ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 8, pp. 933–946.

NONLINEAR SYSTEMS

Continuous Processes with Fuzzy States and Their Applications

V. L. Khatskevich

Military Training and Research Center of the Air Force,
Air Force Academy named after N.E. Zhukovsky and Yu.A. Gagarin, Voronezh, Russia
e-mail: vlkhats@mail.ru

February 25, 2022
April 3, 2023
June 9, 2023

Abstract—Scalar characteristics of continuous processes with fuzzy states—mean and correlation functions—are introduced and studied. Their algebraic properties as well as some properties related to the differentiation and integration of fuzzy functions of a real argument are established. The dependence between the characteristics of a fuzzy signal at the input and output of a dynamic system described by a high-order differential equation with constant coefficients is shown.

Keywords: continuous fuzzy processes, means, correlation functions, fuzzy dynamic systems

DOI: 10.25728/arcRAS.2023.75.63.001

1. INTRODUCTION

When studying dynamic processes under limited initial information, a possible approach is to treat their parameters as realizations of some random processes [1]. However, the distribution of random variables at the time instants under consideration often has a weakly formalizable law. In this case, it is convenient to treat such processes as those with fuzzy states (fuzzy processes). In particular, an important class of fuzzy dynamic processes consists of automatic and optimal control systems.

Thus, continuous fuzzy processes represent an alternative model for automatic control problems in addition to continuous random processes. A fuzzy process is understood as a parametric system of fuzzy numbers that continuously depends on the parameter (time). At present, the theory of fuzzy sets is used in various applications [2, 3]. In particular, different fuzzy models of controlled objects have been investigated [4].

In this paper, numerical characteristics of continuous processes with fuzzy states and continuous time, namely, mean and correlation functions, are introduced and studied; see Sections 3 and 4. Their properties similar to those of the corresponding characteristics of continuous random processes are established. Section 3 considers the algebraic properties of the mean and correlation functions of continuous fuzzy processes. Section 4 is devoted to the properties of these characteristics with respect to integrals and derivatives of fuzzy processes. Integrals of fuzzy functions are understood as a special case of Aumann integrals [5] of multivalued functions (as integrals of α-cutoffs). They were studied in [6, 7] and other publications. Various definitions of derivatives of fuzzy functions were presented, e.g., in [6–8]. Here, we employ the definition in terms of Hukuhara's difference of sets (H-difference) [9]. The results of Sections 3 and 4 rest on the definition and covariance properties of fuzzy numbers discussed in the author's paper [10]; see Section 2.

Nowadays, researchers are actively investigating fuzzy differential equations and their applications; for example, see [3, Chapters 7 and 8; 7, 8, 11–13]. Among the recent works, we mention [14, 15]. Section 5 of this paper considers fuzzy dynamic systems described by nth-order linear differential equations with constant coefficients. The dependence between the numerical characteristics of a fuzzy signal at the output of a fuzzy dynamic system and the corresponding characteristics of its input fuzzy signal is obtained. In contrast to the well-known frameworks [12–15], the approach below develops the Green function method, widely used in the theory of ordinary differential equations [16, Chapter II; 17, Chapter 1], to the class of fuzzy differential equations.

2. THE MEAN, QUASI-SCALAR PRODUCT, AND COVARIANCE OF FUZZY NUMBERS

A fuzzy number is understood as a fuzzy subset of the universal set of real numbers that has a compact support and a normal, convex, and upper semicontinuous membership function; for details, e.g., see [1]. Let J denote the set of all such fuzzy numbers.

The interval representation of fuzzy numbers will be used below.

As is known, the α-level intervals (α-levels) of a fuzzy number $\tilde{z}$ with a membership function $\mu_{\tilde{z}}(x)$ are defined as

$$z_\alpha = \{x|\mu_{\tilde{z}}(x) \geqslant \alpha\}, \quad (\alpha \in (0,1]),\ z_0 = cl\{x|\mu_{\tilde{z}}(x) > 0\},$$

where cl indicates the closure of an appropriate set. Assume that all α-levels of a fuzzy number are closed and bounded intervals on the entire real axis. Let $z^-(\alpha)$ and $z^+(\alpha)$ denote the left and right bounds of an α-interval: $z_\alpha = [z^-(\alpha), z^+(\alpha)]$. The values $z^-(\alpha)$ and $z^+(\alpha)$ are called the left and right α-indices of a fuzzy number, respectively. A real number $x \in R$ is treated as a fuzzy number with the left and right α-indices equal to x.

The sum of fuzzy numbers with indices $z^-(\alpha)$, $z^+(\alpha)$ and $u^-(\alpha)$, $u^+(\alpha)$ is understood as a fuzzy number with the α-level intervals $[z^-(\alpha) + u^-(\alpha), z^+(\alpha) + u^+(\alpha)]$.

Multiplication by a positive real number c is characterized by the α-level intervals $[cz^-(\alpha), cz^+(\alpha)]$. Multiplication by a negative real number c is characterized by the α-level intervals $[cz^+(\alpha), cz^-(\alpha)]$. Equality for fuzzy numbers is understood as equality for all the corresponding α-indices $\forall\, \alpha \in [0,1]$.

According to [18], the mean value of a fuzzy number $\tilde{z}$ can be defined through the interval representation as follows:

$$m(\tilde{z}) = \frac{1}{2}\int_0^1 (z^-(\alpha) + z^+(\alpha))\, d\alpha. \tag{1}$$

Note that the mean (1) is linear.

Example 1. Consider a fuzzy triangular number $\tilde{z}$ characterized by a real-valued triple (a,b,c) with $a < b < c$ defining the membership function

$$\mu_{\tilde{z}}(x) = \begin{cases} \dfrac{x-a}{b-a} & \text{if}\, x \in [a,b] \\ \dfrac{x-c}{b-c} & \text{if}\, x \in [b,c] \\ 0 & \text{otherwise.} \end{cases}$$

In this case, the lower and upper bounds of the α-interval have the form

$$z^-(\alpha) = (b-a)\alpha + a, \quad z^+(\alpha) = -(c-b)\alpha + c.$$

As is easily verified, the mean (1) of the fuzzy triangular number (a,b,c) is $m(\tilde{z}) = \frac{1}{4}(a+2b+c)$.

The distances between fuzzy numbers can be defined on the set of such numbers in different ways. The interval approach often involves the Hausdorff distances between the α-level sets of fuzzy numbers: for fuzzy numbers $\tilde{z}$ and $\tilde{u}$ with α-levels z_α and u_α, respectively, the corresponding metric [19] is given by

$$\rho(\tilde{z},\tilde{u}) = \operatorname*{supp}_{0<\alpha\leqslant 1} \max\left\{\operatorname*{supp}_{z\in z_\alpha} \inf_{u\in u_\alpha} |z-u|,\ \operatorname*{supp}_{u\in u_\alpha} \inf_{z\in z_\alpha} |z-u|\right\}. \quad (2)$$

Definition (2) induces the equality

$$\rho(\tilde{z},\tilde{u}) = \operatorname*{supp}_{0<\alpha\leqslant 1} \max\left\{|z^-(\alpha)-u^-(\alpha)|,\ |z^+(\alpha)-u^+(\alpha)|\right\}. \quad (3)$$

Here, $[z^-(\alpha), z^+(\alpha)]$ and $[u^-(\alpha), u^+(\alpha)]$ are the α-level intervals of the fuzzy numbers $\tilde{z}$ and $\tilde{u}$.

Note that, due to (3), the condition $\rho(\tilde{z},\tilde{u}) = 0$ matches the definition of equality for fuzzy numbers $\tilde{z}$ and $\tilde{u}$ given above.

Consider a fuzzy number $\tilde{z}$ with α-levels $z_\alpha = [z^-(\alpha), z^+(\alpha)]$. Following interval analysis, let

$$mid\, z_\alpha = \frac{1}{2}(z^+(\alpha) + z^-(\alpha)), \quad rad\, z_\alpha = \frac{1}{2}(z^+(\alpha) - z^-(\alpha)).$$

Here, $mid\, z_\alpha$ characterizes the midpoint for each $\alpha \in [0,1]$ and $rad\, z_\alpha$ the range. For fuzzy numbers $\tilde{z}$ and $\tilde{u}$ from J, we define the quasi-scalar product [10]

$$\begin{aligned} \langle \tilde{z},\tilde{u}\rangle &= \int_0^1 (mid\, z_\alpha mid\, u_\alpha + rad\, z_\alpha rad\, u_\alpha)\, d\alpha \\ &= 0.5\int_0^1 (z^+(\alpha)u^+(\alpha) + z^-(\alpha)u^-(\alpha))\, d\alpha. \end{aligned} \quad (4)$$

The quasi-norm is $\|\tilde{z}\| = \langle \tilde{z},\tilde{z}\rangle^{1/2}$.

Example 2. Consider two triangular numbers $\tilde{z}_1$ and $\tilde{z}_2$ characterized by real-valued triples a_i, b_i, c_i with $a_i < b_i < c_i$ $(i = 1,2)$. According to the definition of their right and left indices (see Example 1) and (4), the quasi-scalar product $\langle \tilde{z}_1,\tilde{z}_2\rangle$ is given by

$$\langle \tilde{z}_1,\tilde{z}_2\rangle = \frac{2}{3}b_1b_2 + \frac{1}{3}(a_1a_2 + c_1c_2) + \frac{1}{6}(a_1b_2 + b_1a_2 + b_1c_2 + b_2c_1).$$

Proposition 1 [10]. *The quasi-scalar product* (4) *possesses the following properties:*

1) $\langle \tilde{z},\tilde{u}\rangle = \langle \tilde{u},\tilde{z}\rangle\ \forall\, \tilde{u},\tilde{z} \in J$.

2) $\langle c_1\tilde{z}, c_2\tilde{u}\rangle = c_1c_2\langle \tilde{z},\tilde{u}\rangle$ *provided that* $c_1c_2 > 0$.

3) $\langle \tilde{z}_1 + \tilde{z}_2,\tilde{u}\rangle = \langle \tilde{z}_1,\tilde{u}\rangle + \langle \tilde{z}_2,\tilde{u}\rangle\ \forall\, \tilde{u},\tilde{z}_1,\tilde{z}_2 \in J$.

4) $\langle \tilde{z},\tilde{z}\rangle \geqslant 0$, *and the condition* $\langle \tilde{z},\tilde{z}\rangle = 0$ *is equivalent to the zero left and right indices of* $\tilde{z}$.

5) *The generalized Cauchy–Bunyakovsky–Schwarz inequality* $|\langle \tilde{z},\tilde{u}\rangle| \leqslant \langle \tilde{z},\tilde{z}\rangle^{1/2}\langle \tilde{u},\tilde{u}\rangle^{1/2}$ *holds* $\forall\, \tilde{u},\tilde{z} \in J$.

For fuzzy numbers $\tilde{z}_1$ and $\tilde{z}_2$ with means m_1 and m_2, respectively, we define their covariance by the formula [10]

$$\begin{aligned} cov[\tilde{z}_1,\tilde{z}_2] &= \langle \tilde{z}_1 - m_1, \tilde{z}_2 - m_2\rangle \\ &= 0.5\int_0^1 \Big((z_1^+ - m_1)(z_2^+ - m_2) + (z_1^- - m_1)(z_2^- - m_2)\Big)\, d\alpha. \end{aligned} \quad (5)$$

The variance is denoted by $D(\tilde{z}) = cov|\tilde{z},\tilde{z}|$.

Proposition 2 [10]. *The covariance* (5) *possesses the following properties*:

1) $cov[\tilde{z}_1 + \tilde{z}_2, \tilde{u}] = cov[\tilde{z}_1, \tilde{u}] + cov[\tilde{z}_2, \tilde{u}]$ $(\forall\, \tilde{u}, \tilde{z}_1, \tilde{z}_2 \in J)$.

2) $cov[c_1\tilde{z}, c_2\tilde{u}] = c_1c_2cov[\tilde{z}, \tilde{u}]$ $(\forall\, \tilde{u}, \tilde{z} \in J)$ *for any real numbers* c_1 *and* c_2 *such that* $c_1c_2 > 0$.

3) $cov\,[\tilde{z}_1, \tilde{z}_2] = \langle \tilde{z}_1, \tilde{z}_2 \rangle - m_1m_2$, $(\forall\, \tilde{z}_1, \tilde{z}_2 \in J)$, *where* m_1 *and* m_2 *are the mean values of fuzzy numbers* $\tilde{z}_1$ *and* $\tilde{z}_2$, *respectively (the specific covariance property).*

Proposition 3 [10]. *The variance possesses the following properties*:

1) $D(c\tilde{z}) = c^2D(\tilde{z})$ *for any real number* c.

2) $D(\tilde{z} + \tilde{u}) = D(\tilde{z}) + D(\tilde{u}) + 2cov[\tilde{z}, \tilde{u}]$ $\forall\, \tilde{u}, \tilde{z} \in J$.

In several works (e.g., see [20]), the covariance of fuzzy numbers $\tilde{z}_1$ and $\tilde{z}_2$ was defined as

$$cov_1[\tilde{z}_1, \tilde{z}_2] = \frac{1}{4}\int\limits_0^1 (z_1^+(\alpha) - z_1^-(\alpha))(z_2^+(\alpha) - z_2^-(\alpha))d\alpha.$$

With this definition, covariance is always nonnegative, which disagrees with standard covariance properties (for random variables).

3. CONTINUOUS FUZZY PROCESSES

Consider a fixed segment $[t_0, T]$ of the real axis, where $t_0 \geqslant 0$. A mapping $\tilde{z} : [t_0, T] \to J$ is called a process with fuzzy states (or a fuzzy process) and continuous time.

Let a fuzzy process $\tilde{z}(t)$, $t \in [t_0, T]$, be characterized a membership function $\mu_{\tilde{z}}(x, t)$. For a fixed number $\alpha \in (0, 1]$, consider the α-interval $z_\alpha(t) = \{x \in R : \mu_{\tilde{z}}(x, t) \geqslant \alpha\}$ and $z_0(\alpha) = cl\{x \in R : \mu_{\tilde{z}}(x, t) > 0\}$. We denote by $z_\alpha^-(t) = z^-(t, \alpha)$ and $z_\alpha^+(t) = z^+(t, \alpha)$ the left and right bounds of the α-interval, respectively: $z_\alpha(t) = [z^-(t, \alpha), z^+(t, \alpha)]$.

Assume that the indices $z^-(t, \alpha)$ and $z^+(t, \alpha)$ are square summable in α for each $t \in [t_0, T]$ and continuous in t for any $\alpha \in [0, 1]$.

For each $t \in [t_0, T]$, let the mean of $\tilde{z}(t)$ be defined as

$$m_{\tilde{z}}(t) = m(\tilde{z}(t)) = \frac{1}{2}\int\limits_0^1 (z^-(t, \alpha) + z^+(t, \alpha))d\alpha. \tag{6}$$

Theorem 1. *The mean of a continuous fuzzy process given by* (6) *possesses the following properties:*

1. *If* $\tilde{z}_1(t)$ *and* $\tilde{z}_2(t)$ *are continuous fuzzy processes, then* $m(\tilde{z}_1(t) + \tilde{z}_2(t)) = m(\tilde{z}_1(t)) + m(\tilde{z}_2(t))$ *(additivity).*

2. *If* $\tilde{z}(t)$ *is a continuous fuzzy process and* $\varphi(t)$ *is a real-valued function, then* $m(\varphi(t)\tilde{z}(t)) = \varphi(t)m(\tilde{z}(t))$ *(homogeneity).*

Indeed, property 1 follows from the definition of interval summation and and the additivity of Lebesgue integrals.

It remains to show property 2. For a fixed number $t \in [t_0, T]$, consider the fuzzy number $\tilde{w}(t) = \varphi(t)\tilde{z}(t)$. Note that its left $w^-(t, \alpha)$ and right $w^+(t, \alpha)$ indices coincide with the expressions $\varphi(t)z^-(t, \alpha)$ and $\varphi(t)z^+(t, \alpha)$, respectively, in the case $\varphi(t) \geqslant 0$ or with the expressions $\varphi(t)z^+(t, \alpha)$ and $\varphi(t)z^-(t, \alpha)$, respectively, in the case $\varphi(t) < 0$. However, their sum $w^-(t, \alpha) + w^+(t, \alpha)$ coincides with the expression $\varphi(t)(z^-(t, \alpha) + z^+(t, \alpha))$, which is independent of the sign of $\varphi(t)$. According to (1), this fact implies property 2.

Corollary 1. *If* $f(t)$ *is a real-valued function, then* $m(\tilde{z}(t) + f(t)) = m(\tilde{z}(t)) + f(t)$.

Suppose that $f^-(t) = f^+(t) = f(t)$ for a real number $f(t)\ \forall\, t \in [t_0, T]$.

Let the correlation function of a continuous fuzzy process $\tilde{z}(t)$ be defined as

$$K_{\tilde{z}}(t_1, t_2) = \frac{1}{2}\int\limits_0^1 \left(z^+(t_1,\alpha) - m(\tilde{z}(t_1))\right)\left(z^+(t_2,\alpha) - m(\tilde{z}(t_2))\right) + \left(z^-(t_1,\alpha) - m(\tilde{z}(t_1))\right)\left(z^-(t_2,\alpha) - m(\tilde{z}(t_2))\right) d\alpha. \tag{7}$$

The variance of a continuous fuzzy process is the value $D_{\tilde{z}}(t) = K_{\tilde{z}}(t,t)$. By definition, $D_{\tilde{z}}(t) \geqslant 0$.

Theorem 2. *The correlation function* (7) *of a continuous fuzzy process possesses the following properties.*

1. *For a continuous fuzzy process $\tilde{z}(t)$, the equality*

$$K_{\tilde{z}}(t_1, t_2) = K_{\tilde{z}}(t_2, t_1)$$

holds $\forall\, t_1, t_2 \in [t_0, T]$ (symmetry).

2. *If $\tilde{z}(t)$ is a continuous fuzzy process and $\varphi(t)$ is a real-valued function, then the correlation function $K_{\tilde{w}}(t_1, t_2)$ of a continuous fuzzy process $\tilde{w}(t) = \varphi(t)\tilde{z}(t)$ has the form $K_{\tilde{w}}(t_1, t_2) = \varphi(t_1)\varphi(t_2)K_{\tilde{z}}(t_1, t_2)\ \forall\, t_1, t_2 \in [t_0, T]$ such that $\varphi(t_1)\varphi(t_2) \geqslant 0$.*

3. *If $\tilde{w}(t) = \tilde{z}(t) + \varphi(t)$, then $K_{\tilde{w}}(t_1, t_2) = K_{\tilde{z}}(t_1, t_2)$.*

4. $|K_{\tilde{z}_1}(t_1, t_2)| \leqslant \sqrt{D_{\tilde{z}}(t_1) D_{\tilde{z}}(t_2)}$.

Theorem 2 is based on the properties of the covariance (5) of fuzzy numbers presented in Section 2.

For continuous fuzzy processes $\tilde{z}_1(t)$ and $\tilde{z}_2(t)$, consider the mutual correlation function

$$K_{\tilde{z}_1\tilde{z}_2}(t, s) = \int\limits_0^1 \Big(z_1^+(t,\alpha) - m(\tilde{z}_1(t))\Big)\Big(z_2^+(s,\alpha) - m(\tilde{z}_2(s))\Big) + \Big(z_1^-(t,\alpha) - m(\tilde{z}_1(t))\Big)\Big(z_2^-(s,\alpha) - m(\tilde{z}_2(s))\Big)\, d\alpha.$$

Theorem 3. *Let $\tilde{z}_1(t)$ and $\tilde{z}_2(t)$ be continuous fuzzy processes. The correlation function of their sum $\tilde{w}(t) = \tilde{z}_1(t) + \tilde{z}_2(t)$ has the form*

$$K_{\tilde{w}}(t, s) = K_{\tilde{z}_1}(t, s) + K_{\tilde{z}_2}(t, s) + K_{\tilde{z}_1,\tilde{z}_2}(t, s) + K_{\tilde{z}_1,\tilde{z}_2}(s, t).$$

Continuous fuzzy processes $\tilde{z}_1(t)$ and $\tilde{z}_2(t)$ are said to be uncorrelated on a segment $[t_0, T]$ if

$$K_{\tilde{z}_1\tilde{z}_2}(t, s) = 0 \quad (\forall\, t, s \in [t_0, T]).$$

Corollary 2. *If continuous fuzzy processes $\tilde{z}_1(t)$, $\tilde{z}_2(t)$ are uncorrelated and $\tilde{w}(t) = \tilde{z}_1(t) + \tilde{z}_2(t)$, then*

$$K_{\tilde{w}}(t, s) = K_{\tilde{z}_1}(t, s) + K_{\tilde{z}_2}(t, s) \quad (\forall\, t, s \in [t_0, T]).$$

4. THE INTEGRATION AND DIFFERENTIATION OF CONTINUOUS FUZZY PROCESSES

The integral of a continuous fuzzy process $\tilde{z}(t)$ between the limits of a segment $[t_0, T]$ is a fuzzy number $\tilde{g}$ with the α-level intervals $g_\alpha = \int_{t_0}^{T} z_\alpha(t)\, dt$ for any $\alpha \in [0, 1]$; for details, see [7]. The integral is denoted by $\int_{t_0}^{T} \tilde{z}(t)\, dt$.

In fact, this is the Aumann integral [5] of a multi-valued mapping $z_\alpha(t)$.

If the integral $\int_{t_0}^{T} \tilde{z}(t)\, dt$ exists, then the process $\tilde{z}(t)$ is said to be integrable on $[t_0, T]$.

The mean of the integral possesses the following property.

Theorem 4. *Let $\tilde{z}(t)$ be an integrable fuzzy process on $[t_0, T]$. Then* $m\left(\int_{t_0}^{T} \tilde{z}(\tau)d\tau\right) = \int_{t_0}^{T} m(\tilde{z}(\tau))\,d\tau$.

By the definition of the integral, its indices satisfy the relation

$$\left(\int_{t_0}^{T} \tilde{z}(\tau)\,d\tau\right)_\alpha^{\pm} = \int_{t_0}^{T} z^{\pm}(\tau,\alpha)\,d\tau.$$

Consequently,

$$m\left(\int_{t_0}^{T} \tilde{z}(\tau)\,d\tau\right) = \frac{1}{2}\int_{t_0}^{1}\left(\int_{t_0}^{T}(z^{-}(\tau,\alpha)) + (z^{+}(\tau,\alpha))\,d\tau\right)d\alpha = \int_{t_0}^{T} m(\tilde{z}(\tau))d\tau.$$

For a continuous fuzzy process $\tilde{z}(t)\ \forall\, t \in [t_0, T]$, we define the continuous fuzzy process $\tilde{g}(t) = \int_{t_0}^{t} \tilde{z}(\tau)\,d\tau$.

Theorem 5. *The integral $\tilde{g}(t)$ of a continuous fuzzy process $\tilde{z}(t)$ has the correlation function* $K_{\tilde{g}}(t_1, t_2) = \int_{t_0}^{t_1}\int_{t_0}^{t_2} K_{\tilde{z}}(\tau_1, \tau_2)\,d\tau_1 d\tau_2$.

Proof. By definition,

$$K_{\tilde{g}}(t_1,t_2) = \frac{1}{2}\int_{0}^{1}\left(\int_{t_0}^{t_1} z^{+}(\tau,\alpha)\,d\tau - \int_{t_0}^{t_1} m(\tilde{z}(\tau,\alpha))\,d\tau\right)\left(\int_{t_0}^{t_2} z^{+}(\tau,\alpha)\,d\tau - \int_{t_0}^{t_2} m(\tilde{z}(\tau))d\tau\right)$$

$$+\left(\int_{t_0}^{t_1} z^{-}(\tau,\alpha)\,d\tau - \int_{t_0}^{t_1} m(\tilde{z}(\tau,\alpha))\,d\tau\right)\left(\int_{t_0}^{t_2} z^{-}(\tau,\alpha)\,d\tau - \int_{t_0}^{t_2} m(\tilde{z}(\tau))\,d\tau\right)d\alpha$$

$$= \frac{1}{2}\int_{0}^{1}\left(\int_{t_0}^{t_1}(z^{+}(\tau,\alpha) - m(\tilde{z}(\tau))\,d\tau)\left(\int_{t_0}^{t_2} z^{+}(\tau,\alpha) - m(\tilde{z}(\tau))\,d\tau\right)\right)d\alpha$$

$$+\frac{1}{2}\int_{0}^{1}\left(\int_{t_0}^{t_1} z^{-}(\tau,\alpha) - m(\tilde{z}(\tau))\,d\tau \int_{t_0}^{t_2} z^{-}(\tau,\alpha) - m(\tilde{z}(\tau))\,d\tau\right)d\alpha.$$

Consider the first integral in this expression. Since the integral's value is independent of the integration variable, it can be written as

$$\frac{1}{2}\int_{0}^{1}\left(\int_{t_0}^{t_1} z^{+}(\tau_1,\alpha) - m(\tilde{z}(\tau_1))\,d\tau_1\right)\left(\int_{t_0}^{t_2} z^{+}(\tau_2,\alpha) - m(\tilde{z}(\tau_2))\,d\tau_2\right)d\alpha$$

$$= \frac{1}{2}\int_{0}^{1}\int_{t_0}^{t_1}\int_{t_0}^{t_2}\left(z^{+}(\tau_1,\alpha) - m(\tilde{z}(\tau_1))\right)\left(z^{+}(\tau_2,\alpha) - m(\tilde{z}(\tau_2))\right)d\tau_1\,d\tau_2\,d\alpha.$$

The same line of reasoning applies to the indices with a minus sign. Thus,

$$K_{\tilde{g}}(t_1,t_2) = \frac{1}{2}\int_{0}^{1}\Bigg(\int_{t_0}^{t_1}\int_{t_0}^{t_2}\left(z^{+}(\tau_1,\alpha) - m(\tilde{z}(\tau_1))\right)\left(z^{+}(\tau_2,\alpha) - m(\tilde{z}(\tau_2))\right)$$

$$+\left(z^{-}(\tau_1,\alpha) - m(\tilde{z}(\tau_1))\right)\left(z^{-}(\tau_2,\alpha) - m(\tilde{z}(\tau_2))\right)d\tau_1\,d\tau_2\Bigg)d\alpha.$$

Interchanging the order of integration finally gives the desired result.

Consider now the derivatives of fuzzy functions. Different definitions are introduced in the literature. A common one involves the concept of Hukuhara's difference (H-difference) [9]. For sets A and B, a set C is called their H-difference if $A = B + C$ and is denoted by $A \overset{h}{-} B$.

A mapping $\tilde{z} : [t_0, T] \to J$ is said to be differentiable at a point $t \in [t_0, T]$ [7] if $\forall\, \alpha[0, 1]$ the multivalued mapping $z_\alpha(t)$ is Hukuhara differentiable at the point t with the derivative $D_H z_\alpha(t)$ and the family $\{D_H z_\alpha(t) : \alpha \in [0, 1]\}$ defines a certain element $\tilde{z}\,'(t)$ belonging to J. The element $\tilde{z}\,'(t)$ is called the fuzzy derivative of $\tilde{z}(t)$ at the point t.

By definition, the fuzzy derivative $\tilde{z}\,'(t)$ satisfies the relation

$$\lim_{\Delta t \to 0} \rho\left(\frac{1}{\Delta t}\left(\tilde{z}(t + \Delta t) \overset{h}{-} \tilde{z}(t)\right), \tilde{z}\,'(t)\right) = 0,$$

where the distance ρ is given by (3).

Proposition 4 [7]. *Let a mapping $\tilde{z} : [t_0, T] \to J$ be differentiable and its fuzzy derivative $\tilde{z}\,'(t)$ be integrable on $[t_0, T]$. Then*

$$\tilde{z}(t) = \tilde{z}(t_0) + \int_{t_0}^{t} \tilde{z}\,'(s) ds. \tag{8}$$

Proposition 5 [11]. *Let a fuzzy process $\tilde{z}(t)$ be differentiable and $z_\alpha(t) = [z_\alpha^-(t), z_\alpha^+(t)]$ be its α-interval for any $\alpha \in [0, 1]$. Then the functions $z_\alpha^-(t)$ and $z_\alpha^+(t)$ are differentiable with respect to t and the α-interval of the derivative $\tilde{z}\,'(t)$ has the form $[\tilde{z}\,'(t)]_\alpha = [(z_\alpha^-)'(t), (z_\alpha^+)'(t)]$.*

Proposition 5 shows the connection between the derivative introduced above and the Seikkala derivative [8].

Theorem 6. *Let $\tilde{z}(t)$ be a differentiable fuzzy process with the integrable derivative $\tilde{z}\,'(t)$. Then the mean of its derivative coincides with the derivative of its mean: $m(\tilde{z}\,'(t)) = \frac{d}{dt} m(\tilde{z}(t))$.*

Proof. Taking the mean of the left- and right-hand sides of formula (8) yields

$$m(\tilde{z}(t)) = m(\tilde{z}(t_0)) + \int_{t_0}^{t} m(\tilde{z}\,'(s))\, ds.$$

This equality is based on the additivity of means and Theorem 4. Let us differentiate its sides. In view of the properties of the integral with a variable upper limit, we obtain $\frac{d}{dt} m(\tilde{z}(t)) = m(\tilde{z}\,'(t))$, and the conclusion follows. The proof of Theorem 6 is complete.

Theorem 7. *The derivative $\tilde{z}\,'(t)$ of a differentiable fuzzy process $\tilde{z}(t)$ has the correlation function*

$$K_{\tilde{z}\,'}(t_1, t_2) = \frac{\partial^2 (K_{\tilde{z}}(t_1, t_2))}{\partial t_1 \partial t_2}.$$

Proof. Denoting $\tilde{z}'(t) = \tilde{w}(t)$, we consider $\tilde{g}(t) = \int_{t_0}^{t} \tilde{w}(s)\, ds$. Due to Theorem 5, the correlation function $K_{\tilde{g}}(t_1, t_2)$ is given by

$$K_{\tilde{g}}(t_1, t_2) = \frac{1}{2} \int_0^1 \left(\int_{t_0}^{t_1} w^+(\tau_1, \alpha) d\tau_1 - m(\tilde{w}(\tau_1)) \right) \left(\int_{t_0}^{t_2} w^+(\tau_2, \alpha) d\tau_2 - m(\tilde{w}(\tau_2)) \right) d\alpha$$
$$+ \frac{1}{2} \int_0^1 \left(\int_{t_0}^{t_1} w^-(\tau_1, \alpha) d\tau_1 - m(\tilde{w}(\tau_1)) \right) \left(\int_{t_0}^{t_2} w^-(\tau_2, \alpha) d\tau_2 - m(\tilde{w}(\tau_2)) \right) d\alpha.$$

Differentiating this equality, first with respect to t_1 and then with respect to t_2, yields

$$\frac{\partial^2 K_{\tilde{g}}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{1}{2}\int_0^1 \left(w^+(\tau_1,\alpha) - m(\tilde{w}(\tau_1))\right)\left(w^+(\tau_2,\alpha) - m(\tilde{w}(\tau_2))\right) d\alpha$$

$$+ \frac{1}{2}\int_0^1 \left(w^-(\tau_1,\alpha) - m(\tilde{w}(\tau_1))\right)\left(w^-(\tau_2,\alpha) - m(\tilde{w}(\tau_2))\right) d\alpha.$$

As a result,

$$\frac{\partial^2 K_{\tilde{g}}(t_1,t_2)}{\partial t_1 \partial t_2} = K_{\tilde{w}}(t_1,t_2). \tag{9}$$

Using formula (8) with $\tilde{z}(t_0) = \tilde{\xi}$, we write

$$\tilde{z}(t) = \tilde{\xi} + \int_{t_0}^{t} \tilde{w}(s)\, ds = \tilde{\xi} + \tilde{g}(t).$$

Letting $\tilde{\eta}(t) = \tilde{\xi} + \tilde{g}(t)$ and calculating the correlation function of the sum of fuzzy processes, we obtain

$$K_{\tilde{z}}(t_1,t_2) = K_{\tilde{\eta}}(t_1,t_2) = K_{\tilde{\xi}}(t_1,t_2) + K_{\tilde{g}}(t_1,t_2) + K_{\xi g}(t_1,t_2) + K_{\xi g}(t_2,t_1).$$

By analogy, differentiating this equality, first with respect to t_1 and then with respect to t_2, yields $\frac{\partial^2 K_{\tilde{z}}(t_1,t_2)}{\partial t_1 \partial t_2} = \frac{\partial^2 K_{\tilde{g}}(t_1,t_2)}{\partial t_1 \partial t_2}$. The other terms on the right-hand side vanish since $K_{\tilde{\xi}}$ is independent of t_1 and t_2 by definition whereas $K_{\xi g}(t_1,t_2)$ and $K_{\xi g}(t_2,t_1)$ depend only on t_2 and t_1, respectively. Considering formula (9), we finally arrive at the equality

$$\frac{\partial^2 K_{\tilde{z}}(t_1,t_2)}{\partial t_1 \partial t_2} = K_{\tilde{w}}(t_1,t_2) = K_{\tilde{z}'}(t_1,t_2),$$

and the proof of Theorem 7 is complete.

5. TRANSFORMATION OF A CONTINUOUS FUZZY PROCESS BY A LINEAR DYNAMIC SYSTEM

Consider some device A with continuous fuzzy signals $\tilde{y}(t)$ and $\tilde{z}(t)$ at its input and output, respectively.

Device A is called a linear dynamic system if the relationship between the input and output signals is described by an nth-order differential equation with constant coefficients. With fuzzy input $\tilde{y}(t)$ and output $\tilde{z}(t)$ signals, the linear dynamic system is described by the fuzzy differential equation

$$\begin{aligned} &a_n \tilde{z}^{(n)}(t) + a_{n-1}\tilde{z}^{(n-1)}(t) + \dots + a_1 \tilde{z}\,'(t) + a_0 \tilde{z}(t) \\ &= b_k \tilde{y}^{(k)}(t) + b_{k-1}\tilde{y}^{(k-1)}(t) + \dots + b_1 \tilde{y}\,'(t) + b_0 \tilde{y}(t) \equiv \tilde{f}(t). \end{aligned} \tag{10}$$

Here, the coefficients a_i $(i = 0, \dots, n)$ and b_i $(i = 0, \dots, k)$ are constant numbers, the second-order derivatives of the fuzzy function are understood as $\tilde{z}\,''(t) = (\tilde{z}\,'(t))'$ (and so on for higher-order derivatives).

The next result characterizes the connection between the mean values of the input and output fuzzy signals.

Lemma 1. *The mean value $z_{mean}(t) = m(\tilde{z}(t))$ of the output fuzzy signal $\tilde{z}(t)$ of the dynamic system* (10) *satisfies the scalar differential equation*

$$a_n x^{(n)} + a_{n-1} x^{(n-1)} + \cdots + a_1 x' + a_0 x = f(t), \tag{11}$$

where f stands for the mean of the right-hand side of (10)*: $f(t) = m\tilde{f}(t)$.*

Indeed, consider the mean of the left- and right-hand sides of equality (10). Using the additivity and homogeneity of means as well as Theorem 6, we obtain

$$a_n (m\tilde{z}(t))^{(n)} + a_{n-1}(m\tilde{z}(t))^{(n-1)} + \cdots + a_1 (m\tilde{z}(t))' + a_0 m\tilde{z}(t)$$
$$= b_k (m\tilde{y}(t))^{(k)} + b_{k-1}(m\tilde{y}(t))^{(k-1)} + \cdots + b_1 (m\tilde{y}(t))' + b_0 (m\tilde{y}(t)) \equiv m\tilde{f}(t).$$

Then the scalar function $z_{mean}(t) = m(\tilde{z}(t))$ satisfies equation (11).

Proposition 6 [16, Chapter II]. *Let the roots of the characteristic equation $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$ contain no points on the imaginary axis. Then for any continuous function $f(t)$ bounded on the entire real axis, there exists a unique solution of equation* (11) *that is bounded on the entire real axis. This solution has the form*

$$x(t) = \int_{-\infty}^{\infty} G(t-s) f(s)\, ds, \tag{12}$$

where $G(t)$ is the Green function of the problem on bounded solutions of equation (11).

Note that the Green function of the problem on bounded solutions of equation (11) is known; for example, see [17, Chapter 1, § 8].

Remark 1. Assume that under the hypotheses of Proposition 6, all roots of the characteristic equation belong to the left half-plane: ($Re\lambda_i < 0$, $i = 1, \ldots, n$). Then the bounded solution of equation (11) is asymptotically Lyapunov stable. In addition, the Green function of the problem on bounded solutions of equation (11) has the form

$$G(t) = \begin{cases} k(t) & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0, \end{cases}$$

where $k(t)$ is the Cauchy function of the homogeneous equation corresponding to (11).

Theorem 8. *Let the input fuzzy process $\tilde{y}(t)$ be continuous and bounded on the entire real axis together with its derivatives $\tilde{y}^i(t)$ ($i = 1, 2, \ldots, k$). Let the roots of the characteristic equation $a_n\lambda^n + a_{n-1}\lambda^{n-1} + \cdots + a_1\lambda + a_0 = 0$ contain no points on the imaginary axis. Then the mean value $m(\tilde{z}(t))$ at the output of the dynamic system* (10) *can be represented as*

$$m(\tilde{z}(t)) = \int_{-\infty}^{\infty} G(t-s) m(\tilde{f}(s))\, ds, \tag{13}$$

where G is the Green function of the problem on bounded solutions of equation (11).

Indeed, under the hypotheses of Theorem 8, the right-hand side of equation (11) is a bounded function on the entire real axis. Then, according to Lemma 1, the function $z_{mean}(t) = m(\tilde{z}(t))$ is the solution of equation (11) bounded on the entire real axis. Hence, Theorem 8 follows from Proposition 6.

Note that the boundedness of the fuzzy signal $\tilde{y}(t)$ (in Theorem 8 and below) is understood as the boundedness of all the corresponding α-indices $y_\alpha^\pm(t)$ in t $\forall\, \alpha \in [0, 1]$.

Corollary 3. *Assume that under the hypotheses of Theorem 8, the input signal is quasi-stationary:* $m(y(t)) = m_{\tilde{y}} = \text{const}$. *Then the output signal is quasi-stationary as well, and its mean value is* $m(\tilde{z}(t)) = m_{\tilde{z}} = \frac{b_0}{a_0} m_{\tilde{y}}$.

Indeed, the arbitrary-order derivative of a constant is zero and, in this case, the right-hand side of equation (11) is $b_0 m_{\tilde{y}}$. Then $m_{\tilde{z}}$ is the solution of the corresponding equation (11): $a_0 m_{\tilde{z}} = b_0 m_{\tilde{y}}$. Equation (11) has no other bounded solutions under the hypotheses of Theorem 8.

The same conclusion can be drawn for the mean value of the fuzzy input signal that stabilizes over time, i.e., $m(y(t)) \to m_{\tilde{y}}$ as $t \to \infty$.

In some cases, the indices of the output fuzzy signal of the dynamic system (10) can be written explicitly.

Theorem 9. *Assume that under the hypotheses of Theorem 8, all coefficients of the dynamic system* (10) *are positive* ($a_i > 0$, $i = 0, \ldots, n$). *Then the indices of the output fuzzy signal* $\tilde{z}(t)$ *of the dynamic system* (10) *have the form*

$$z_\alpha^-(t) = \int_{-\infty}^{\infty} G(t-s) f_\alpha^-(s)\, ds, \quad z_\alpha^+(t) = \int_{-\infty}^{\infty} G(t-s) f_\alpha^+(s)\, ds, \tag{14}$$

where $f_\alpha^\pm(s)$ *are the indices of the function* $\tilde{f}(s)$.

Indeed, equality for fuzzy numbers means equality for all the corresponding α-intervals. Due to the positivity of the coefficients a_i and Theorem (6), by the rules of interval arithmetic, equation (10) $\forall\, \alpha \in [0,1]$ implies

$$a_n (z_\alpha^-)^{(n)}(t) + a_{n-1} (z_\alpha^-)^{(n-1)}(t) + \cdots + a_1 (z_\alpha^-)'(t) + a_0 z_\alpha^-(t) = f_\alpha^-(t); \tag{15}$$

by analogy, for the indices with a plus sign,

$$a_n (z_\alpha^+)^{(n)}(t) + a_{n-1} (z_\alpha^+)^{(n-1)}(t) + \cdots + a_1 (z_\alpha^+)'(t) + a_0 z_\alpha^+(t) = f_\alpha^+(t). \tag{16}$$

According to (15) and (16), equalities (14) hold by Proposition 6.

Proposition 7. *Assume that under the hypotheses of Theorem 9, the Green function G of problem* (10) *is nonnegative. Then the bounded fuzzy signal at the output of the dynamic system* (10) *can be represented as*

$$\tilde{z}(t) = \int_{-\infty}^{\infty} G(t-s) \tilde{f}(s)\, ds. \tag{17}$$

Indeed, by the definition of the integral of a fuzzy function, we have the index relations

$$\left(\int_{-\infty}^{\infty} G(t-s) \tilde{f}(s)\, ds \right)_\alpha^- = \int_{-\infty}^{\infty} G(t-s) f_\alpha^-(s)\, ds,$$

$$\left(\int_{-\infty}^{\infty} G(t-s) \tilde{f}(s)\, ds \right)_\alpha^+ = \int_{-\infty}^{\infty} G(t-s) \tilde{f}_\alpha^+(s)\, ds.$$

In view of (14), they imply the representation (17).

Theorem 10. *Assume that under the hypotheses of Theorem 9, all roots of the characteristic equation have negative real parts* ($Re\lambda_i < 0$, $i = 1, \ldots, n$). *Then the output fuzzy signal* $\tilde{z}(t)$ *of the* *dynamic system* (10) *has the correlation function*

$$K_{\tilde{z}}(t_1,t_2)=\int\limits_{-\infty}^{t_1}\int\limits_{-\infty}^{t_2}G(t_1-\tau_1)G(t_2-\tau_2)K_{\tilde{f}}(\tau_1,\tau_2)d\tau_1 d\tau_2, \qquad (18)$$

where $K_{\tilde{f}}(\tau_1,\tau_2)$ *is the correlation function of the input signal* $\tilde{f}=\sum_{i=1}^{n}b_i\tilde{y}^{(i)}$ *and* G *is the Green function of the problem on bounded solutions of equation* (11).

Proof. Considering Remark 1, by definition (7) and formulas (13) and (14), we write

$$K_{\tilde{z}}(t_1,t_2)=\frac{1}{2}\int\limits_0^1\Bigg[\Bigg(\int\limits_{-\infty}^{t_1}G(t_1-s)(f_\alpha^+(s)-m(\tilde{f}(s)))ds\Bigg)\Bigg(\int\limits_{-\infty}^{t_2}G(t_2-s)(f_\alpha^+(s)-m(\tilde{f}(s)))ds\Bigg)$$
$$+\Bigg(\int\limits_{-\infty}^{t_1}G(t_1-s)(f_\alpha^-(s)-m(\tilde{f}(s)))ds\Bigg)\Bigg(\int\limits_{-\infty}^{t_2}G(t_2-s)(f_\alpha^-(s)-m(\tilde{f}(s)))ds\Bigg)\Bigg]d\alpha$$
$$=\frac{1}{2}\int\limits_0^1\int\limits_{-\infty}^{t_1}\int\limits_{-\infty}^{t_2}G(t_1-\tau_1)G(t_2-\tau_2)\Big[(f_\alpha^+(\tau_1)-m(\tilde{f}(\tau_1)))(f_\alpha^+(\tau_2)-m(\tilde{f}(\tau_2)))$$
$$+(f_\alpha^-(\tau_1)-m(\tilde{f}(\tau_1)))(f_\alpha^-(\tau_2)-m(\tilde{f}(\tau_2)))\Big]d\tau_1 d\tau_2 d\alpha.$$

Interchanging the order of integration gives

$$K_{\tilde{z}}(t_1,t_2)=\int\limits_{-\infty}^{t_1}\int\limits_{-\infty}^{t_2}G(t_1-\tau_1)G(t_2-\tau_2)\Bigg(\frac{1}{2}\int\limits_0^1(f_\alpha^-(\tau_1)-m(\tilde{f}(\tau_1)))(f_\alpha^-(\tau_2)-m(\tilde{f}(\tau_2)))$$
$$+(f_\alpha^+(\tau_1)-m(\tilde{f}(\tau_1)))(f_\alpha^+(\tau_2)-m(\hat{f}(\tau_2)))d\alpha\Bigg)d\tau_1 d\tau_2,$$

directly leading to (18).

Note that the assumption $Re\lambda_i<0$, $i=1,\ldots,n$, in Theorem 10 serves only for clarity when comparing with Theorem 5. Without this assumption, formula (18) becomes

$$K_{\tilde{z}}(t_1,t_2)=\int\limits_{-\infty}^{\infty}\int\limits_{-\infty}^{\infty}G(t_1-\tau_1)G(t_2-\tau_2)K_{\tilde{f}}(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

Example 3. Consider a linear dynamic system described by the first-order differential equation with constant coefficients

$$\tilde{z}'(t)+\beta\tilde{z}(t)=\tilde{y}'(t), \quad \beta>0.$$

Let a fuzzy signal $\tilde{y}'(t)$ bounded on the entire real axis be supplied to the input of this system. It is required to find the numerical characteristics of the bounded output fuzzy signal $\tilde{z}(t)$.

Note that the Green function of the problem on bounded solutions of the scalar equation $x'+\beta x=y(t)$ is represented as

$$G_1(t)=\begin{cases} e^{-\beta t} & \text{for } t\geqslant 0\\ 0 & \text{for } t<0.\end{cases}$$

Then, according to Theorem 8, the mean value at the system output has the form

$$m(\tilde{z}(t)) = \int\limits_{-\infty}^{t} e^{-\beta(t-s)} m(\tilde{y}\,'(s))ds = e^{-\beta t} \int\limits_{-\infty}^{t} e^{\beta s} m(\tilde{y}(s))' ds.$$

Integrating by parts the right-hand side gives

$$m(\tilde{z}(t)) = m(\tilde{y}(t)) - \beta e^{-\beta t} \int\limits_{-\infty}^{t} e^{\beta s} m(\tilde{y}(s))ds.$$

Using Theorem 10 and property 2 from Theorem 2, we write the correlation function at the output as follows:

$$K_{\tilde{z}}(t_1, t_2) = e^{-\beta(t_1+t_2)} \int\limits_{-\infty}^{t_1} \int\limits_{-\infty}^{t_2} e^{\beta(\tau_1+\tau_2)} \frac{\partial^2 K_{\tilde{y}}(\tau_1, \tau_2)}{\partial \tau_1 \partial \tau_2} d\tau_1 d\tau_2,$$

where $K_{\tilde{y}}(\tau_1, \tau_2)$ is the correlation function of the input signal.

Example 4. Consider a linear dynamic system described by the second-order differential equation with constant coefficients

$$\tilde{z}\,''(t) + a_1 \tilde{z}\,'(t) + a_0 \tilde{z}(t) = \tilde{y}(t).$$

Let a continuous fuzzy signal $\tilde{y}\,(t)$ bounded on the entire real axis be supplied to the input of this system. It is required to find the numerical characteristics of the bounded output fuzzy signal $\tilde{z}(t)$.

Suppose that the coefficients of this equation satisfy the conditions $a_1, a_0 > 0$ and $a_1^2 - 4a_0 > 0$. Then the roots λ_1 and λ_2 of the characteristic equation $\lambda^2 + a_1\lambda + a_0 = 0$ are real and $\lambda_1 < \lambda_2 < 0$. In this case, the Green function G_2 of the problem on bounded solutions of the equation $a_2x'' + a_1x' + a_0x = f(t)$ has the form

$$G_2(t) = \begin{cases} (e^{\lambda_2 t} - e^{\lambda_1 t})(\lambda_2 - \lambda_1)^{-1} & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0. \end{cases}$$

Then, according to Theorems 8 and 10, the output fuzzy signal $\tilde{z}(t)$ satisfies the relations

$$m(\tilde{z}(t)) = \int\limits_{-\infty}^{t} G_2(t-s) m(\tilde{y}(s))ds,$$

$$K_{\tilde{z}}(t_1, t_2) = \int\limits_{-\infty}^{t_1} \int\limits_{-\infty}^{t_2} G_2(t_1 - \tau_1) G_2(t_2 - \tau_2) K_{\tilde{y}}(\tau_1, \tau_2) d\tau_1 d\tau_2.$$

Note that the Green functions G_1 and G_2 in Examples 3 and 4 are nonnegative. Hence, the representation (17) holds in these examples.

Example 5. Consider a linear dynamic system described by the third-order differential equation with constant coefficients

$$\tilde{z}\,'''(t) + a_2 \tilde{z}\,''(t) + a_1 \tilde{z}\,''(t) + a_0 \tilde{z}(t) = \tilde{y}(t).$$

Let a continuous fuzzy signal $\tilde{y}(t)$ bounded on the entire real axis be supplied to the input of this system. It is required to find the numerical characteristics of the output fuzzy signal $\tilde{z}(t)$.

Suppose that $a_2 > 0$, $a_1 > 0$, $a_0 > 0$, and $a_2a_1 - a_0 > 0$. Then, by the Hurwitz criterion, the equation $\lambda^3 + a_2\lambda^2 + a_1 + a_0 = 0$ has the roots with $Re\lambda_i < 0$ $(i = 1, 2, 3)$. Therefore, according to Theorems 8 and 10, the output fuzzy signal $\tilde{z}(t)$ satisfies the relations

$$m(\tilde{z}(t)) = \int\limits_{-\infty}^{t} G_3(t-s)m(\tilde{y}(s))ds,$$

$$K_{\tilde{z}}(t_1,t_2) = \int\limits_{-\infty}^{t_1}\int\limits_{-\infty}^{t_2} G_3(t_1-\tau_1)G_3(t_2-\tau_2)K_{\tilde{y}}(\tau_1,\tau_2)d\tau_1 d\tau_2.$$

Here, $G_3(t)$ is the Green function of the problem on bounded solutions of the equation

$$x'''(t) + a_2x''(t) + a_1x'(t) + a_0x(t) = f(t),$$

which has the form $G_3(t) = \begin{cases} k(t) & \text{for } t \geqslant 0 \\ 0 & \text{for } t < 0, \end{cases}$ where $k(t)$ is the Cauchy function representing the solution of the homogeneous equation

$$k'''(t) + a_2k''(t) + a_1k'(t) + a_0k(t) = 0$$

with the initial conditions

$$k(0) = k'(0) = 0, \quad k''(0) = 1.$$

(For details, see [17, Chapter 2, § 8].)

For example, if the characteristic equation has different roots, the Cauchy function is given by

$$k(t) = C_1e^{\lambda_1 t} + C_2e^{\lambda_2 t} + C_3e^{\lambda_3 t},$$

where

$$C_1 = \frac{1}{(\lambda_1-\lambda_2)(\lambda_1-\lambda_3)}, \quad C_2 = \frac{1}{(\lambda_2-\lambda_3)(\lambda_2-\lambda_1)}, \quad C_3 = \frac{1}{(\lambda_3-\lambda_1)(\lambda_3-\lambda_2)}.$$

6. CONCLUSIONS

The results of Sections 3 and 4 of this paper—the properties of numerical characteristics of fuzzy processes—are similar to the well-known counterparts for continuous random processes. However, despite their significance, they have not been established before.

The main results of this paper concern fuzzy dynamic systems described by nth-order linear differential equations with bounded input fuzzy signals (Section 5). They are based on the new properties of the mean and correlation functions of continuous fuzzy processes (Sections 3 and 4) as well as on the development of the Green function method to the class of fuzzy differential equations.

The approach outlined here is an alternative to the conventional one used to study linear dynamic systems with constant coefficients in terms of frequency response and direct and inverse Fourier transform. Unlike the known approaches, it does not assume stationarity (in any sense) for the processes under consideration. Note that this approach can be extended to continuous processes with fuzzy random states.

REFERENCES

1. Venttsel', E.S. and Ovcharov, L.A., *Teoriya sluchainykh protsessov i ikh inzhenernye prilozheniya* (Theory of Random Processes and Their Engineering Applications), Moscow: Knorus, 2016.
2. Averkin, A.N., *Nechetkie mnozhestva v modelyakh upravleniya i iskusstvennogo intellekta* (Fuzzy Sets in Models of Control and Artificial Intelligence), Moscow: Nauka, 1986.
3. Buckley, J.J., Eslami, E., and Feuring, T., *Fuzzy Mathematics in Economics and Engineering*, Heidelberg–New York: Physica-Verl., 2002.
4. Pegat, A., *Nechetkoe modelirovanie i upravlenie* (Fuzzy Modeling and Control), Moscow: BINOM. Laboratoriya Znanii, 2015.
5. Aumann, R.J., Integrals of Set-Valued Functions, *J. Math. Anal. Appl.*, 1965, no. 12, pp. 1–12.
6. Puri, M.L. and Ralescu, D.A., Differential of Fuzzy Functions, *J. Math. Anal. Appl.*, 1983, vol. 91, pp. 552–558.
7. Kaleva, O., Fuzzy Differential Equations, *Fuzzy Sets and Syst.*, 1987, vol. 24, no. 3, pp. 301–317.
8. Seikkala, S., On the Fuzzy Initial Value Problem, *Fuzzy Sets Syst.*, 1987, vol. 24, no. 3, pp. 319–330.
9. Hukuhara, M., Integration des applications mesurables dont la valeur est un compact convexe, *Func. Ekvacioj.*, 1967, no. 11, pp. 205–223.
10. Khatskevich, V.L., Means, Quasi-scalar Product and Covariance of Fuzzy Numbers, *Journal of Physics: Conference Series*, 2021, vol. 1902(1), art. no. 012136.
11. Park, J.Y. and Han, H., Existence and Uniqueness Theorem for a Solution of Fuzzy Differential Equations, *Int. J. Math. Mathem. Sci.*, 1999, no. 22(2), pp. 271–280.
12. Ahmad, L., Farooq, M., and Abdullah, S., Solving nth Order Fuzzy Differential Equation by Fuzzy Laplace Transform, *Ind. J. Pure Appl. Math.*, 2014, no. 2, pp. 1–20.
13. Mochalov, I.A., Khrisat, M.S., and Shihab Eddin, M.Ya., Fuzzy Differential Equations in Control. Part II, *Information Technologies*, 2015, vol. 21, no. 4, pp. 243–250.
14. Demenkov, N.P., Mikrin, E.A., and Mochalov, I.A., Fuzzy Optimal Control of Linear Systems. Part 1. Positional Control, *Information Technologies*, 2019, vol. 25, no. 5, pp. 259–270.
15. Esmi, E., Sanchez, D.E., Wasques, V.F., and de Barros, L.C., Solutions of Higher Order Linear Fuzzy Differential Equations with Interactive Fuzzy Values, *Fuzzy Sets and Systems*, 2021, vol. 419, pp. 122–140.
16. Daletskii, Yu.L. and Krein, M.G., *Ustoichivost' reshenii differentsial'nykh uravnenii v banakhovom prostranstve* (Stability of Solutions of Differential Equations in a Banach Space), Moscow: Nauka, 1970.
17. Krasnosel'skii, M.A., Burd, V.Sh., and Kolesov, Yu.S., *Nonlinear Almost Periodic Oscillations*, New York: J. Wiley, 1973.
18. Dubois, D. and Prade, H., The Mean Value of Fuzzy Number, *Fuzzy Sets and Syst.*, 1987, vol. 24, no. 3, pp. 279–300.
19. Kaleva, O. and Seikkala, S., On Fuzzy Metric Spaces, *Fuzzy Sets and Systems*, 1984, vol. 12, pp. 215–229.
20. Fuller, R. and Majlender, P., On Weighted Possibilistic Mean Value and Variance of Fuzzy Numbers, *Fuzzy Sets and Systems*, 2003, vol. 136, pp. 363–374.

This paper was recommended for publication by D.V. Vinogradov, a member of the Editorial Board