m/s}^2$ is the gravitational acceleration at the Earth surface.

In addition to the central Newtonian gravitational field, we consider the motion of the spacecraft's center of mass in the gravitational field with the second zonal harmonic:

$$\begin{gathered}\dot{x}(t) = v_x(t), \quad \dot{y}(t) = v_y(t), \quad \dot{z}(t) = v_z(t), \\ \dot{v}_x(t) = -\frac{\mu x(t)}{r^3(t)} + \frac{3}{2}J_2\mu\frac{R_0^2}{r^5(t)}\left(\frac{5x(t)z^2(t)}{r^2(t)} - x(t)\right) + \frac{P_x(t)}{m(t)}, \\ \dot{v}_y(t) = -\frac{\mu y(t)}{r^3(t)} + \frac{3}{2}J_2\mu\frac{R_0^2}{r^5(t)}\left(\frac{5y(t)z^2(t)}{r^2(t)} - y(t)\right) + \frac{P_y(t)}{m(t)}, \\ \dot{v}_z(t) = -\frac{\mu z(t)}{r^3(t)} + \frac{3}{2}J_2\mu\frac{R_0^2}{r^5(t)}\left(\frac{5z^3(t)}{r^2(t)} - 3z(t)\right) + \frac{P_z(t)}{m(t)}, \\ \dot{m}(t) = -\frac{P(t)}{c},\end{gathered}$$

where $J_2 = 1082.636023 \times 10^{-6}$ is the coefficient of the second zonal harmonic.

Controls are supposed to be piecewise continuous functions:

$$P(t) = \sqrt{(P_x(t))^2 + (P_y(t))^2 + (P_z(t))^2} \leqslant P_{\max},$$

where $P_{\max} = g_{\text{Ear}} n m_0$ is the limit on the magnitude of the control thrust vector, n is the initial thrust-to-weight ratio of the spacecraft, and m_0 is the initial weight of the spacecraft.

At the initial time instant ($t = 0$) the spacecraft is in the circular reference orbit of a given radius R_0. Due to the chosen frame, the ascending node has the longitude $\Omega_0 = 0$, and the spacecraft's motion in the initial circular orbit can be formalized by the conditions

$$\begin{gathered} x(0)^2 + y(0)^2 + z(0)^2 = R_0^2, \quad x(0)C_{0x} + y(0)C_{0y} + z(0)C_{0z} = 0, \\ v_x(0) + \frac{v_0}{R_0}\left(y(0)\cos i_0 + z(0)\sin i_0\right) = 0, \;\; v_y(0) - \frac{v_0}{R_0} x(0)\cos i_0 = 0, \\ v_z(0) - \frac{v_0}{R_0} x(0)\sin i_0 = 0, \end{gathered} \tag{2}$$

where

$$C_{0x} = 0, \quad C_{0y} = -C_0 \sin i_0, \quad C_{0z} = C_0 \cos i_0, \quad C_0 = \sqrt{\mu R_0},$$
$$v_0 = \sqrt{\frac{\mu}{R_0}}, \quad R_0 = R_{\text{Ear}} + h_0.$$

In these formulas, C_{0x}, C_{0y}, and C_{0z} are the components of the kinetic momentum vector of the spacecraft relative to the Earth's center; C_0 is the magnitude of this vector; v_0 is the magnitude of the velocity vector in the reference orbit; R_0 is the radius of the reference orbit; $R_{\text{Ear}} = 6378.25$ km is the Earth's radius; finally, $h_0 = 200$ km is the altitude of the reference orbit above the Earth's surface.

The mass of the spacecraft is considered dimensionless and therefore equals 1 at the initial time instant:

$$m(0) = m_0 = 1. \tag{3}$$

The perigee radius $r_{\text{p}}(x, y, z, v_x, v_y, v_z)$ of an instantaneous elliptical orbit is a function of spacecraft coordinates and velocities and is calculated using the following formulas [20]:

$$\begin{gathered} r = \sqrt{x^2 + y^2 + z^2}, \quad V = \sqrt{v_x^2 + v_y^2 + v_z^2}, \\ \cos\varphi = \frac{x v_x + y v_y + z v_z}{rV}, \quad V_{\text{cir}}^2 = \frac{\mu}{r}, \\ e = \sqrt{\left[\left(\frac{V}{V_{\text{cir}}}\right)^2 - 1\right]^2 + \frac{r}{a}\left(\frac{V}{V_{\text{cir}}}\right)^2 \cos^2\varphi}, \\ a = \frac{r}{2 - \left(\dfrac{V}{V_{\text{cir}}}\right)^2}, \quad r_{\text{p}} = a(1 - e), \end{gathered} \tag{4}$$

where a is the semi-major axis; e is eccentricity; V_{cir} is the circular velocity at the distance r from the Earth's center; finally, φ is the angle between the radius vector $\vec{r} = (x,\, y,\, z)$ and the velocity vector $\overrightarrow{V} = (v_x,\, v_y,\, v_z)$.

In what follows, the perigee radius of the orbit is denoted by

$$r_{\text{p}}\,(\tau) := r_{\text{p}}\left(x(\tau), y(\tau), z(\tau), v_x(\tau), v_y(\tau), v_z(\tau)\right),$$

where τ is an arbitrary time instant.

After performing the first series of maneuvers at the time instant $\tau_{\text{rel1}}^{\text{AFT}}$, the spacecraft must be in the instantaneous Keplerian orbit touching the conditional boundary of the atmosphere. (In fact, the conditions of touching the atmosphere of the instantaneous Keplerian orbit do not ensure touching the orbit's atmosphere in the Earth's real field; by assumption in this paper, such conditions are sufficient for the rapid elimination of space debris.) The orbit perigee altitude is lowered to 100 km (the conditional boundary of the atmosphere) by activating the CB engine on the residual fuel from the AFT. At the time instant $\tau_{\text{rel1}}^{\text{AFT}}$ we have the conditions

$$r_{\text{p}}(\tau_{\text{rel1-}}^{\text{AFT}}) = R_{\text{Ear}} + 100\,\text{km}, \tag{5}$$

$$x(\tau_{\text{rel1+}}^{\text{AFT}}) - x(\tau_{\text{rel1-}}^{\text{AFT}}) = 0, \quad y(\tau_{\text{rel1+}}^{\text{AFT}}) - y(\tau_{\text{rel1-}}^{\text{AFT}}) = 0,$$

$$z(\tau_{\text{rel1+}}^{\text{AFT}}) - z(\tau_{\text{rel1-}}^{\text{AFT}}) = 0, \quad v_x(\tau_{\text{rel1+}}^{\text{AFT}}) - v_x(\tau_{\text{rel1-}}^{\text{AFT}}) = 0,$$

$$v_y(\tau_{\text{rel1+}}^{\text{AFT}}) - v_y(\tau_{\text{rel1-}}^{\text{AFT}}) = 0, \quad v_z(\tau_{\text{rel1+}}^{\text{AFT}}) - v_z(\tau_{\text{rel1-}}^{\text{AFT}}) = 0,$$

$$\tau_{\text{rel1+}}^{\text{AFT}} - \tau_{\text{rel1-}}^{\text{AFT}} = 0.$$

After the spacecraft reaches the AFT release orbit, the passive segment $[\tau_{\text{rel1}}^{\text{AFT}}, \tau_{\text{rel2}}^{\text{AFT}}]$ begins (AFT release). On this segment, the mass is neglected in the system of differential equations. By assumption, undocking the AFT takes a given time:

$$\tau_{\text{rel2+}}^{\text{AFT}} - \tau_{\text{rel1-}}^{\text{AFT}} = 120 \text{ s}.$$

We consider two different but similar problem statements. Within the first one, by assumption, the tanks contain exactly as much fuel as is necessary to perform the corresponding maneuvers, the dry mass of the AFT and the mass of the CB's main tank are proportional to the mass of their fuel with a coefficient α, and the engine mass (including the additional CB structures) is proportional to the thrust-to-weight ratio with a coefficient β. Within the second statement, the mass characteristics of the booster are given.

The mass of the spacecraft after AFT release is calculated as follows:

$$m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) = m\left(\tau_{\text{rel1-}}^{\text{AFT}}\right) - \alpha\left(m_0 - m\left(\tau_{\text{rel1-}}^{\text{AFT}}\right)\right) \tag{6}$$

(the first problem statement) and

$$m\left(\tau_{\text{rel2}}^{\text{AFT}}\right) = m\left(\tau_{\text{rel1}}^{\text{AFT}}\right) - m^{\text{AFT}} \tag{7}$$

(the second problem statement), where m^{AFT} is the given dry dimensionless mass of the AFT. (This value can be supposed to include the mass of unreduced fuel residue.)

The constraint on the AFT fuel mass in the second problem statement has the form

$$m_0 - m\left(\tau_{\text{rel1}}^{\text{AFT}}\right) \leqslant m_{\text{fuel}}^{\text{AFT}}. \tag{8}$$

After releasing the AFT, the spacecraft performs a transition maneuver to the safe orbit. This maneuver ends at the time instant τ_{safe}. Fuel from the main tank is used to perform it. As before, the pericenter radius $r_{\text{p}}(\cdot)$ is a function of the coordinates and components of the spacecraft velocity vector (4). At the time instant τ_{safe} we have the conditions

$$r_{\text{p}}\left(\tau_{\text{safe-}}\right) = R_{\text{Ear}} + 200\,\text{km}, \tag{9}$$

$$x(\tau_{\text{safe+}}) - x(\tau_{\text{safe-}}) = 0, \; y(\tau_{\text{safe+}}) - y(\tau_{\text{safe-}}) = 0, \; z(\tau_{\text{safe+}}) - z(\tau_{\text{safe-}}) = 0,$$

$$v_x(\tau_{\text{safe+}}) - v_x(\tau_{\text{safe-}}) = 0, \; v_y(\tau_{\text{safe+}}) - v_y(\tau_{\text{safe-}}) = 0, \; v_z(\tau_{\text{safe+}}) - v_z(\tau_{\text{safe-}}) = 0,$$

$$\tau_{\text{safe+}} - \tau_{\text{safe-}} = 0.$$

After reaching the safe orbit, the second series of maneuvers begins to transfer the spacecraft to the target orbit. In the target orbit, the satellite is undocked from the CB. The payload mass of the satellite remaining in the target orbit has to be optimized:

$$m_{\mathrm{p}} = m\left(\tau_{\text{tar-}}\right) - m\left(\tau_{\text{tar+}}\right) \to \max,$$

where $m\left(\tau_{\text{tar-}}\right)$ is the mass of the spacecraft in the target orbit before undocking the satellite; $m\left(\tau_{\text{tar+}}\right)$ is the CB mass in the target orbit after undocking the satellite. The satellite moves to the GEO using its engines. By assumption, the characteristic velocity of the final ascent maneuver from the target orbit to the GEO is limited by a given value Δv^* and the apsidal line of the target orbit lies in the equatorial plane, i.e., the z-component of the Laplace vector is zero. At the time instant τ_{tar} we have the conditions

$$\begin{gathered}
\Delta v_{\text{fa}}(\tau_{\text{tar-}}) := \Delta v_{\text{fa}}(x(\tau_{\text{tar-}}), y(\tau_{\text{tar-}}), z(\tau_{\text{tar-}}), v_x(\tau_{\text{tar-}}), v_y(\tau_{\text{tar-}}), v_z(\tau_{\text{tar-}})) \leqslant \Delta v^*, \\
\mathcal{A}\left(\tau_{\text{tar-}}\right) := C_y\left(\tau_{\text{tar-}}\right) v_x\left(\tau_{\text{tar-}}\right) - C_x\left(\tau_{\text{tar-}}\right) v_y\left(\tau_{\text{tar-}}\right) - \frac{\mu z\left(\tau_{\text{tar-}}\right)}{r\left(\tau_{\text{tar-}}\right)} = 0, \\
x(\tau_{\text{tar+}}) - x(\tau_{\text{tar-}}) = 0,\ y(\tau_{\text{tar+}}) - y(\tau_{\text{tar-}}) = 0,\ z(\tau_{\text{tar+}}) - z(\tau_{\text{tar-}}) = 0, \\
v_x(\tau_{\text{tar+}}) - v_x(\tau_{\text{tar-}}) = 0,\ v_y(\tau_{\text{tar+}}) - v_y(\tau_{\text{tar-}}) = 0,\ v_z(\tau_{\text{tar+}}) - v_z(\tau_{\text{tar-}}) = 0, \\
\tau_{\text{tar+}} - \tau_{\text{tar-}} = 0,
\end{gathered} \tag{10}$$

where τ_{tar} is the time instant of reaching the target orbit; $C_x\left(\tau_{\text{tar-}}\right)$, $C_y\left(\tau_{\text{tar-}}\right)$, and $C_z\left(\tau_{\text{tar-}}\right)$ are the components of the kinetic momentum vector of the spacecraft orbital motion at the time instant $\tau_{\text{tar-}}$.

Note that the characteristic velocity of the final ascent maneuvers of the satellite from the target orbit to the GEO is considered by the simplified scheme. (It is considered within the central Newtonian field, with apsidal impulse actions for maneuvering only, orbit rotation by the second impulse action only, and the first accelerating and the last setting impulse actions not changing the orbit plane.) Used together, these conditions simplify the problem statement very significantly. The value $R_{\max}$ (the distance to the Earth) is chosen, on the one hand, to be large enough, and on the other hand, to be appropriate for neglecting the influence of other bodies of the Solar System. First of all, the matter concerns the influence of the Moon and the Sun; for example, consideration of the Moon will even avoid activation of the engine at the remote point [21]. Of course, this influence can be modeled in the next steps of the problem hierarchy methodology. In this paper, the influence of other bodies is omitted. The final ascent of the satellite is implemented using three impulse actions:

$$\Delta v_{\text{fa}}\left(\tau_{\text{tar}}\right) = \Delta v_{\text{fa1}}\left(\tau_{\text{tar}}\right) + \Delta v_{\text{fa2}}\left(\tau_{\text{tar}}\right) + \Delta v_{\text{fa3}}\left(\tau_{\text{tar}}\right).$$

The first impulse action $\Delta v_{\text{fa1}}\left(\tau_{\text{tar}}\right)$ is applied at the perigee of the target orbit; without changing the inclination, it raises the apogee to the maximum possible distance $R_{\max}$ of the spacecraft from the Earth:

$$\begin{gathered}
\Delta v_{\text{fa1}}\left(\tau_{\text{tar}}\right) = \sqrt{V_{\text{tar p}}^2 + V_{\text{1p}}^2 - 2V_{\text{tar p}}V_{\text{1p}}}, \\
V_{\text{tar p}} = \sqrt{\frac{2\mu R_{\text{tar a}}}{R_{\text{tar p}}(R_{\text{tar a}} + R_{\text{tar p}})}}, \quad V_{\text{1p}} = \sqrt{\frac{2\mu R_{\max}}{R_{\text{tar p}}(R_{\max} + R_{\text{tar p}})}},
\end{gathered} \tag{11}$$

where $R_{\text{tar p}}$ is the perigee radius of the target orbit and $V_{\text{tar p}}$ is the velocity at the perigee of the target orbit.

The second impulse action $\Delta v_{\text{fa2}}(\tau_{\text{tar}})$ is applied at the apogee; it increases the perigee to the GEO radius R_{GEO} and decreases the inclination to zero:

$$\Delta v_{\text{fa2}}(\tau_{\text{tar}}) = \sqrt{V_{1\text{a}}^2 + V_{2\text{a}}^2 - 2V_{1\text{a}}V_{2\text{a}}\cos i_{\text{tar}}},$$
$$V_{1\text{a}} = \sqrt{\frac{2\mu R_{\text{tar p}}}{R_{\max}(R_{\max}+R_{\text{tar p}})}}, \quad V_{2\text{a}} = \sqrt{\frac{2\mu R_{\text{GEO}}}{R_{\max}(R_{\max}+R_{\text{GEO}})}}, \tag{12}$$

where i_{tar} is the inclination angle of the target orbit to the equatorial plane. At the time instant of passing the apogee, this value can be calculated as

$$\cos i_{\text{tar}} = \frac{\sqrt{v_x^2(\tau_{\text{tar a}}) + v_y^2(\tau_{\text{tar a}})}}{\sqrt{v_x^2(\tau_{\text{tar a}}) + v_y^2(\tau_{\text{tar a}}) + v_z^2(\tau_{\text{tar a}})}}. \tag{13}$$

The third impulse action $\Delta v_{\text{fa3}}(\tau_{\text{tar}})$ is applied at the perigee; without changing the inclination, it reduces the apogee to the GEO radius, thus moving the satellite to a non-predetermined point in the GEO:

$$\Delta v_{\text{fa3}}(\cdot) = V_{2\text{p}} - v_{\text{GEO}},$$
$$V_{2\text{p}} = \sqrt{\frac{2\mu R_{\max}}{R_{\text{GEO}}(R_{\max}+R_{\text{GEO}})}}, \quad v_{\text{GEO}} = \sqrt{\frac{\mu}{R_{\text{GEO}}}}. \tag{14}$$

Note that Δv_{fa3} is actually a constant (depends on the given value R_{GEO} and the problem parameter $R_{\max}$).

After undocking the satellite, the CB maneuver continues. Due to additional activation of the engine, the perigee altitude of the CB's orbit is lowered to 100 km (the conditional boundary of the atmosphere):

$$r_{\text{p}}(T) = R_{\text{Ear}} + 100\,\text{km}. \tag{15}$$

At the final time instant T all the fuel contained in the CB's main tank is exhausted. Within the first problem statement, the tanks are filled with exactly as much fuel as is necessary to perform the corresponding maneuvers, the dry mass of the CB's main tank is proportional to the mass of fuel contained in it with the coefficient α, and the engine mass is proportional to the thrust-to-weight ratio with the coefficient β [11]. Therefore, we obtain

$$m(T) - \alpha m_{\text{fuel}} - \beta n = 0,$$
$$m_{\text{fuel}} = \left(m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) - m(\tau_{\text{tar-}})\right) + \left(m(\tau_{\text{tar+}}) - m(T)\right). \tag{16}$$

Within the second problem statement, the CB's dry mass and the fuel constraint in the CB's main tank are given. In this case,

$$m(T) - m^{\text{CB}} = 0, \tag{17}$$
$$\left(m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) - m(\tau_{\text{tar-}})\right) + \left(m(\tau_{\text{tar+}}) - m(T)\right) \leqslant m_{\text{fuel}}^{\text{CB}},$$

where m^{CB} is the given dry dimensionless mass of the CB (the engine and additional structures) and $m_{\text{fuel}}^{\text{CB}}$ is the maximum dimensionless mass of fuel that can be filled into the CB's main tank.

Note that the spacecraft coordinates and velocities are continuous at all time instants. In addition, the problem under consideration has another peculiarity: its objective functional is a function of the phase variables at an intermediate time instant.

3. PONTRYAGIN'S MAXIMUM PRINCIPLE

The problem under consideration is an optimal control problem with intermediate conditions. It can be solved using Pontryagin's maximum principle [12].

In the case of the central Newtonian gravitational field, the Pontryagin function has the form

$$H = p_x v_x + p_y v_y + p_z v_z + p_m \left(-\frac{P}{c}\right)$$
$$+ p_{vx}\left(-\frac{\mu x}{r^3} + \frac{P_x}{m}\right) + p_{vy}\left(-\frac{\mu y}{r^3} + \frac{P_y}{m}\right) + p_{vz}\left(-\frac{\mu z}{r^3} + \frac{P_z}{m}\right);$$

in the problems with the second zonal harmonic, the form

$$H = p_x v_x + p_y v_y + p_z v_z + p_m \left(-\frac{P}{c}\right) + p_{vx}\left(-\frac{\mu x}{r^3} + \frac{3}{2}J_2\mu\frac{R_0^2}{r^5}\left(\frac{5xz^2}{r^2} - x\right) + \frac{P_x}{m}\right)$$
$$+ p_{vy}\left(-\frac{\mu y(t)}{r^3(t)} + \frac{3}{2}J_2\mu\frac{R_0^2}{r^5(t)}\left(\frac{5yz^2}{r^2} - y\right) + \frac{P_y}{m}\right) + p_{vz}\left(-\frac{\mu z}{r^3} + \frac{3}{2}J_2\mu\frac{R_0^2}{r^5}\left(\frac{5z^3}{r^2} - 3z\right) + \frac{P_z}{m}\right).$$

For the first problem statement, the terminant is given by

$$l = l_0 + l_{\text{rel1}} + l_{\text{rel2}} + l_{\text{safe}} + l_{\text{tar}} + l_T - \lambda_0\left(m\left(\tau_{\text{tar}}-\right) - m\left(\tau_{\text{tar}}+\right)\right),$$

where

$$l_0 = \lambda_{R0}\left(x(0)^2 + y(0)^2 + z(0)^2 - R_0^2\right) + \lambda_{C0}\left(x(0)C_{0x} + y(0)C_{0y} + z(0)C_{0z}\right)$$
$$+\lambda_{vx0}\left(v_x(0) + \frac{v_0}{R_0}\left(y(0)\cos i_0 + z(0)\sin i_0\right)\right) + \lambda_{vy0}\left(v_y(0) - \frac{v_0}{R_0}x(0)\cos i_0\right)$$
$$+\lambda_{vz0}\left(v_z(0) - \frac{v_0}{R_0}x(0)\sin i_0\right) + \lambda_{m0}\left(m(0) - m_0\right),$$

$$l_{\text{rel1}} = \sum_{\xi=(x,y,z,v_x,v_y,v_z)} \lambda_{\xi\text{rel1}}\left(\xi\left(\tau_{\text{rel1+}}^{\text{AFT}}\right) - \xi\left(\tau_{\text{rel1-}}^{\text{AFT}}\right)\right) + \lambda_{\tau\text{rel1}}\left(\tau_{\text{rel1+}}^{\text{AFT}} - \tau_{\text{rel1-}}^{\text{AFT}}\right)$$
$$+ \lambda_{\text{rel1}}\left(r_{\text{p}}\left(\tau_{\text{rel1-}}^{\text{AFT}}\right) - R_{\text{Ear}} - 100\right),$$

$$l_{\text{rel2}} = \lambda_{m\tau}\left(m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) - m\left(\tau_{\text{rel1-}}^{\text{AFT}}\right) + \alpha\left(m_0 - m\left(\tau_{\text{rel1-}}^{\text{AFT}}\right)\right)\right)$$
$$+ \lambda_\tau\left(\tau_{\text{rel2+}}^{\text{AFT}} - \tau_{\text{rel1-}}^{\text{AFT}} - 120\right) + \lambda_{\tau\text{rel2}}\left(\tau_{\text{rel2+}}^{\text{AFT}} - \tau_{\text{rel2-}}^{\text{AFT}}\right),$$

$$l_{\text{safe}} = \sum_{\xi=(x,y,z,v_x,v_y,v_z)} \lambda_{\xi\text{safe}}\left(\xi\left(\tau_{\text{safe+}}\right) - \xi\left(\tau_{\text{safe-}}\right)\right) + \lambda_{\tau\text{safe}}\left(\tau_{\text{safe+}} - \tau_{\text{safe-}}\right)$$
$$+ \lambda_{\text{safe}}\left(r_{\text{p}}\left(\tau_{\text{safe-}}\right) - R_{\text{Ear}} - 200\right),$$

$$l_{\text{tar}} = \sum_{\xi=(x,y,z,v_x,v_y,v_z)} \lambda_{\xi\text{tar}}\left(\xi\left(\tau_{\text{tar+}}\right) - \xi\left(\tau_{\text{tar-}}\right)\right) + \lambda_{\tau\text{tar}}\left(\tau_{\text{tar+}} - \tau_{\text{tar-}}\right)$$
$$+ \lambda_{\text{tar}}\left(C_y(\tau_{\text{tar-}})v_x(\tau_{\text{tar-}}) - C_x(\tau_{\text{tar-}})v_y(\tau_{\text{tar-}}) - \frac{\mu z(\tau_{\text{tar-}})}{r(\tau_{\text{tar-}})}\right)$$
$$+ \lambda_{\text{fa}}(\Delta v_{\text{fa}}(x(\tau_{\text{tar-}}), y(\tau_{\text{tar-}}), z(\tau_{\text{tar-}}), v_x(\tau_{\text{tar-}}), v_y(\tau_{\text{tar-}}), v_z(\tau_{\text{tar-}})) - \Delta v^*),$$

$$l_T = \lambda_T\left(r_{\text{p}}(T) - R_{\text{Ear}} - 100\right)$$
$$+ \lambda_{mT}\left(m(T) - \alpha\left(\left(m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) - m\left(\tau_{\text{tar-}}\right)\right) + \left(m\left(\tau_{\text{tar+}}\right) - m(T)\right)\right) - \beta n\right).$$

For the second problem statement, l_{rel2} and l_T are given by

$$l_{\text{rel2}} = \lambda_{m\tau 1}\left(m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) - m\left(\tau_{\text{rel1-}}^{\text{AFT}}\right) + m^{\text{AFT}}\right) + \lambda_{m\tau 2}\left(m_0 - m\left(\tau_{\text{rel1-}}^{\text{AFT}}\right) - m_{\text{fuel}}^{\text{AFT}}\right)$$
$$+ \lambda_\tau\left(\tau_{\text{rel2+}}^{\text{AFT}} - \tau_{\text{rel1-}}^{\text{AFT}} - 120\right) + \lambda_{\tau\text{rel2}}\left(\tau_{\text{rel2+}}^{\text{AFT}} - \tau_{\text{rel2-}}^{\text{AFT}}\right),$$

$$l_T = \lambda_T\left(r_{\text{p}}(T) - R_{\text{Ear}} - 100\right) + \lambda_{mT1}\left(m(T) - m^{\text{CB}}\right)$$
$$+ \lambda_{mT2}\left(\left(m\left(\tau_{\text{rel2+}}^{\text{AFT}}\right) - m\left(\tau_{\text{tar-}}\right)\right) + \left(m\left(\tau_{\text{tar+}}\right) - m(T)\right) - m_{\text{fuel}}^{\text{CB}}\right).$$

Here, $p_x(\cdot)$, $p_y(\cdot)$, $p_z(\cdot)$, $p_{v_x}(\cdot)$, $p_{v_y}(\cdot)$, $p_{v_z}(\cdot)$, and $p_m(\cdot)$ are the conjugate variables (the functional Lagrange multipliers) at each of the trajectory segments; λ_0, λ_{R0}, λ_{C0}, λ_{vx0}, λ_{vy0}, λ_{vz0}, λ_{m0}, $\lambda_{\xi\text{rel1}}$, $\lambda_{\xi\text{safe}}$, $\lambda_{\xi\text{tar}}$ ($\xi = x, y, z, v_x, v_y, v_z$), $\lambda_{\tau\text{rel1}}$, λ_{rel1}, $\lambda_{m\tau}$, λ_τ, $\lambda_{\tau\text{rel2}}$, $\lambda_{\tau\text{safe}}$, λ_{safe}, $\lambda_{\tau\text{tar}}$, λ_{tar}, λ_{fa}, λ_T, λ_{mT}, $\lambda_{m\tau 1}$, $\lambda_{m\tau 2}$, λ_{mT1}, and λ_{mT2} are the numerical Lagrange multipliers.

Note that the term corresponding to the differential equation $\dot{m} = -\frac{P}{c}$ is absent in the Pontryagin function on the segment $[\tau_{\text{rel1}}^{\text{AFT}}, \tau_{\text{rel2}}^{\text{AFT}}]$.

The stationarity conditions with respect to the phase variables (the Euler–Lagrange equations) have the form

$$\dot{p}_x = \frac{\mu}{r^3}\left[p_{vx} - \frac{3x}{r^2}\left(xp_{vx} + yp_{vy} + zp_{vz}\right)\right],$$
$$\dot{p}_y = \frac{\mu}{r^3}\left[p_{vy} - \frac{3y}{r^2}\left(xp_{vx} + yp_{vy} + zp_{vz}\right)\right],$$
$$\dot{p}_z = \frac{\mu}{r^3}\left[p_{vz} - \frac{3z}{r^2}\left(xp_{vx} + yp_{vy} + zp_{vz}\right)\right],$$
$$\dot{p}_{vx} = -p_x, \quad \dot{p}_{vy} = -p_y, \quad \dot{p}_{vz} = -p_z,$$
$$\dot{p}_m = \frac{P_x p_{vx} + P_y p_{vy} + P_z p_{vz}}{m^2}.$$

In the case of a transfer in a gravitational field with the second zonal harmonic, the Euler–Lagrange equations are not presented explicitly here. Their right-hand sides were calculated using numerical-analytical differentiation [19].

Due to their bulkiness, we formally write the transversality conditions as

$$p_\xi(0) = \frac{\partial l}{\partial \xi(0)}, \quad p_\xi(T) = -\frac{\partial l}{\partial \xi(T)},$$
$$p_\xi(\beta_+) = \frac{\partial l}{\partial \xi(\beta_+)}, \quad p_\xi(\beta_-) = -\frac{\partial l}{\partial \xi(\beta_-)},$$
$$\xi = x, y, z, v_x, v_y, v_z, \quad \beta = \tau_{\text{rel1}}^{\text{AFT}}, \tau_{\text{rel2}}^{\text{AFT}}, \tau_{\text{safe}}, \tau_{\text{tar}}.$$

The transversality conditions at the initial time instant imply

$$p_x(0) = 2\lambda_{R0}x(0) + \lambda_{C0}C_{0x} - \frac{v_0}{R_0}\left(p_{vy}(0)\cos i_0 + p_{vz}(0)\sin i_0\right),$$
$$p_y(0) = 2\lambda_{R0}y(0) + \lambda_{C0}C_{0y} + \frac{v_0}{R_0}p_{vx}(0)\cos i_0, \quad (18)$$
$$p_z(0) = 2\lambda_{R0}z(0) + \lambda_{C0}C_{0z} + \frac{v_0}{R_0}p_{vy}(0)\sin i_0.$$

At the time instants $\tau_{\text{rel1}}^{\text{AFT}}$ and τ_{safe}, these conditions yield

$$p_\xi\left(\gamma_-\right) - p_\xi\left(\gamma_+\right) + \lambda_i \frac{\partial r_{\text{p}}\left(\gamma_-\right)}{\partial \xi\left(\gamma_-\right)} = 0,$$
$$\xi = x, y, z, v_x, v_y, v_z, \quad \gamma = \tau_{\text{rel1}}^{\text{AFT}}, \tau_{\text{safe}}, \quad i = \text{rel1}, \text{safe}. \tag{19}$$

Finally, at the time instant τ_{tar}, from the transversality conditions it follows that

$$p_\xi\left(\tau_{\text{tar-}}\right) - p_\xi\left(\tau_{\text{tar+}}\right) + \lambda_{\text{fa}} \frac{\Delta v_{\text{fa}}\left(\tau_{\text{tar-}}\right)}{\partial \xi\left(\tau_{\text{tar-}}\right)} + \lambda_{\text{tar}} \frac{\partial \mathcal{A}\left(\tau_{\text{tar-}}\right)}{\partial \xi\left(\tau_{\text{tar-}}\right)} = 0,$$
$$\xi = x, y, z, v_x, v_y, v_z. \tag{20}$$

The derivatives of the functions $r_{\text{p}}(\cdot)$, $\Delta v_{\text{fa}}(\cdot)$, and $\mathcal{A}(\cdot)$ (see (19), (20), and the transversality conditions at the final time instant T) are calculated using numerical-analytical differentiation.

In the first problem statement, the transversality conditions with respect to the variable m at the time instants $\tau_{\text{rel1}}^{\text{AFT}}$, $\tau_{\text{rel2}}^{\text{AFT}}$, and T imply the equality

$$(1+\alpha) p_m(\tau_{\text{rel2+}}^{\text{AFT}}) - p_m(\tau_{\text{rel1-}}^{\text{AFT}}) - \alpha p_m(T) = 0. \tag{21}$$

Let us prove this equality. The transversality conditions with respect to the variable m at the time instants $\tau_{\text{rel1}}^{\text{AFT}}$, $\tau_{\text{rel2}}^{\text{AFT}}$, and T have the form

$$p_m(\tau_{\text{rel1-}}^{\text{AFT}}) = -\frac{\partial l}{\partial m(\tau_{\text{rel1-}}^{\text{AFT}})} = \lambda_{m\tau}(1+\alpha),$$
$$p_m(\tau_{\text{rel2+}}^{\text{AFT}}) = \frac{\partial l}{\partial m(\tau_{\text{rel2+}}^{\text{AFT}})} = \lambda_{m\tau} - \alpha \lambda_{mT},$$
$$p_m(T) = -\frac{\partial l}{\partial m(T)} = -\lambda_{mT}(1+\alpha).$$

We obtain the following chain of equalities:

$$p_m(\tau_{\text{rel2+}}^{\text{AFT}}) - \lambda_{m\tau} + \alpha \lambda_{mT} = 0, \quad \lambda_{m\tau} = \frac{p_m(\tau_{\text{rel1-}}^{\text{AFT}})}{1+\alpha}, \quad \lambda_{mT} = -\frac{p_m(T)}{1+\alpha}$$
$$\Rightarrow p_m(\tau_{\text{rel2+}}^{\text{AFT}}) - \frac{p_m(\tau_{\text{rel1-}}^{\text{AFT}})}{1+\alpha} - \alpha \frac{p_m(T)}{1+\alpha} = 0$$
$$\Rightarrow (1+\alpha) p_m(\tau_{\text{rel2+}}^{\text{AFT}}) - p_m(\tau_{\text{rel1-}}^{\text{AFT}}) - \alpha p_m(T) = 0.$$

In the second problem statement, the transversality conditions with respect to the variable m at the time instants $\tau_{\text{rel1}}^{\text{AFT}}$ and $\tau_{\text{rel2}}^{\text{AFT}}$ imply the equality

$$p_m(\tau_{\text{rel2+}}^{\text{AFT}}) - p_m(\tau_{\text{rel1-}}^{\text{AFT}}) = \lambda_{mT2} - \lambda_{m\tau 2}. \tag{22}$$

Let us prove this equality. The transversality conditions with respect to the variable m at the time instants $\tau_{\text{rel1}}^{\text{AFT}}$ and $\tau_{\text{rel2}}^{\text{AFT}}$ have the form

$$p_m(\tau_{\text{rel2+}}^{\text{AFT}}) = \lambda_{m\tau 1} + \lambda_{mT2}, \quad p_m(\tau_{\text{rel1-}}^{\text{AFT}}) = \lambda_{m\tau 1} + \lambda_{m\tau 2}.$$

Subtracting the second equality from the first one yields (22).

The transversality conditions with respect to the variable m at the time instant τ_{tar} imply the continuity of the conjugate variable:

$$p_m(\tau_{\text{tar+}}) = p_m(\tau_{\text{tar-}}). \tag{23}$$

Indeed, $p_m(\tau_{\text{tar-}}) = -\alpha\lambda_{mT}$ and $p_m(\tau_{\text{tar+}}) = -\alpha\lambda_{mT}$ (the first problem statement) and $p_m(\tau_{\text{tar-}}) = -\alpha\lambda_{mT2}$ and $p_m(\tau_{\text{tar+}}) = -\alpha\lambda_{mT2}$ (the second problem statement).

At the initial time instant the stationarity condition is absent. The stationarity conditions at the time instants $\tau_{\text{rel1}}^{\text{AFT}}$ and $\tau_{\text{rel2}}^{\text{AFT}}$ imply $H(\tau_{\text{rel2+}}^{\text{AFT}}) = H(\tau_{\text{rel1-}}^{\text{AFT}})$. At the time instants τ_{safe} and τ_{tar}, the Pontryagin function is continuous: $H(\tau_{\text{safe+}}) = H(\tau_{\text{safe-}})$ and $H(\tau_{\text{tar+}}) = H(\tau_{\text{tar-}})$. The stationarity condition at a time instant T (unknown in advance) has the form $H(T) = 0$.

Let $\vec{e}$ be a unit vector. Then the optimality conditions with respect to the control actions P_x, P_y, and P_z have the form

$$\vec{P} = P\vec{e}, \quad \vec{e} = (\cos\alpha, \cos\beta, \cos\gamma),$$
$$P_x = P\cos\alpha, \quad P_y = P\cos\beta, \quad P_z = P\cos\gamma,$$

$$\begin{aligned}\vec{P}_{opt} &= \underset{0\leqslant P\leqslant P_{\max}}{arg\,abs\,max}\left[\frac{p_{vx}P_x + p_{vy}P_y + p_{vz}P_z}{m} - \frac{p_m}{c}P\right]\\ &= \underset{0\leqslant P\leqslant P_{\max}}{arg\,abs\,max}\left[\frac{p_{vx}P\cos\alpha + p_{vy}P\cos\beta + p_{vz}P\cos\gamma}{m} - \frac{p_m}{c}P\right]\\ &= \underset{0\leqslant P\leqslant P_{\max}}{arg\,abs\,max}\left[P\left(\frac{p_{vx}\cos\alpha + p_{vy}\cos\beta + p_{vz}\cos\gamma}{m} - \frac{p_m}{c}\right)\right],\end{aligned}$$

where $\cos\alpha$, $\cos\beta$, and $\cos\gamma$ are the direction cosines.

If $\left(\frac{p_{vx}\cos\alpha + p_{vy}\cos\beta + p_{vz}\cos\gamma}{m} - \frac{p_m}{c}\right) > 0$, then $P_{opt} = P_{\max}$; in the case $\left(\frac{p_{vx}\cos\alpha + p_{vy}\cos\beta + p_{vz}\cos\gamma}{m} - \frac{p_m}{c}\right) < 0$, $P_{opt} = 0$. Thus,

$$P_{opt} = \begin{cases} P_{\max}, & \chi > 0\\ 0, & \chi < 0,\end{cases}$$

where $\chi \equiv \frac{\rho}{m} - \frac{p_m}{c}$ is the switching function.

Note that $p_{vx}\cos\alpha + p_{vy}\cos\beta + p_{vz}\cos\gamma$ is the inner product of the vectors $\vec{p}_v = (p_{vx}, p_{vy}, p_{vz})$ and $\vec{e} = (\cos\alpha, \cos\beta, \cos\gamma)$. It achieves maximum for the codirectional vectors $\vec{p}_v$ and $\vec{e}$:

$$\cos\alpha_{opt} = \frac{p_{vx}}{\rho}, \quad \cos\beta_{opt} = \frac{p_{vy}}{\rho}, \quad \cos\gamma_{opt} = \frac{p_{vz}}{\rho},$$

where $\rho = \sqrt{p_{vx}^2 + p_{vy}^2 + p_{vz}^2}$. Thus, due to the optimal direction of the thrust vector, we find

$$(P_x)_{opt} = P_{opt}\frac{p_{vx}}{\rho}, \quad (P_y)_{opt} = P_{opt}\frac{p_{vy}}{\rho}, \quad (P_z)_{opt} = P_{opt}\frac{p_{vz}}{\rho}.$$

Special control regimes, potentially possible in the problems under study, are not considered in this paper.

For the first problem statement, the complementary slackness and nonnegativity conditions have the form

$$\begin{gathered}\lambda_{\text{fa}}(\Delta v_{\text{fa}}(\tau_{\text{tar-}}) - \Delta v^*) = 0,\\ \lambda_0 \geqslant 0, \quad \lambda_{\text{fa}} \geqslant 0.\end{gathered} \tag{24}$$

For the second problem statement, in addition to (24), we get the following complementary slackness and nonnegativity conditions:

$$\lambda_{m\tau 2}\left(m_0 - m\left(\tau^{\text{AFT}}_{\text{rel1-}}\right) - m^{\text{AFT}}_{\text{fuel}}\right) = 0,$$

$$\lambda_{mT2}\left(\left(m\left(\tau^{\text{AFT}}_{\text{rel2+}}\right) - m\left(\tau_{\text{tar-}}\right)\right) + \left(m\left(\tau_{\text{tar+}}\right) - m(T)\right) - m^{\text{CB}}_{\text{fuel}}\right) = 0,$$

$$\lambda_{m\tau 2} \geqslant 0, \quad \lambda_{mT2} \geqslant 0. \quad (25)$$

The normalization condition is

$$p^2_{vx}(0) + p^2_{vy}(0) + p^2_{vz}(0) = 1. \quad (26)$$

4. TRAJECTORY STRUCTURE AND NUMERICAL RESULTS

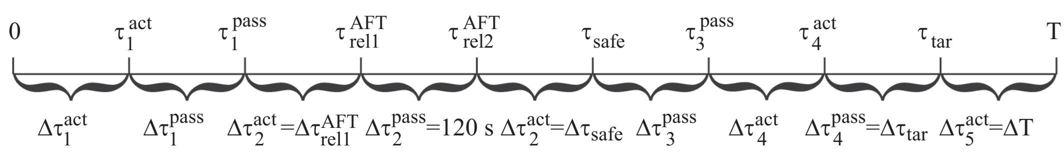
The trajectory structure is determined based on the previous studies [1, 8–10]. The main advantage of the given-structure trajectory approach is that it yields Pontryagin extremals: under a "good" computational scheme for the shooting method and a "good" initial approximation, the modified Newton's method converges in a few iterations.

Other possible methods for solving the corresponding boundary-value problems were tested as well, but without success (the Pontryagin extremals were not constructed).

The initial approximation to the values of the phase and conjugate variables in the shooting parameter vector is chosen using the solution obtained previously in the modified pulse statement [1] in accordance with [12]: at the thrust activation time instants they correspond to the values of the phase and conjugate variables before the impulse action; at the thrust deactivation time instants, to the values of the phase and conjugate variables after the impulse action. The duration of active segments is estimated based on fuel consumption for a given engine activation; the duration of passive segments is equal to the corresponding duration of passive segments between the impulse actions. First, the problem with high limited thrust is solved in the first statement with $n = 10$. Then the parameter continuation method is applied for the thrust-to-weight ratio to obtain the solution for $n = 0.1$. The transition from the first problem statement to the second one is also carried out using the parameter continuation method: the corresponding equations from the first and second problem statements are multiplied by $(1 - \gamma)$ and γ, respectively, where $\gamma \in [0, 1]$.

Let us describe the computational scheme of the shooting method (see the figure). The shooting parameter vector consists of the following components:

- the numerical Lagrange multipliers λ_{R0}, λ_{C0}, λ_{rel1}, λ_{safe}, λ_{tar}, λ_{fa}, and λ_T; in the second problem statement, also the numerical Lagrange multiplier λ_{mT2};
- the angular position of the spacecraft in the reference circular orbit, φ_0, and the values of the four conjugate variables at the initial time instant, $p_{vx}(0)$, $p_{vy}(0)$, $p_{vz}(0)$, and $p_m(0)$; (The coordinates and velocities of the spacecraft at the initial time instant are calculated by the angular position; the values $p_x(0)$, $p_y(0)$, and $p_z(0)$ of the conjugate variables are calculated using (18). By the condition, $m(0) = 1$ and, therefore, $m(0)$ is not included in the shooting parameter vector. Thus, we obtain the starting point for solving the Cauchy problem.)
- the duration of the first active segment, $\Delta\tau^{\text{act}}_1$;



The computational scheme of the shooting method.

- the coordinates and velocities as well as the values of the conjugate variables after engine deactivation, $x(\tau_{1+}^{\rm act})$, $y(\tau_{1+}^{\rm act})$, $z(\tau_{1+}^{\rm act})$, $v_x(\tau_{1+}^{\rm act})$, $v_y(\tau_{1+}^{\rm act})$, $v_z(\tau_{1+}^{\rm act})$, $p_x(\tau_{1+}^{\rm act})$, $p_y(\tau_{1+}^{\rm act})$, $p_z(\tau_{1+}^{\rm act})$, $p_{vx}(\tau_{1+}^{\rm act})$, $p_{vy}(\tau_{1+}^{\rm act})$, and $p_{vz}(\tau_{1+}^{\rm act})$;
- the duration of the first passive segment, $\Delta\tau_1^{\rm pass}$;
- the coordinates and velocities as well as the values of the conjugate variables after engine activation, $x(\tau_{1+}^{\rm pass})$, $y(\tau_{1+}^{\rm pass})$, $z(\tau_{1+}^{\rm pass})$, $v_x(\tau_{1+}^{\rm pass})$, $v_y(\tau_{1+}^{\rm pass})$, $v_z(\tau_{1+}^{\rm pass})$, $p_x(\tau_{1+}^{\rm pass})$, $p_y(\tau_{1+}^{\rm pass})$, $p_z(\tau_{1+}^{\rm pass})$, $p_{vx}(\tau_{1+}^{\rm pass})$, $p_{vy}(\tau_{1+}^{\rm pass})$, and $p_{vz}(\tau_{1+}^{\rm pass})$;
- the duration of the second active segment, $\Delta\tau_2^{\rm act} = \Delta\tau_{\rm rel1}^{\rm AFT}$, where the perigee altitude of the spacecraft orbit is lowered to the conditional boundary of the atmosphere;
- the coordinates and velocities as well as the values of the conjugate variables after engine deactivation, $x(\tau_{\rm rel1+}^{\rm AFT})$, $y(\tau_{\rm rel1+}^{\rm AFT})$, $z(\tau_{\rm rel1+}^{\rm AFT})$, $v_x(\tau_{\rm rel1+}^{\rm AFT})$, $v_y(\tau_{\rm rel1+}^{\rm AFT})$, $v_z(\tau_{\rm rel1+}^{\rm AFT})$, $p_x(\tau_{\rm rel1+}^{\rm AFT})$, $p_y(\tau_{\rm rel1+}^{\rm AFT})$, $p_z(\tau_{\rm rel1+}^{\rm AFT})$, $p_{vx}(\tau_{\rm rel1+}^{\rm AFT})$, $p_{vy}(\tau_{\rm rel1+}^{\rm AFT})$, and $p_{vz}(\tau_{\rm rel1+}^{\rm AFT})$; (The duration of the second passive segment (AFT release), $\Delta\tau_2^{\rm pass}$, is a parameter of the problem (120 s), being therefore not included in the shooting parameter vector.)
- the coordinates and velocities as well as the values of the conjugate variables after engine activation, $x(\tau_{\rm rel2+}^{\rm AFT})$, $y(\tau_{\rm rel2+}^{\rm AFT})$, $z(\tau_{\rm rel2+}^{\rm AFT})$, $v_x(\tau_{\rm rel2+}^{\rm AFT})$, $v_y(\tau_{\rm rel2+}^{\rm AFT})$, $v_z(\tau_{\rm rel2+}^{\rm AFT})$, $p_x(\tau_{\rm rel2+}^{\rm AFT})$, $p_y(\tau_{\rm rel2+}^{\rm AFT})$, $p_z(\tau_{\rm rel2+}^{\rm AFT})$, $p_{vx}(\tau_{\rm rel2+}^{\rm AFT})$, $p_{vy}(\tau_{\rm rel2+}^{\rm AFT})$, and $p_{vz}(\tau_{\rm rel2+}^{\rm AFT})$, including the conjugate variable corresponding to the mass, $p_m(\tau_{\rm rel2}^{\rm AFT}) = p_m(\tau_{\rm rel2+}^{\rm AFT})$; (The mass of the spacecraft after AFT release is not included in the shooting parameter vector and is calculated by formulas (6) (the first problem statement) and (7) (the second problem statement).)
- the duration of the third active segment, $\Delta\tau_3^{\rm act} = \Delta\tau_{\rm safe}$, where the perigee altitude of the spacecraft orbit is increased to 200 km;
- the coordinates and velocities as well as the values of the conjugate variables after engine deactivation, $x(\tau_{\rm safe+})$, $y(\tau_{\rm safe+})$, $z(\tau_{\rm safe+})$, $v_x(\tau_{\rm safe+})$, $v_y(\tau_{\rm safe+})$, $v_z(\tau_{\rm safe+})$, $p_x(\tau_{\rm safe+})$, $p_y(\tau_{\rm safe+})$, $p_z(\tau_{\rm safe+})$, $p_{vx}(\tau_{\rm safe+})$, $p_{vy}(\tau_{\rm safe+})$, and $p_{vz}(\tau_{\rm safe+})$;
- the duration of the third passive segment, $\Delta\tau_3^{\rm pass}$;
- the coordinates and velocities as well as the values of the conjugate variables after engine activation, $x(\tau_{3+}^{\rm pass})$, $y(\tau_{3+}^{\rm pass})$, $z(\tau_{3+}^{\rm pass})$, $v_x(\tau_{3+}^{\rm pass})$, $v_y(\tau_{3+}^{\rm pass})$, $v_z(\tau_{3+}^{\rm pass})$, $p_x(\tau_{3+}^{\rm pass})$, $p_y(\tau_{3+}^{\rm pass})$, $p_z(\tau_{3+}^{\rm pass})$, $p_{vx}(\tau_{3+}^{\rm pass})$, $p_{vy}(\tau_{3+}^{\rm pass})$, and $p_{vz}(\tau_{3+}^{\rm pass})$;
- the duration of the fourth active segment, $\Delta\tau_4^{\rm act}$, at the end of which the spacecraft reaches the target orbit;
- the coordinates and velocities as well as the values of the conjugate variables after engine deactivation, $x(\tau_{4+}^{\rm act})$, $y(\tau_{4+}^{\rm act})$, $z(\tau_{4+}^{\rm act})$, $v_x(\tau_{4+}^{\rm act})$, $v_y(\tau_{4+}^{\rm act})$, $v_z(\tau_{4+}^{\rm act})$, $p_x(\tau_{4+}^{\rm act})$, $p_y(\tau_{4+}^{\rm act})$, $p_z(\tau_{4+}^{\rm act})$, $p_{vx}(\tau_{4+}^{\rm act})$, $p_{vy}(\tau_{4+}^{\rm act})$, and $p_{vz}(\tau_{4+}^{\rm act})$;
- the duration of the fourth passive segment, $\Delta\tau_4^{\rm pass} = \Delta\tau_{\rm tar}$, where the spacecraft moves in the target orbit; (For convenience of calculations, $\tau_{\rm tar}$ is the last engine activation point for the CB release instead of the first point in the target orbit of the spacecraft; this is possible because the points are connected by the passive segment.)
- the coordinates and velocities as well as the values of the conjugate variables after engine activation, $x(\tau_{\rm tar+})$, $y(\tau_{\rm tar+})$, $z(\tau_{\rm tar+})$, $v_x(\tau_{\rm tar+})$, $v_y(\tau_{\rm tar+})$, $v_z(\tau_{\rm tar+})$, $p_x(\tau_{\rm tar+})$, $p_y(\tau_{\rm tar+})$, $p_z(\tau_{\rm tar+})$, $p_{vx}(\tau_{\rm tar+})$, $p_{vy}(\tau_{\rm tar+})$, and $p_{vz}(\tau_{\rm tar+})$, and the CB mass $m(\tau_{\rm tar+})$ after undocking the satellite; (The conjugate variable $p_m(\tau_{\rm tar+})$ is not included in the shooting parameter vector since p_m is continuous at the point $\tau_{\rm tar}$ by (23).)
- the duration of the fifth active segment, $\Delta\tau_5^{\rm act} = \Delta T$, where the CB perigee is lowered to 100 km (the conditional boundary of the atmosphere).

The residual vector function includes the following elements:

- the twelve continuity conditions of the phase and conjugate variables at the time instant $\tau_1^{\rm act}$;

- the twelve continuity conditions of the phase and conjugate variables at the time instant τ_1^{pass};
- the six continuity conditions of the phase variables and the six implications of the transversality conditions at the time instant $\tau_{\text{rel1}}^{\text{AFT}}$ (19);
- the twelve continuity conditions of the phase and conjugate variables at the time instant $\tau_{\text{rel2}}^{\text{AFT}}$;
- the six continuity conditions of the phase variables and the six implications of the transversality conditions at the time instant τ_{safe} (19);
- the twelve continuity conditions of the phase and conjugate variables at the time instant τ_3^{pass};
- the twelve continuity conditions of the phase and conjugate variables at the time instant τ_4^{act};
- the six continuity conditions of the phase variables and the six implications of the transversality conditions at the time instant τ_{tar} (20);
- the zero value of the z-component of the Laplace vector (the second condition from (10));
- the condition of exhausting all fuel from the CB's main tank at the final time instant: formulas (16) and (17) in the first and second problem statements, respectively;
- the complementary slackness condition: a given value of the final ascent impulse from the target orbit to the geostationary orbit (the first condition from (24));
- the three conditions on the perigee of the spacecraft orbit at the time instants $\tau_{\text{rel1}}^{\text{AFT}}$, τ_{safe}, and T: formulas (5), (9), and (15), respectively;
- the four conditions on the switching function: $\chi(\tau_{1-}^{\text{act}}) = 0$, $\chi(\tau_{1+}^{\text{pass}}) = 0$, $\chi(\tau_{3+}^{\text{pass}}) = 0$, and $\chi(\tau_{4-}^{\text{act}}) = 0$;
- the six transversality conditions at the final time instant T;
- the implication of the transversality conditions with respect to the variable m: formulas (21) and (22) in the first and second problem statements, respectively (in the latter case, with $\lambda_{m\tau 2} = 0$);
- the three implications of the stationarity conditions, $H(\tau_{\text{rel2+}}^{\text{AFT}}) = H(\tau_{\text{rel1-}}^{\text{AFT}})$, $H(\tau_{\text{safe+}}) = H(\tau_{\text{safe-}})$, and $H(\tau_{\text{tar+}}) = H(\tau_{\text{tar-}})$;
- the stationarity condition at the final time instant, $H(T) = 0$;
- the normalization condition (26);
- in the second problem statement, also the second complementary slackness condition from (25). (The first complementary slackness condition from (25) is not included in the residual vector function, and the corresponding inequality (8) is verified after solving the problem: the strict inequality holds on the Pontryagin extremal, which matches the case $\lambda_{m\tau 2} = 0$.)

Thus, the first problem statement has one hundred and eighteen shooting parameters and one hundred and eighteen residuals; the second problem statement, one hundred and nineteen shooting parameters and one hundred and nineteen residuals. In other words, in both statements, the number of unknown parameters coincides with the number of equations for their determination.

In the Appendix, we present the Pontryagin extremal in the second problem statement with the second zonal harmonic and $n = 0.1$, $P_{\text{spe}} = 350$ s, $i_0 = 0.9$ rad, $\Delta v^* = 1.5$ km/s, $m(0) = 1$ ($M(0) = 22\,500$ kg), the AFT's dry mass $m^{\text{AFT}} = 0.052$ (which corresponds to the mass 1170 kg), the CB's dry mass $m^{\text{CB}} = 0.0635556$ (which corresponds to the mass 1430 kg), the maximum AFT fuel mass $m_{\text{fuel}}^{\text{AFT}} = 0.6488889$ (which corresponds to the mass 14 600 kg), the maximum CB fuel mass $m_{\text{fuel}}^{\text{CB}} = 0.2266667$ (which corresponds to the mass 5100 kg), and $R_{\max} = 280\,000$ km.

5. CONCLUSIONS

One result of the previous studies in the pulse statement was the possibility of releasing the additional fuel tank and the booster of the central block into the Earth's atmosphere at low cost. The same result has been confirmed above in the case of spacecraft with a high limited-thrust engine.

As it has turned out, the solution of the spacecraft transfer problem with a high limited-thrust engine is, to some extent, close to that obtained in the pulse statement. In the case under consideration, the method for passing from the latter solution to the former one [12] is effective: the Pontryagin extremal has been constructed.

The problems in the first and second statements (with optimizable design and fixed mass characteristics) have been successfully included in the parametric family. The transition from the solution of the first problem (with the chosen constants $\alpha = 0.08$ and $\beta = 0.01$) to the second one (see the extremal in the Appendix) has been effectively implemented the parameter continuation method.

The difference between the extremal considering the second zonal harmonic and the one without such consideration is small in the sense of convergence of Newton's method. (This method converges in 11 iterations.)

The numerical-analytical differentiation technique has demonstrated its effectiveness and advantages (the simplified program code and the reduced probability of programming errors).

The problem hierarchy methodology has been adopted to solve the original problem, choose an appropriate computational scheme and a good initial approximation, and cope with the difficulties of numerical solution due to the complexity and bulkiness of the problem statement, thus demonstrating its effectiveness. The candidate's dissertation by A.S. Samokhin [22] is another complete example of the effective application of this methodology. Note that when the maximum possible distance $R_{\max}$ of the spacecraft to the Earth increases, the problem statement changes: it will be necessary to consider the influence of the Moon's gravitational field [21]; moreover, the resulting problem will be in the next steps of the problem hierarchy methodology.

APPENDIX

The Pontryagin Extremal in the Second Problem Statement with the Second Zonal Harmonic

Here, we present the Pontryagin extremal in the second problem statement with the second zonal harmonic and $n = 0.1$, $P_{\text{spe}} = 350$ s, $i_0 = 0.9$ rad, $\Delta v^* = 1.5$ km/s, $m(0) = 1$ ($M(0) = 22\,500$ kg), the AFT's dry mass $m^{\text{AFT}} = 0.052$ (which corresponds to the mass 1170 kg), the CB's dry mass $m^{\text{CB}} = 0.0635556$ (which corresponds to the mass 1430 kg), the maximum AFT fuel mass $m^{\text{AFT}}_{\text{fuel}} = 0.6488889$ (which corresponds to the mass 14 600 kg), the maximum CB fuel mass $m^{\text{CB}}_{\text{fuel}} = 0.2266667$ (which corresponds to the mass 5100 kg), and $R_{\max} = 280\,000$ km.

The numerical Lagrange multipliers are:

$$\lambda_{R0} = 0.000143381, \quad \lambda_{C0} = 0.000591830,$$
$$\lambda_{vx0} = 0.454024806, \quad \lambda_{vy0} = 0.573821987, \quad \lambda_{vz0} = 0.681608247,$$
$$\lambda_{m0} = 0.003461557, \quad \lambda_{x\text{rel}1} = -0.000341292, \quad \lambda_{y\text{rel}1} = 7.242061031 \times 10^{-7},$$
$$\lambda_{z\text{rel}1} = -4.016946803 \times 10^{-7}, \quad \lambda_{vx\text{rel}1} = -0.006462504, \quad \lambda_{vy\text{rel}1} = -0.749532226,$$
$$\lambda_{vz\text{rel}1} = -0.815268562, \quad \lambda_{x\text{safe}} = -4.192288919 \times 10^{-5}, \quad \lambda_{y\text{safe}} = 1.862663610 \times 10^{-6},$$
$$\lambda_{z\text{safe}} = 3.375542260 \times 10^{-6}, \quad \lambda_{vx\text{safe}} = 0.005756096, \quad \lambda_{vy\text{safe}} = 0.122589304,$$
$$\lambda_{vz\text{safe}} = 0.271310189, \quad \lambda_{x\text{tar}} = 2.712182075 \times 10^{-7}, \quad \lambda_{y\text{tar}} = -1.138471581 \times 10^{-9},$$
$$\lambda_{z\text{tar}} = -6.622593433 \times 10^{-13}, \quad \lambda_{vx\text{tar}} = 0.000524475, \quad \lambda_{vy\text{tar}} = 0.124996420,$$
$$\lambda_{vz\text{tar}} = 0.154483787, \quad \lambda_{\tau\text{rel}1} = 1.017515559 \times 10^{-10}, \quad \lambda_{\text{rel}1} = 0.000457417,$$
$$\lambda_{m\tau 1} = 0.005674790, \quad \lambda_{m\tau 2} = 0, \quad \lambda_{\tau} = -7.385740053 \times 10^{-11},$$
$$\lambda_{\tau\text{rel}2} = 8.771225602 \times 10^{-11}, \quad \lambda_{\tau\text{safe}} = 1.237010821 \times 10^{-11}, \quad \lambda_{\text{safe}} = -0.000291125,$$
$$\lambda_{\tau\text{tar}} = -1.948360860 \times 10^{-19}, \quad \lambda_{\text{tar}} = -0.020742502, \quad \lambda_{\text{fa}} = 1.164436713,$$
$$\lambda_{T} = 4.669488715 \times 10^{-6}, \quad \lambda_{mT1} = -0.009927899, \quad \lambda_{mT2} = 0.000804002, \quad \lambda_0 = 0.009915863.$$

The spacecraft engine is activated at the initial time instant $t = 0$ under the angular position $\varphi_0 = -0.7817944$ rad in the reference orbit:

$$x(0) = 4668.258 \text{ km}, \quad y(0) = -2880.996 \text{ km}, \quad z(0) = -3630.510 \text{ km},$$
$$v_x(0) = 5.484390 \text{ km/s}, \quad v_y(0) = 3.433812 \text{ km/s}, \quad v_z(0) = 4.327147 \text{ km/s},$$
$$p_x(0) = 0.000284790, \quad p_y(0) = -0.000515932, \quad p_z(0) = -0.000601403,$$
$$p_{vx}(0) = 0.454024806, \quad p_{vy}(0) = 0.573821987, \quad p_{vz}(0) = 0.681608247,$$
$$m(0) = 1, \quad p_m(0) = 0.003461557.$$

The duration of the first active segment is $\Delta\tau_1^{\text{act}} = 1234.190$ s. The spacecraft moves to the elliptic orbit with the apogee $r_{\text{a1}} = 15\,500.572$ km, the perigee $r_{\text{p1}} = 6702.795$ km, and the inclination angle $i_1 = 0.8956402$ rad. The coordinates, velocities, and mass of the spacecraft at the engine activation time instant τ_1^{act} are

$$x(\tau_{1-}^{\text{act}}) = x(\tau_{1+}^{\text{act}}) = 5360.198 \text{ km}, \quad y(\tau_{1-}^{\text{act}}) = y(\tau_{1+}^{\text{act}}) = 3045.731 \text{ km},$$
$$z(\tau_{1-}^{\text{act}}) = z(\tau_{1+}^{\text{act}}) = 3807.202 \text{ km}, \quad v_x(\tau_{1-}^{\text{act}}) = v_x(\tau_{1+}^{\text{act}}) = -4.376498 \text{ km/s},$$
$$v_y(\tau_{1-}^{\text{act}}) = v_y(\tau_{1+}^{\text{act}}) = 4.635010 \text{ km/s}, \quad v_z(\tau_{1-}^{\text{act}}) = v_z(\tau_{1+}^{\text{act}}) = 5.786158 \text{ km/s},$$
$$p_x(\tau_{1-}^{\text{act}}) = p_x(\tau_{1+}^{\text{act}}) = 0.000331052, \quad p_y(\tau_{1-}^{\text{act}}) = p_y(\tau_{1+}^{\text{act}}) = 0.000386951,$$
$$p_z(\tau_{1-}^{\text{act}}) = p_z(\tau_{1+}^{\text{act}}) = 0.000452488, \quad p_{vx}(\tau_{1-}^{\text{act}}) = p_{vx}(\tau_{1+}^{\text{act}}) = -0.371919208,$$
$$p_{vy}(\tau_{1-}^{\text{act}}) = p_{vy}(\tau_{1+}^{\text{act}}) = 0.637424282, \quad p_{vz}(\tau_{1-}^{\text{act}}) = p_{vz}(\tau_{1+}^{\text{act}}) = 0.754898302,$$
$$m(\tau_1^{\text{act}}) = 0.6473743, \quad p_m(\tau_1^{\text{act}}) = 0.005597245.$$

The duration of the first passive segment is $\Delta\tau_1^{\text{pass}} = 5219.504$ s. At the end of this segment, the spacecraft is in the orbit with the apogee $r_{\text{a2}} = 15\,497.241$ km, the perigee $r_{\text{p2}} = 6704.141$ km, and the inclination angle $i_2 = 0.8956703$ rad. The coordinates, velocities, and mass of the spacecraft at the engine activation time instant τ_1^{pass} are

$$x(\tau_{1-}^{\text{pass}}) = x(\tau_{1+}^{\text{pass}}) = -15495.958 \text{ km}, \quad y(\tau_{1-}^{\text{pass}}) = y(\tau_{1+}^{\text{pass}}) = 133.386 \text{ km},$$
$$z(\tau_{1-}^{\text{pass}}) = z(\tau_{1+}^{\text{pass}}) = 131.434 \text{ km}, \quad v_x(\tau_{1-}^{\text{pass}}) = v_x(\tau_{1+}^{\text{pass}}) = -0.061410 \text{ km/s},$$
$$v_y(\tau_{1-}^{\text{pass}}) = v_y(\tau_{1+}^{\text{pass}}) = -2.462966 \text{ km/s}, \quad v_z(\tau_{1-}^{\text{pass}}) = v_z(\tau_{1+}^{\text{pass}}) = -3.076413 \text{ km/s},$$
$$p_x(\tau_{1-}^{\text{pass}}) = p_x(\tau_{1+}^{\text{pass}}) = 0.000121015, \quad p_y(\tau_{1-}^{\text{pass}}) = p_y(\tau_{1+}^{\text{pass}}) = -2.300740522 \times 10^{-6},$$
$$p_z(\tau_{1-}^{\text{pass}}) = p_z(\tau_{1+}^{\text{pass}}) = -3.527557800 \times 10^{-6}, \quad p_{vx}(\tau_{1-}^{\text{pass}}) = p_{vx}(\tau_{1+}^{\text{pass}}) = 0.007053866,$$
$$p_{vy}(\tau_{1-}^{\text{pass}}) = p_{vy}(\tau_{1+}^{\text{pass}}) = 0.600617145, \quad p_{vz}(\tau_{1-}^{\text{pass}}) = p_{vz}(\tau_{1+}^{\text{pass}}) = 0.868167234,$$
$$m(\tau_1^{\text{pass}}) = 0.6473743, \quad p_m(\tau_1^{\text{pass}}) = 0.005597245.$$

The duration of the second active segment is $\Delta\tau_2^{\text{act}} = \Delta\tau_{\text{rel1}}^{\text{AFT}} = 30.961$ s. The spacecraft moves to the elliptic orbit with the apogee $r_{\text{a3}} = 15\,497.241$ km, the perigee $r_{\text{p3}} = 6478.25$ km, and the inclination angle $i_3 = 0.8948234$ rad. This orbit touches the conditional boundary of the atmosphere. The coordinates, velocities, and mass of the spacecraft at the engine deactivation time instant $\tau_{\text{rel1}}^{\text{AFT}}$ are

$$x(\tau_{\text{rel1-}}^{\text{AFT}}) = x(\tau_{\text{rel1+}}^{\text{AFT}}) = -15\,497.060 \text{ km}, \quad y(\tau_{\text{rel1-}}^{\text{AFT}}) = y(\tau_{\text{rel1+}}^{\text{AFT}}) = 57.540 \text{ km},$$
$$z(\tau_{\text{rel1-}}^{\text{AFT}}) = z(\tau_{\text{rel1+}}^{\text{AFT}}) = 36.780 \text{ km}, \quad v_x(\tau_{\text{rel1-}}^{\text{AFT}}) = v_x(\tau_{\text{rel1+}}^{\text{AFT}}) = -0.009780 \text{ km/s},$$
$$v_y(\tau_{\text{rel1-}}^{\text{AFT}}) = v_y(\tau_{\text{rel1+}}^{\text{AFT}}) = -2.436415 \text{ km/s}, \quad v_z(\tau_{\text{rel1-}}^{\text{AFT}}) = v_z(\tau_{\text{rel1+}}^{\text{AFT}}) = -3.037856 \text{ km/s},$$
$$p_x(\tau_{\text{rel1-}}^{\text{AFT}}) = 0.000121065, \quad p_y(\tau_{\text{rel1-}}^{\text{AFT}}) = -3.087854169 \times 10^{-7}, \quad p_z(\tau_{\text{rel1-}}^{\text{AFT}}) = -6.465598107 \times 10^{-7},$$
$$p_{vx}(\tau_{\text{rel1-}}^{\text{AFT}}) = 0.003306216, \quad p_{vy}(\tau_{\text{rel1-}}^{\text{AFT}}) = 0.600657543, \quad p_{vz}(\tau_{\text{rel1-}}^{\text{AFT}}) = 0.868231852,$$
$$p_x(\tau_{\text{rel1+}}^{\text{AFT}}) = -0.000341292, \quad p_y(\tau_{\text{rel1+}}^{\text{AFT}}) = 7.242061031 \times 10^{-7}, \quad p_z(\tau_{\text{rel1+}}^{\text{AFT}}) = -4.016946803 \times 10^{-7},$$
$$p_{vx}(\tau_{\text{rel1+}}^{\text{AFT}}) = -0.006462504, \quad p_{vy}(\tau_{\text{rel1+}}^{\text{AFT}}) = -0.749532226, \quad p_{vz}(\tau_{\text{rel1+}}^{\text{AFT}}) = -0.815268562,$$
$$m(\tau_{\text{rel1-}}^{\text{AFT}}) = 0.6385284, \quad p_m(\tau_{\text{rel1-}}^{\text{AFT}}) = 0.005674790.$$

The duration of the second passive segment is $\Delta\tau_2^{\text{pass}} = 120$ s. On this segment, the AFT is undocked from the spacecraft. At the end of the second passive segment, the spacecraft is in the orbit with the apogee $r_{\text{a}4} = 15\,497.245$ km, the perigee $r_{\text{p}4} = 6478.246$ km, and the inclination angle $i_4 = 0.8948232$ rad. The coordinates, velocities, and mass of the spacecraft at the engine activation time instant $\tau_{\text{rel}2}^{\text{AFT}}$ are

$$x(\tau_{\text{rel2-}}^{\text{AFT}}) = x(\tau_{\text{rel2+}}^{\text{AFT}}) = -15486.279 \text{ km}, \quad y(\tau_{\text{rel2-}}^{\text{AFT}}) = y(\tau_{\text{rel2+}}^{\text{AFT}}) = -234.799 \text{ km},$$
$$z(\tau_{\text{rel2-}}^{\text{AFT}}) = z(\tau_{\text{rel2+}}^{\text{AFT}}) = -327.697 \text{ km}, \quad v_x(\tau_{\text{rel2-}}^{\text{AFT}}) = v_x(\tau_{\text{rel2+}}^{\text{AFT}}) = 0.189472 \text{ km/s},$$
$$v_y(\tau_{\text{rel2-}}^{\text{AFT}}) = v_y(\tau_{\text{rel2+}}^{\text{AFT}}) = -2.435275 \text{ km/s}, \quad v_z(\tau_{\text{rel2-}}^{\text{AFT}}) = v_z(\tau_{\text{rel2+}}^{\text{AFT}}) = -3.035984 \text{ km/s},$$
$$p_x(\tau_{\text{rel2-}}^{\text{AFT}}) = p_x(\tau_{\text{rel2+}}^{\text{AFT}}) = -0.000341191, \quad p_y(\tau_{\text{rel2-}}^{\text{AFT}}) = p_y(\tau_{\text{rel2+}}^{\text{AFT}}) = -8.913952605 \times 10^{-6},$$
$$p_z(\tau_{\text{rel2-}}^{\text{AFT}}) = p_z(\tau_{\text{rel2+}}^{\text{AFT}}) = -1.089032490 \times 10^{-5}, \quad p_{vx}(\tau_{\text{rel2-}}^{\text{AFT}}) = p_{vx}(\tau_{\text{rel2+}}^{\text{AFT}}) = 0.034488810,$$
$$p_{vy}(\tau_{\text{rel2-}}^{\text{AFT}}) = p_{vy}(\tau_{\text{rel2+}}^{\text{AFT}}) = -0.749040916, \quad p_{vz}(\tau_{\text{rel2-}}^{\text{AFT}}) = p_{vz}(\tau_{\text{rel2+}}^{\text{AFT}}) = -0.814591115,$$
$$m(\tau_{\text{rel2+}}^{\text{AFT}}) = 0.5865284, \quad p_m(\tau_{\text{rel2+}}^{\text{AFT}}) = 0.006478792.$$

The duration of the third active segment is $\Delta\tau_3^{\text{act}} = \Delta\tau_{\text{safe}} = 12.584$ s. The spacecraft moves to the elliptic orbit with the apogee $r_{\text{a}5} = 15\,497.362$ km, the perigee $r_{\text{p}5} = 6578.25$ km, and the inclination angle $i_5 = 0.8944602$ rad. This is the safe orbit. The coordinates, velocities, and mass of the spacecraft at the engine deactivation time instant τ_{safe} are

$$x(\tau_{\text{safe-}}) = x(\tau_{\text{safe+}}) = -15\,483.759 \text{ km}, \quad y(\tau_{\text{safe-}}) = y(\tau_{\text{safe+}}) = -265.532 \text{ km},$$
$$z(\tau_{\text{safe-}}) = z(\tau_{\text{safe+}}) = -365.996 \text{ km}, \quad v_x(\tau_{\text{safe-}}) = v_x(\tau_{\text{safe+}}) = 0.211071 \text{ km/s},$$
$$v_y(\tau_{\text{safe-}}) = v_y(\tau_{\text{safe+}}) = -2.449215 \text{ km/s}, \; v_z(\tau_{\text{safe-}}) = v_z(\tau_{\text{safe+}}) = -3.051042 \text{ km/s},$$
$$p_x(\tau_{\text{safe-}}) = -0.000341167, \quad p_y(\tau_{\text{safe-}}) = -9.925271032 \times 10^{-6},$$
$$p_z(\tau_{\text{safe-}}) = -1.199086427 \times 10^{-5}, \quad p_{vx}(\tau_{\text{safe-}}) = 0.038782187,$$
$$p_{vy}(\tau_{\text{safe-}}) = -0.748922381, \quad p_{vz}(\tau_{\text{safe-}}) = -0.814447148,$$
$$p_x(\tau_{\text{safe+}}) = -4.192288919 \times 10^{-5}, \quad p_y(\tau_{\text{safe+}}) = 1.862663610 \times 10^{-6},$$
$$p_z(\tau_{\text{safe+}}) = 3.375542260 \times 10^{-6}, \quad p_{vx}(\tau_{\text{safe+}}) = 0.005756096,$$
$$p_{vy}(\tau_{\text{safe+}}) = 0.122589304, \quad p_{vz}(\tau_{\text{safe+}}) = 0.271310189,$$
$$m(\tau_{\text{safe}}) = 0.5829330, \quad p_m(\tau_{\text{safe}}) = 0.006518753.$$

The duration of the third passive segment is $\Delta\tau_3^{\text{pass}} = 5213.308$ s. At the end of this segment, the spacecraft is in the orbit with the apogee $r_{\text{a}6} = 15\,511.458$ km, the perigee $r_{\text{p}6} = 6576.991$ km, and the inclination angle $i_6 = 0.8945149$ rad. The coordinates, velocities, and mass of the spacecraft at the engine activation time instant τ_3^{pass} are

$$x(\tau_{3-}^{\text{pass}}) = x(\tau_{3+}^{\text{pass}}) = 5800.915 \text{ km}, \quad y(\tau_{3-}^{\text{pass}}) = y(\tau_{3+}^{\text{pass}}) = -2325.058 \text{ km},$$
$$z(\tau_{3-}^{\text{pass}}) = z(\tau_{3+}^{\text{pass}}) = -2873.476 \text{ km}, \quad v_x(\tau_{3-}^{\text{pass}}) = v_x(\tau_{3+}^{\text{pass}}) = 3.552179 \text{ km/s},$$
$$v_y(\tau_{3-}^{\text{pass}}) = v_y(\tau_{3+}^{\text{pass}}) = 5.123342 \text{ km/s}, \quad v_z(\tau_{3-}^{\text{pass}}) = v_z(\tau_{3+}^{\text{pass}}) = 6.398453 \text{ km/s},$$
$$p_x(\tau_{3-}^{\text{pass}}) = p_x(\tau_{3+}^{\text{pass}}) = 0.000705371, \quad p_y(\tau_{3-}^{\text{pass}}) = p_y(\tau_{3+}^{\text{pass}}) = -0.000362590,$$
$$p_z(\tau_{3-}^{\text{pass}}) = p_z(\tau_{3+}^{\text{pass}}) = -0.000422128, \quad p_{vx}(\tau_{3-}^{\text{pass}}) = p_{vx}(\tau_{3+}^{\text{pass}}) = 0.375643830,$$
$$p_{vy}(\tau_{3-}^{\text{pass}}) = p_{vy}(\tau_{3+}^{\text{pass}}) = 0.672228633, \quad p_{vz}(\tau_{3-}^{\text{pass}}) = p_{vz}(\tau_{3+}^{\text{pass}}) = 0.795432792,$$
$$m(\tau_3^{\text{pass}}) = 0.5829330, \quad p_m(\tau_3^{\text{pass}}) = 0.006518753.$$

The duration of the fourth active segment is $\Delta\tau_4^{\text{act}} = 780.500$ s. The spacecraft moves to the target orbit with the apogee $r_{\text{a}7} = 227\,835.611$ km, the perigee $r_{\text{p}7} = 6644.321$ km, and the inclination angle $i_7 = 0.8906535$ rad. The coordinates, velocities, and mass of the spacecraft at the engine deactivation time instant τ_4^{act} are

$$x(\tau_{4-}^{\text{act}}) = x(\tau_{4+}^{\text{act}}) = 6084.753 \text{ km}, \quad y(\tau_{4-}^{\text{act}}) = y(\tau_{4+}^{\text{act}}) = 2384.542 \text{ km},$$
$$z(\tau_{4-}^{\text{act}}) = z(\tau_{4+}^{\text{act}}) = 2973.927 \text{ km}, \quad v_x(\tau_{4-}^{\text{act}}) = v_x(\tau_{4+}^{\text{act}}) = -2.938015 \text{ km/s},$$
$$v_y(\tau_{4-}^{\text{act}}) = v_y(\tau_{4+}^{\text{act}}) = 6.263591 \text{ km/s}, \quad v_z(\tau_{4-}^{\text{act}}) = v_z(\tau_{4+}^{\text{act}}) = 7.730739 \text{ km/s},$$
$$p_x(\tau_{4-}^{\text{act}}) = p_x(\tau_{4+}^{\text{act}}) = 0.000708160, \quad p_y(\tau_{4-}^{\text{act}}) = p_y(\tau_{4+}^{\text{act}}) = 0.000286427,$$
$$p_z(\tau_{4-}^{\text{act}}) = p_z(\tau_{4+}^{\text{act}}) = 0.000340122, \quad p_{vx}(\tau_{4-}^{\text{act}}) = p_{vx}(\tau_{4+}^{\text{act}}) = -0.318610391,$$
$$p_{vy}(\tau_{4-}^{\text{act}}) = p_{vy}(\tau_{4+}^{\text{act}}) = 0.697696303, \quad p_{vz}(\tau_{4-}^{\text{act}}) = p_{vz}(\tau_{4+}^{\text{act}}) = 0.821832874,$$
$$m(\tau_4^{\text{act}}) = 0.3599331, \quad p_m(\tau_4^{\text{act}}) = 0.010719865.$$

In the target orbit, the satellite is separated from the CB. The satellite mass in the target orbit (payload mass) is $m_{\text{p}} = 0.2963061$ (6666.888 kg). At the time instant τ_{tar} the last engine activation occurs to lower the perigee altitude of the CB orbit to the conditional boundary of the atmosphere. For convenience of calculations, the mass jump (after undocking the satellite) is considered at the last engine activation instant. The duration of the fourth passive section (passive flight of the CB in the target orbit) is $\Delta\tau_4^{\text{pass}} = \Delta\tau_{\text{tar}} = 197\,376.995$ s. At the end of this passive segment, the spacecraft is in the orbit with the apogee $r_{\text{a}8} = 226\,259.913$ km, the perigee $r_{\text{p}8} = 6643.293$ km, and the inclination angle $i_8 = 0.8905128$ rad. The coordinates and mass of the CB at the engine activation time instant τ_{tar} are

$$x(\tau_{\text{tar-}}) = x(\tau_{\text{tar+}}) = -226\,257.921 \text{ km}, \quad y(\tau_{\text{tar-}}) = y(\tau_{\text{tar+}}) = 949.323 \text{ km},$$
$$z(\tau_{\text{tar-}}) = z(\tau_{\text{tar+}}) = 0.031 \text{ km}, \quad v_x(\tau_{\text{tar-}}) = v_x(\tau_{\text{tar+}}) = -0.000838 \text{ km/s},$$
$$v_y(\tau_{\text{tar-}}) = v_y(\tau_{\text{tar+}}) = -0.199407 \text{ km/s}, \quad v_z(\tau_{\text{tar-}}) = v_z(\tau_{\text{tar+}}) = -0.246448 \text{ km/s},$$
$$p_x(\tau_{\text{tar-}}) = -7.173006000 \times 10^{-7}, \quad p_y(\tau_{\text{tar-}}) = 1.993046144 \times 10^{-9},$$
$$p_z(\tau_{\text{tar-}}) = -3.571982769 \times 10^{-8}, \quad p_{vx}(\tau_{\text{tar-}}) = -0.003979552,$$
$$p_{vy}(\tau_{\text{tar-}}) = -0.672797915, \quad p_{vz}(\tau_{\text{tar-}}) = 0.617436221,$$
$$p_x(\tau_{\text{tar+}}) = 2.712182075 \times 10^{-7}, \quad p_y(\tau_{\text{tar+}}) = -1.138471581 \times 10^{-9},$$
$$p_z(\tau_{\text{tar+}}) = -6.622593433 \times 10^{-13}, \quad p_{vx}(\tau_{\text{tar+}}) = 0.000524475,$$
$$p_{vy}(\tau_{\text{tar+}}) = 0.124996420, \quad p_{vz}(\tau_{\text{tar+}}) = 0.154483787,$$
$$m(\tau_{\text{tar-}}) = 0.3599331, \quad m(\tau_{\text{tar+}}) = 0.0636269,$$
$$p_m(\tau_{\text{tar-}}) = p_m(\tau_{\text{tar+}}) = 0.010719865.$$

The duration of the fifth (last) active segment is $\Delta\tau_5^{\text{act}} = \Delta T = 0.250$ s. The spacecraft moves to the orbit touching the conditional boundary of the atmosphere with the apogee $r_{\text{a}9} = 226\,259.913$ km, the perigee $r_{\text{p}9} = 6478.25$ km, and the angle of inclination $i_9 = 0.8905128$ rad. The coordinates, velocities, and mass of the CB at the engine deactivation time instant T are

$$x(T) = -226\,257.922 \text{ km}, \quad y(T) = 949.274 \text{ km},$$
$$z(T) = -0.030 \text{ km}, \quad v_x(T) = -0.000826 \text{ km/s},$$
$$v_y(T) = -0.196984 \text{ km/s}, \quad v_z(T) = -0.243454 \text{ km/s},$$
$$p_x(T) = 2.712182120 \times 10^{-7}, \quad p_y(T) = -1.137397235 \times 10^{-9},$$
$$p_z(T) = 6.655346255 \times 10^{-13}, \quad p_{vx}(T) = 0.000524407,$$
$$p_{vy}(T) = 0.124996420, \quad p_{vz}(T) = 0.154483787,$$
$$m(T) = 0.0635556, \quad p_m(T) = 0.010731902.$$

The final ascent impulses to transfer the satellite from the target orbit to the GEO are

$$\Delta v_{\text{fa}1} = 0.029677 \text{ km/s}, \quad \Delta v_{\text{fa}2} = 0.491271 \text{ km/s}, \quad \Delta v_{\text{fa}3} = 0.979052 \text{ km/s}.$$

The fuel consumed to lower the perigee altitude to 100 km (to release the AFT) is 199.034 kg (0.0088460). The fuel consumed to raise the perigee altitude to 200 km (to reach the safe orbit) is 80.897 kg (0.0035954). The fuel consumed to lower the perigee altitude to 100 km (to release the CB) is 1.606 kg (7.1361228×10^{-5}). The total fuel consumption for releasing the AFT and CB constitutes 281.536 kg (0.0125127).

The correspondence of the phase and conjugate variables at the starts and ends of the passive segments can be verified by numerical integration. The conditions of Pontryagin's maximum principle can be verified by substituting the phase and conjugate variables and the numerical Lagrange multipliers into the corresponding formulas, and numerical-analytical differentiation can be used to verify the transversality conditions. The basic dimensional units in the calculations are 1000 km and 1 s. When passing to other dimensional units, the conjugate variables must be recalculated by appropriate formulas. The 8(7)th order Dorman–Prince method was used for numerical integration.

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