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Resolvents of the Ito Differential Equations Multiplicative with Respect to the State Vector

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Abstract—Integral representations of solutions of linear multiplicatively perturbed differential equations are obtained, the diffusion part of which is bilinear on the state vector and the vector of independent Wiener processes. Equations of such class serve as models of stochastic systems with control functioning under conditions of parametric uncertainty or undesirable influence of external disturbances. The concepts and analytical apparatus of the theory of Lie algebras are used to find integral representations and fundamental matrices of the equations.

Keywords: multiplicative stochastic system, fundamental matrix, Fisk–Stratonovich differential, group-theoretic method, matrix Lie algebra, Wei–Norman theorem, stochastic resolvent

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1. INTRODUCTION

In the theory of optimization of dynamical systems an important place is given to the control problems of objects functioning under conditions of parametric uncertainty or undesirable influence of external disturbances. The simplest models of such systems in the stochastic section of the theory are linear, called multiplicative, Ito equations, the diffusion components of which are linear on vectors of state, control and external or parametric perturbation. Multiplicative equations are simple enough mathematical objects, and it is hoped to obtain in closed analytic form their solutions or integral representations for them.

Consider a stochastic Ito system (1.1), (1.2), whose *dynamics* is given by the multiplicative Markov equation

$$dx_t = a(t, x_t)dt + b(t)(x_t; dw(t)), \quad x_t \in R^d, \quad w(t) \in R^r, \quad x_0 = \text{const} \tag{1.1}$$

(coefficients depend on t), *driving force* is determined by a random function f with a differential

$$df(t) = (B_1(t)u_t + B_2(t)\upsilon_t)dt + B_{01}(t)u_t dw_1(t) + B_{02}(t)\upsilon_t dw_2(t), \tag{1.2}$$

where u_t and υ_t are vector signals of control and external perturbation respectively; $w(t)$ with or without indices denotes the vector Wiener process. Equation (1.1) is assumed to be linear in the state vector x_t such that $a(t, x) = A(t)x$, where $A(t) \in R^{d\times d}$ is the matrix $d \times d$ at each t, the diffusion component is defined by the function $b(t)(\cdot\,; \cdot)$ of two variables $(x, h) \in R^d \times R^r$ taking values in R^d, and the mapping $R^d \times R^r \to R^d$ is bilinear. The operator $B(t)h$ defined by the relation $(B(t)h)x = b(t)(x; h)$ is linear $R^d \to R^d$ at fixed h. All matrix functions in (1.1), (1.2) are assumed to be continuous on each finite interval of values of the parameter t. The system (1.1), (1.2) is called below (x, u, υ)-*multiplicative*; in particular, the system (1.1)—(x)-multiplicative. Multiplicative models of the type (1.1), (1.2) are used, in particular, in the theory of H_2/H_∞—optimization of stochastic systems [1].

The purpose of the paper is to obtain in integral form the solution of the linear (x, u, v)-*multiplicative* equation or the stochastic analog of its fundamental matrix. Let's make it clear what kind of fundamental matrix and which solution in integral form is talking about. The solution in the deterministic case of the linear differential equation $\dot{x} = A(t)x + B(t)$ has the following form

$$x(t) = R(t, t_0)x_0 + \int_{t_0}^{t} R(t, \tau)B(\tau)d\tau, \tag{1.3}$$

where $R(t, t_0)$ is the *resolvent* (or fundamental matrix) of the homogeneous at $B = 0$ equation [2, p. 144]. The function $R(t, t_0)x_0$ is a general solution of the homogeneous equation taking the value x_0 at $t = t_0$, and the integral in (1.3) is the solution of the perturbed equation going to zero at $t = t_0$. The fundamental matrix of equation (1.1) in the *stochastic* case is a matrix *random* function $\Phi(t, \tau)$, and the general solution of the perturbed equation, following the analogy with (1.3), should be given by the formula

$$x(t) = \Phi(t, t_0)x_0 + \int_{t_0}^{t} \Phi(t, \tau) \circ df(\tau), \tag{1.4}$$

where integral is stochastic; $\circ df$ is denoted the Stratonovich differential [3, p. 105–109]. The integral is chosen stochastic in the Stratonovich sense for the reason that the differentiation rule of a complex function $t \mapsto f(\xi^1(t), \dots, \xi^d(t))$ is represented in the in the same form as in the classical calculus, that is, as $df = \sum_{i=1}^{d} \frac{\partial f}{\partial x^i} \circ d\xi^i$ [3]. This integral in Stratonovich form makes it possible to extend some group-theoretic methods to the stochastic case. In the deterministic case, the group-theoretical concepts allow to overcome the difficulties of studying multidimensional systems caused by the *non-commutativity* of the matrix coefficients defining the dynamics of the system [4]. Perhaps, the same concepts can be useful in the problem of multiplicativity.

Some examples of the application of group-theoretic methods to statistical research are known in the literature. Here is a small list of publications thematically close to the problem of analyzing multiplicative systems [5–11]. In [5] the problem of numerical approximation of the solution of the stochastic equation is considered in the following form

$$dx_t = (Ax_t + f(x_t))dt + \sum_{i=1}^{n}(B_i x_t + g_i(x_t))dw_i, \quad x(0) = x_0 \in R^d$$

with nonlinear functions $f, g_i : R^d \to R^d$ and matrices $A, B_i \in R^{d\times d}$ satisfying the following conditions: A, B_i take values in the matrix Lie algebra $\mathfrak{g}$ with commutator relations $[A, B_i] = 0$, $[B_i, B_j] = 0$ for all i, j. On the background of works on the group-theoretic analysis of *deterministic* equations, the number of which has clearly decreased recently [6], the analysis of solution properties and numerical algorithms for finding solutions (so-called exponential integrators) for *stochastic* equations remains an active field of research on multiplicative and additive noise equations [7, 8]. The question of the mean-square stability of numerical methods for the calculation of exponential integrators is investigated in [9]. As shown in [10], group-theoretic methods are also effective for the numerical integration of partial equations. Among the works of Russian authors we note the research of multiplicative stochastic differential-operator equation with operators A, B acting in a separable Hilbert space [11]. In this paper, it is assumed that the operator A gives rise to a a semigroup of operators $S(t)$, $t > 0$ of class C_0; it guarantees the correctness of the Cauchy problem for the unperturbed equation $\dot{X}(t) = AX(t)$.

The problem solved in this paper considers a finite-dimensional multiplicative equation, for the computation of its resolvent analog the group-theoretic method is applied, which is a generalization to the stochastic case of the deterministic Wei–Norman method [12] of finding the resolvents of linear differential equations. Wei–Norman method: if in the matrix equation $\dot{\Phi}(t) = B(t)\Phi(t)$, $\Phi(0) = E$ (E is a unit matrix), the non-random function $B(t)$ takes values in the matrix Lie algebra $\mathfrak{g}$, then the solution $\Phi(t)$ belongs to the corresponding Lie group $\mathcal{G}$. In this case, one way to construct the solution of $\Phi(t)$ is to represent by a finite product of matrix exponentials

$$\Phi(t) = \exp(s_1(t)A_1)\dots\exp(s_m(t)A_m), \tag{1.5}$$

where $\{A_1, \dots, A_m\}$ is the basis of the minimal Lie algebra $\mathfrak{g}$ generated by matrices $A(t)$ for all t, and $s_i(t)$, $i = 1, \dots, m$ are some real functions. Finding the desired $s_i(t)$ is reduced to the solution of some system of nonlinear differential equations [12]. The basis of the Wei–Norman method proposed here for the case of the multiplicative Ito equation is to write the latter in the form of the Fisk–Stratonovich equation and to find a solution of the latter in the form of the product of matrix exponents $\exp(A_i s_i(t))$ with the needed *semimartingales* (in the terminology adopted in the [3]) $s_i(t)$. Regarding the matrices A_i, $i = 1, \dots, m$, it is assumed, as in the deterministic case, that they form the basis of some matrix Lie algebra.

Applications of group theory to the problems of analyzing and finding solutions of deterministic differential equations are widely known from the monographic literature [4, 13, 14]. Applications to the theory of stochastic differential equations are much more modest; from the textbook literature we mention [3, 15, 16]. An exposition of the group-theoretic method of Wei–Norman to the problem of computing the of the resolvents of multiplicative Ito equations has not been found in the literature.

2. PROBLEM FORMULATION

By characterizing the stochastic system in the previous section as being given by the (x, u, v)-multiplicative Ito equation was separation of the equation into its dynamical part and the forcing force, which does not depend on the state vector of the system. This is dictated by the character of the problem to compute the fundamental matrix (resolvent) of the stochastic Ito equation, which is defined by its homogeneous x_t-dependent part. Having calculated the resolvent, it is not difficult to obtain then an integral representation of the solution of the equation. Following this consideration, it is possible to pass from the general (x, u, v)-multiplicative system to its dynamic part, i.e., to equation (1.1), which is multiplicative only on the state.

Let us list the tasks solved in the paper. The first problem is determination of the Wiener and martingale species of the diffusion component $b(t)(x_t; dw(t))$ of equation (1.1). The second problem in Sections 3, 4 is to write the multiplicative equation (1.1) in the symmetrized Fisk–Stratonovich form. The third task is to obtain the *integral* representation of the solution of the multiplicative equation (1.1). The more general case of the diffusion equation with the matrix $\sigma(t, x)$, depending *affinely* (not simply linearly) on x, see Section 5, the interesting phenomenon of the appearance of an additional forcing force in the integral representation for the solution of the equation. When solving the following two problems in Sections 6 and 7, there arise group-theoretic aspects of solving a multiplicative equation written in a symmetrized form, with *solvable* (in Section 6) Lie algebra and with arbitrary Lie algebra for the matrix coefficients of the diffusion component of the equation in Section 7. The equation in Section 7 is given in the unsymmetrized martingale form instead of Wiener processes. In a separate section we give *example* of finding the resolvent of the equation by the group-theoretic method. Concluding remarks and a list of cited references conclude the paper.

3. WIENER AND MARTINGALE REPRESENTATIONS OF THE DIFFUSION COMPONENT

Both the Wiener process and martingale representations of the differential equation for perturbing forces are quite interesting in multiplicative theory. The martingale equation is discussed in more detail in Section 7.

Proposition 1. *The diffusion component* $b(t)(x_t; dw(t))$ *of the homogeneous equation* (1.1) *admits the following equivalent representations:*

(a) $b(t)(x_t; dw(t)) = (B_1(t)x_t, \ldots, B_r(t)x_t)dw(t)$, *where* $B_j(t)$, $j = 1, \ldots, r$ *is a matrix with size* $d \times d$;

(b) $b(t)(x_t; dw(t)) = \sum\limits_{i=1}^{m} A_i x_t d\zeta^i(t)$, *where* A_i, $i = 1, \ldots, m$ *are matrices* $d \times d$ *and* $d\zeta^i(t) = \sum\limits_{j=1}^{r} b_j^i(t) dw^j(t)$, $b_j^i(t) \in R$, *where* $\zeta^i(t)$ *are martingales.*

Proof. As noted in Section 1, the diffusion part $b(t)(x_t; dw(t))$ of the linear equation at each t is given by the bilinear mapping $b(t)$ of the product $V \times H$, where $V = R^d$, $H = R^r$, of vector spaces into the space V. When $h \in H$ is fixed, the operator $B(t)h$, defined by the equality $(B(t)h)x = b(t)(x; h)$, is an element of the space $End\,V$ of linear operators from V to V. Let $\{h_j, j = 1, \ldots, r\}$ be a basis in H such that in the decomposition $w(t) = \sum_j w^j(t) h_j$ the Wiener processes $w^j(t)$ are mutually independent. There is

$$b(t)(x; dw(t)) = b(t)\left(x; \sum_{j=1}^{r} dw^j(t)(b(t)h_j)\right) = \sum_{j=1}^{r} dw^j(t)(B(t)h_j)x,$$

where $B(t)h_j \in End\,V$.

Denoting $B_j(t) := B(t)h_j$, we obtain statement (a) $b(t)(x_t; dw(t)) = (B_1(t)x_t, \ldots, B_r(t)x_t)dw(t)$ Proposition 1. Thus, the dependence of $b(t)$ on x is given by a set of r arbitrary square $d \times d$ matrices $B_j(t)$, not necessarily linearly independent [1, 17].

Further, let $\{A_i, i = 1, \ldots, m\}$ be the basis of a linear subspace $L \subset End\,V$, generated by the operators $B(t)h_j$. Assuming $B(t)h_j = \sum_{i=1}^{m} b_j^i(t) A_i$, $j = 1, \ldots, m$, where $b_j^i(t) \in R$, and introducing the notations $d\zeta^i(t) := \sum_{j=1}^{r} b_j^i(t) dw^j(t)$, $i = 1, \ldots, m$, we get $b(t)(x; dw(t)) = \sum_{i=1}^{m} d\zeta^i(t) A_i x$, which finishes the the proof Proposition 1. Below, without loss of generality, we assume $\dim L = m = r$.

In the proof of Proposition 1, the drift $a(t, x_t)dt$ in the in equation (1.1) was not taken into account. Implicitly, it was assumed to be zero. It can indeed be converted to zero by the well-known transformation (of course, in this case (1.1) will be replaced by an equation with another bilinear mapping $b(t)$). Indeed, let $y_t = \Lambda_t^{-1} x_t$, where Λ_t is a matrix exponent satisfying, as is known, the integral equation $\Lambda_t = E + \int_0^t a(s)\Lambda_s ds$ with initial condition $\Lambda_0 = E$. Since $dy_t = (d\Lambda_t^{-1})x_t + \Lambda_t^{-1} dx_t$ and $d\Lambda_t^{-1} = -\Lambda_t^{-1} a(t)dt$, then

$$dy_t = \Lambda_t^{-1} a(t) x_t dt + \Lambda_t^{-1} b_t(x_t; dw(t)) - \Lambda_t^{-1} a(t) x_t dt$$

(note that the matrices $a(t)$ and Λ_t commute), thus we obtain the equation $dy_t = \Lambda_t^{-1} b_t(\Lambda_t y_t; dw(t))$ with zero drift. See that the matrices defined above in Proposition 1 $B_j(t)$ are replaced by the matrices $\tilde{\Lambda}_t = \Lambda_t^{-1} B_j(t)\,\Lambda_t$, $j = 1, \ldots, r$.

Let us now find out how to transform the multiplicative equation (1.1) to the symmetrized Fisk–Stratonovich form. It has been noted above that such transformation is a necessary requirement of the methodology proposed here.

Proposition 2. *In the symmetrized Fisk–Stratonovich form the equation of state* (1.1) *written in the form*

$$dx_t = a(t)x_t dt + (B_1(t)x_t, \ldots, B_r(t)x_t)dw(t) \tag{3.1}$$

(*Proposition* 1,(a)) *takes the form*

$$dx_t = a(t)x_t dt + B_0(t)x_t dt + \sum_{j=1}^{r} B_j(t)x_t \circ dw^j(t), \tag{3.2}$$

where $B_0(t) := -1/2 \sum_{j=1}^{r} B_j^2(t)$.

Proof. Starting from the theory of Markov type equations

$$dx_t = a(t, x_t)dt + \sigma(t, x_t)dw(t), \tag{3.3}$$

which does not even assume linearity on x_t of the functions $a(t, x_t)$ and $\sigma(t, x_t)$ [3], let us write (3.3) in coordinate form

$$dx_t^i = a^i(t, x_t)dt + \sum_{j=1}^{r} b_j^i(t, x_t)dw^j(t), \quad i = 1, \ldots, d.$$

According to the general theory, equation (3.3), using the Fisk–Stratonovich differential, is represented as

$$dx_t = \bar{a}(t, x_t)dt + \sigma(t, x_t) \circ dw(t), \tag{3.4}$$

where the vector $\bar{a}(t, x)$ has components

$$\bar{a}^i(t, x) = a^i(t, x) - 1/2 \sum_{j=1}^{d} \sum_{k=1}^{r} \left(\frac{\partial}{\partial x^j} b_k^i(t, x) \right) b_k^j(t, x). \tag{3.5}$$

Recall that the stochastic Ito differential $dw^j(t)$ and the differential $\circ dw^j(t)$ are related by the formula

$$x_t dw^q(t) = x_t \circ dw^q(t) - 1/2 dx_t dw^q(t). \tag{3.6}$$

Consider equation (3.1) in the form

$$dx_t = a(t)x_t dt + \sum_{j=1}^{r} B_j(t)x_t dw^j(t) \tag{3.7}$$

and refer to formula (3.6). Since $x_t dw^j(t) = x_t \circ dw^j(t) - 1/2 dx_t dw^j(t)$, we have, ignoring for now the drift in (3.7), the equation $dx_t = \sum_j B_j(t)x_t \circ dw^j(t) - 1/2 \sum_j B_q(t)dx_t dw^j(t)$. Noting that

$$dx_t dw^j(t) = \sum_k B_k(t)x_t dw^k(t)dw^j(t) = \sum_k B_k(t)x_t \delta_{jk} dt = B_j(t)x_t dt,$$

equation (3.1) in the transformed form can be written as

$$dx_t = a(t)x_t dt + \sum_{j=1}^{r} B_j(t)x_t \circ dw^j(t) - 1/2 \sum_{j=1}^{r} B_j^2(t)x_t \, dt, \tag{3.8}$$

which is what was required. The drift in this equation is determined by the matrix $A(t) := a(t) - 1/2\sum_{j=1}^{r} B_j^2(t)$; it can be converted to zero by passing to the state vector $y_t = \Lambda_t^{-1}x_t$, where $\Lambda_t = E + \int_0^t A(s)\Lambda_s ds$.

In particular, if the matrices $B_j(t)$, $j = 1, \ldots, r$, commute, then the solution of the last equation is written as products

$$x_t = \prod_{q=1}^{r} \exp\left\{\int_0^t B_q(s)dw^q(s) - 1/2\int_0^t B_q^2(s)ds\right\} x_0.$$

It is also clear that the matrices $B_q(t)$ and $B_q^2(t)$ commute, so that the the multipliers in the product can be represented as

$$\exp\left\{\int_0^t B_q(s)dw^q(s)\right\} \exp\left\{-1/2\int_0^t B_q^2(s)ds\right\}, \quad q = 1, \ldots, r.$$

The solution of the equation $dx_t = B_q(t)x_t \circ dw^q(t)$ with zero drift, with initial condition x_0 is the function $U_q(t)x_0 = \exp\left\{\int_0^t B_q(s)dw^q(s)\right\} x_0$. The mapping $t \mapsto U_q(t)$ is the stochastic resolvent of this equation.

4. STOCHASTIC RESOLVENT OF MULTIPLICATIVE EQUATION

The non-random component $a(t)x_t dt$ in a multiplicative equation of the type

$$dx_t = a(t)x_t dt + \sum_{j=1}^{r} B_j(t)x_t \circ dw^j(t) \tag{4.1}$$

can be converted to zero (Section 3) and, without loss of generality, one can consider the equation to be given in the form $dx_t = \sum_{j=1}^{r} B_j(t)x_t \circ dw^j(t)$ with new matrix coefficients. To do this, let us put $y_t = \Lambda_t^{-1}x_t$ and then $dy_t = \Lambda_t^{-1}b(t)(x_t; dw(t))$. Since it is realized that $b(t)(x_t; dw(t)) = \sum_{j=1}^{r} B_j(t)x_t dw^j(t)$, then

$$dy_t = \sum_{j=1}^{r} B_j(t)\Lambda_t y_t \circ dw^j(t). \tag{4.2}$$

Within the group-theoretic formalism, it is the matrices $\tilde{B}_j(t) = \Lambda_t^{-1}B_j(t)\Lambda_t$, not the original matrices B_j, $j = 1, \ldots, r$, must give rise to the Lie algebra basic to the Wei–Norman method, where $\Lambda_t = \exp\int_0^t a(s)ds$ is the exponent of the matrix function $t \mapsto \int_0^t a(s)ds$.

Let us now define the (stochastic) resolvent of an equation linear in x_t. If Ψ_t is the solution of

$$d\Psi_t = \sum_{j=1}^{r} \Lambda_t^{-1}B_j(t)\Lambda_t\Psi_t \circ dw^j(t), \quad \Psi_t|_{t=0} = E \tag{4.3}$$

with zero drift, then the the fundamental matrix (resolvent) $\Phi(t)$ of the initial equation (4.1) is defined by the formula $\Phi(t) = \Lambda_t\Psi_t$. But since equation (4.1) is equivalent to Ito's equation $dx_t = \sum_{j=1}^{r} dw^j(t)B_j(t)x_t$, hence, $\Phi(t)$ satisfies the Stratonovich equation

$$d\Phi_t = a(t)\Phi_t dt + \sum_{j=1}^{r} B_j(t)\Phi_t \circ dw^j(t), \quad \Phi_t|_{t=0} = E. \tag{4.4}$$

If $f(t)$ is the driving force in an inhomogeneous stochastic equation, then the solution of the latter must have an integral representation (1.4).

5. INTEGRAL CAUCHY REPRESENTATION OF THE SOLUTION OF THE MULTIPLICATIVE EQUATION WITH AFFINE COEFFICIENTS

Equations with affine coefficients are necessary in the theory of such linear controllable systems that are multiplicative not only on the state vector, but also on the vectors of control and external perturbation. The stochastic resolvent theory of the previous section dealt with a multiplicative equation with linear but not *affine* coefficients. Now consider a vector equation (with a vector Wiener process) of the form

$$dx_t = a(t, x_t)dt + \sigma(t, x_t)d\,w(t), \quad x_0 \in R^d, \tag{5.1}$$

where $a(t,x) = a(t)x + a_0(t)$, $\sigma(t,x) = B(t,x) + b_0(t)$, $a(t) \in R^{d\times d}$, $a_0(t) \in R^d$, $B(t,x)$, $b_0(t) \in R^{d\times r}$, $w(t) \in R^r$. If (5.1) is a multiplicative system, then $B(t,x) = (B_1(t)x, \ldots, B_r(t)x)$ is a matrix with columns $B_j(t)x$, where $B_j(t) \in R^{d\times d}$, $j = 1, \ldots, r$, which is established in (4.1). Then, $b_0(t)$ is a matrix with columns $b_{0j}(t)$, $j = 1, \ldots, r$. A special case of a one-dimensional ($x_t \in R$) system with affine coefficients and scalar $w(t)$ is considered by in [15]. In the vector case $x_t \in R^d$, let us find the fundamental matrix of the of equation (5.1).

Proposition 3. *The solution of the multiplicative equation with coefficients affine with respect to x_t has the following integral representation:*

$$x_t = \Phi_t \left(x_0 + \int_0^t \Phi_s^{-1} \left(a_0(s) - \sum_{j=1}^r B_j(s) b_{0j}(s) \right) ds + \int_0^t \Phi_s^{-1} \sum_{j=1}^r b_{0j}(s) dw^j(s) \right). \tag{5.2}$$

Here Φ_t defined in the in the previous section, the resolvent (4.1) *written for* (5.1) *with the conditions $a_0(t) = 0$, $b_{0j}(t) = 0$, $j = 1, \ldots, r$.*

Note that the appearance under the integrals in (5.2), in addition to $a_0(s)ds + \sum_{j=1}^r b_{0j}(s)dw^j(s)$, *additional* driving force $B(s)b_0(s) := -\sum_{j=1}^r B_j(s)b_{0j}(s)ds$ (it is caused by the "affine" additive $b_0(s)$) could not be foreseen in advance, but a direct check shows that the function (5.2) indeed satisfies equation (5.1).

Proof. Let us again turn to equation (5.1). In stochastic case, let us apply a method analogous to the deterministic method of constant variation. Let's put $x_t = \Phi_t \eta_t$ and consider η_t as the new unknown instead of x_t. Differentiating $x_t = \Phi_t \eta_t$ stochastically, we obtain

$$dx_t = (d\Phi_t)\eta_t + \Phi_t d\eta_t + d\Phi_t d\eta_t,$$

or by the definition of Φ_t as the equation (4.4)

$$dx_t = \left(a(t)dt + \sum_{j=1}^r dw^j B_j(t) \right) x_t + \Phi_t d\eta_t + \left(a(t)dt + \sum_{j=1}^r dw^j B_j(t) \right) \Phi_t d\eta_t.$$

Equating the right-hand sides of this equation and the original equation (5.1)

$$dx_t = (a(t)x_t + a_0(t))dt + \sum_{j=1}^r dw^j \Big(B_j(t)x_t + b_{0j}(t) \Big),$$

we obtain after abbreviations

$$\left(E + a(t)dt + \sum_{j=1}^r dw^j(t) B_j(t) \right) \Phi_t d\eta_t = a_0(t)dt + \sum_{j=1}^r b_{0j}(t) dw^j(t). \tag{5.3}$$

Because, if we're being formal,

$$\left(E + a(t)dt + \sum_{j=1}^{r} dw^j B_j(t)\right)^{-1} = E - \left(a(t)dt + \sum_{j=1}^{r} dw^j B_j(t)\right) + \left(a(t)dt + \sum_{j=1}^{r} dw^j B_j(t)\right)^2 + \ldots$$

and

$$\left(a(t)dt + \sum_{j=1}^{r} dw^j B_j(t)\right)^2 = \sum_{j=1}^{r} B_j^2 d(t),$$

we obtain from (5.3)

$$d\eta_t = \Phi_t^{-1}\left(a_0(t)dt - \sum_{j=1}^{r} B_j(t)b_{0j}(t)dt + \sum_{j=1}^{r} dw^j(t)b_{0j}(t)\right). \tag{5.4}$$

This equation expresses the fact that η_t is primitive for the right-hand side in (5.3), so that integrating (5.4) gives exactly the formula (5.2). Being expressed in terms of resolvent $\mathcal{R}(t,s) = \Phi_t\Phi_s^{-1}$, the same formula gives the integral Cauchy representation of the solution of the multiplicative equation (4.1) with affine coefficients. This was required to prove.

6. MULTIPLICATIVE EQUATION WITH SOLVABLE LIE ALGEBRA

In this section, an exhaustive solution to the problem of the integral of the solution of the multiplicative equation is obtained at the cost of a strong assumption that the Lie algebra associated to the equation is solvable. A result with a solvable Lie algebra is obtained by H. Kunita [18] and is given in [3] as as one of the examples.[1] The equation of state is assumed here to be given a priori in the symmetrized Fisk–Stratonovich form, the coefficients of the equation do not depend on t.

So, the equation is considered (with constant coefficients)

$$\begin{gathered} dx_t = (B_0x_t + b_0)dt + \sum_{p=1}^{r}(B_px_t + b_p) \circ dw^p(t), \\ B_p \in R^{d\times d}, \quad b_p \in R^d, \quad p = 0, 1, \ldots, d. \end{gathered} \tag{6.1}$$

Lie algebra generated by vector fields fields $L_p = \sum_{i=1}^{d}(B_px + b_p)^i \frac{\partial}{\partial x^i}$, $p = 0, 1, \ldots, r$, is solvable, which holds when $(B_p)^i_j = 0$ for $i > j$, $p = 0, 1, \ldots, r$. This condition means that in each of the matrices B_p, its elements under the of the main diagonal are zero. In particular, the only non-zero element of the last dth row is only the diagonal element $(B_p)^d_d$, $p = 0, 1, \ldots, r$ at $i = d$. It follows from equation (6.1)

$$dx_t^d = \left((B_0)^d_d x_t^d + b_0^d\right) dt + \sum_{p=1}^{r}\left((B_p)^d_d x_t^d + b_p^d\right) \circ dw^p(t) \tag{6.2}$$

when $i = d$. This (scalar) stochastic equation is similar to a deterministic equation in the sense that it is written using the Fisk–Stratonovich differential, so its solution has the form

$$x_t^d = e^{c_d(t)}\left(x_0^d + \int_0^t e^{-c_d(s)} \circ df_d(s)\right), \tag{6.3}$$

[1] A simple example of a solvable Lie algebra is generated by the group of translations of the plane R^2 and rotations about an axis perpendicular to it. The Lie algebra is three-dimensional, its commutation relations are $[X_1, X_2] = 0$, $[X_1, X_3] = X_2$, $[X_3, X_2] = X_1$ [19].

where the function $c_d(t)$ under the exponent sign and the driving force $f_d(t)$ are given respectively by the formulas

$$c_d(t) = (B_0)^d_d\, t + \sum_{p=1}^{r} (B_p)^d_d w^p(t), \quad f_d(t) = b^d_0\, t + \sum_{p=1}^{r} b^d_p w^p(t).$$

By proceeding analogously, let us consider the equation for x^{d-1}_t:

$$\begin{aligned} dx^{d-1}_t &= \left((B_0)^{d-1}_{d-1} x^{d-1}_t + (B_0)^{d-1}_d x^d_t + b^{d-1}_0\right) dt \\ &+ \sum_{p=1}^{r} \left((B_p)^{d-1}_{d-1} x^{d-1}_t + (B_p)^{d-1}_d x^d_t + b^{d-1}_p\right) \circ dw^p(t). \end{aligned} \tag{6.4}$$

In the right-hand side of equation (6.4) depends on x^{d-1}_t the sum of

$$(B_0)^{d-1}_{d-1} x^{d-1}_t dt + \sum_{p=1}^{r} (B_p)^{d-1}_{d-1} x^{d-1}_t \circ dw^p(t).$$

Let us denote the integral of the coefficient at x^{d-1}_t in this sum by

$$c_{d-1}(t) := (B_0)^{d-1}_{d-1} t + \sum_{p=1}^{r} (B_p)^{d-1}_{d-1} w^p(t).$$

The summands in the right-hand side of equation (6.4) independent of x^{d-1}_t form the sum

$$df_{d-1}(t) := \left((B_0)^{d-1}_d x^d_t + b^{d-1}_0\right) dt + \sum_{p=1}^{r} \left((B)^{d-1}_d x^d_t + b^{d-1}_p\right) \circ dw^p(t), \tag{6.5}$$

in which x^d_t is already known as the solution (6.3) of equation (6.2). The solution of equation (6.4) is written, therefore, in the form

$$x^{d-1}_t = e^{c_{d-1}(t)} \left(x^{d-1}_0 + \int_0^t e^{-c_{d-1}(s)} \circ df_{d-1}(s) \right). \tag{6.6}$$

The procedure of sequential solution of scalar equations for components x^k_t, $k = n, n-1, \ldots, 1$ of the vector $x_t \in R^n$ is quite obvious from the above. However, the equivalence between this form of solution and the one given in [3] is not obvious. To establish the equivalence, let us write the original equation (6.1) by components:

$$dx^i_t = \sum_{j \geqslant i} \left((B_0)^i_j x^j_t + b^i_0\right) dt + \sum_{p=1}^{r} \sum_{j \geqslant i} \left((B_p)^i_j x^j_t + b^i_p\right) \circ dw^p(t). \tag{6.7}$$

When i is fixed, the summands in the of the right-hand side, depending on x^j_t with $j \geqslant i$, play a special role. First, the differential dx^i_t is related to the variable x^j_t by the coefficient $(B_0)^i_j dt + \sum_{p=1}^{r} (B_p)^i_j dw^p(t)$, the integral of which is denoted by c^i_j:

$$c^i_j(t) := (B_0)^i_j t + \sum_{p=1}^{r} (B_p)^i_j w^p(t), \quad j = i, i+1, \ldots, d.$$

The diagonal element $c_i^i(t)$ of this matrix coincides with the function, which was denoted above by $c_i(t)$. Second, the sum of summands on the right-hand side in (6.7) with in $\sum_{j=i+1}^{d} x_t^j \circ dc_j^i(t)$. Finally, the terms that do not depend on the components of the vector x_t at all form the sum $\sum_{p=1}^{r} b_p^i dw^p(t) + b_0^i dt$. Thus, the solution of the system of equations (6.7) is given by the formulas

$$x_t^d = e^{c_d^d(t)} \left(x_0^d + \int_0^t e^{-c_d^d(s)} \circ df_d(s) \right), \quad f_d(t) = b_0^d t + \sum_{p=1}^{r} b_p^d dw^p(t),$$

if $i = d$, and by the formulas

$$x_t^i = e^{c_i^i(t)} \left(x_0^i + \int_0^t e^{-c_i^i(s)} \circ df_i(s) \right),$$

where

$$f_i(t) := \sum_{j=i+1}^{d} \int_0^t x^j(s) \circ dc_j^i(s) + b_0^i t + \sum_{p=1}^{r} b_p^i dw^p(t),$$

if $i = d-1, d-2, \ldots, 1$ [3].

7. LIE ALGEBRA OF MULTIPLICATIVE EQUATION WITH CONTINUOUS SEMIMARTINGALES

Let us consider the slightly more general case of Ito's equation

$$dx_t = \sum_{p=1}^{m} A_p x_t d\zeta^p(t), \quad \zeta(0) = 0 \tag{7.1}$$

with continuous semimartingales $\zeta^i(t) = \int_0^t \sum_{j=1}^m b_j^i(t) dw^j(t)$ instead of the Wiener processes $w^j(t)$, $j = 1, \ldots, m$. The matrices A_p are assumed to be non-commutative, giving rise to an arbitrary finite-dimensional Lie algebra. It should be noted that the topic of the interaction of the *stochastic* structure of the differential equation with the *algebraic* group-theoretic structure of its coefficients remains to date insufficiently studied.

Suppose that the equation has a single solution x_t, $t > 0$, then x_t depends linearly on x_0. We will let $x_t = U(t)x_0$. The solution of the equation $dx_t = A_p x_t d\zeta^p(t)$, $\zeta(0) = 0$ with a single matrix A_p and supermartingale $\zeta^p(t)$ is an exponential supermartingale (if $\sum_j (b_j^p)^2 < \infty$)

$$\sigma_p(t) = e^{A_p \zeta^p(t) - 1/2 A_p^2 < \zeta^p > (t)} \sigma_p(0),$$

where $< \zeta^p > (t) = \sum_j \int_0^t (b_j^p)^2(s) ds$; [20, Section 2.7]. To obtain the solution, let us again use the method, already described in the introduction, namely: we write the original Ito equation (7.1) in the symmetrized Fisk–Stratonovich form and apply to the obtained equation an analogue of the deterministic Wei–Norman method [12]. After that, it will not be difficult to obtain the integral representation of the solution of equation (7.1).

Theorem. *Let*

$$dx_t = \sum_{p=1}^{m} A_p x_t d\zeta^p(t), \quad \zeta(0) = 0 \tag{7.2}$$

be a stochastic system with semimartingales

$$\zeta^i(t) = \int\limits_0^t \sum_{j=1}^m b_j^i(t) dw^j(t).$$

The functions $b_j^p(t)$ are known. Consider the functions

$$F_i = \prod_{k=1}^{i-1} e^{X_k s_k} X_i \prod_{k=i-1}^{1} e^{-X_k s_k},$$

where $X_1, \ldots, X_n$ is a basis of the Lie algebra generated by the matrix coefficients $A_p(t)$, $p = 1, \ldots, m$, and $s^i(t)$ are the desired functions. Then for the differentials $\circ ds^i(t)$ of the unknown functions $s^i(t)$ are valid the system of equations $s^i(t)$:

$$\sum_{i=1}^n F_i(t) \circ ds^i(t) = \sum_{p=1}^n \tilde{A}_p(t) \circ d\zeta^p(t). \tag{7.3}$$

Through the functions $s^i(t)$ a solution $x_t = U(t)x_0$ of the original equation (7.1) is expressed.

Proof. Keeping in mind the above remarks, let us write down equation (7.1) in the symmetrized form. There are $x_t \times d\zeta^p(t) = x_t \circ d\zeta^p(t) - 1/2\, dx_t \times d\zeta^p(t)$, where

$$dx_t \times d\zeta^p(t) = \sum_{q=1}^m A_q x_t\, d\zeta^q(t) \times d\zeta^p(t).$$

Since $d\zeta^q(t) \times d\zeta^p(t) = \sum_{j=1}^r b_j^q(t) b_j^p(t)\, dt =: c^{qp}(t)dt$, where $c^{qp}(t)$ are elements of the matrix $c(t) = b^*(t)b(t)$ of order $m \times m$, and matrix $b(t) = (b_j^p(t))$ of order $r \times m$, then

$$x_t \times d\zeta^p(t) = x_t \circ d\zeta^p(t) - 1/2 \sum_{q=1}^m A_q\, x_t c^{qp}(t)dt,$$

and equation (7.1) is written in the form of

$$dx_t = a(t)x_t dt + \sum_{p=1}^m A_p x_t \circ d\zeta^p(t) \tag{7.4}$$

with the drift coefficient $a(t) = -1/2 \sum_{p,q=1}^m A_p A_q\, c^{qp}(t)$. The fundamental matrix of equation (7.3), as above in Section 5 above, let us find it as a product of $\Phi_t = \Lambda_t \Psi_t$, where matrix $\Lambda_t = \exp\{\int_0^t a(s)ds\}$ satisfies the matrix equation $d\Lambda_t = a(t)\Lambda_t dt$. Given that $\Psi_t = \Lambda_t^{-1}\Phi_t$ and $d\Psi_t = (d\Lambda_t^{-1})\Phi + \Lambda_t^{-1} d\Phi_t$ and $d\Lambda_t^{-1} = -\Lambda_t^{-1}(d\Lambda_t)\Lambda_t^{-1}$ for the unknown function Ψ_t one gets the matrix differential equation

$$d\Psi_t = \sum_{p=1}^m (\Lambda_t^{-1} A_p \Lambda_t) \Psi_t \circ d\zeta_t^p. \tag{7.5}$$

The matrix drift coefficient turns here to zero, and the matrix coefficients A_p of the initial equation (7.1) turn into coefficients $\tilde{A}_p(t) = \Lambda_t^{-1} A_p \Lambda_t$. In such a case, from the Campbell-Baker-Hausdorff theorem [21], according to which $a, b \in L \Rightarrow e^a b e^{-a} \in L$, it follows that also $\tilde{A}_p \in L$, $p = 1, \ldots, m$. Here there arises a limitation for the application of group-theoretic methods caused by the necessity to transform the Ito equation to its symmetrized form. Probably, for this reason, in most statistical applications, group analysis is applied to equations given immediately in the Fisk–Stratonovich form.

To continue the topic of group-theoretic analysis of equation (7.5), here let us also assume that the matrix coefficients $\tilde{A}_p(t) = \Lambda_t^{-1} A_p \Lambda_t$ in equation (7.5) are known a priori and $\tilde{L}$ is a Lie algebra generated by them for all t with some basis $\{X_1, \dots, X_n\}$, then the Wei–Norman method can be applied to the algebra $\tilde{L}$. Below, the assumption of the existence Lie algebra $\tilde{L}$ for equation (7.5) is considered to be satisfied.

Purposing to search for the fundamental matrix Ψ_t in the form of the product $\prod_{i=1}^n e^{X_i s^i(t)}$ with unknown scalar functions $s^i(t)$, consider the matrix function

$$u(s) = u(s_1, \dots, s_n) = e^{X_1 s_1} \cdots e^{X_n s_n}, \quad s_i \in R, \quad i = 1, \dots, n$$

(caution against confusing the numeric variable s_i with the function $s^i(t)$). The partial derivative $\frac{\partial}{\partial s_i} u(s)$ equals $u_{s_i} = \prod_{k=1}^{i-1} e^{X_k s_k} X_i \prod_{k=i}^{n} e^{X_k s_k}$, which can be be written in the form $u_{s_i} = F_i u(x)$, where denoted by $F_i = \prod_{k=1}^{i-1} e^{X_k s_k} X_i \prod_{k=i-1}^{1} e^{-X_k s_k}$. Therefore, the Fisk–Stratonovich differential of the function $t \mapsto \Psi_t = u(s^1(t), \dots, s^n(t))$ is equal to

$$d\Psi_t = \sum_{i=1}^n F_i(t) \Psi_t \circ ds^i(t),$$

where $F_i(t)$ is obtained from the formula for F_i by substituting into it $s^i(t)$ instead of s_i for all $i = 1, \dots, n$. Comparing $d\Psi_t$ with the differential for Ψ_t from equation (7.5), which (by replacing m by n) we rewrite as $d\Psi_t = \sum_{p=1}^n \tilde{A}_p(t) \Psi_t \circ d\zeta^p(t)$, we obtain, after reduction by the to the special matrix Ψ_t, the basic equation for the differentials $\circ\, ds^i(t)$ of the desired processes $s^i(t)$:

$$\sum_{i=1}^n F_i(t) \circ ds^i(t) = \sum_{p=1}^n \tilde{A}_p(t) \circ d\zeta^p(t). \tag{7.6}$$

Let us remind once again that there are relations

$$d\zeta^p(t) = \sum_{j=1}^r b_j^p(t) dw^j(t), \ \ p = 1, \dots, n, \quad ds^q(t) = \sum_{j=1}^r g_j^q(t) dw^j(t), \ \ q = 1, \dots, n,$$

where the functions $b_j^p(t)$ are known and the functions $g_j^q(t)$ are sought, where $n = \dim \tilde{L}$. If one decomposes both parts of the basic equation (7.6) by the basis $\{X_1, \dots, X_n\}$ of the Lie algebra $\tilde{L}$, then we obtain a system of equations relating the unknowns functions g_j^p to the known b_j^p. Equation (7.6) is obtained by assumption that the drift coefficient $a(t)$ (7.4) is zero. The latter is ensured by transforming the original equation (7.4) to equation (7.5). The proof is now complete.

8. EXAMPLE

Let us consider an example of solving equation Ito of type (7.1), in which the assumption $a(t) = 0$ is violated, but still $a(t) \in \tilde{L}$. The fundamental matrix of the equation of the state is in the form of a product of exponential semimartingales. This is an example of using a modification of the Wei–Norman method (its stochastic version).

Let us find the fundamental matrix U_t of the stochastic equation $dx_t = \sum_{p=1}^3 X_p x_t d\zeta^p(t)$, $d\zeta^p(t) = \sum_{j=1}^3 b_j^p(t) dw^j(t)$, $\zeta_i(0) = 0$. Let $L = L_3$ be a Lie algebra of dimension $\dim L = 3$ with basis (X_i) and multiplication table $[X_1, X_2] = X_3$, $[X_2, X_3] = [X_1, X_3] = 0$. The algebra L_3 admits a representation of (3×3)-matrices $X_1 = E_{12}$, $X_2 = E_{23}$, $X_3 = E_{13}$ (E_{ij}—matrix canonical units), with $X_i^2 = 0$ for all i. The matrix $U_t$will be found as the product $U_t = \prod_{i=1}^3 \exp\{s^i(t) X_i\}$, where the components $Z_k(t) = s^k(t)X_k$ are absent by virtue of $X_k^2 = 0$, and the functions $s^i(t)$ are suitably chosen random processes with differentials $ds^k(t) = \sum_{j=1}^{3} g_j^k(t)dw^j(t)$, $s^k(0) = 0$. We assume

$$F_1(t) = X_1, \; F_2(t) = e^{Z_1(t)}X_2e^{-Z_1(t)}, \; F_3 = e^{Z_1(t)}e^{Z_2(t)}X_3e^{-Z_2(t)}e^{-Z_1(t)}.$$

It is directly verified that

$$X_i^2 = 0 \quad \forall i, \; F_1 = X_1, \; F_2 = X_2 + s_1X_3, \; F_3 = X_3, \; F_1F_2 = X_3,$$

the remaining F_iF_j are zero. Using the modification of the method outlined in Section 3. Wei–Norman method of formulating equations for the unknown functions (in this example they are $s^i(t)$), one obtains the equation

$$\sum_{i=1}^{3} F_i ds^i = \sum_{i=1}^{3} X_i d\zeta^i.$$

Taking into account the formulas for F_i in the X_i basis decomposition, from this equation we get

$$ds^1 = d\zeta^1, \; ds^2 = d\zeta^2, \; ds^3 = d\zeta^3 - s^1ds^2 - ds^1ds^2. \tag{8.1}$$

To check the correctness of the obtained solution, let us find the the stochastic differential of the function U_t by calculating the function itself.

Since

$$\exp\{s^1X_1\} = I + s^1X_1, \; \exp\{s^2X_2\} = I + s^2X_2, \; \exp\{s^3X_3\} = I + s^3X_3,$$

then, by multiplication we find $U_t = I + s^1X_1 + s^2X_2 + (s^3 + s^1s^2)X_3$ and it follows that, $dU_t = X_1ds^1 + X_2ds^2 + X_3(ds^3 + d(s^1s^2))$, where $d(s^1s^2) = s^1ds^2 + s^2ds^1 + ds^1\,ds^2$. Substituting here the expressions for ds^i from (8.1), after the reduction we obtain $X_1d\zeta^1 + X_2d\zeta^2 + X_3d\zeta^3$, which coincides with the coefficient in the right-hand side of the original Ito equation. Thus, the solution of the stochastic Ito equation is found in the form of the product of the of stochastic semimartingales ("stochastic exponents").

9. CONCLUSION

The base of the integral representation of the solution of the linear of a stochastic equation is, as in the deterministic case, the fundamental matrix of solutions, through which the Green's function for the inhomogeneous equation is expressed. During the finding of the fundamental matrix of a multivariate equation, the main difficulty belongs to the noncommutativity of matrix coefficients of drift and diffusion components.

The non-commutativity of matrices is overcome in a known way if they are in involution. Turning to the methodology of group theory, we should assume that the coefficients of the equation belong to a certain matrix Lie algebra L closed with respect to a matrix commutator. For a linear system with diffusion components, depending only on the Wiener processes, but independent of the state vector, the Lie algebra associated to the system is organized quite simply: it is generated by the diffusion and drift coefficients. In the case of diffusion depending linearly on the state vector, it is necessary to preliminary transformation of the initial equation to the form, using the Fisk–Stratonovich differential. The drift coefficient becomes in this case depending on squares of diffusion coefficients, and diffusion coefficients, in their turn, undergoes transformations depending on the drift coefficient. And only in the commutative case (or in the case of a solvable algebra), it is possible to avoid the difficulties noted above. Thus, the situation with the application of standard group-theoretic concepts to the stochastic equation is satisfactory. Perhaps some algebraic structure other than Lie algebra, would be more appropriate in this problem, but the clarification of this question requires further study.

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