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Parametric Algorithm for Finding a Guaranteed Solution to a Quantile Optimization Problem

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Abstract—The problem of stochastic programming with a quantile criterion for a normal distribution is studied in the case of a loss function that is piecewise linear in random parameters and convex in strategy. Using the confidence method, the original problem is approximated by a deterministic minimax problem parameterized by the radius of a ball inscribed in a confidence polyhedral set. The approximating problem is reduced to a convex programming problem. The properties of the measure of the confidence set are investigated when the radius of the ball changes. An algorithm is proposed for finding the radius of a ball that provides a guaranteeing solution to the problem. A method for obtaining a lower estimate of the optimal value of the criterion function is described. The theorems are proved on the convergence of the algorithm with any predetermined probability and on the accuracy of the resulting solution.

Keywords: stochastic programming, quantile criterion, confidence method, quantile optimization, guaranteeing solution

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1. INTRODUCTION

Stochastic programming problems with a quantile criterion are optimization problems in which the minimum point of the quantile of the loss function is sought, depending on the optimization strategy and random parameters. Similar problems arise when modeling technical and economic systems, in which the requirements for the reliability of the decision being made play an important role. The quantile function describes the level of loss that cannot be exceeded with a given fixed probability, usually close to one. The monographs [1, 2] are devoted to problems of this class.

An effective way to solve the problem of minimizing the quantile function is the confidence method [1, 2]. The essence of this method is that the original quantile optimization problem is approximated by a minimax problem. In this problem, we first consider the maximum of the objective function on a certain set of values of random parameters (confidence set) as a function of the confidence set and the optimization strategy. Then, the minimum of the obtained maximum function is searched for by the optimization strategy and the confidence set. The choice of the optimal confidence set is not an easy task. However, with a properly chosen fixed confidence set, one can obtain a fairly accurate upper estimation of the quantile function. In particular, it is shown [2], that for a Gaussian distribution of random factors, the choice of a confidence set in the form of a ball for large values of the reliability level ensures high accuracy of the resulting estimate. This article discusses the loss functions that are presented as the maximum of a finite number of linear (with respect to random parameters) functions. For this class of loss functions, the optimal confidence set is a polyhedron. In this regard, the estimate on the ball can be improved by performing an additional optimization over the class of confidence sets in the form of polyhedra, parametrized by the radius of the inscribed ball. This idea was implemented for the Gaussian distribution in [3]. In [4], this algorithm was extended to the case of an arbitrary distribution of random factors, and an algorithm was proposed for further improving the guaranteeing solution by moving the faces of a convex polyhedral confidence set while maintaining its probability measure. It should be noted that in [3, 4] the loss function was assumed to be linear in the optimization strategy. This allowed the approximating minimax problem to be reduced to a linear programming problem.

A feature of the approximating problem obtained by using the algorithms [3, 4] is the fact that in the case of a Gaussian distribution, it can be used to obtain not only the upper, but also the lower estimate of the optimal value of the quantile function. To do this, in the approximating problem, instead of the confidence set, take the kernel of the probability measure [2], which, in the case of a standard Gaussian distribution, is a ball of radius calculated as a quantile of the standard normal distribution of the same level as the quantile function. It should be noted that the kernel of a probability measure is not a confidence set.

Of special interest is the case of a loss function that is linear in random parameters. In [1] it is proved that, under the condition of regularity of the kernel, the quantile function can be calculated as a maximum by random parameters of the loss function on the core. Later, the regularity conditions for the kernel were loosen in [5]. The said kernel property was used in [6] to construct an algorithm for solving a stochastic programming problem with a quantile criterion and a bilinear loss function, as well as in [7] for approximating probabilistic constraints.

Stochastic programming problems with a quantile criterion are a special case of problems with probabilistic constraints [8, 9]. The review of methods for solving problems with probabilistic constraints can be found in [10]. In particular, we should note the approach based on the use of p-efficient points [11, 12]. However, problems with a quantile criterion have a number of properties that are not characteristic of problems with arbitrary probabilistic constraints, which makes it possible to use special methods of analysis, in particular, the confidence method. Problems with a quantile criterion and additional probabilistic constraints were studied in detail in [1].

This article considers a stochastic programming problem with a loss function that is piecewise linear in random parameters and convex in terms of the optimization strategy, which makes it possible to approximate the problem under study by a convex programming problem. For this problem, an algorithm is developed based on the ideas of constructing algorithms in [3, 4] for piecewise linear problems. Estimates are given for the accuracy of the proposed algorithm.

2. FORMULATION OF THE PROBLEM

Let X be the random vector (column) with realizations $x \in \mathbb{R}^m$, given on the probability space $(\Omega, \mathcal{F}, \mathbf{P})$. It is assumed that the distribution X is standard normal. We assume that the loss function Φ is piecewise linear in random parameters:

$$\Phi(u,x) \triangleq \max_{i=\overline{1,k_1}} \{B_{1i}(u)x + b_{1i}(u)\}.$$

The constraints in the problem are described by the function

$$Q(u,x) \triangleq \max_{j=\overline{1,k_2}} \{B_{2j}(u)x + b_{2j}(u)\},$$

where $u \in U \subset \mathbb{R}^n$ is the strategy; $B_{1i}(u)$, $B_{2j}(u)$ are rows of matrices $B_1(u)$, $B_2(u)$ respectively, $b_{1j}(u)$, $i = \overline{1,k_1}$, and $b_{2j}(u)$, $j = \overline{1,k_2}$, are elements of vectors (columns) $b_1(u)$ and $b_2(u)$ respectively. This article assumes that the functions $u \mapsto B_1(u)$, $u \mapsto B_2(u)$ are linear (i.e., $B_l(u) = D_l u + a_l$, where D_l is a matrix, a_l is a vector, $l \in \{1,2\}$), and functions $u \mapsto b_1(u)$, $u \mapsto b_2(u)$ are convex and continuous on a convex closed set U. Note that the linear transformation of the random vector X does not change the structure of the functions Φ and Q. Moreover, any normal vector can be obtained by a linear transformation of the vector X of suitable dimension. For these reasons, the case of an arbitrary normal distribution of the vector X reduces to the case under consideration.

Define the probability function as

$$P_\varphi(u) \triangleq \mathbf{P}\{\Phi(u,X) \leqslant \varphi, \quad Q(u,X) \leqslant 0\},$$

where $\varphi \in \mathbb{R}$ is a given value of the loss function, and the quantile function as

$$\Phi_\alpha(u) \triangleq \min\{\varphi \mid P_\varphi(u) \geqslant \alpha\}, \quad \alpha \in (0, P^*),$$

where

$$P^* \triangleq \sup_{u \in U} \mathbf{P}\{Q(u,X) \leqslant 0\}.$$

The article considers the problem of quantile optimization

$$U_\alpha \triangleq \operatorname{Arg}\min_{u \in U} \Phi_\alpha(u). \tag{1}$$

Since the functions Φ and Q are continuous and measurable, according to the result of [13, Theorem 6], which is a generalization of a similar result in [1], the function $u \mapsto \Phi_\alpha(u)$ is lower semicontinuous. Therefore, a solution to the problem (1) exists if the set U is compact. Let us determine the optimal value of the criterion function as

$$\varphi_\alpha \triangleq \Phi_\alpha(u_\alpha),$$

where $u_\alpha \in U_\alpha$. In what follows, we will assume that a solution to the problem (1) exists. In this case, the boundedness of the set U, generally speaking, is not required.

3. CONSTRUCTION OF SOLUTION ESTIMATES

According to the confidence method, [1] the problem (1) is equivalent to

$$\varphi_\alpha = \min_{S \in \mathcal{E}_\alpha, u \in U} \left\{ \sup_{x \in S} \Phi(u,x) \mid \sup_{x \in S} Q(u,x) \leqslant 0 \right\}, \tag{2}$$

where $\mathcal{E}_\alpha$ is the family of all confidence sets $S \subset \mathbb{R}^m$ of level α, i.e. Borel sets such that $\mathbf{P}\{X \in S\} \geqslant \alpha$.

Denote by B_r the ball of radius r:

$$B_r \triangleq \{x \in \mathbb{R}^m \mid \|x\| \leqslant r\},$$

where $\|x\| \triangleq \sqrt{x^\top x}$ is the Euclidean norm of the vector x.

Let us consider a problem similar to the problem (2), in which the set $S = B_r$ is fixed:

$$\psi(r) \triangleq \min_{u \in U} \left\{ \max_{x \in B_r} \Phi(u,x) \mid \max_{x \in B_r} Q(u,x) \leqslant 0 \right\}. \tag{3}$$

We will assume that the minimum in u in problem (3) is reached, which is true, for example, in the case of compact set U. In the problem (3) the supremum is replaced by the maximum, because

$$\max_{x \in B_r} \Phi(u,x) = \max_{x \in B_r} \max_{i=\overline{1,k_1}} \{B_{1i}(u)x + b_{1i}(u)\}$$
$$= \max_{i=\overline{1,k_1}} \max_{x \in B_r} \{B_{1i}(u)x + b_{1i}(u)\} = \max_{i=\overline{1,k_1}} \{b_{1i}(u) + \|B_{1i}(u)\| r\}.$$

In a similar way is $\max_{x \in B_r} Q(u,x)$. Thus, the problem (3) can be rewritten as

$$\psi(r) = \min_{u \in U} \left\{ \max_{i=\overline{1,k_1}} \{b_{1i}(u) + \|B_{1i}(u)\| r\} \mid \max_{j=\overline{1,k_2}} \{b_{2j}(u) + \|B_{2j}(u)\| r\} \leqslant 0 \right\}. \tag{4}$$

If the constraints of this problem are inconsistent, we will assume that $\psi(r) = +\infty$. From the monotonic nondecreasing of the objective function and the narrowing of the set of admissible strategies as r increases it follows that the function ψ is non-decreasing. Problem (4) is equivalent to the convex programming problem

$$\varphi \to \min_{u \in U,\, \varphi \in \mathbb{R}} \tag{5}$$

under constraints

$$b_{1i}(u) + \|B_{1i}(u)\| r \leqslant \varphi, \quad i = \overline{1,k_1},$$
$$b_{2j}(u) + \|B_{2j}(u)\| r \leqslant 0, \quad j = \overline{1,k_2}.$$

Equivalence is understood here in the sense that the optimal value of the variable φ coincides with $\psi(r)$, and the sets of admissible values u coincide. The total number of constraints in this problem will be denoted by $k = k_1 + k_2$. Problem (5) can be solved with high accuracy using convex optimization methods [14].

Let R_α be the ball of probabilistic measure α, i.e. the solution of the equation

$$\mathbf{P}\{X \in B_{R_\alpha}\} = \alpha.$$

Let us fix in the problem (2) a confidence set S in the form of a ball B_{R_α}. Thus, an upper estimate of the quantile function can be found.

To search for a lower estimate, the kernel of the probability measure can be used, defined as the intersection of all closed half-spaces A such that $\mathbf{P}\{X \in A\} = \alpha$. It is known that for $\alpha > \frac{1}{2}$ the kernel of the distribution of the standard normal Gaussian vector is a ρ_α-radius ball centered at zero, where ρ_α is the quantile of the standard normal distribution of the α level. In [1, Section 3.4.3, Corollary 2] it is shown that $\psi(\rho_\alpha) \leqslant \varphi_\alpha$, when X distributed normally.

Thus, we have obtained the estimate

$$\psi(\rho_\alpha) \leqslant \varphi_\alpha \leqslant \psi(R_\alpha). \tag{6}$$

The upper estimate for $\psi(R_\alpha)$ can be improved. Let $(u(r), \psi(r))$ be some solution to the problem (5). Let us define the set

$$\begin{aligned} C_r &\triangleq \left\{x \in \mathbb{R}^m \mid \Phi(u(r),x) \leqslant \psi(r),\ Q(u(r),x) \leqslant 0\right\} \\ &= \left\{x \in \mathbb{R}^m \mid B_{1i}(u(r))x + b_{1i}(u(r)) \leqslant \psi(r),\ B_{2j}(u(r))x + b_{2j}(u(r)) \leqslant 0,\ i = \overline{1,k_1},\ j = \overline{1,k_2}\right\}. \end{aligned} \tag{7}$$

We introduce the notation $h(r) \triangleq \mathbf{P}\{X \in C_r\}$ for the probability measure of the set C_r. Note that $h(r)$ and C_r depend on the choice of $u(r)$. Therefore, in what follows, the choice of $u(r)$ is assumed to be fixed.

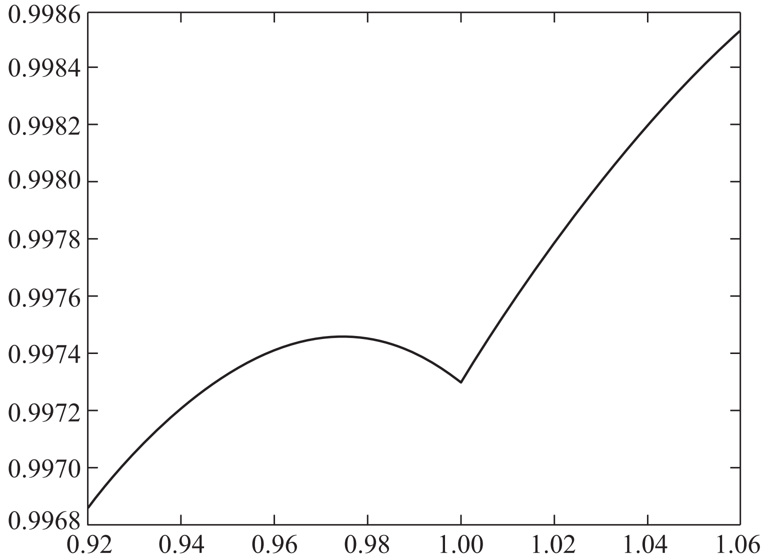
Because

$$\max_{x \in B_r} \Phi(u(r),x) = \psi(r), \quad \max_{x \in B_r} Q(u(r),x) \leqslant 0, \tag{8}$$

the inclusion $B_r \subset C_r$ is valid. Besides,

$$\max_{x \in B_r} \Phi(u(r),x) = \max_{x \in C_r} \Phi(u(r),x), \quad \max_{x \in C_r} Q(u(r),x) \leqslant 0. \tag{9}$$

It follows from (8) and (9) that if $h(r) \geqslant \alpha$, then C_r is a confidence set and $\psi(r) \geqslant \varphi_\alpha$.



Dependency graph for $h(r) = \mathbf{P}\{X \in C_r\}$ of r.

It follows from the monotonicity of ψ that the upper bound for the quantile function can be improved by finding r close to $r^* \triangleq \inf\{r \mid h(r) \geqslant \alpha\}$, such that $h(r) \geqslant \alpha$. If the function $r \mapsto h(r)$ is monotonic, then the dichotomy method can be used to find r^* . Unfortunately, the function h can be non-monotone, as the following example demonstrates.

Example 1. Let the loss function be

$$\Phi(u, x) = \max\{u + 4x, -u + 2x + 2, -11u - 4x\},$$

$u \in \mathbb{R}$, x is a realization of a random variable $X \sim \mathcal{N}(0, \sigma^2)$, $\sigma^2 = \frac{1}{9}$.

It is easy to check that the problem (5) has a solution

$$u(r) = 1 - r, \quad \psi(r) = 1 + 3r \quad \text{if} \quad r \in [0, 1];$$
$$u(r) = 0, \quad \psi(r) = 4r \quad \text{if} \quad r \in [1, +\infty).$$

Therefore,

$$C_r = \{x \mid \Phi(u(r), x) \leqslant \psi(r)\} = \begin{cases} [-3 + 2r, r], & \text{if } r \in [0, 1], \\ [-r, r], & \text{if } r \in [1, +\infty). \end{cases}$$

Let us calculate the measure of the set C_r if $r \in [0, 1]$:

$$h(r) = \mathbf{P}\{X \in C_r\} = \int\limits_{-3+2r}^{r} \frac{3}{\sqrt{2\pi}} e^{-\frac{3x^2}{2}}\, dx.$$

Let us calculate the derivative of the obtained function:

$$\frac{dh}{dr}(r) = \frac{3}{\sqrt{2\pi}} e^{-\frac{3r^2}{2}} - 2\frac{3}{\sqrt{2\pi}} e^{-\frac{3(2r-3)^2}{2}}.$$

Let us calculate the left-hand limit

$$\lim_{r \to 1-} \frac{dh}{dr}(r) = -\frac{3}{\sqrt{2\pi}} e^{-\frac{3}{2}} < 0.$$

This means that on some interval $(1 - \varepsilon, 1)$, where $\varepsilon > 0$, function h is decreasing. Moreover, $h(1) \approx 0.9973$. The dependence graph for $h(r)$ is shown in the figure.

Table 1. Dependence of R_α on m

$\alpha \setminus m$	1	2	3	4	5	6	7	8	9	10	50
0.95	1.96	2.45	2.80	3.08	3.32	3.55	3.75	3.94	4.11	4.28	8.22
0.99	2.58	3.03	3.37	3.64	3.88	4.10	4.30	4.48	4.65	4.82	8.73

Table 2. Dependence of ρ_β on k

$\alpha \setminus k$	1	2	3	4	5	6	7	8	9	10	50
0.95	1.64	1.96	2.13	2.24	2.33	2.39	2.45	2.50	2.54	2.58	3.09
0.99	2.33	2.58	2.71	2.81	2.88	2.93	2.98	3.02	3.06	3.09	3.54

As can be seen from the example above, the function h may turn out to be nonmonotone. In this connection, we propose sufficient conditions that ensure the monotonicity of the function h.

Theorem 1. *Let $U = \mathbb{R}^n$ and the conditions are fulfilled:*

1) $b_{1i}(u) = A_{1i}u + c_{1i}$, *$A_{1i}$ are the rows of the matrix A_1*, $b_{2j}(u) = A_{2j}u + c_{2j}$, *$A_{2j}$ are the rows of the matrix A_2, matrices $B_1(u)$ and $B_2(u)$ do not depend on u*;

2) *the rows of the block matrix*

$$\begin{pmatrix} A_1 & e_{k_1} \\ A_2 & 0_{k_2} \end{pmatrix}$$

are linearly independent, where $e_{k_1}, 0_{k_2}$ are the columns of ones and zeros respectively (if $Q(u,x) \equiv 0$, then there are no rows corresponding to A_2 in the above matrix);

3) *for some $r = R$ the solution to the problem* (5) *exists.*

Then the function h is non-decreasing on the interval $[0, R]$.

The proof of the 1 and all subsequent theorems are in the Appendix.

Note that in Theorem 1 the set U is not compact. Unfortunately, it is difficult to propose more general conditions for the monotonicity of the function h since the monotonicity measures can only be guaranteed under the assumption that the set C_r expands as r increases. However, only the distance from the origin of the faces touching the ball B_r can be guaranteed. The remaining faces can both move away and approach the origin of coordinates.

In connection with the nonmonotonicity of the function h it is necessary to indicate as accurately as possible the interval in which it is necessary to look for r^*. For this we get the following result.

Theorem 2. *Let $k = k_1 + k_2$. The inequality $h(r) \geqslant \alpha$ holds if $r \geqslant \rho_\beta$ and the set C_r is defined, where ρ_β is the quantile of standard normal distribution of the level $\beta = 1 - \frac{1-\alpha}{k}$.*

From the Theorem 2 and the inequality (6) it follows that

$$\psi(\rho_\alpha) \leqslant \varphi_\alpha \leqslant \min\{\psi(R_\alpha), \psi(\rho_\beta)\} = \psi\left(\min\{R_\alpha, \rho_\beta\}\right). \tag{10}$$

It follows from the definition of a confidence ball that $R_\alpha = \sqrt{\chi^2_\alpha(m)}$, where $\chi^2_\alpha(m)$ is the chi-square distribution quantile with m degrees of freedom. In contrast to R_α the value of ρ_β does not depend on the dimension of the random vector, but depends only on the number of constraints k. It is known [2], that $R_\alpha - \rho_\alpha \to 0$ for $\alpha \to 1$, but the rate of convergence depends on dimension n. It is easy to see that $\rho_\beta \to +\infty$ for $k \to 1$. However, it turns out that for small values k the inequality $\rho_\beta < R_\alpha$ can be satisfied. The dependence of R_α on m is given in Table 1, and the dependence of ρ_β on k is given in Table 2. The levels $\alpha = 0.95$ and $\alpha = 0.99$ are considered. Let, for example, $m = 8$, $\alpha = 0.95$. Then $R_\alpha = 3.94$, and $\rho_\beta < R_\alpha$ even for $k = 50$.

Note that for $k = 1$ we have the equality $\rho_\beta = \rho_\alpha$. Therefore, $\varphi_\alpha = \psi(\rho_\alpha)$, and the optimal strategy u_α can be found from problem (5) for $r = \rho_\alpha$, which agrees with the known result [5].

4. ALGORITHM FOR SEARCHING FOR A GUARANTEEING SOLUTION

A strategy $u \in U$, satisfying the relation $\varphi_\alpha(u) \leqslant \psi(\min\{R_\alpha, \rho_\beta\})$, will be called a guaranteeing solution. Thus, a guaranteeing solution can be found from the problem (5) for $r = \bar{R}_\alpha$, where $\bar{R}_\alpha \triangleq \min\{R_\alpha, \rho_\beta\}$. Denote this guaranteeing solution by u^0. In this section, we propose an algorithm for improving the guaranteeing solution u^0, i.e. providing a smaller value of the criterion function $\varphi_\alpha(u)$ than $\varphi_\alpha(u^0)$.

As noted in the previous section, the dichotomy method can be used to find the radius of the ball r^*, inscribed in the confidence polyhedron C_r. In this case, the following difficulties arise: first, the continuity and monotonicity of $h(r) = \mathbf{P}\{X \in C_r\}$ is not guaranteed in the general case, secondly, the calculation of the probability of X falling into the polyhedron C_r requires the use of approximate methods. Nevertheless, we will use the dichotomy method to find an improved guaranteeing solution. Due to the fact that $h(r)$ will be calculated approximately using the Monte Carlo procedure, we will look for a value of r, such that $h(r) \geqslant \alpha + \varepsilon$, where ε is a small positive constant ($\varepsilon < 1 - \alpha$). Approximate calculation of the measure can lead to the fact that an unacceptable solution of the problem will be found, therefore it is necessary to specify the probability p of finding an acceptable solution. Since the quantile setting implies finding a solution that guarantees a given level of objective function value with a probability α, it is recommended to choose $p \geqslant \alpha$.

Algorithm 1.

1. Set algorithm parameters $\varepsilon \in (0, 1-\alpha)$ (accuracy parameter for measure calculation), $\delta > 0$ (accuracy parameter for radius calculation) and $p \in [\alpha, 1)$ (probability of finding an acceptable solution).

2. Calculate ρ_α being the α level quantile of the standard normal distribution and $\bar{R}_\alpha \triangleq \min\{R_\alpha, \rho_\beta\}$, where $R_\alpha = \sqrt{\chi^2_\alpha(m)}$, $\chi^2_\alpha(m)$ being chi-square distribution quantile with m degrees of freedom, $\beta = 1 - \frac{1-\alpha}{k}$.

3. Calculate sample size

$$N = \left\lceil \frac{\ln(1/(1 - \sqrt[K]{p}))}{2\varepsilon^2} \right\rceil,$$

where $K = \left\lceil \log_2 \dfrac{|\bar{R}_\alpha - \rho_\alpha|}{\delta} \right\rceil$, $\lceil a \rceil$ is the rounding of a up to the nearest integer.

4. Set $r_1 := \rho_\alpha$, $r_2 := \bar{R}_\alpha$.

5. Find the lower estimate for the solution $\psi(r_1)$ and the upper estimate $\psi(r_2)$ of the optimal value of the criterion function, as well as the initial guaranteeing solution $u(r_2)$, by solving the problem (5) for $r = r_1$ and $r = r_2$.

6. While $|r_1 - r_2| > \delta$ repeat the following steps:

6.1. Assign $r := \frac{r_1 + r_2}{2}$.

6.2. Calculate $u(r)$ and $\psi(r)$, by solving the problem (5).

6.3. Simulate N independent realizations of a random vector X.

6.4. Calculate $\mu(r) \triangleq \mathbf{P}\{X \in B_r\} = F_{\chi^2(m)}(r^2)$, where $F_{\chi^2(m)}(r^2)$ is the value of the distribution function of the chi-square law with m degrees of freedom at point r^2.

6.5. Find $\hat{h}(r)$ being an estimate of the measure of the set C_r, defined by the formula (7):

$$\hat{h}(r) = \mu(r) + \frac{s(r)}{N},$$

where $s(r)$ is the number of sample elements included in the set $C_r \setminus B_r$.

6.6. If $\hat{h}(r) \geqslant \alpha + \varepsilon$, then $r_2 := r$. Otherwise $r_1 := r$.

7. As a guaranteeing solution, take $u(r_2)$.

Note that to improve the accuracy of the algorithm, one can use not the dichotomy method, but divide the segment of the search for a solution into several equal parts. In this case, at step 6.1 of the algorithm, it will be necessary to take several values of r in the segment $[r_1, r_2]$. It should also be noted that in the case of a nonmonotonic dependence of $r \mapsto h(r)$ the algorithm may not find the root of the equation $h(r) = \alpha + \varepsilon$, but some guaranteeing solution will be found.

Let us formulate a theorem on the convergence of the algorithm.

Theorem 3. *Let the problem* (5) *have a solution for* $r \in [\rho_\alpha, \bar{R}_\alpha]$. *Then application of the algorithm ensures finding a guaranteeing solution with a probability not less than* p.

The following theorem characterizes the accuracy of the solution found using the proposed algorithm 1. This result is a refinement of [2, Theorem 3.13] for optimization problems of the class under consideration.

Theorem 4. *Let the function* ψ *be defined and takes finite values on the segment* $[\rho, R]$, *and let the loss function be Lipschitz with constant* L, *i.e.*

$$|\Phi(u, x) - \Phi(u, y)| \leqslant L\|x - y\|.$$

Also suppose that

$$\max_{j=\overline{1,k_2}} \{b_{2j}(u(\rho)) + \|B_{2j}(u(\rho))\| R\} \leqslant 0. \tag{11}$$

Then $0 \leqslant \psi(R) - \psi(\rho) \leqslant (R - \rho)L$.

This inequality indicates the closeness of the found upper estimate of the criterion function to its optimal value. Theorem 4 gives an estimate of the bounds in these inequalities, which can be obtained even before applying the Algorithm 1. According to this estimation

$$0 \leqslant \psi(\bar{R}_\alpha) - \psi(\rho_\alpha) \leqslant L|\bar{R}_\alpha - \rho_\alpha|,$$

if the conditions of Theorem 4 are satisfied. Note that these conditions are satisfied for a Lipschitz loss function, for example, for $Q(u, x) \equiv 0$.

5. NUMERICAL EXPERIMENT

Example 2. Let us find a guaranteeing solution to the problem (1) for

$$\begin{aligned}\Phi(u, x) = \max\{&u_1 + 3u_3 + 2u_5 + x_1 + 2x_3 + 4,\\ &-u_1 + 2u_2 - u_3 + 3u_4 + 2u_5 + 2x_1 - x_2 + 2x_3,\\ &2u_1 + u_2 + 2u_3 - 2u_4 - u_5 + 3x_1 + x_2 + 2x_3 + 2,\\ &3u_1 - 2u_2 + u_3 + 3u_4 - 3u_5 - 2x_1 + 3x_2 - 3x_3 + 5,\\ &0.1u_1^2 - 0.02u_1u_2 - 0.03u_1u_3 + 0.2u_2^2 + 0.05u_3^2 + 0.3u_4^2 +\\ &+ 0.1u_5^2 - 0.2u_1 - 0.3u_2 - 0.1u_3 - 0.2u_5 - 3x_1 - 2x_2 + x_3 + 6\},\\ Q(u, x) = {}&3u_2 + u_1 + 4u_3 - 2u_5 - x_1 - 3x_2 - 4x_3 - 10,\end{aligned}$$

$U = \{u \in \mathbb{R}^5 \mid u_i \in [0; 10], i = \overline{1,5}\}$, $\alpha = 0.95$. For this level α, $\rho_\alpha = 1.645$, $R_\alpha = 2.796$, $\beta = 0.992$, $\rho_\beta = 2.394$, $\bar{R}_\alpha = 2.394$. Therefore, the function h must be considered on the segment $[1.645; 2.394]$. Solving problem (5) for $r = \rho_\alpha$ and $r = \bar{R}_\alpha$, find an estimate

$$\varphi_\alpha \in [\psi(\rho_\alpha), \psi(\bar{R}_\alpha)] = [11.813; 14.754].$$

Table 3. Application of the Algorithm 1

Iteration	r	$\hat{h}(r)$	$\psi(r)$
1	2.019	0.949	13.267
2	2.207	0.970	14.007
3	2.113	0.961	13.635
4	2.066	0.956	13.451
5	2.043	0.952	13.359
6	2.031	0.950	13.313
7	2.037	0.9507	13.336

The initial guaranteeing solution has the form

$$u(\bar{R}_\alpha) = (0.139; 0.602; 0.000; 0.004; 1.613)^\top.$$

Let us set the algorithm parameters: $\varepsilon = 0.001$, $\delta = 0.01$, $p = 0.99$. These parameters require a sample size of $N = 3\,273\,389$. The application of Algorithm 1 is shown in Table 3. Improved guaranteeing solution complies with $r = r^* \triangleq 2.043$ and it has the form

$$u(r^*) = (0.536; 0.688; 0.000; 0.003; 1.356)^\top.$$

At the same time

$$\varphi_\alpha \in [\psi(\rho_\alpha), \psi(r^*)] = [11.813; 13.359].$$

Thus, the use of Algorithm 1 made it possible to reduce the length of the uncertainty interval of the optimal value of the criterion function on $(1 - \frac{13.359-11.813}{14.754-11.813})100\% = 47\%$, which indicates the efficiency of the proposed algorithm.

All calculations were carried out on a computer with Intel(R) Core(TM) i5-6300U CPU, 2.40 GHz, RAM 8 GB RAM in Matlab system using program for solving quadratic Gurobi optimization problems. The counting time was 1035 s. The bulk of the calculation was the calculation of the measure of the polyhedron C_r using the Monte Carlo method.

6. CONCLUSION

The paper proposes an algorithm for solving a stochastic programming problem with a quantile criterion in the case of a loss function that is piecewise linear in random parameters and convex in strategy. The advantage of the proposed algorithm is the ease of constructing approximating problems, which can later be solved using convex optimization methods. The main computational difficulty in its application is the need to estimate the measure using the Monte Carlo method. The proposed algorithm for choosing a confidence set parameterized by the radius of the inscribed ball, as the example showed, can be successfully applied to solve stochastic optimization problems with a quantile criterion in the case of a convex piecewise quadratic linear loss function. It can be seen that this algorithm can also be applied to the case of discrete optimization strategies. The form of Algorithm 1 will not change, but in the course of applying the algorithm, it will be necessary to solve not a convex continuous optimization problem, but a discrete optimization problem. Algorithms for solving such problems may be the subject of further research.

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APPENDIX

Proof of Theorem 1. Conditions 2 and 3 ensure that all constraints in the problem (5) are active. This means that all faces of the set C_r touch the ball B_r. As r increases on the segment $[0, R]$ the faces of the set C_r are transferred in parallel, touching the ball B_r. This means that the set C_r expands as r increases. Therefore, the function h, defined as the measure C_r, is non-decreasing. Theorem 1 is proved.

Proof of Theorem 2. Let $\gamma \in (0,1)$. The set C_{ρ_γ} is defined as the intersection of k half-planes of measure no less than γ. Denote these half-planes by L_i, $i = \overline{1,k}$. Then

$$h(\rho_\gamma) = \mathbf{P}\left\{X \in \bigcap_{i=1}^{k} L_i\right\} = 1 - \mathbf{P}\left\{X \in \bigcup_{i=1}^{k} (\mathbb{R}^m \setminus L_i)\right\} \geqslant 1 - \sum_{i=1}^{k} \mathbf{P}\{X \notin L_i\} = 1 - (1-\gamma)k.$$

Thus, $h(\rho_\gamma) \geqslant \alpha$ for $\alpha \leqslant 1 - (1-\gamma)k$, which is equivalent to $\gamma \geqslant \beta = 1 - \frac{1-\alpha}{k}$. Theorem 2 is proved.

Proof of Theorem 3. Since at each iteration the segment of the search for a solution narrows two times, the number of iterations K of the algorithm can be found as the minimum natural number K, that satisfies the inequality

$$\frac{|\bar{R}_\alpha - \rho_\alpha|}{2^K} \leqslant \delta.$$

It follows from this inequality that $K = \left\lceil \log_2 \frac{|\bar{R}_\alpha - \rho_\alpha|}{\delta} \right\rceil$. The algorithm can make an error in its work only if at some iteration it turns out that $\hat{h}(r) \geqslant \alpha + \varepsilon$, although in fact $h(r) < \alpha$. It is easy to see that the random variable $s(r)$ is distributed according to the binomial law with the success probability $h(r) - \mu(r)$. The inequality is known [15, ch. 1, § 6]:

$$\mathbf{P}\{\hat{h}(r) - h(r) \geqslant \varepsilon\} = \mathbf{P}\left\{\frac{s(r)}{N} - (h(r) - \mu(r)) \geqslant \varepsilon\right\} \leqslant e^{-2N\varepsilon^2}.$$

Therefore, if we assume that $h(r) < \alpha$, then $\mathbf{P}\{\hat{h}(r) \geqslant \alpha + \varepsilon\} \leqslant e^{-2N\varepsilon^2}$. Since the samples used to evaluate the measure are independent, the probability that the algorithm will work correctly is at least $(1 - e^{-2N\varepsilon^2})^K$. Hence it follows that in order to ensure the probability p of successful operation of Algorithm he inequality

$$p \leqslant (1 - e^{-2N\varepsilon^2})^K \Longleftrightarrow N \geqslant \frac{\ln(1/(1 - \sqrt[K]{p}))}{2\varepsilon^2}.$$

must be satisfied.

Theorem 3 is proved.

Proof of Theorem 4. Let $\Psi(u,r) \triangleq \max_{x \in B_r} \Phi(u,x) = \Phi(u, x^0(r))$, where x^0 is the point on the boundary of the ball B_r, where the specified maximum is reached. Since $B_\rho \subset B_R$, $\Psi(u,\rho) \leqslant \Psi(u,R)$ holds. Since the point $y = \frac{\rho}{R} x^0(R)$ lies on the boundary of the ball B_ρ, $\Phi(u,y) \leqslant \Psi(u,\rho)$. That's why

$$0 \leqslant \Psi(u,R) - \Psi(u,\rho) \leqslant \Phi(u, x^0(R)) - \Phi(u,y) \leqslant L\|x^0(R) - y\| = (R-\rho)L.$$

Thus, the inequalities

$$\Psi(u,\rho) \leqslant \Psi(u,R) \leqslant \Psi(u,\rho) + (R-\rho)L \tag{A.1}$$

are true. Minimizing the left and right parts of the first inequality in (A.1) with respect to $u \in U$ so that $\max_{j=\overline{1,k_2}} \{b_{2j}(u) + \|B_{2j}(u)\|R\} \leqslant 0$ (constraints of the problem (4) for $r = R$), we obtain the first inequality to be proved $\psi(\rho) \leqslant \psi(R)$ (here we take into account that $\psi(\rho)$ is defined at least on a wider set). From (11) and the second inequality in (A.1) it follows that

$$\psi(R) \leqslant \Psi(u(\rho), R) \leqslant \Psi(u(\rho), \rho) + (R - \rho)L = \psi(\rho) + (R - \rho)L.$$

This estimate implies the second inequality to be proved. Theorem 4 is proved.

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