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Velocity of Flow on Regular Non-Homogeneous Open One-Dimensional Net with Non-Symmetrical Arrangement of Nodes

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Abstract—A system is studied such that this system belongs to the class of dynamical systems called the Buslaev nets. This class has been developed for the purpose of creating traffic models on network structures such that, for these models, analytical results can be obtained. There may be other network applications of Buslaev nets. The considered system is called an open chain of contours. Segments called clusters move along circumferences (contours) according to prescribed rules. For each contour (except the leftmost and rightmost contours) there are two adjacent contours. Each of the leftmost and rightmost contours has one adjacent contour. There is a common point (node) for any two adjacent contours. Results have been obtained on the average velocity of cluster movement, taking into account delays during the passage through nodes. These results generalize the results obtained previously for a particular case of the system under consideration.

Keywords: dynamical systems, mathematical traffic models, Buslaev nets

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1. INTRODUCTION

A class of mathematical traffic models consists of models in which particles move in a one-dimensional or two-dimensional lattice. These models can be interpreted as cellular automata [1] or exclusion processes [2]. In [3], the Nagel-Schreckenberg traffic model has been introduced. This model were studied by a number of authors. In this model, an infinite or closed lattice is a sequence of cells, and particles move along the lattice according to prescribed rules. Analytical results for simple versions of models of this class have been obtained, e.g., in [4–10]. Models with network structures, belonging to this class of models, were studied mainly by simulation.

The paper [7] (a preprint of this paper was published in 1999), the movement of particles along closed lattice is considered. It is assumed that, at each step, each particle moves onto one cell forward if the cell ahead is vacant, and the particle does not move if this cell is occupied. Suppose a cellular automaton corresponds to the system. If a cell of the automaton is in the state 1, then the cell corresponds to an occupied cell of the system, and, if a cell of the automaton is in the state 0, then the cell corresponds to a vacant cell. As it noted in [7], this automaton is the elementary cellular automaton 184 in terms of S Wolfram classification [1]. According to the results obtained in [7], if the ratio of the number of particles to the number of cells (particle density) does not exceed 1/2, then, for any initial state, from a certain point in time, all particles move at each moment without delays (free movement, self-organization). If the density is greater than 1/2, then the average velocity of particles (the ratio of average number of moving particles per a time unit) is equal to $(1-\rho)/\rho$ where ρ is the density. Analogous results were obtained independently in [5], where moreover an upper bound for the time it takes for the system to reach limit mode. In [6], analytical results were obtained for more general model. In this model, with prescribed probability, a particle moves from the cell i to the cell $i+1$ provided that the cell $i+1$ is vacant. This probability depends on the states of cells $i-1$ and $i+2$ (cells are numbered in the direction of movement). The behavior of the system was studied in [6] for particular cases. In [8], a formula has been obtained for the average velocity of particles in the stochastic traffic model such that, in this model, at any step, with prescribed probability, each particle moves onto one cell forward if the cell ahead is vacant. Some generalisations of results found in [5, 7, 8] were obtained in [9] where a dynamical system with continuous state space was studied. In particular cases, this system is equivalent to systems considered in [5, 7, 8]. In [10], a stochastic traffic model is studied. In this model, particles move along a not closed lattice containing a finite number of cells. Particles appear at one end of the lattice and move in the direction to the opposite end. After reaching this end, the particles leave the lattice. In [10], a matrix approach has been developed to analyze the system.

In [11], a two-dimensional traffic model was proposed (the Biham–Middleton–Levine model, BML model). In this model, particles move along two-dimensional lattice in orthogonal directions. Particles of the first type move along rows, and particles of the second type move along columns. The rule of particle movement is a two-dimensional counterpart of the elementary automaton 184. This analogy is noted in [11]. In [11–18], different versions of BML model was considered, and analytical results have obtained mainly regarding conditions for self-organization and conditions for jam).

In [19], a graph with a variable configuration of particles has been introduced. The developed approach makes it possible to simulate phenomena that arise in complex networks (for example, in transport and social network).

The book [20] is a monograph on mathematical modeling of traffic flows. Based on the material presented in the book, one can see that quite few approaches are known to analytical study of traffic flows with network structure. This determines that it is relevant to develop new such approaches. One of these approaches is the development of models based on Buslaev nets.

In [21], the concept of cluster movement in transport models was introduced. In discrete version, clusters are groups of particles located in adjacent cells and moving simultaneously. In this case, the movement of particles corresponds to the rule of an elementary cellular automaton 240. In the continuous version, clusters are moving segments and are called clusters by analogy with a discrete version.

A.P. Buslaev has developed a class of dynamical systems, which now are called Buslaev nets [22]. A Buslaev net is a dynamic system containing a system of contours. Adjacent contours have common points, called nodes. In the discrete version, a contour is a closed sequence of particles. In the continuous version, the contour is represented as a circumference. Segments called clusters move along contours according to prescribed rules. As particles pass through nodes, delays occur due to the fact that more than one particle cannot pass through a node at the same time. The main problems in study of Buslaev nets are to find the average velocity of particles (clusters), conditions for the system to enter a state of free movement (starting from a certain moment, all clusters move without delay at the current moment and in the future) or collapse (from a certain moment no particle moves). Analytical results have been obtained for two-contour nets with one [23] or two [24] common nodes and Buslaev nets with regular periodic structures [25–30].

In [26], a Buslaev net called an open chain of contours is considered. In [26], a version of open chain is studied such that this net is an open chain with continuous time scale. There is a common point (node) for a contour and each adjacent contour except for the leftmost and rightmost contours. Each of two later contours have one adjacent contour. There is a common node for two adjacent contours. In each contour, there is a cluster. The length of each contour is the same. Each contour is divided by two nodes into parts of the same length. It has been proved that if the length of cluster is not greater than half the length of the contour, then the system from a certain moment is in a state free movement, i.e., all clusters move without delay, and if the cluster length is greater than half contour length, then all clusters move with the same average velocity, which is less than the velocity of free movement and independent of the initial state of the system. A formula for the average velocity of clusters was obtained. In [27], the average velocity of clusters has been obtained for a version of open chain with discrete state space and discrete time provided that the length of each contour is the same (the same number of cells in contour) and each contour is divided by nodes into two parts of the same length. Examples are given showing that, in the general case, clusters can move with unequal average velocities, and the average velocity can depend on the initial state of the system. In [30], the limit distribution (invariant measure) has been found for an open chain with contours of the same length and clusters of unequal length.

In [23], an asymmetrical two-contour system with one node was considered. This system is a particular case of a heterogeneous open chain of contours, for which the number of contours is equal to two. The following example of a possible application of the results on contour networks is given. Let, during the working day, raw materials or fuel are constantly delivered to two departments of the enterprise from the warehouse. The cargo is delivered by vehicles such as trucks along narrow-gauge tracks. The warehouse is located at the intersection of these paths. A vehicle that arrives to a warehouse while another vehicle is loading waits for service to complete and then begins loading. Suppose that the ith vehicle travels from the warehouse to the department and back in time $c_i - l_i$, taking into account the unloading time in the department, and loading time for the vehicle at the warehouse lasts l_i units of time, $i = 1, 2$. Then the process of vehicles movement is modeled by a system of the type under consideration with contour lengths equal to c_1 and c_2 and cluster lengths equal to l_1 and l_2, respectively, under the assumption that, in the absence of delays, each cluster moves at unit velocity.

This example can be generalized. Let us assume that there are three departments of the enterprise, two warehouses and three vehicles, each of which delivers cargo to the department to which the vehicle is related. Cargo is delivered to the department 1 from the warehouse 1. Cargo is delivered to the department 3 from the warehouse 2. Cargo is delivered to the department 2 alternately from both warehouses. The transport work process may be modeled by a heterogeneous three-contour open chain of contours. Each contour corresponds to a vehicle path. Each of the two nodes corresponds to one of the warehouses.

In this paper, we prove a theorem regarding the behavior of the system in the case when the length of the contour and the length of moving cluster depend on the contour number, and contours can be divided by the nodes into parts of unequal lengths. It is assumed that the clusters have sufficiently large lengths. The formula is obtained for the cluster velocity. This formula is a generalization of the formula obtained in the paper [27] for the particular case considered in that paper.

2. DESCRIPTION OF SYSTEM

Suppose a dynamical system, Fig. 1. The system contains N circumferences called *contours.* The length of the cluster i is equal to c_i, $i = 1, \ldots, N$. The coordinate system $[0, c_i)$ is given for the contour i, $i = 1, \ldots, N$. There is a common point of the nodes i and $i+1$ called the *node*

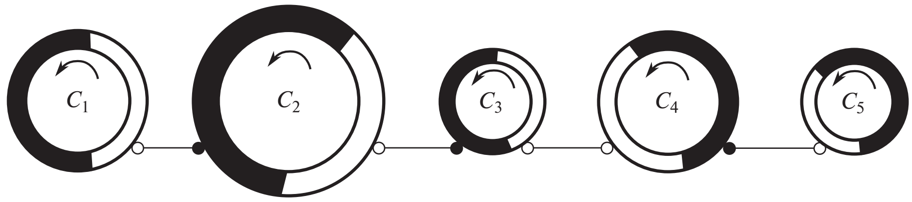

Fig. 1. An open chain of contours, $N = 5$, c_i is the length of the contour i, l_i is the length of the cluster i, $i = 1, \ldots, 5$.

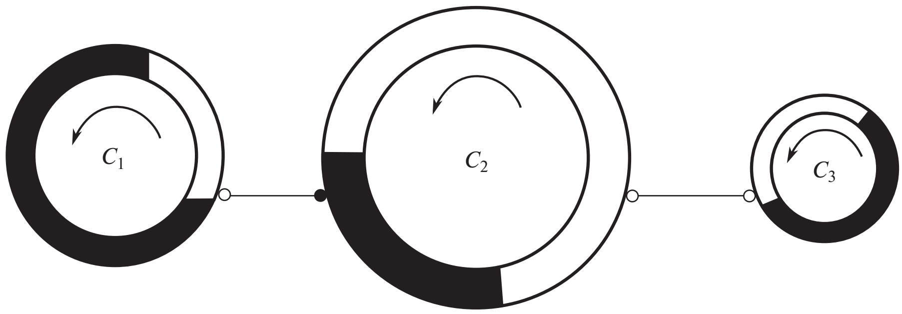

Fig. 2. The cluster 2 occupies the node $(1, 2)$, $l_2 + d_2 < c_2$. A delay of the cluster 1 at the node (1,2).

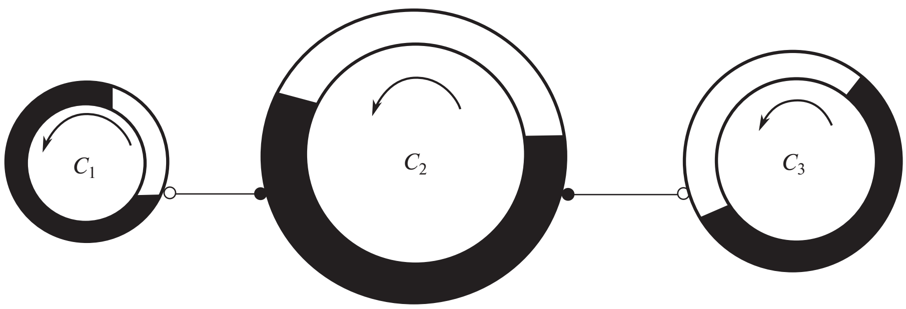

Fig. 3. The cluster 2 occupies the node $(1, 2)$, $l_2 + d_2 > c_2$. A delay of the cluster 1 at the node (1,2).

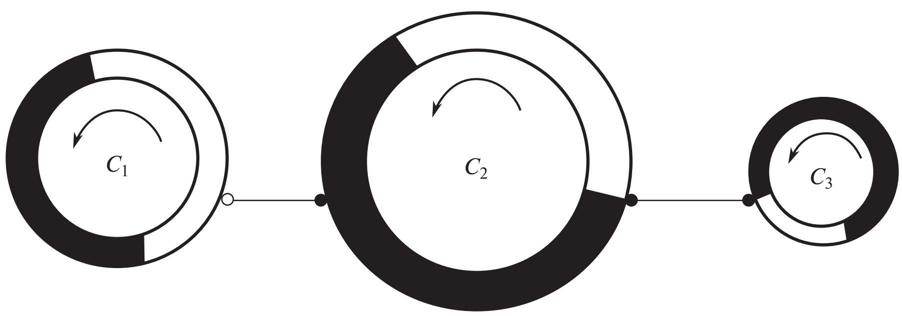

Fig. 4. A competition of the clusters 2 and 3.

$(i, i+1)$, $i = 1, \ldots, N-1$. For the contour i, the coordinate of the node $(i, i+1)$ is equal to 0, and, for the contour $i+1$, the coordinate of the node $(i, i+1)$ is equal to $d_{i+1} > 0$, $i = 1, \ldots, N-2$. Suppose the coordinate of the node $(N-1, N)$ is equal to 0 in both the contour $N-1$ and the contour N. There is a moving segment of the length l_i, $i = 1, \ldots, N$. The segment is called a *cluster*. The direction of the coordinate axis i is the same as the direction of the cluster movement, $i = 1, \ldots, N$. If delays of a cluster do not occur, then the cluster moves with velocity 1, i.e., the cluster i makes a full revolution in c_i units of time, $i = 1, \ldots, N$. If, at time $t \geq 0$, the coordinate i

is equal to $x_i(t)$, then the cluster i is located on the arc $(x_i(t) - l_i, x_i(t))$ (subtraction modulo c_i), $i = 1, \dots, N$. The *state* of the system at time t is the vector $x(t) = (x_1(t), \dots, x_N(t))$, where $x_i(t)$ is the coordinate of the leading point of the cluster i, $i = 1, \dots, N$. We say that, at time t, the cluster i occupies the node $(i, i+1)$ if $0 < x_i(t) < l_i$, $i = 1, \dots, N-1$. We say that, at time t, the cluster i occupies the node $(i-1, i)$ if $d_i < x_i(t) < d_i + l_i$ (for $d_i + l_i < c_i$), Fig. 2, or $0 \le x_i(t) < d_i + l_i - c_i$ (for $d_i + l_i \ge c_i$), Fig. 3, $i = 2, \dots, N-1$, Fig. 4, $0 < x_i(t) < l_i$, $i = N$. We say that, at time t, the cluster i is at the node $(i, i+1)$ if $x_i(t) = 0$, $i = 1, \dots, N-1$. We say that, at time t, the cluster i is at the node $(i-1, i)$ if $x_i(t) = d_i$, $i = 2, \dots, N-1$; $x_N(t) = 0$. The state is *admissible* if no node is occupied by two clusters. The system state at time $t = 0$ (*the initial state*) prescribed and must be admissible. A delay of cluster is at a node such that, at the moment, this node is occupied by the cluster of adjacent contour. The delay ends when this node stops being busy. If clusters i and $i+1$ are simultaneously located at node $(i, i+1)$, then *competition* between these clusters occurs (Fig. 4) and the cluster passes through the node first, chosen in accordance with deterministic or stochastic competition resolution rule. If the conditions of Theorem 1, which is proved in Section 4, hold, then, for any competition resolution rule, and the cycle is implemented such that no competitions occur, and the cycle is independent of the initial state.

3. LIMIT CYCLES. THE AVERAGE VELOCITY

The system is deterministic, and its behavior in the future is determined by its state at present. Therefore, if at some moment a state is repeated, then from the moment the states of the system will be periodically repeated, forming a cycle (limit cycle).

Let $H_i(t)$ be a total distance traveled by the cluster i in the time interval $(0, t)$. Then, if there exists the limit

$$v_i = \lim_{t \to \infty} \frac{H_i(t)}{t},$$

then this limit is called the average velocity of the cluster i, $i = 1, \dots, N$. It is obvious that, if a cycle is implemented, then this limit exists, and the limit equals the ratio of the distance traveled by the cluster during the cycle to the period.

4. BEHAVIOR OF THE SYSTEM

In Section 4, statements about the behavior of the system are proved.

Lemma 1. *If*

$$\frac{l_i}{c_i} + \frac{l_{i+1}}{c_{i+1}} > 1 \tag{1}$$

i *and* $i+1$ $(1 \le i \le N-1)$, *then the average velocity of at least one of the clusters* i *and* $i+1$ *is less than 1.*

Proof. Suppose there exists i such that $v_i = v_{i+1} = 1$, and therefore the clusters i and $i+1$ $(1 \le i \le N-1)$ move without delays. Then the first term in the left side of the inequality (1) is the limit of the ratio of the total time in the interval $(0, t)$ during that the node $(i, i+1)$ is occupied by the cluster i to t as $t \to \infty$, and the second term is the limit of the ratio of the total time in the interval $(0, t)$ during that the node $(i, i+1)$ is occupied by the cluster $i+1$ to t as $t \to \infty$. Therefore the sum of these terms is not greater than 1 since the node cannot be occupied by two clusters simultaneously. The contradiction proves Lemma 1.

The following theorem characterizes the behavior of the system for sufficiently large cluster lengths (heavy load).

Theorem 1. *Suppose the following conditions hold*

$$l_i > \max(d_i, c_i - d_i), \quad i = 2, \dots, N-1, \tag{2}$$

$$l_1 + l_i > c_i, \quad i = 2, \dots, N, \tag{3}$$

$$l_i + l_N > c_i, \quad i = 1, \dots, N-1, \tag{4}$$

and at least one of two conditions holds

$$\frac{l_1}{c_1} + \frac{l_2}{c_2} > 1, \tag{5}$$

$$\frac{l_{N-1}}{c_{N-1}} + \frac{l_N}{c_N} > 1. \tag{6}$$

Then, for any deterministic or stochastic competition resolution rule, the same limit cycle is implemented, in which no competitions occur. The period of the cycle is equal to

$$T = l_1 + l_N + 2\sum_{j=2}^{N-1} l_j - \sum_{j=2}^{N-1} c_j. \tag{7}$$

The average velocity of the cluster i equals

$$v_i = \frac{c_i}{l_1 + l_N + 2\sum\limits_{j=2}^{N-1} l_j - \sum\limits_{i=2}^{N-1} c_i}, \quad i = 1, \dots, N. \tag{8}$$

Proof. Suppose the conditions (2)–(5) hold. According to Lemma 1, a delay of at least one of the clusters 1 and 2 occur.

Suppose, at time t_1, a delay of the cluster 2 begins at the node $(1,2)$. Then we have $0 < x_1(t_1) < l_1$, and the movement of cluster 2 resumes at time $t_0 = t_1 + l_1 - x_1(t_1)$, and

$$x_1(t_0) = l_1,\ x_2(t_0) = d_2. \tag{9}$$

Suppose, at time t_1, a delay of the cluster 1 at the node $(1,2)$. If, at time $t_2 > t_1$, this delay ends, then $x_1(t_2) = 0$, $x_2(t_2) = d_2 + l_2 - c_2$. For $t_3 = t_2 + c_2 - l_2$, we have $x_2(t_3) = d_2$. According to (3), we have $c_2 - l_2 < l_1$. Therefore, at time t_2, the cluster 1 occupies the node $(1,2)$, and a delay of the cluster 2 begins at the node $(1,2)$. Let us make sure that the delay of the cluster 1, starting at time t_1, will end. If the delay of the cluster 1 never ends, then, from time t, the cluster 1 is at the node $(1,2)$. We can prove by induction that, for $2 \le i \le N-1$, from some moment, the cluster i is, at the node $(i, i+1)$ at present and in the future. There exists t such that, at time t, the system is in the state such that $x_{N-1}(t) = 0$, $x_N(t) = l_N - \frac{c_N}{2}$. After this moment, the cluster $N-1$ begins to move. Hence there exists a moment such that the movement of a cluster is resumed. Let t_4 be the minimum value $t_4 > t_0$ such that, at time t_4, the movement of the cluster $1 \le i_0 \le N-1$ is resumed, and the clusters $1, 2, \dots, i_0 - 1$ do not move. Then, at time

$$t_4 + \sum_{k=0}^{i_0-j-1} (d_{i_0-k} + l_{i_0-j} - c_{i_0-k}),$$

the movement of the cluster $j = 1, \dots, i_0 - 1$ is resumed. In particular, the movement of the cluster 1 is resumed.

Thus, there exists t_0 such that (9) holds.

At time $t_0 - l_1$, the cluster 2 was at the node $(1, 2)$, and, from this time, the cluster occupied this node. If (2) holds, then at least each of the clusters $i = 2, \dots, N-1$ occupies at least one node. Hence, at time $t_0 - l_1$, the cluster 2 occupies the node $(2, 3)$. At time $t_0 - l_1$, each of the clusters $i = 2, \dots, N-1$ occupies the node $(i, i+1)$. The proof is by induction. Therefore,

$$d_i + l_i - c_i \leqslant x_i(t_0) \leqslant d_i, \quad i = 3, \dots, N-1. \tag{10}$$

Combining (2), (3), (9), (10), we get that, at time t_0, the system is in the state

$$x(t_0) = (l_1, d_2, \dots, d_{N-1}, 0).$$

At time t_0, the movement of the cluster 2 begins and the movement of the cluster 1 continues. The movement of the cluster i is resumed at time

$$t(i) = t_0 + \sum_{j=2}^{i-1}(l_j - d_j),\ i = 3, \dots, N.$$

The cluster this cluster finds the node occupied and waits for it to become free. According to (2), (4), each of the clusters $i = 1, \dots, N$, approaching the node $(i, i+1)$, waits for its release. At time $u_0 + a$

$$a = l_N + \sum_{j=2}^{N-1}(l_j - d_j), \tag{11}$$

the system is in the state

$$x(t_0 + a) = (0, \dots, 0, l_N).$$

The movement of the cluster i is resumed at time

$$u(i) = t_0 + a + \sum_{j=i+1}^{N-1}(d_j + l_j - c_j), \quad i = 2, \dots, N-1.$$

According (2), (3), at time

$$u(1) = t_0 + \sum_{i=2}^{N-1}(d_i + l_i - c_i),$$

the movement of the cluster 1 is resumed at time $u_0 = t_0 + a + b$, the system enters the state

$$x(u_0) = x(t_0 + a + b) = (l_1, d_2, \dots, d_{N-1}, 0) = u(t_0), \tag{12}$$

$$b = l_1 + \sum_{i=2}^{N-1}(d_i + l_i - c_i). \tag{13}$$

Combining (11), (13), we obtain

$$a + b = l_1 + l_N + 2\sum_{j=2}^{N-1} l_i - \sum_{i=2}^{N-1} c_i. \tag{14}$$

Therefore a cycle with period $a + b$ is implemented. During the cycle, there is no competition between clusters simultaneously located at the same node. Each cluster makes one revolution during the cycle. Using (14), we obtain the theorem under the conditions (2)–(5).

The proof does not take into account the competition resolution rule. Therefore, due to symmetry, condition (5) should be replaced by condition (6). The theorem has been proved.

Corollary 1. *Under the conditions of the theorem, the period* (7) *of the implemented cycle and the average velocities* (8) *of the clusters do not depend on* $d_2, \ldots, d_{N-1}$.

The statement follows from Theorem 1.

Suppose

$$c_1 = \cdots = c_N = 1, \quad d_2 = \cdots = d_{N-1} = \frac{1}{2}, \tag{15}$$
$$l_i > \frac{1}{2}, \quad i = 1, \ldots, N,$$

i.e., the length of each contours is the same(without loss of generality, we assume that this length equals 1), the nodes divide the contour into two equal parts, and the length of each cluster is greater than half the length of the contour. If (15) holds, then (7), (8) has form [21]

$$T = l_1 + l_N + 2\sum_{i=2}^{N-1} l_i - N + 2, \tag{16}$$

$$v_i = \frac{1}{l_1 + l_N + 2\sum\limits_{i=2}^{N-1} l_i - N + 2}, \quad i = 1, \ldots, N. \tag{17}$$

Note that, in this case, the average velocity of any cluster is the same. In [22], it has been shown by examples that, for a discrete open chain with the contours of the same length and the clusters of different lengths, among which there can be clusters of lengths greater than half the length of the contours, and clusters of lengths not greater than half the length of the contours the average velocity can depend on the initial state, and the average velocities of clusters can be different for the same initial state of the system. Let us give an example for the considered continuous chain.

Example 1. Suppose the conditions (15), $N = 3$, $l_1 = l_3 = 0.75$, $l_2 = 25$ hold. Let $(0.75, 0.5, 0.25)$ be the initial state. Then any cluster comes to a node when the node is not occupied. Indeed, we have $x(0) = (0.75, 0.5, 0.75)$, $x(0,5) = (0.25, 0, 0.25)$, $x(1) = (0.75, 0.5, 0.75) = x(0)$. Hence the initial state belongs to the cycle with the velocity 1. If the initial state is $(0.25, 0.5, 0.75)$, then we have $x(0) = (0.25, 0.5, 0.75)$, $x(0.5) = (0.75, 0.5, 0.25)$, $x(1) = (0.25, 0, 0.75)$, $x(1.5) = (0.75, 0, 0.25)$, $x(2) = (0.25, 0.5, 0.75) = x(0)$. Hence, during the cycle with period 2, the clusters 1 and 3 move without delays and make two revolutions, and the cluster 2 makes only one revolution. Therefore, $v_1 = 1$, $v_2 = 1/2$, $v_3 = 1$. Hence, there exists an initial state such that the velocity of any cluster is equal to 1, and there exists an initial state such that the velocity of the clusters 1 and 3 is equal to 1, and the average velocity of the cluster 2 is equal to 1/2.

If the conditions (15) and $l_1 = \cdots = l_n = l > 1/2$ hold, then (16), (17) have the form [18]

$$T = 2(N-1)l - N + 2,$$
$$v_i = \frac{1}{2(N-1)l - N + 2}, \quad i = 1, \ldots, N.$$

Under the conditions (15) and $l_1 = \cdots = l_N = l \leq 1/2$, it has been proved in [19] that the system results in the state of free movement from any initial state.

Thus, for an open chain with contours of the same length and clusters of the same length, the average velocity is the same for all clusters and does not depend on the initial state of the system in contrast to a closed chain with contours of equal length and clusters of different lengths [27], for which the average velocity of clusters depends on the initial state. In [23], the limit state distribution (invariant measure) for an open chain with contours of the same length and clusters with the same length under the condition $l > 1/2$ has been found.

5. CONCLUSION

A theorem has been proved about the behavior of a dynamical system called an open chain of contours. The system belongs to the class of Buslaev nets. Previously, the system was considered under the assumption that all contours have the same length and the nodes divide the contours into parts of the same length. In this paper, it is supposed that the lengths of the contours may be different. Each contour can be divided into parts of different lengths. The system is considered under the assumption that clusters located in the contours have sufficiently large lengths. It has been proved that under the considered assumptions the limit cycle of the system is unique. The average cluster velocities and the period of the limit cycle are found. The results of the work can be used in traffic modeling, and also have other applications, in particular, in infocommunication systems modeling.

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REFERENCES

1. Wolfram, S., Statistical mechanics of cellular automata, *Rev. Mod. Phys.*, 1983, vol. 55, pp. 601–644. https://doi.org/10.1103/RevModPhys.55.601
2. Spitzer, F., Interaction of Markov processes, *Advances in Mathematics*, 1970, vol. 5, no. 2, pp. 246–290.
3. Nagel, K. and Schreckenberg, M., A cellular automaton model for freeway traffic, *J. Phys. I*, 1992, vol. 2, no. 12, pp. 2221–2229. https://doi.org/10.1051/jp1:1992277
4. Schreckenberg, M., Schadschneider, A., Nagel, K., and Ito, N., Discrete stochastic models for traffic flow, *Phys. Rev. E*, 1995, vol. 51, pp. 2939–2949. https://doi.org/10.1103/PhysRevE.51.2939
5. Blank, M.L., Exact analysis of dynamical systems arising in models of traffic flow, *Russian Mathematical Surveys*, 2000, vol. 55, no. 3, pp. 562–563. https://doi.org/10.1070/RM2000v055n03ABEH000295
6. Gray, L. and Griffeath, D., The ergodic theory of traffic jams, *J. Stat. Phys.*, 2001, vol. 105, no. 3/4, pp. 413–452.
7. Belitsky, V. and Ferrari, P.A., Invariant measures and convergence properties for cellular automation 184 and related processes, *J. Stat. Phys.*, 2005, vol. 118, no. 3/4, pp. 589–623. https://doi.org/10.1007/s10955-004-8822-4
8. Kanai, M., Nishinari, K., and Tokihiro, T., Exact solution and asymptotic behaviour of the asymmetric simple exclusion process on a ring, *J. Phys. A: Mathematical and General*, 2006, vol. 39, no. 29, 9071. https://doi.org/10.1088/0305-4470/39/29/004
9. Blank, M., Metric properties of discrete time exclusion type processes in continuum, *J. Stat. Phys.*, 2010, vol. 140, no. 1, pp. 170–197. https://doi.org/10.1007/s10955-010-9983-y
10. Evans, M.R., Rajewsky, N., and Speer, E.R., Exact solution of a cellular automaton for traffic, *J. Stat. Phys.*, 2010, vol. 95, pp. 45–56. https://doi.org/10.1023/A:1004521326456
11. Biham, O., Middleton, A.A., and Levine, D., Self-organization and a dynamic transition in traffic-flow models, *Phys. Rev. A*, 1992, vol. 46, no. 10, pp. R6124–6127. https://doi.org/10.1103/PhysRevA.46.R6124
12. Angel, O., Holroyd, A.E., and Martin, J.B., The Jammed Phase of the Biham-Middleton-Levine Traffic Model, *Electronic Communications in Probability*, 2005, vol. 10, paper 17, pp. 167–178. https://doi.org/10.48550/arXiv.math/0504001
13. D'Souza, R.M., Coexisting phases and lattice dependence of a cellular automata model for traffic flow, *Physical Review E*, 2005, vol. 71, 0066112.
14. D'Souza, R.M., BML revisited: Statistical physics, computer simulation and probability, *Complexity*, 2006, vol. 12, no. 2, pp. 30–39.

15. Austin, T. and Benjamini, I., For what number must self organization occur in the Biham-Middleton-Levine traffic model from any possible starting configuration?, *arXiv preprint math/0607759*, 2006.

16. Pan Wei, Xue Yu, Zhao Rui, and Lu Wei-Zhen, Biham–Middleton–Levine model in consideration of cooperative willingness, *Chin. Phys. B*, 2014, vol. 23, no. 5, 058902. https://doi.org/10.1088/1674-1056/23/5/058902

17. Wenbin Hu, Liping Yan, Huan Wang, Bo Du, and Dacheng Tao, Real-time traffic jams prediction inspired by Biham, Middeleton and Levine (BML), *Information Sciences*, 2017, pp. 209–228. https://doi.org/10.1016/j.ins.2016.11.023

18. Moradi, H.R., Zardadi, A., and Heydarbeygi, Z., The number of collisions in Biham–Middleton–Levine model on a square lattice with limited number of cars, *Appl. Math. E-Notes*, 2019, vol. 19, pp. 243–249.

19. Malecky, K., Graph cellular automata with relation-based neighbourhoods of cells for complex systems modelling: A case of traffic simulation, *Symmetry*, 2017, vol. 9, 322. https://doi.org/10.3390/sym9120322

20. Gasnikov, A.V. et al., *Introduction to mathematical modeling of traffic flows*, 2nd ed., Gasnikov, A.V., Ed., Moscow: MTsNMO, 2013.

21. Bugaev, A.S., Buslaev, A.P., Kozlov, V.V., and Yashina, M.V., Distributed problems of monitoring and modern approaches to traffic modeling, *2011 14th International IEEE Conference on Intelligent Transportation Systems (ITSC)*, Washington, USA, 5–7 October 2011, pp. 477–481. https://doi.org/10.1109/ITSC.2011.6082805

22. Kozlov, V.V., Buslaev, A.P., and Tatashev, A.G., On synergy of totally connected flows on chainmails, *Proc. of the 13th International Conference of Computational and Applied Methods in Science and Engineering*, Almeria, Spain, 24–27 June 2013, vol. 3, pp. 861–874.

23. Myshkis, P.A., Tatashev, A.G., and Yashina, M.V., Cluster motion in a two-contour system with priority rule for conflict resolution, *Journal of Computer and Systems Sciences International*, 2020, vol. 59, no. 3, pp. 311–321. Translated from *Izvestiya RAN. Teoriya i Sistemy Upravleniya*, 2020, vol. 20, no. 3, pp. 3–13. https://doi.org/10.1134/S1064230730030119

24. Yashina, M. and Tatashev, A., Spectral cycles and average velocity of clusters in discrete two-contours system with two nodes, *Math. Meth. Appl. Sci.*, 2020, vol. 43, no. 7, pp. 4303–4316. https://doi.org/10.1002/mma.6194

25. Buslaev, A.P., Tatashev, A.G., and Yashina, M.V., Qualitative properties of dynamical system on toroidal chainmails, *AIP Conference Proceedings*, 2013, vol. 1558, pp. 1144–1147. https://doi.org/10.1063/1.4825710

26. Buslaev, A.P. and Tatashev, A.G., Spectra of local cluster flows on open chain of contours, *Eur. J. Pure Appl. Math.*, 2018, vol. 11, no. 3, pp. 628–641. https://doi.org/10.29020/nybg.ejpam.11i3.3292

27. Yashina, M. and Tatashev, A., Discrete open Buslaev chain with heterogeneous loading, *2019 7th International Conference on Control, Mechatronics and Automation (ICCMA)*, 6–8 Nov. 2019, Delft, Netherlands, pp. 283–288. https://doi.org/10.1109/ICCMA46720.2019.8988654

28. Bugaev, A.S., Yashina, M.V., Tatashev, A.G., and Fomina, M.Yu., On velocity spectrum for saturated flows on a regular open one-dimensional network, *XI All-Russian Multiconference on Management Problems MKPU-2021, Material of the XIV Multiconference: in 4 volumes*, Rostov-on-Don, 2021, pp. 41–44.

29. Bugaev, A.S., Tatashev, A.G., and Yashina, M., Spectrum of a continuous closed symmetric chain with an arbitrary number of contours, *Mathematical Models and Computer Simulation*, 2021, vol. 13, no. 6, pp. 1014–1027. Translated from *Matematicheskoe Modelirovanie*, 2021, vol. 33, no. 14, pp. 21–44. https://doi.org/10.1134/S207004822106003X

30. Yashina, M.V. and Tatashev, A.G., Invariant measure for continuous open chain of contours with discrete time, *Computational and Mathematical Methods. e1197*, First published: 28 September 2021. https://doi.org/10.1002/cmm4.1197

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