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Design of Generalized H_∞-Suboptimal Controllers Based on Experimental and A Priori Data

M. M. Kogan[*,a] **and A. V. Stepanov**[**,b]

[*]*Lobachevsky University, Nizhny Novgorod, Russia*
[**]*Nizhny Novgorod State University of Architecture and Civil Engineering, Nizhny Novgorod, Russia*
e-mail: [a]*mkogan@nngasu.ru,*[b] *andrey8st@yahoo.com*

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Abstract—This paper considers a linear continuous- or discrete-time dynamic object in the absence of its mathematical model. As is demonstrated below, a control law that suboptimally damps initial and (or) exogenous disturbances of such objects can be implemented based on experimental and a priori data. The approach involves the methods of robust control design and duality theory as well as the technique of linear matrix inequalities.

Keywords: generalized H_∞ norm, uncertainty, robust control, experimental data, dual systems, linear matrix inequalities

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1. INTRODUCTION

Recently, increasing attention in control theory has been paid to the design of control laws for dynamic objects with highly uncertain mathematical models, exogenous disturbances, and unknown initial conditions. Within this line of research, by assumption, a series of experiments can be conducted with an object by setting input actions and measuring output variables. The problem is to determine the feedback parameters ensuring a given quality of the closed-loop control system directly, i.e., based on available measurements and a priori data without identifying the unknown parameters of the object.

As was established in [1], a single trajectory can be used to fully characterize a linear time-invariant dynamic system under the so-called persistency of excitation. In view of this fundamental result, different direct control design schemes based on experimental data were proposed in [2] for objects with unknown state dynamics matrices and given target output matrices under the persistency of excitation. According to [3], it suffices to fulfill the data informativity condition in order to construct control laws from experimental data, which is less restrictive than the persistency of excitation. For a fully uncertain object, H_2- and H_∞-optimal control laws were constructed based on input and output measurements using a matrix version of S-lemma [5] in the publication [4] and using Petersen's lemma [7] in the publication [6]. In [8, 9], the state feedback parameters were calculated from a priori data and open-loop measurements of the input and output of a discrete-time uncertain object subjected to an unmeasured disturbance from a definite class.

In this paper, generalized H_∞-suboptimal control laws that damp initial and (or) exogenous disturbances (as a special case, linear-quadratic control laws) for continuous- or discrete-time objects with completely unknown state dynamics and target output matrices are designed from a priori and experimental data. The design procedure is based on the approach used in [9]: the uncertain system is "immersed" into an artificial system with known equations and an additional disturbance whose influence corresponds to that of the unknown terms in the original equation. The idea of such an artificial immersion (in other words, the representation of an uncertain system as a system whose feedback loop contains a block with unknown bounded parameters or an unknown bounded operator) was actively employed in robust control based on H_∞ optimization; see the survey [10]. However, the direct application of this approach to the design of control laws based on experimental data caused difficulties. This problem is solved below by passing from the original uncertain system to a dual uncertain system immersed into the corresponding augmented system. Implementing such an approach requires establishing a connection between the generalized H_∞ norms of the primal and dual systems.

This paper is organized as follows. After the Introduction, Section 2 gives the general problem statement; in particular, two quadratic inequalities for the unknown object parameter matrices (state and target output) are derived from a priori information and experimental data. In Section 3, a necessary background is provided on the generalized H_∞ norm, and this norm is calculated in terms of the dual system; see Lemma 3.1. Section 4 describes the design procedure for the generalized H_∞-suboptimal control laws based on a priori and experimental data, including the main theorem and its proof. Several experiments with an uncertain system are presented in Section 5 to illustrate the effectiveness of this control approach. Finally, Section 6 summarizes the results and draws conclusions.

2. PROBLEM STATEMENT

Consider an uncertain system described by

$$\begin{aligned} &\partial x(t) = Ax(t) + Bu(t) + w(t), \quad x(0) = x_0, \\ &z(t) = Cx(t) + Du(t) \end{aligned} \tag{2.1}$$

with the following notations: ∂ is the differentiation operator in the continuous-time case or the shift operator in the discrete-time case; $x(t) \in \mathrm{R}^{n_x}$ is the state vector, $u(t) \in \mathrm{R}^{n_u}$ is the control vector (input), $w(t) \in \mathrm{R}^{n_w}$ is an exogenous disturbance, and $z(t) \in \mathrm{R}^{n_z}$ is the target output. By assumption, the disturbance $w(t) \in L_2(l_2)$ and the system matrices A, B, C, and D are unknown. In general, it is required to design linear state-feedback control laws based on a priori and experimental data so that the damping level of the disturbances in the closed loop system does not exceed a specified value.

The information about the unknown parameters of system (2.1) is extracted from a finite set of measurements of its trajectory. For the discrete-time system, there are available measurements of its state and target output, $x_0, x_1, \ldots, x_N$ and $z_0, \ldots, z_{N-1}$, respectively, under chosen controls $u_0, \ldots, u_{N-1}$ and some unknown disturbance $w_0, \ldots, w_{N-1}$. We compile the matrices

$$\begin{gathered} \Phi = (x_0 \cdots x_{N-1}), \quad \Phi_+ = (x_1 \cdots x_N), \\ U = (u_0 \cdots u_{N-1}), \quad W = (w_0 \cdots w_{N-1}), \quad Z = (z_0 \cdots z_{N-1}). \end{gathered}$$

In the continuous-time case, there are measurements of the system state, its derivative, and the target output, $x(t_0), \ldots, x(t_{N-1})$, $\dot{x}(t_0), \ldots, \dot{x}(t_{N-1})$, and $z(t_0), \ldots, z(t_{N-1})$, respectively, under chosen controls $u(t_0), \ldots, u(t_{N-1})$ and some unknown disturbances $w(t_0), \ldots, w(t_{N-1})$ at time instants $t_0, \ldots, t_{N-1}$. By analogy, we compile the matrices

$$\begin{gathered} \Phi = (x(t_0) \cdots x(t_{N-1})), \quad \Phi_+ = (\dot{x}(t_0) \cdots \dot{x}(t_{N-1})), \\ U = (u(t_0) \cdots u(t_{N-1})), \quad W = (w(t_0) \cdots w(t_{N-1})), \quad Z = (z(t_0) \cdots z(t_{N-1})). \end{gathered}$$

The experimental data matrices in both cases satisfy the relations

$$\begin{aligned} \Phi_+ &= A_{real}\Phi + B_{real}U + W, \\ Z &= C_{real}\Phi + D_{real}U, \end{aligned} \tag{2.2}$$

where A_{real}, B_{real}, C_{real}, and D_{real} are the real (unknown) system matrices. With the notations

$$\Delta_{real} = \begin{pmatrix} A_{real} & B_{real} \\ C_{real} & D_{real} \end{pmatrix}, \quad \widehat{\Phi} = \begin{pmatrix} \Phi \\ U \end{pmatrix}, \quad \widetilde{\Phi} = \begin{pmatrix} \Phi_+ \\ Z \end{pmatrix}, \quad \widehat{W} = \begin{pmatrix} W \\ 0 \end{pmatrix},$$

equations (2.2) can be written as the linear matrix regression

$$\widetilde{\Phi} = \Delta_{real}\widehat{\Phi} + \widehat{W}. \tag{2.3}$$

Assume that the disturbance in the experiment satisfies the condition

$$\sum_{i=0}^{N-1} w(t_i)w^{\mathrm{T}}(t_i) = WW^{\mathrm{T}} \leqslant \Omega. \tag{2.4}$$

In particular, if $\|w(t)\|_\infty \leqslant d_w$ for all t and a given value d_w (the damping level), then $\Omega = d_w^2 n_w N I_{n_x}$. In the case $\sum_{i=0}^{N-1} |w(t_i)|^2 \leqslant \alpha^2$ (i.e., the total energy of the disturbance is bounded during the experiment), we obtain $\Omega = \alpha^2 I$. If $w(t)$ in (2.1) has the form $w(t) = B_v v(t)$, where $v(t) \in \mathrm{R}^{n_v}$ for some matrix B_v and $\|v(t)\|_\infty \leqslant d_v$, then $\Omega = d_v^2 n_v N B_v B_v^{\mathrm{T}}$.

From (2.4) it follows that

$$\widehat{W}\widehat{W}^{\mathrm{T}} \leqslant \begin{pmatrix} \Omega & \star \\ 0 & 0 \end{pmatrix} = \widehat{\Omega}. \tag{2.5}$$

We define the set $\mathbf{\Delta_p}$ of matrices Δ of dimensions $(n_x + n_z) \times (n_x + n_u)$ that could generate the experimental matrices Φ, Φ_+, and Z under the chosen controls U and some admissible disturbances W satisfying the constraint (2.4). For these matrices, the quality $\widetilde{\Phi} = \Delta\widehat{\Phi} + \widehat{W}$ must hold with some matrix $\widehat{W}$ satisfying (2.5). Consequently,

$$\mathbf{\Delta_p} = \left\{ \Delta : \widetilde{\Phi} = \Delta\widehat{\Phi} + \widehat{W}, \quad \widehat{W}\widehat{W}^{\mathrm{T}} \leqslant \widehat{\Omega} \right\}$$

and $\Delta \in \mathbf{\Delta_p}$ iff

$$(\widetilde{\Phi} - \Delta\widehat{\Phi})(\widetilde{\Phi} - \Delta\widehat{\Phi})^{\mathrm{T}} \leqslant \widehat{\Omega}. \tag{2.6}$$

It is obvious that $\Delta_{real} \in \mathbf{\Delta_p}$. For further use, we represent this inequality as

$$(\Delta \quad I_{n_x+n_z})\, \Psi_1 \,(\Delta \quad I_{n_x+n_z})^{\mathrm{T}} \leqslant 0, \tag{2.7}$$

where the symmetric matrix Ψ_1 of order $(2n_x + n_u + n_z)$ is partitioned into appropriate blocks as follows:

$$\Psi_1 = \left(\begin{array}{cc|cc} \Phi\Phi^{\mathrm{T}} & * & * & * \\ U\Phi^{\mathrm{T}} & UU^{\mathrm{T}} & * & * \\ \hline -\Phi_+\Phi^{\mathrm{T}} & -\Phi_+U^{\mathrm{T}} & \Phi_+\Phi_+^{\mathrm{T}} - \Omega & * \\ -Z\Phi^{\mathrm{T}} & -ZU^{\mathrm{T}} & Z\Phi_+^{\mathrm{T}} & ZZ^{\mathrm{T}} \end{array} \right). \tag{2.8}$$

Thus, the set of all matrices Δ consistent with the available experimental data satisfies inequality (2.7). The lemma below formulates boundedness conditions for the set $\mathbf{\Delta_p}$. Its proof is provided in the Appendix.

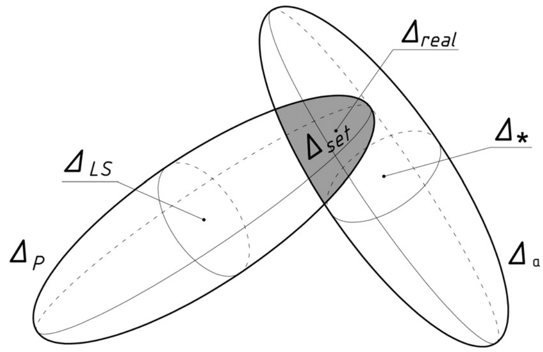

Fig. 1. The set $\mathbf{\Delta_{set}}$ of unknown parameters Δ consistent with experimental and a priori data.

Lemma 2.1. *If the information matrix* $\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}}$ *is nonsingular, then the set* $\mathbf{\Delta_p}$ *is a nondegenerate "matrix ellipsoid" centered at* Δ_{LS} *given by*

$$(\Delta - \Delta_{LS})(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})(\Delta - \Delta_{LS})^{\mathrm{T}} \leqslant \Gamma, \tag{2.9}$$

where

$$\Gamma = \widehat{\Omega} + \widetilde{\Phi}[\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}\widehat{\Phi} - I]\widetilde{\Phi}^{\mathrm{T}} \geqslant 0, \tag{2.10}$$

and $\Delta_{LS} = \widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}$ *is the optimal least-squares estimate of the unknown matrix* Δ_{real} *in* (2.3) *that minimizes the squared matrix norm of the residual* $\|\widetilde{\Phi} - \Delta\widehat{\Phi}\|_F^2$ *with respect to* Δ.

According to this lemma, given a nonsingular information matrix, the "size" of the set $\mathbf{\Delta_p}$ is determined by the regressor matrices $\widehat{\Phi}$ and ultimately depends on the real object, the controls U chosen in the experiment, and the disturbances W.

Now consider an additional information that the unknown matrix Δ_{real} satisfies the constraint

$$(\Delta - \Delta_*)(\Delta - \Delta_*)^{\mathrm{T}} \leqslant \rho^2 I, \quad \Delta_* = \begin{pmatrix} A_* & B_* \\ C_* & D_* \end{pmatrix} = \begin{pmatrix} \Delta_*^{(1)} \\ \Delta_*^{(2)} \end{pmatrix}, \tag{2.11}$$

where Δ_* and ρ are given matrix and parameter characterizing the center and size of the uncertainty domain. We write this inequality as

$$(\Delta \quad I_{n_x+n_z})\,\Psi_2\,(\Delta \quad I_{n_x+n_z})^{\mathrm{T}} \leqslant 0, \tag{2.12}$$

where

$$\Psi_2 = \left(\begin{array}{cc|cc} I_{n_x} & \star & \star & \star \\ 0_{n_u\times n_x} & I_{n_u} & \star & \star \\ \hline -A_* & -B_* & \Delta_*^{(1)}\Delta_*^{(1)\mathrm{T}} - \rho^2 I_{n_x} & \star \\ -C_* & -D_* & \Delta_*^{(2)}\Delta_*^{(1)\mathrm{T}} & \Delta_*^{(2)}\Delta_*^{(2)\mathrm{T}} - \rho^2 I_{n_z} \end{array}\right). \tag{2.13}$$

We introduce the following notations: $\mathbf{\Delta_a}$ is the set of matrices satisfying inequality (2.12), and $\mathbf{\Delta_{set}} = \mathbf{\Delta_p} \bigcap \mathbf{\Delta_a}$ is the set of matrices satisfying inequalities (2.7) and (2.12). Obviously, $\Delta_{real} \in \mathbf{\Delta_{set}}$ (see Fig. 1).

The quality of the closed loop system (2.1) with the linear state-feedback control law $u(t) = \Theta x(t)$ and a given matrix Δ will be evaluated by the damping level of the exogenous and initial disturbances, i.e., by the generalized H_∞ norm

$$\gamma_{g\infty}(\Delta, \Theta) = \sup_{x_0,\, w} \frac{\|z\|}{\left(x_0^{\mathrm{T}} R^{-1} x_0 + \|w\|^2\right)^{1/2}},$$

where $R = R^{\mathrm{T}} > 0$ is a weight matrix and $\|\xi\|^2 = \sum_{t=0}^{\infty} |\xi(t)|^2$ (in the discrete-time case) or $\|\xi\|^2 = \int_{t=0}^{\infty} |\xi(t)|^2$ (in the continuous-time case). If $w(t) \equiv 0$ (no exogenous disturbance), the generalized H_∞ norm turns into the so-called γ_0 norm given by

$$\gamma_0(\Delta, \Theta) = \sup_{x_0 \neq 0} \frac{\|z\|}{\left(x_0^{\mathrm{T}} R^{-1} x_0\right)^{1/2}}.$$

This norm characterizes the "worst" value of the quadratic functional on the system trajectories provided that the initial state is inside the ellipsoid $x^{\mathrm{T}} R^{-1} x \leqslant 1$. Under zero initial state, the generalized H_∞ norm (with $R \to 0$) turns into the conventional H_∞ norm:

$$\gamma_\infty(\Delta, \Theta) = \sup_{w \neq 0} \frac{\|z\|}{\|w\|}.$$

The quality of the closed-loop uncertain system (2.1) with the control law $u(t) = \Theta x(t)$ will be evaluated by the minimum upper bound of the damping level of the exogenous and initial disturbances, i.e., by the minimum upper bound of the generalized H_∞ norm for all object matrices consistent with experimental and a priori data:

$$\gamma_*(\Theta) = \sup_{\Delta \in \boldsymbol{\Delta}_{\mathbf{set}}} \gamma_{g\infty}(\Delta, \Theta). \tag{2.14}$$

The robust generalized H_∞-optimal control law is defined as a control law with the parameter matrix Θ_* minimizing this bound, i.e., with the solution of the minimax problem

$$\inf_{\Theta} \sup_{\Delta \in \boldsymbol{\Delta}_{\mathbf{set}}} \gamma_{g\infty}(\Delta, \Theta) = \inf_{\Theta} \gamma_*(\Theta) = \gamma_*(\Theta_*). \tag{2.15}$$

The problem is to design, directly from input and state measurements, a robust generalized H_∞-suboptimal control law with a parameter matrix Θ under which the generalized H_∞ norm of the closed loop system will be bounded by a given constant: $\gamma_*(\Theta) < \gamma$.

3. THE GENERALIZED H_∞ NORM IN TERMS OF THE DUAL SYSTEMS

Recall that

$$\gamma_{g\infty} = \sup_{x_0,\, v} \frac{\|z\|}{\left(x_0^{\mathrm{T}} R^{-1} x_0 + \|v\|^2\right)^{1/2}}, \tag{3.1}$$

the generalized H_∞ norm from the input v to the output z of a stable system

$$\begin{aligned} \partial x(t) &= \mathcal{A} x(t) + \mathcal{B} v(t), \\ z(t) &= \mathcal{C} x(t), \end{aligned} \tag{3.2}$$

satisfies the condition $\gamma_{g\infty} < \gamma$ iff the following LMIs are solvable in the matrix $Y = Y^{\mathrm{T}} > 0$:

$$\begin{pmatrix} Y\mathcal{A}^T + \mathcal{A}Y & \star & \star \\ \mathcal{B}^{\mathrm{T}} & -\gamma^2 I & \star \\ \mathcal{C}Y & 0 & -I \end{pmatrix} < 0, \quad \begin{pmatrix} Y & \star \\ I & \gamma^2 R^{-1} \end{pmatrix} > 0 \tag{3.3}$$

(for the continuous-time system) or

$$\begin{pmatrix} -Y & \star & \star & \star \\ Y\mathcal{A}^{\mathrm{T}} & -Y & \star & \star \\ \mathcal{B}^{\mathrm{T}} & 0 & -\gamma^2 I & \star \\ 0 & \mathcal{C}Y & 0 & -I \end{pmatrix} < 0, \quad \begin{pmatrix} Y & \star \\ I & \gamma^2 R^{-1} \end{pmatrix} > 0 \tag{3.4}$$

(for the discrete-time system). According to [11, 12], inequalities (3.3) and (3.4) mean that

$$\dot{V}(x) + |z|^2 - \gamma^2|v|^2 < 0 \text{ and } \Delta V(x) + |z|^2 - \gamma^2|v|^2 < 0 \quad \forall\, x, v, \text{ respectively,} \tag{3.5}$$

for a positive definite function $V(x) = x^{\mathrm{T}}Y^{-1}x$ with $Y > \gamma^{-2}R$ along the trajectories of system (3.2).

The next auxiliary result, proved in the Appendix, characterizes the generalized H_∞ norm of system (3.2) in terms of the dual system.

Lemma 3.1. *The generalized H_∞ norm of system* (3.2) *satisfies the condition $\gamma_{g\infty} < \gamma$ iff there exists a positive definite quadratic form $V_a(x_a) = x_a^{\mathrm{T}}Px_a$ with $P > R$ such that*

$$\begin{aligned} &\dot{V}_a(x_a(t)) + |z_a(t)|^2 - \gamma^2|v_a(t)|^2 < 0 \ \textit{or} \\ &\Delta V_a(x_a(t)) + |z_a(t)|^2 - \gamma^2|v_a(t)|^2 < 0, \ \textit{respectively,} \end{aligned} \tag{3.6}$$

along the trajectories of the dual system

$$\begin{aligned} &\partial x_a(t) = \mathcal{A}^{\mathrm{T}}x_a(t) + \mathcal{C}^{\mathrm{T}}v_a(t), \\ &z_a(t) = \mathcal{B}^{\mathrm{T}}x_a(t). \end{aligned} \tag{3.7}$$

Corollary 3.1. *For $v(t) \equiv 0$, the γ_0 norm of system* (3.2) *satisfies the condition $\gamma_0 < \gamma$ iff there exists a quadratic form $V_a(x_a) = x_a^{\mathrm{T}}Px_a$ with $P > R$ such that the corresponding inequality in* (3.6) *is valid for $z_a(t) \equiv 0$ along the trajectories of the dual system*

$$\partial x_a(t) = \mathcal{A}^{\mathrm{T}}x_a(t) + \mathcal{C}^{\mathrm{T}}v_a(t).$$

Remark 1. Formally, the dual system is described by the equations

$$\begin{aligned} &\dot{\widehat{x}}_a = -\mathcal{A}^{\mathrm{T}}\widehat{x}_a - \mathcal{C}^{\mathrm{T}}\widehat{v}_a, \\ &\widehat{z}_a = \mathcal{B}^{\mathrm{T}}\widehat{x}_a \end{aligned} \tag{3.8}$$

(in the continuous-time case) or

$$\begin{aligned} &\widehat{x}_a(t) = \mathcal{A}^{\mathrm{T}}\widehat{x}_a(t+1) + \mathcal{C}^{\mathrm{T}}\widehat{v}_a(t), \\ &\widehat{z}_a(t) = \mathcal{B}^{\mathrm{T}}\widehat{x}_a(t+1) \end{aligned} \tag{3.9}$$

(in the discrete-time case). By the proof of this lemma, from systems (3.8) and (3.9) we can pass to system (3.7), also called dual, which satisfies the corresponding inequality of (3.6).

Remark 2. The matrices of the quadratic forms $V(x) = x^{\mathrm{T}}Y^{-1}x$ and $V_a(x_a) = x_a^{\mathrm{T}}Px_a$ of the primal and dual systems have the relation $P = \gamma^2 Y$; see the proof of Lemma 3.1.

4. DESIGN OF GENERALIZED H_∞-SUBOPTIMAL CONTROLLERS

We describe the main steps for obtaining an upper bound of the generalized H_∞ norm and the corresponding parameter matrices Θ of control laws for the uncertain closed-loop system

$$\begin{aligned} \partial x(t) &= (A + B\Theta)x(t) + w(t), \\ z(t) &= (C + D\Theta)x(t). \end{aligned} \tag{4.1}$$

Assume that the closed loop system with the parameters Θ is stable. With the notations introduced above, these equations can be written as

$$\begin{aligned} \partial x(t) &= (I_{n_x}\ 0_{n_x\times n_z})\,\Delta \begin{pmatrix} I_{n_x} \\ \Theta \end{pmatrix} x(t) + w(t), \\ z(t) &= (0_{n_z\times n_x}\ I_{n_z})\,\Delta \begin{pmatrix} I_{n_x} \\ \Theta \end{pmatrix} x(t), \end{aligned} \tag{4.2}$$

where Δ is an unknown matrix of dimensions $(n_x+n_z)\times(n_x+n_u)$ and Θ is the controller's parameter matrix of dimensions $(n_u \times n_x)$. Due to Lemma 3.1, the dual continuous- and discrete-time systems are described by the equations

$$\begin{aligned} &\partial x_a(t) = \begin{pmatrix} I \\ \Theta \end{pmatrix}^{\mathrm T} \Delta^{\mathrm T} \begin{pmatrix} I \\ 0 \end{pmatrix} x_a(t) + \begin{pmatrix} I \\ \Theta \end{pmatrix}^{\mathrm T} \Delta^{\mathrm T} \begin{pmatrix} 0 \\ I \end{pmatrix} w_a(t), \\ &z_a(t) = x_a(t). \end{aligned} \tag{4.3}$$

We define an augmented system with an additional artificial input $w_\Delta(t) \in L_2(l_2)$ and an output $z_\Delta(t)$ in both cases as follows:

$$\begin{aligned} &\partial \widehat{x}(t) = \begin{pmatrix} I \\ \Theta \end{pmatrix}^{\mathrm T} w_\Delta(t), \\ &\widehat{z}(t) = \widehat{x}(t), \quad z_\Delta(t) = \begin{pmatrix} I \\ 0 \end{pmatrix} \widehat{x}(t) + \begin{pmatrix} 0 \\ I \end{pmatrix} \widehat{w}(t), \end{aligned} \tag{4.4}$$

where $\widehat{x}(t)$ is the state variable, $\widehat{w}(t)$ is a disturbance, and $\widehat{z}(t)$ is the target output. Note that for $w_\Delta(t) = \Delta^{\mathrm T} z_\Delta(t)$, equations (4.4) coincide with the equations of system (4.3). For all $t \geqslant 0$, let the additional input and output signals in system (4.4) satisfy the two inequalities

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm T} \Psi_1 \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0, \quad \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm T} \Psi_2 \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0 \tag{4.5}$$

where the matrices Ψ_1 and Ψ_2 are given by (2.8) and (2.13). We denote by $\mathbf{W}_{\boldsymbol\Delta}$ the set of all such signals $w_\Delta(t)$. According to (2.7) and (2.12), for $w_\Delta(t) = \Delta^{\mathrm T} z_\Delta(t)$ and all $\Delta \in \boldsymbol{\Delta}_{\mathbf{set}}$,

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm T} \Psi_1 \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} = z_\Delta^{\mathrm T}(t) \begin{pmatrix} \Delta^{\mathrm T} \\ I \end{pmatrix}^{\mathrm T} \Psi_1 \begin{pmatrix} \Delta^{\mathrm T} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0,$$

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm T} \Psi_2 \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} = z_\Delta^{\mathrm T}(t) \begin{pmatrix} \Delta^{\mathrm T} \\ I \end{pmatrix}^{\mathrm T} \Psi_2 \begin{pmatrix} \Delta^{\mathrm T} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0.$$

Thus, $w_\Delta(t) = \Delta^{\mathrm T} z_\Delta(t) \in \mathbf{W}_{\boldsymbol\Delta}$ and consequently, system (4.3) with $\Delta \in \boldsymbol{\Delta}_{\mathbf{set}}$, dual to the original uncertain system, is immersed into the augmented system (4.4), (4.5). In view of Lemma 3.1, this fact can be used to derive an upper bound of the generalized H_∞ norm of the uncertain system through the corresponding property of the augmented system.

Theorem 4.1. *The upper bound of the generalized H_∞ norm of the uncertain system (2.1) with the control law $u(t) = \Theta x(t)$, $\Theta = QP^{-1}$, is less than γ if the following LMIs are solvable in $P = P^{\mathrm{T}} > 0$, Q, $\mu_1 \geqslant 0$, and $\mu_2 \geqslant 0$:*

$$\begin{pmatrix} I - \sum\limits_{i=1}^{2} \mu_i \, \Xi_{11}^{(i)} & \star & \star & \star \\ -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{21}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{22}^{(i)} - \gamma^2 I & \star & \star \\ P - \sum\limits_{i=1}^{2} \mu_i \, \Xi_{31}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{32}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{33}^{(i)} & \star \\ Q - \sum\limits_{i=1}^{2} \mu_i \, \Xi_{41}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{42}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{43}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{44}^{(i)} \end{pmatrix} < 0 \tag{4.6}$$

(for the continuous-time system) or

$$\begin{pmatrix} -P & \star & \star & \star & \star \\ 0 & -P + I - \sum\limits_{i=1}^{2} \mu_i \, \Xi_{11}^{(i)} & \star & \star & \star \\ 0 & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{21}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{22}^{(i)} - \gamma^2 I & \star & \star \\ P & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{31}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{32}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{33}^{(i)} & \star \\ Q & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{41}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{42}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{43}^{(i)} & -\sum\limits_{i=1}^{2} \mu_i \, \Xi_{44}^{(i)} \end{pmatrix} < 0 \tag{4.7}$$

(for the discrete-time system), where $P > R$,

$$\begin{gathered} \Xi_{11}^{(1)} = \Phi_+ \Phi_+^{\mathrm{T}} - \Omega, \quad \Xi_{21}^{(1)} = Z \Phi_+^{\mathrm{T}}, \quad \Xi_{22}^{(1)} = Z Z^{\mathrm{T}}, \\ \Xi_{31}^{(1)} = -\Phi \Phi_+^{\mathrm{T}}, \quad \Xi_{32}^{(1)} = -\Phi Z^{\mathrm{T}}, \quad \Xi_{33}^{(1)} = \Phi \Phi^{\mathrm{T}}, \\ \Xi_{41}^{(1)} = -U \Phi_+^{\mathrm{T}}, \quad \Xi_{42}^{(1)} = -U Z^{\mathrm{T}}, \quad \Xi_{43}^{(1)} = U \Phi^{\mathrm{T}}, \quad \Xi_{44}^{(1)} = U U^{\mathrm{T}}, \\ \Xi_{11}^{(2)} = \Delta_*^{(1)} \Delta_*^{(1)\mathrm{T}} - \rho^2 I_{n_x}, \quad \Xi_{21}^{(2)} = \Delta_*^{(2)} \Delta_*^{(1)\mathrm{T}}, \quad \Xi_{22}^{(2)} = \Delta_*^{(2)} \Delta_*^{(2)\mathrm{T}} - \rho^2 I_{n_z}, \\ \Xi_{31}^{(2)} = -A_*^{\mathrm{T}}, \quad \Xi_{32}^{(2)} = -C_*^{\mathrm{T}}, \quad \Xi_{33}^{(2)} = I_{n_x}, \\ \Xi_{41}^{(2)} = -B_*^{\mathrm{T}}, \quad \Xi_{42}^{(2)} = -D_*^{\mathrm{T}}, \quad \Xi_{43}^{(2)} = 0_{n_u \times n_x}, \quad \Xi_{44}^{(2)} = I_{n_u}. \end{gathered}$$

Proof of Theorem 4.1. We establish conditions for the existence of a positive definite quadratic function $\widehat{V}(\widehat{x}) = \widehat{x}^{\mathrm{T}} P \widehat{x}$ with $P > R$ that satisfies the corresponding inequality in (3.6) along the trajectories of the augmented system (4.4) for all $w_\Delta(t)$ with (4.5). By the S-procedure, a sufficient condition is the existence of a function $\widehat{V}(\widehat{x}) = \widehat{x}^{\mathrm{T}} P \widehat{x}$ with $P > R$ that satisfies the corresponding inequality

$$\begin{aligned} &\dot{\widehat{V}}(\widehat{x}) + |\widehat{z}|^2 - \gamma^2|\widehat{w}|^2 - \sum_{i=1}^{2}\mu_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm{T}} \Psi_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} < 0, \\ &\Delta\widehat{V}(\widehat{x}) + |\widehat{z}|^2 - \gamma^2|\widehat{w}|^2 - \sum_{i=1}^{2}\mu_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm{T}} \Psi_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} < 0 \end{aligned} \tag{4.8}$$

along the trajectories of system (4.4) for all $\widehat{x}$, $\widehat{w}$, w_Δ, and some $\mu_1 \geqslant 0$ and $\mu_2 \geqslant 0$.

These inequalities reduce to the following inequalities for the quadratic forms in the variables $\widehat{x}$, $\widehat{w}$, and w_Δ :

$$\begin{aligned} &2\widehat{x}^{\mathrm{T}} P \begin{pmatrix} I \\ \Theta \end{pmatrix}^{\mathrm{T}} w_\Delta + |\widehat{z}|^2 - \gamma^2|\widehat{w}|^2 - \sum_{i=1}^{2}\mu_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm{T}} \Psi_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} < 0, \\ &w_\Delta^{\mathrm{T}} \begin{pmatrix} I \\ \Theta \end{pmatrix} P \begin{pmatrix} I \\ \Theta \end{pmatrix}^{\mathrm{T}} w_\Delta - \widehat{x}^{\mathrm{T}} P \widehat{x} + |\widehat{z}|^2 - \gamma^2|\widehat{w}|^2 - \sum_{i=1}^{2}\mu_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm{T}} \Psi_i \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} < 0, \end{aligned} \tag{4.9}$$

where $\widehat{z} = \widehat{x}$ and $z_\Delta = \operatorname{col}(\widehat{x}, \widehat{w})$. System (4.3), dual to the original one (4.2), is immersed into the augmented system, and condition (4.5) holds. Therefore, we have inequality (3.6) along the trajectories of (4.3) for all $\Delta \in \mathbf{\Delta_{set}}$. By Lemma 3.1, for any $\Delta \in \mathbf{\Delta_{set}}$, the original uncertain system satisfies $\gamma_{g\infty}(\Delta, \Theta) < \gamma$ and consequently, $\gamma_*(\Theta) < \gamma$. Finally, we write inequalities (4.9) for the quadratic forms as matrix inequalities, introduce the new matrix variable $Q = \Theta P$, and apply Schur's complement lemma to get the LMIs (4.6) and (4.7), respectively. The proof of Theorem 4.1 is complete.

Remark 3. To find the upper bound of the γ_0 norm, it is necessary to eliminate the term I from the block located in the first row and first column of inequalities (4.6) (for the continuous-time system) or from the block in the second row and second column of inequalities (4.7) (for the discrete-time system). This follows from the fact that in the case of the γ_0 norm, the term $|\widehat{z}|^2$ vanishes in inequalities (4.8) and, accordingly, in inequalities (4.9). To find the upper bound of the conventional H_∞ norm, we should use Theorem 4.1 with $R = 0$.

Remark 4. According to the lossless S-procedure under two quadratic constraints (Theorem 4.1 in [13]), if $\mu_1\Psi_1 + \mu_2\Psi_2 > 0$ for some μ_1 and μ_2 (this LMI can be directly solved with respect to μ_1 and μ_2), then the corresponding inequality (4.8) is a sufficient and also necessary condition for the existence of the above function $\widehat{V}(\widehat{x}) = \widehat{x}^{\mathrm{T}} P \widehat{x}$ for the augmented system.

The minimum value of γ for which each of inequalities (4.6) or (4.7) is solvable will be denoted by $\gamma_{rob}(\Theta_{rob})$, where Θ_{rob} is the corresponding control parameter matrix. Since

$$\gamma_*(\Theta_*) \leqslant \gamma_*(\Theta_{rob}) \leqslant \gamma_{rob}(\Theta_{rob}),$$

where Θ_* is the parameter matrix of the robust generalized H_∞-optimal control law (2.15), then $\gamma_{rob}(\Theta_{rob})$ is the upper bound of the minimum damping level of the disturbances in the uncertain system with the robust generalized H_∞-optimal control law under given a priori and experimental data. In addition, Theorem 4.1 can be used to find out whether the guaranteed generalized H_∞ norm of the closed-loop uncertain system (4.1) with the feedback parameter matrix $\widehat{\Theta}$ is less than a given number γ^2. For this purpose, we should let $Q = \widehat{\Theta} P$ in inequality (4.6) for the continuous-time system or inequality (4.7) for the discrete-time system and solve the resulting inequality with respect to the variables P, μ_1, and μ_2.

5. AN ILLUSTRATIVE EXAMPLE

To illustrate the approach, we consider a discrete-time object of the form (2.1) of the fifth order ($n_x = 5$) with two control actions ($n_u = 2$), a five-dimensional disturbance ($n_w = 5$), and two target outputs ($n_z = 2$) with matrices whose elements were chosen randomly on the interval $[-1, 1]$. Thus, the system contains 49 unknown parameters. In the experiment, the initial conditions and the components of the control vector were chosen randomly on the interval $[-1, 1]$, and the disturbance was chosen randomly on the interval $[-d, d]$. In total, $N = 50$ measurements were taken. The weight matrix of the initial disturbance is $R = 0.01I_5$. Figure 2 shows three typical graphs of the squared damping levels of the disturbances in the closed loop system with the control law designed from the experimental data only, depending on the disturbance level d in the experiment. The solid curve corresponds to the square $\gamma_{rob}^2(\Theta_{rob})$ of the guaranteed generalized H_∞ norm under the control law with the parameter matrix Θ_{rob} obtained by solving the LMIs (4.7) with the minimum value of γ^2. The dashed-dotted curve corresponds to the square of the damping level $\gamma_{real} = \gamma_{g\infty}(\Delta_{real}, \Theta_{rob})$ of the disturbances, i.e., the generalized H_∞ norm of the closed loop system composed of the real object with the parameter matrix Δ_{real} (if it were known) and the feedback loop with the parameters Θ_{rob}. The dotted curve corresponds to the square of the damping level $\gamma_{prob} = \gamma_{g\infty}(\Delta_{prob}, \Theta_{rob})$ of the disturbances, i.e., the generalized H_∞ norm of the closed loop system composed of the trial object with the matrix $\Delta_{prob} = \Delta_{LS} + \Gamma^{1/2}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1/2}$, which lies on the boundary of the uncertainty ellipsoid $\boldsymbol{\Delta}_{\mathbf{set}}$ (see Lemma 2.1), and the feedback loop with the parameters Θ_{rob}. The growth of these curves with increasing the disturbance level d in the experiment can be explained as follows: for a higher value of d, we obtain a greater ellipsoid $\boldsymbol{\Delta}_{\mathbf{p}}$ of the unknown parameters Δ consistent with the experimental data.

According to Fig. 2, first, the curve γ_{rob}^2 majorizes with some margin the damping levels of the disturbances in the closed loop system for particular objects with the matrices Δ_{real} and Δ_{prob} from the set $\boldsymbol{\Delta}_{\mathbf{set}}$; second, under the control law with the parameter matrix Θ_{rob}, the generalized H_∞ norms of the closed loop systems slightly exceed (especially at small perturbation levels d) their minimum values $\gamma^2 \simeq 0.26$ for the completely known model. Note that the margin by which γ_{rob}^2

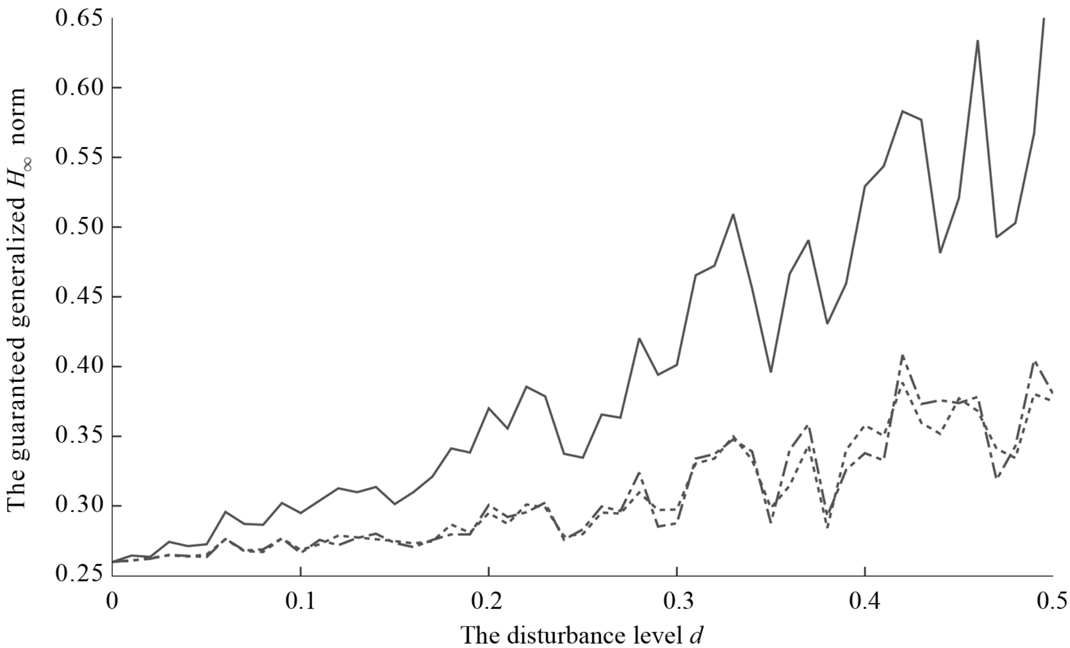

Fig. 2. The guaranteed generalized H_∞ norm and the generalized H_∞ norms for the real and trial objects as functions of the disturbance level in measurements.

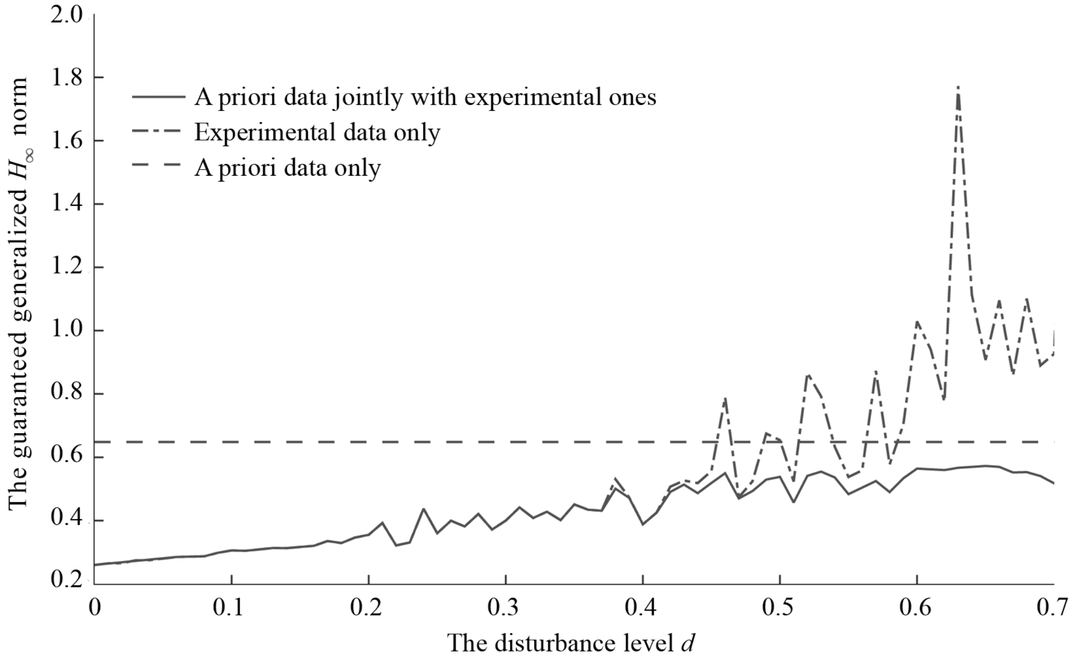

Fig. 3. The guaranteed estimates of the H_∞ norm as functions of the disturbance level in experimental data for different types of available information.

exceeds γ^2_{real} and γ^2_{prob} substantially depends on the experimental data and can be much smaller than on the graphs in Fig. 2.

Figure 3 presents the three guaranteed estimates of the generalized H_∞ norm based on different information (a priori data only, experimental data only, and a priori data jointly with experimental ones) as a function of the disturbance level d in the experiment. The a priori information was that the unknown matrices of the system satisfy condition (2.11) with $\rho = 0.1$ and $A_* = A_{real} + (\rho/2)I$, $B_* = B_{real}$, $C_* = C_{real}$, and $D_* = D_{real}$. Starting from some disturbance level in the experiment, the guaranteed estimates of the norms of the closed-loop uncertain system designed using both a priori and experimental data are much smaller than the corresponding estimates of the norms of the closed loop system with the control laws designed using only a priori or only experimental data.

Finally, we note the following aspect as well. Consider the object with the matrices A_{LS}, B_{LS}, C_{LS}, and D_{LS} constituting the parameter matrix Δ_{LS} obtained by the least squares method from the same experimental data. For this object, let us find the parameter matrix $\Theta_{LS} = QP^{-1}$ of the generalized H_∞-optimal feedback loop by solving the LMIs (3.4) with $Y = P$, $\mathcal{A}Y = A_{LS}P + B_{LS}Q$, $\mathcal{C}Y = C_{LS}P + D_{LS}Q$, and $\mathcal{B} = I$. This is essentially the so-called indirect H_∞-suboptimal adaptive control, i.e., the control law determined by estimating the unknown parameters of the object. If there are sufficiently many measurements and the information matrix is nonsingular, the generalized H_∞ norm of the closed loop system consisting of the real object and the feedback loop with $\Theta = \Theta_{LS}$ may be smaller than the corresponding guaranteed generalized H_∞ norm under the feedback loop with $\Theta = \Theta_{rob}$. In the latter case, we have an upper bound of the generalized H_∞ norm of the closed loop system for any object from the set $\mathbf{\Delta_{set}}$ consistent with the experimental data; however, for the feedback loop with $\Theta = \Theta_{LS}$, such an estimate can be obtained from inequalities (4.7) only under very small disturbance levels d (see the experimental results). In the example under consideration, for $d = 0.02$ we have $\gamma^2_{rob}(\Theta_{rob}) = 0.27$, whereas $\gamma^2_{rob}(\Theta_{LS}) = 41.77$; for $d > 0.03$, inequality (4.7) with $Q = \Theta_{LS}P$ becomes unsolvable.

6. CONCLUSIONS

This paper has been devoted to constructing generalized H_∞-suboptimal (as a special case, linear-quadratic) control laws for linear continuous- and discrete-time dynamic objects without precise mathematical models. As has been demonstrated above, for dynamic objects whose equations contain unknown parameters in some bounded sets, classical robust control methods based on a priori data can be applied, after an appropriate modification, to control design from a priori and experimental data. These methods consist in immersing an uncertain system into some enlarged system with additional input and output satisfying a quadratic inequality, applying the S-procedure, and reducing the problem to the design of H_∞-optimal control for the enlarged system. The modification is to characterize the control criterion (the generalized H_∞ norm of the system under initial and exogenous disturbances or the value of the quadratic functional under the initial disturbance only) in terms of the dual system, immerse the dual uncertain system into some enlarged system, and apply the technique of LMIs. As a result, the parameters of linear suboptimal feedback loops are expressed in terms of solutions of LMIs containing only a priori and experimental data. An illustrative example with a randomly generated fifth-order object has been provided to demonstrate that when a priori and experimental data are applied together, the quality of the control system is improved significantly.

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APPENDIX

Proof of Lemma 2.1. For the unknown matrix Δ_{real} in (2.3), we define the least-squares estimate Δ_{LS} minimizing the squared matrix norm of the residual with respect to Δ, i.e., the function $\|\widetilde{\Phi} - \Delta\widehat{\Phi}\|_F^2 = \mathrm{tr}\,(\widetilde{\Phi} - \Delta\widehat{\Phi})^{\mathrm{T}}(\widetilde{\Phi} - \Delta\widehat{\Phi})$. Equating the gradient of this function with respect to Δ to zero, $-2\widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}} + 2\Delta\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}} = 0$, yields the optimal estimate $\Delta_{LS} = \widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}$ under the assumption that the information matrix $\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}}$ is nonsingular. Next, we transform inequality (2.6) to

$$\Delta\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}}\Delta^{\mathrm{T}} - \widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}}\Delta^{\mathrm{T}} - \Delta\widehat{\Phi}\widetilde{\Phi}^{\mathrm{T}} + \widetilde{\Phi}\widetilde{\Phi}^{\mathrm{T}} - \widehat{\Omega} \leqslant 0,$$

writing the result as

$$[\Delta - \widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}](\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})[\Delta - \widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}]^{\mathrm{T}} \leqslant \Gamma,$$

where $\Gamma = \widehat{\Omega} + \widetilde{\Phi}[\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}\widehat{\Phi} - I]\widetilde{\Phi}^{\mathrm{T}}$. Substituting the expression for $\widetilde{\Phi}$ (2.3) into Γ and using (2.5) finally give

$$\Gamma = \widehat{\Omega} + \widehat{W}[\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}\widehat{\Phi} - I]\widehat{W}^{\mathrm{T}} \geqslant \widehat{W}\widehat{\Phi}^{\mathrm{T}}(\widehat{\Phi}\widehat{\Phi}^{\mathrm{T}})^{-1}\widehat{\Phi}\widehat{W}^{\mathrm{T}} \geqslant 0.$$

Proof of Lemma 3.1. We define a linear operator Γ mapping the pair $(x(0), v(t)) \in \mathrm{R}^n \times L_2(l_2) = \Xi$ (the initial state of the system and the input disturbance) into the target output $z(t) \in L_2(l_2) = \Upsilon$, i.e.,

$$\Gamma : \Xi = \mathrm{R}^{n_x} \times L_2(l_2) \to \Upsilon = L_2(l_2) : (x(0), v) \to z.$$

The inner products in these spaces are given by

$$\langle \cdot, \cdot \rangle_\Xi = x_1^{\mathrm{T}}(0)R^{-1}x_2(0) + \langle v_1(t), v_2(t)\rangle_{L_2(l_2)}, \quad \langle \cdot, \cdot \rangle_\Upsilon = \langle z_1(t), z_2(t)\rangle_{L_2(l_2)}.$$

Moreover, the generalized H_∞ norm coincides with the induced norm of this operator since

$$\|\Gamma\| = \sup_{(x_0,\, v)\neq 0} \frac{\|\Gamma(x_0,\, v)\|}{\|(x_0,\, v)\|} = \sup_{x_0,\, v\neq 0} \frac{\|z\|}{\left(x_0^{\mathrm{T}} R^{-1} x_0 + \|v\|^2\right)^{1/2}} = \gamma_{g\infty}.$$

We show that the adjoint operator Γ^* is given by

$$\Gamma^* : \Upsilon \to \Xi : \widehat{v}_a(t) \to (R\widehat{x}_a(0), \widehat{z}_a(t)),$$

where $\widehat{x}_a(t)$ and $\widehat{z}_a(t)$ satisfy equations (3.8) and (3.9) in the continuous- and discrete-time cases, respectively.

Indeed, for the continuous-time system, from equations (3.8) it follows that

$$\frac{d(x^{\mathrm{T}}\widehat{x}_a)}{dt} = v^{\mathrm{T}}\widehat{z}_a - z^{\mathrm{T}}\widehat{v}_a;$$

for the discrete-time system (see equations (3.9)),

$$x^{\mathrm{T}}(t+1)\widehat{x}_a(t+1) - x^{\mathrm{T}}(t)\widehat{x}_a(t) = v^{\mathrm{T}}(t)\widehat{z}_a(t) - z^{\mathrm{T}}(t)\widehat{v}_a(t).$$

Integrating in the former case or summing in the latter one, we obtain

$$< z, \widehat{v}_a > = x^{\mathrm{T}}(0)R^{-1}[R\widehat{x}_a(0)] + < v, \widehat{z}_a > .$$

Thus,

$$< \Gamma(x(0), v), \widehat{v}_a >_\Upsilon = < (x(0), v), \Gamma^*(\widehat{v}_a) >_\Xi .$$

Because the norms of the adjoint operators are equal,

$$\|\Gamma\| = \|\Gamma^*\| = \sup_{\widehat{v}_a\neq 0} \frac{\left(\|\widehat{z}_a\|^2 + \widehat{x}_a^{\mathrm{T}}(0)R\widehat{x}_a(0)\right)^{1/2}}{\|\widehat{v}_a\|}.$$

Next, we establish that $\|\Gamma^*\| < \gamma$ iff there exists a function $V(\widehat{x}_a) = \widehat{x}_a^{\mathrm{T}} P \widehat{x}_a$ with $P > R$ such that

$$\begin{gathered} \dot{V}(\widehat{x}_a(t)) - |\widehat{z}_a(t)|^2 + \gamma^2|\widehat{v}_a(t)|^2 > 0 \text{ or} \\ \Delta V(\widehat{x}_a(t)) - |\widehat{z}_a(t)|^2 + \gamma^2|\widehat{v}_a(t)|^2 > 0 \end{gathered} \tag{A.1}$$

along the trajectories of the continuous-time system (3.8) or along the trajectories of the discrete-time system (3.9), respectively.

Indeed, integrating the former inequality or summing the latter one with $P > R$, we arrive at $\|\widehat{z}_a\|^2 + \widehat{x}_a^{\mathrm{T}}(0)R\widehat{x}_a(0) < \gamma^2\|\widehat{v}_a\|^2$ for all $\widehat{v}_a(t)$, i.e., $\|\Gamma^*\| < \gamma$. Conversely, let $\|\Gamma^*\| < \gamma$, which implies $\|\Gamma\| < \gamma$. According to [11, 12], this means the existence of a function $V(x) = x^{\mathrm{T}}Y^{-1}x$ with a matrix Y satisfying inequalities (3.3) in the continuous-time case or inequalities (3.4) in the discrete-time case. Here, we consider the former case only: the proof for the discrete-time system is analogous. Using Schur's complement lemma, the first inequality in (3.3) can be transformed to

$$\begin{pmatrix} Y\mathcal{A}^T + \mathcal{A}Y + \gamma^{-2}\mathcal{B}\mathcal{B}^{\mathrm{T}} & \star \\ \mathcal{C}Y & -I \end{pmatrix} < 0.$$

With the change of variables $Y = \gamma^{-2}P$, this condition is equivalently written as the following inequality for the quadratic form in the abstract variables $\widehat{x}_a$ and $\widehat{v}_a$:

$$2\widehat{x}_a^{\mathrm{T}} P(-\mathcal{A}^T\widehat{x}_a - \mathcal{C}^{\mathrm{T}}\widehat{v}_a) - \widehat{x}_a^{\mathrm{T}}\mathcal{B}\mathcal{B}^{\mathrm{T}}\widehat{x}_a + \gamma^2\widehat{v}_a^{\mathrm{T}}\widehat{v}_a > 0.$$

It obviously coincides with the first inequality in (A.1). Due to $Y > \gamma^{-2}R$, the function $V_a(\widehat{x}_a) = \widehat{x}_a^{\mathrm{T}} P \widehat{x}_a$ with $P > R$ satisfies the first inequality in (A.1) along the trajectories of system (3.8). Thus, $\|\Gamma^*\| < \gamma$ and consequently, $\|\Gamma\| = \gamma_{g\infty} < \gamma$ iff the corresponding inequality in (A.1) holds along the trajectories of system (3.8) or (3.9). Reverting the time, we finally pass from equations (3.8) or (3.9) to system (3.7), along whose trajectories the function $V(x_a) = x_a^{\mathrm{T}} P x_a$ will satisfy the corresponding inequality in (3.6).

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