ISSN 0005-1179 (print), ISSN 1608-3032 (online), Automation and Remote Control, 2023, Vol. 84, No. 8, pp. 918–932.

LINEAR SYSTEMS

Design of Suboptimal Robust Controllers Based on A Priori and Experimental Data

M. M. Kogan[*,a] **and A. V. Stepanov**[*,b]

[*]*Nizhny Novgorod State University of Architecture and Civil Engineering, Nizhny Novgorod, Russia*
e-mail: [a]*mkogan@nngasu.ru,* [b]*andrey8st@yahoo.com*

Received March 21, 2023
Revised May 2, 2023
Accepted June 9, 2023

Abstract—This paper develops a novel unified approach to designing suboptimal robust control laws for uncertain objects with different criteria based on a priori information and experimental data. The guaranteed estimates of the γ_0, generalized H_2, and H_∞ norms of a closed loop system and the corresponding suboptimal robust control laws are expressed in terms of solutions of linear matrix inequalities considering a priori knowledge and object modeling data. A numerical example demonstrates the improved quality of control systems when a priori and experimental data are used together.

Keywords: robust control, a priori data, experimental data, γ_0 norm, generalized H_2 norm, H_∞ norm, linear matrix inequalities

DOI: 10.25728/arcRAS.2023.29.47.001

1. INTRODUCTION

In a rich variety of control design approaches for objects with an incomplete mathematical model, there exist two main ones as follows. Within one approach, the controller's parameters are found from a priori information about the possible ranges of the object's uncertain parameters. Following the other approach, the controller's parameters are tuned recursively using current information or are calculated based on experimental data. Traditionally, the former approach is associated with robust control (see [1] and the survey [2]); the latter approach, with adaptive control (see the surveys [3, 4]).

In recent years, researchers have been actively developing the data-driven design of control systems without any explicit mathematical model of the object [5–9]. The paper [10] was pioneering in this area: it was discovered that a single trajectory can be used to fully characterize a linear time-invariant dynamic system under the so-called persistency of excitation. If this condition holds, linear quadratic control of objects without disturbances and without measurement noises can be implemented without knowledge of the object's mathematical model directly from input and output measurement data [5]. According to [6], it suffices to fulfill the less restrictive condition of data informativity for the property of interest in order to construct control laws from experimental data. (Examples of such properties are stabilizability by linear state feedback control or linear quadratic control with a given performance criterion). In [7], the state feedback parameters were derived from open-loop measurements of the input and output of an uncertain object subjected to an unmeasured disturbance from a definite class. For a fully uncertain object, H_2- and H_∞-optimal control laws were constructed based on input and output measurements using a matrix version of S-lemma [11] in the publication [8] and using Petersen's lemma [12] in the publication [9].

This paper develops a novel robust control design approach for uncertain dynamic objects based on the joint use of a priori information about the structure of the uncertain object parameter matrix and the upper bound of its norm (on the one hand) and experimental data obtained by observing the object on some time interval (on the other hand). The quality of robust control is evaluated by upper bounds for one of the three performance indices: the γ_0 norm (the damping level of stochastic disturbances in the closed-loop uncertain system or the maximum value of the quadratic functional of the target output under a pulse disturbance), the generalized H_2 norm (the time-maximum deviation of the Euclidean norm of the system's target output under all deterministic disturbances bounded in the l_2 norm), and the H_∞ norm (the maximum value of the ratio of the l_2 norms of the target output and the exogenous disturbance).

The design procedure includes several basic steps. First, the set of unknown matrices consistent with a priori information is characterized by a quadratic inequality. Then, an experiment is conducted to measure the system trajectory under given initial conditions and controls and an unknown exogenous disturbance with known bounds of its components. This step yields another quadratic inequality satisfied by all unknown matrices consistent with the experimental results. Next, an extended and completely defined system with additional artificial input and output satisfying the two quadratic inequalities is determined; this system "incorporates" the original uncertain system. Finally, upper bounds are found for the damping levels of the disturbances of the original uncertain system as those of the disturbances of the extended system under all additional inputs satisfying the two quadratic inequalities.

This paper is organized as follows. After the Introduction, Section 2 gives the general problem statement; in particular, two quadratic inequalities for the unknown object parameter matrix are derived from a priori information and experimental data. In Section 3, necessary background is provided on the γ_0, generalized H_2, and H_∞ norms as well as their relations in the primal and dual systems. Section 4 describes the robust control design procedure, including the main theorem and its proof. Several experiments with an uncertain third-order system are presented in Section 5; they show the advantages of robust control laws based on a priori information and experimental data over the counterparts designed using a priori information or experimental data only. Section 6 summarizes the results and draws conclusions.

2. ROBUST CONTROL BASED ON A PRIORI AND EXPERIMENTAL DATA: PROBLEM STATEMENT

Consider an uncertain system described by

$$\begin{aligned} x(t+1) &= (A + B_\Delta \Delta C_\Delta)x(t) + (B_u + B_\Delta \Delta D_\Delta)u(t) + Bw(t), \\ z(t) &= Cx(t) + Du(t) \end{aligned} \tag{2.1}$$

with the following notations: $x(t) \in \mathrm{R}^{n_x}$ is the state vector, $z(t) \in \mathrm{R}^{n_z}$ is the target output, $w(t) \in \mathrm{R}^{n_w}$ is an exogenous disturbance, and $u(t) \in \mathrm{R}^{n_u}$ is the control vector (input). All matrices except the unknown parameter matrix Δ are given. In general, it is required to design linear state-feedback control laws based on information about the unknown parameters of the system so that the damping levels of the exogenous disturbances from different classes in the closed loop system do not exceed specified values.

The information about the unknown matrix Δ is divided into a priori one and the one obtained by a preliminary experiment. Assume that the matrix Δ has a block-diagonal structure and

$$\Delta = \mathrm{diag}\,(\Delta_1, \ldots, \Delta_l) = \sum_{i=1}^{l} L_i \Delta_i R_i^{\mathrm{T}}, \quad \Delta_i \Delta_i^{\mathrm{T}} \leqslant \eta_i^2 I, \tag{2.2}$$

where $\Delta_i \in \mathrm{R}^{m_i \times n_i}$ is a complete matrix block or a diagonal square matrix block $\Delta_i = \delta_i I_{n_i}$; L_i and R_i are matrices composed of unit column vectors corresponding to the location of the ith matrix block such that $L_i^{\mathrm{T}} L_j = 0$ and $R_i^{\mathrm{T}} R_j = 0$, $i \neq j$, and η_i are given values. In accordance with the structure of the matrix Δ, the matrix B_Δ can be written as $B_\Delta = (B_1 \ \dots \ B_l)$, where $B_i = B_\Delta L_i$. With the notation $\widehat{\Delta} = B_\Delta \Delta$, we have

$$\widehat{\Delta} = B_\Delta \sum_{i=1}^{l} L_i \Delta_i R_i^{\mathrm{T}} = \sum_{i=1}^{l} B_i \Delta_i R_i^{\mathrm{T}}. \tag{2.3}$$

Since $\widehat{\Delta} R_j = B_j \Delta_j$, $j = 1, \dots, l$, it follows that $\widehat{\Delta} = \left(\widehat{\Delta}_1 \ \widehat{\Delta}_2 \ \cdots \ \widehat{\Delta}_l\right)$, where $\widehat{\Delta}_i = B_i \Delta_i$.

In particular, if the state and control matrices in the object's equation are completely unknown, then

$$A = 0, \quad B_u = 0, \quad B_\Delta = I, \quad C_\Delta = (I \quad 0)^{\mathrm{T}}, \quad D_\Delta = (0 \quad I)^{\mathrm{T}} \tag{2.4}$$

in (2.1); in this case, $\widehat{\Delta} = \Delta = (A^{(real)} \ B_u^{(real)})$, where $A^{(real)}$ and $B_u^{(real)}$ are the unknown state and control matrices, respectively. This case without using a priori information was studied in the papers [5, 6, 8, 9]).

Next we express the a priori information about the matrix Δ in terms of the matrix $\widehat{\Delta}$. Following the well-known robust control design approach under structured uncertainty [13, 14], let us define the set $\mathbf{\Lambda} = \mathrm{diag}\,(\mathbf{\Lambda_1}, \dots, \mathbf{\Lambda_l})$ consisting of all $\Lambda = \mathrm{diag}\,(\Lambda_1, \dots, \Lambda_l)$ for which $\Lambda_i = \lambda_i I_{n_i}$, $\lambda_i \geqslant 0$, if the matrix block Δ_i is complete and all symmetric nonnegative definite matrices $\Lambda_i \in \mathrm{R}^{n_i \times n_i}$ if $\Delta_i = \delta_i I_{n_i}$. Due to (2.2), the inequality $\lambda_i \Delta_i \Delta_i^{\mathrm{T}} \leqslant \lambda_i \eta_i^2 I$ holds for a complete matrix block $\Delta_i \in \mathrm{R}^{n_i \times n_i}$ for all $\lambda_i \geqslant 0$ and the inequality $\Delta_i \Lambda_i \Delta_i^{\mathrm{T}} \leqslant \eta_i^2 \Lambda_i$ holds for a block $\Delta_i = \delta_i I_{n_i}$ for all symmetric nonnegative definite matrices $\Lambda_i \in \mathrm{R}^{n_i \times n_i}$. Hence, as is easily verified,

$$\Delta \Lambda \Delta^{\mathrm{T}} - \eta \Lambda \eta^{\mathrm{T}} \leqslant 0 \quad \forall\, \Lambda \in \mathbf{\Lambda} \tag{2.5}$$

with $\eta = \mathrm{diag}\,(\eta_1 I_{n_1}, \dots, \eta_l I_{n_l})$ for all the matrices Δ satisfying (2.2).

Multiplying this inequality by the matrices B_Δ and B_Δ^{T} on the left and right, respectively, yields

$$\widehat{\Delta} \Lambda \widehat{\Delta}^{\mathrm{T}} - B_\Delta \eta \Lambda \eta^{\mathrm{T}} B_\Delta^{\mathrm{T}} \leqslant 0 \quad \forall\, \Lambda \in \mathbf{\Lambda}. \tag{2.6}$$

This condition can be written as

$$\left(\widehat{\Delta} \quad I\right) \Upsilon \left(\widehat{\Delta} \quad I\right)^{\mathrm{T}} \leqslant 0 \quad \forall\, \Lambda \in \mathbf{\Lambda}, \tag{2.7}$$

where $\Upsilon = \mathrm{diag}(\Lambda, -B_\Delta \eta \Lambda \eta^{\mathrm{T}} B_\Delta^{\mathrm{T}})$. Let $\mathbf{\Delta}$ denote the set of given-structure matrices Δ satisfying (2.5) and $\widehat{\mathbf{\Delta_a}}$ denote the set of matrices $\widehat{\Delta} = (\widehat{\Delta}_1, \dots, \widehat{\Delta}_l)$ satisfying inequality (2.6). Clearly, for any $\Delta \in \mathbf{\Delta}$ there exists $\widehat{\Delta} = B_\Delta \Delta \in \widehat{\mathbf{\Delta_a}}$. The converse is also true as follows.

Lemma 2.1. *If matrices $B_i = B_\Delta L_i$, $i = 1, \dots, l$, have full column rank, then for any $\widehat{\Delta} \in \widehat{\mathbf{\Delta_a}}$ there exists $\Delta \in \mathbf{\Delta}$ such that $\widehat{\Delta} = B_\Delta \Delta$.*

Proof of Lemma. Assume that $\widehat{\Delta} \in \widehat{\mathbf{\Delta_a}}$. Due to (2.6), we have $\widehat{\Delta}^{\mathrm{T}} a = 0$ for any vector $a \neq 0$ with $B_\Delta^{\mathrm{T}} a = 0$. This means that the columns of the matrix $\widehat{\Delta}$ belong to the image of the matrix B_Δ. Hence, the linear matrix equation $B_\Delta \Delta = \widehat{\Delta}$ is solvable in the matrix Δ. It remains to show inequality (2.5) for this solution. From (2.6) it follows that

$$B_i (\Delta_i \Lambda_i \Delta_i^{\mathrm{T}} - \eta_i^2 \Lambda_i) B_i^{\mathrm{T}} \leqslant 0$$

for each block. Since the matrices B_i have full column rank, $\Delta_i \Lambda_i \Delta_i^{\mathrm{T}} - \eta_i^2 \Lambda_i \leqslant 0$ holds for all i, i.e., $\Delta \in \mathbf{\Delta}$, and the desired result is established.

According to this lemma, there is no loss of information when passing from the matrix Δ that satisfies inequality (2.5) to the matrix $\widehat{\Delta}$ that satisfies inequality (2.7). In view of this fact, we write the original uncertain system (2.1) as

$$\begin{aligned} x(t+1) &= (A + \widehat{\Delta} C_\Delta) x(t) + (B_u + \widehat{\Delta} D_\Delta) u(t) + B w(t), \\ z(t) &= C x(t) + D u(t), \end{aligned} \tag{2.8}$$

where the unknown parameter matrix $\widehat{\Delta} = (\widehat{\Delta}_1, \ldots, \widehat{\Delta}_l)$ of the corresponding structure satisfies inequality (2.7).

Additional information about the unknown parameters of system (2.8) is extracted from a finite set of its trajectory measurements. More precisely put, it is possible to measure the system states $x_0, x_1, \ldots, x_N$ under given controls $u_0, \ldots, u_{N-1}$ and some unknown disturbance $w(t)$ whose components satisfy the constraint

$$|w_i(t)| \leqslant d, \quad t = 0, \ldots, N-1, \quad i = 1, \ldots, n_w \tag{2.9}$$

for some given d (the disturbance level), i.e., $\max_{0 \leqslant t \leqslant N-1} \|w(t)\|_\infty \leqslant d$. Following conventional notations (e.g., see [6]), we compile the matrices

$$\begin{aligned} \Phi &= (x_0 \quad x_1 \quad \cdots \quad x_{N-1}), \quad \Phi_+ = (x_1 \quad x_2 \quad \cdots \quad x_N), \\ W &= (w_0 \quad w_1 \quad \cdots \quad w_{N-1}), \quad U = (u_0 \quad u_1 \quad \cdots \quad u_{N-1}) \end{aligned}$$

and introduce

$$C_\Delta \Phi + D_\Delta U = \widehat{\Phi}.$$

Due to the object's equation,

$$\widetilde{\Phi} = \widehat{\Delta}^{(real)} \widehat{\Phi} + BW, \tag{2.10}$$

where $\widetilde{\Phi} = \Phi_+ - A\Phi - B_u U$ and $\widehat{\Delta}^{(real)}$ is the real unknown parameter matrix of the object (2.8). According to (2.9) and (2.10),

$$(\widetilde{\Phi} - \widehat{\Delta}\widehat{\Phi})(\widetilde{\Phi} - \widehat{\Delta}\widehat{\Phi})^{\mathrm{T}} = BWW^{\mathrm{T}}B^{\mathrm{T}} \leqslant d^2 n_w N B B^{\mathrm{T}}$$

for $\widehat{\Delta} = \widehat{\Delta}^{(real)}$.

Let $\widehat{\boldsymbol{\Delta}}_{\mathbf{p}}$ denote the set of given-structure matrices $\widehat{\Delta}$ satisfying this inequality. Obviously, $\widehat{\Delta}^{(real)} \in \widehat{\boldsymbol{\Delta}}_{\mathbf{p}}$. Introducing the matrix

$$\Psi = \begin{pmatrix} \Psi_{11} & * \\ \Psi_{12}^{\mathrm{T}} & \Psi_{22} \end{pmatrix} = \begin{pmatrix} \widehat{\Phi}\widehat{\Phi}^{\mathrm{T}} & * \\ -\widetilde{\Phi}\widehat{\Phi}^{\mathrm{T}} & \widetilde{\Phi}\widetilde{\Phi}^{\mathrm{T}} - d^2 n_w N B B^{\mathrm{T}} \end{pmatrix}, \tag{2.11}$$

we write this inequality as

$$\begin{pmatrix} \widehat{\Delta} & I \end{pmatrix} \Psi \begin{pmatrix} \widehat{\Delta} & I \end{pmatrix}^{\mathrm{T}} \leqslant 0. \tag{2.12}$$

Let $\widehat{\boldsymbol{\Delta}} = \widehat{\boldsymbol{\Delta}}_{\mathbf{a}} \bigcap \widehat{\boldsymbol{\Delta}}_{\mathbf{p}}$ denote the set of matrices $\widehat{\Delta}$ satisfying the constraints (2.7) and (2.12).

The quality of the closed-loop uncertain system (2.8) with a linear state-feedback control law will be evaluated by its response to stochastic and deterministic disturbances under zero initial state, measured by three performance indices: the guaranteed estimates of the γ_0, generalized H_2, and H_∞ norms. The guaranteed estimate of the γ_0 norm is defined as the damping level of a stochastic disturbance from the class $\mathcal{G}_{n_w}$ of vector Gaussian white noises of dimension n_w, equal to the maximum value of the square root of the ratio of the steady-state time-averaged variances of the output z and input w under all nonzero covariance matrices K_w of the input [15]:

$$\gamma_0 = \sup_{\widehat{\Delta} \in \widehat{\mathbf{\Delta}}} \gamma_0(\widehat{\Delta}), \quad \gamma_0(\widehat{\Delta}) = \text{ess} \sup_{w \in \mathcal{G}_{n_w}} \frac{\|z\|_{\mathcal{P}}}{\|w\|_{\mathcal{P}}},$$

where $\|s\|^2_{\mathcal{P}} = \lim\limits_{N\to\infty}(1/N)\sum_{t=0}^{N-1}|s(t)|^2$ and ess stands for essential supremum (the least upper bound with probability 1). The guaranteed estimates of the generalized H_2 and H_∞ norms characterize, respectively, the relative maximum values of the time-maximal deviation and the quadratic functional of the target output under deterministic disturbances from the class l_2. They are defined as

$$\gamma_{g2} = \sup_{\widehat{\Delta} \in \widehat{\mathbf{\Delta}}} \gamma_{g2}(\widehat{\Delta}), \quad \gamma_{g2}(\widehat{\Delta}) = \sup_{w(t)\neq 0} \frac{\sup_{t\geqslant 0}|z(t)|}{\|w\|},$$

$$\gamma_\infty = \sup_{\widehat{\Delta} \in \widehat{\mathbf{\Delta}}} \gamma_\infty(\widehat{\Delta}), \quad \gamma_\infty(\widehat{\Delta}) = \sup_{w(t)\neq 0} \frac{\|z\|}{\|w\|},$$

where $\|s\|^2 = \sum_{t=0}^{\infty}|s(t)|^2$. The problem is to obtain upper bounds for these norms and finally design control laws ensuring the required system quality estimates.

3. NECESSARY BACKGROUND ON THE γ_0, GENERALIZED H_2, AND H_∞ NORMS

Before deriving the guaranteed estimates of the above norms, we clarify the calculation of the norms $\gamma_0(\widehat{\Delta})$, $\gamma_{g2}(\widehat{\Delta})$, and $\gamma_\infty(\widehat{\Delta})$ for the closed loop system (2.8), $u(t) = \Theta x(t)$ under a fixed matrix $\widehat{\Delta}$ given by the equations

$$\begin{aligned} x(t+1) &= [A + B_u\Theta + \widehat{\Delta}(C_\Delta + D_\Delta\Theta)]x(t) + Bw(t), \\ z(t) &= (C + D\Theta)x(t). \end{aligned} \tag{3.1}$$

With the notations

$$A_\Theta = A + B_u\Theta, \quad C_{\Delta\Theta} = C_\Delta + D_\Delta\Theta, \quad A_\Delta = A_\Theta + \widehat{\Delta}C_{\Delta\Theta}, \quad C_\Theta = C + D\Theta,$$

these equations can be written as

$$\begin{aligned} x(t+1) &= A_\Delta x(t) + Bw(t), \\ z(t) &= C_\Theta x(t). \end{aligned} \tag{3.2}$$

The damping level of the stochastic disturbance, i.e., the γ_0 norm of this system, is found by solving a semidefinite programming problem in the covariance matrices $K_w = K_w^{\mathrm{T}} \geqslant 0$ (the disturbance) and $K_x = K_x^{\mathrm{T}} \geqslant 0$ (the state) [15]:

$$\gamma_0^2(\widehat{\Delta}) = \max \operatorname{tr} C_\Theta K_x C_\Theta^{\mathrm{T}}: \quad A_\Delta K_x A_\Delta^{\mathrm{T}} - K_x + BK_wB^{\mathrm{T}} = 0, \quad \operatorname{tr} K_w \leqslant 1. \tag{3.3}$$

Here, we need the following auxiliary result, proved in the Appendix.

Lemma 3.1. *Problem* (3.3) *is Lagrange dual to the problem*

$$\gamma_0^2(\widehat{\Delta}) = \min \gamma^2: \quad A_\Delta^{\mathrm{T}} P A_\Delta - P + C_\Theta^{\mathrm{T}} C_\Theta \leqslant 0, \quad B^{\mathrm{T}} P B \leqslant \gamma^2 I. \tag{3.4}$$

According to problem (3.4), the increment of the function $V(x) = x^{\mathrm{T}}Px$ along the trajectories of (3.2) with the initial disturbance $w(0) = w_0$, $w(t) \equiv 0$, $t > 0$, and zero initial conditions satisfies the inequalities

$$\triangle V + |z|^2 \leqslant 0,\ t \geqslant 1, \quad V(x_1) = w_0^{\mathrm{T}} B^{\mathrm{T}} P B w_0 \leqslant \gamma^2 |w_0|^2 \quad \forall x \in \mathrm{R}^{n_x}, \forall\, w_0 \in \mathrm{R}^{n_w}. \tag{3.5}$$

In other words, the damping level of stochastic disturbances coincides with that of a deterministic initial disturbance, understood as the maximum value of the ratio of the l_2 norm of the output under the "pulse" disturbance $w(0) = w_0,\ w(t) \equiv 0,\ t \geqslant 1$, and zero initial conditions to the Euclidean norm of the disturbance:

$$\gamma_0^2(\widehat{\Delta}) = \max_{w_0 \neq 0} \frac{\|z\|^2}{|w_0|^2}.$$

The next characteristic—the maximum deviation of the output (the generalized H_2 norm [16, 17])—is found by solving the problem

$$\gamma_{g2}^2(\widehat{\Delta}) = \min \gamma^2 : \quad A_\Delta Q A_\Delta^{\mathrm{T}} - Q + BB^{\mathrm{T}} \leqslant 0, \quad C_\Theta Q C_\Theta^{\mathrm{T}} \leqslant \gamma^2 I. \tag{3.6}$$

With the change of variables $P = Q^{-1}$, it can be written as

$$\gamma_{g2}^2(\widehat{\Delta}) = \min \gamma^2 : \quad \begin{pmatrix} A_\Delta^{\mathrm{T}} P A_\Delta - P & * \\ B^{\mathrm{T}} P A_\Delta & B^{\mathrm{T}} P B - I \end{pmatrix} \leqslant 0, \quad \begin{pmatrix} P & * \\ C_\Theta & \gamma^2 I \end{pmatrix} \geqslant 0.$$

This means that the increment of the function $V(x) = x^{\mathrm{T}}Px$ along the trajectories of (3.2) with zero initial conditions satisfies the inequality

$$\triangle V - |w|^2 \leqslant 0, \quad \forall\, x \in \mathrm{R}^{n_x}, \quad \forall\, w \in \mathrm{R}^{n_w}, \quad P \geqslant \gamma^{-2} C_\Theta^{\mathrm{T}} C_\Theta. \tag{3.7}$$

As is well known, the system under consideration has the H_∞ norm below γ if and only if the linear matrix inequality (LMI)

$$\begin{pmatrix} A_\Delta^{\mathrm{T}} P A_\Delta - P & * & * \\ B^{\mathrm{T}} P A_\Delta & B^{\mathrm{T}} P B - \gamma^2 I & * \\ C_\Theta & 0 & -I \end{pmatrix} < 0 \tag{3.8}$$

is solvable in the matrix $P = P^{\mathrm{T}} > 0$. According to this LMI, the increment of the positive definite function $V(x) = x^{\mathrm{T}}Px$ along the trajectories of (3.2) satisfies the inequality

$$\triangle V + |z|^2 - \gamma^2 |w|^2 < 0 \tag{3.9}$$

for all x and w.

Direct comparison of problems (3.4) and (3.6) shows that the γ_0 and generalized H_2 norms of system (3.1), respectively, coincide with the generalized H_2 and γ_0 norms of the dual system

$$\begin{aligned} \widehat{x}(t+1) &= (A_\Theta + \widehat{\Delta} C_{\Delta\Theta})^{\mathrm{T}} \widehat{x}(t) + C_\Theta^{\mathrm{T}} \widehat{w}(t), \quad \widehat{x}(0) = 0, \\ \widehat{z}(t) &= B^{\mathrm{T}} \widehat{x}(t). \end{aligned} \tag{3.10}$$

In addition, the dual systems (3.1) and (3.10) obviously have the same H_∞ norm.

4. ROBUST γ_0-, GENERALIZED H_2-, AND H_∞-SUBOPTIMAL CONTROL LAWS

Now we present the main steps to obtain the guaranteed estimates of the γ_0, γ_{g2}, and γ_∞ norms of the uncertain system (3.1) and to find the parameters of the corresponding suboptimal robust control laws. Let $\widehat{\gamma}_0$, $\widehat{\gamma}_{g2}$, and $\widehat{\gamma}_\infty$ denote the corresponding guaranteed estimates of the norms of the dual system (3.10). According to the previous section,

$$\gamma_0 = \widehat{\gamma}_{g2}, \quad \gamma_{g2} = \widehat{\gamma}_0, \quad \gamma_\infty = \widehat{\gamma}_\infty.$$

Consider an extended system with the additional artificial input $w_\Delta(t)$ and output $z_\Delta(t)$ described by

$$\begin{aligned} x_a(t+1) &= A_\Theta^{\mathrm{T}} x_a(t) + C_{\Delta\Theta}^{\mathrm{T}} w_\Delta(t) + C_\Theta^{\mathrm{T}} w_a(t), \quad x_a(0) = 0, \\ z_a(t) &= B^{\mathrm{T}} x_a(t), \quad z_\Delta(t) = x_a(t), \end{aligned} \tag{4.1}$$

where $x_a(t)$, $w_a(t)$, and $z_a(t)$ are the state, disturbance, and target output, respectively. Suppose that for all $t \geqslant 0$, the additional input signal $w_\Delta(t)$ in system (4.1) satisfies the two inequalities

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0, \quad \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Upsilon \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} \leqslant 0, \tag{4.2}$$

where the matrices Ψ and Υ are given by (2.11) and (2.7). The set of all such signals will be denoted by $\mathbf{W_\Delta}$. System (3.10) is "immersed" in system (4.1), (4.2): for $w_\Delta(t) = \widehat{\Delta}^{\mathrm{T}} z_\Delta(t)$, equations (4.1) turn into equations (3.10); as follows from (2.12) and (2.7), for all $\widehat{\Delta} \in \widehat{\boldsymbol{\Delta}}$ we have

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Psi \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} = z_\Delta^{\mathrm{T}}(t) \begin{pmatrix} \widehat{\Delta}^{\mathrm{T}} \\ I \end{pmatrix}^{\mathrm{T}} \Psi \begin{pmatrix} \widehat{\Delta}^{\mathrm{T}} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0,$$

$$\begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix}^{\mathrm{T}} \Upsilon \begin{pmatrix} w_\Delta(t) \\ z_\Delta(t) \end{pmatrix} = z_\Delta^{\mathrm{T}}(t) \begin{pmatrix} \widehat{\Delta}^{\mathrm{T}} \\ I \end{pmatrix}^{\mathrm{T}} \Upsilon \begin{pmatrix} \widehat{\Delta}^{\mathrm{T}} \\ I \end{pmatrix} z_\Delta(t) \leqslant 0,$$

i.e., $w_\Delta(t) = \widehat{\Delta}^{\mathrm{T}} z_\Delta(t) \in \mathbf{W_\Delta}$.

For the extended system (4.1), (4.2), we define the γ_0, generalized H_2, and H_∞ norms with respect to the input w_a and output z_a under all admissible inputs w_Δ as

$$\begin{aligned} \widetilde{\gamma}_0 &= \sup_{w_\Delta(t) \,\in\, \mathbf{W_\Delta}} \operatorname{ess} \sup_{w_a \,\in\, \mathcal{G}_{n_w}} \frac{\|z_a\|_{\mathcal{P}}}{\|w_a\|_{\mathcal{P}}}, \\ \widetilde{\gamma}_{g2} &= \sup_{w_\Delta(t) \,\in\, \mathbf{W_\Delta}} \sup_{w_a(t) \neq 0} \frac{\sup_{t \geqslant 0} |z_a(t)|}{\|w_a\|}, \\ \widetilde{\gamma}_\infty &= \sup_{w_\Delta(t) \,\in\, \mathbf{W_\Delta}} \sup_{w_a(t) \neq 0} \frac{\|z_a\|}{\|w_a\|}. \end{aligned} \tag{4.3}$$

They obviously restrict from above the guaranteed estimates of the corresponding norms of system (3.10). In view of the relations between the norms of dual systems (see above), the guaranteed estimates of the norms of the original uncertain system (3.1) satisfy the inequalities

$$\gamma_0 \leqslant \widetilde{\gamma}_{g2}, \quad \gamma_{g2} \leqslant \widetilde{\gamma}_0, \quad \gamma_\infty \leqslant \widetilde{\gamma}_\infty.$$

The performance indices (4.3) will be below a given value γ if there exists a positive definite quadratic function $V(x_a) = x_a^{\mathrm{T}} P x_a$ whose increment along the trajectories of (4.1) satisfies the following conditions for each norm (similar to conditions (3.5), (3.7), and (3.9) for system (3.2)):

$$(A_\Theta^{\mathrm T}x_a + C_{\Delta\Theta}^{\mathrm T}w_\Delta)^{\mathrm T}P(A_\Theta^{\mathrm T}x_a + C_{\Delta\Theta}^{\mathrm T}w_\Delta) - x_a^{\mathrm T}Px_a + |z_a|^2 \leqslant 0, \quad C_\Theta PC_\Theta^{\mathrm T} < \gamma^2 I;$$

$$(A_\Theta^{\mathrm T}x_a + C_{\Delta\Theta}^{\mathrm T}w_\Delta + C_\Theta^{\mathrm T}w_a)^{\mathrm T}P(A_\Theta^{\mathrm T}x_a + C_{\Delta\Theta}^{\mathrm T}w_\Delta + C_\Theta^{\mathrm T}w_a) - x_a^{\mathrm T}Px_a - |w_a|^2 \leqslant 0,$$

$$\begin{pmatrix} P & * \\ B^{\mathrm T} & \gamma^2 I \end{pmatrix} > 0;$$

$$(A_\Theta^{\mathrm T}x_a + C_{\Delta\Theta}^{\mathrm T}w_\Delta + C_\Theta^{\mathrm T}w_a)^{\mathrm T}P(A_\Theta^{\mathrm T}x_a + C_{\Delta\Theta}^{\mathrm T}w_\Delta + C_\Theta^{\mathrm T}w_a) - x_a^{\mathrm T}Px_a + |z_a|^2 - \gamma^2|w_a|^2 < 0$$

for all x_a, w_a, and all $w_\Delta \in \mathbf{W}_{\boldsymbol\Delta}$, i.e., those obeying the constraints (4.2). A sufficient condition for this is the existence of a matrix $P = P^{\mathrm T} > 0$ and nonnegative numbers $\mu \geqslant 0$ and $\nu \geqslant 0$ such that, for all x_a, w_a, and w_Δ,

$$\triangle V + |z_a|^2 - \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm T} (\mu\Psi + \nu\Upsilon) \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} \leqslant 0, \quad C_\Theta PC_\Theta^{\mathrm T} < \gamma^2 I;$$

$$\triangle V - |w_a|^2 - \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm T} (\mu\Psi + \nu\Upsilon) \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} \leqslant 0, \quad \begin{pmatrix} P & * \\ B^{\mathrm T} & \gamma^2 I \end{pmatrix} > 0;$$

$$\triangle V + |z_a|^2 - \gamma^2|w_a|^2 - \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix}^{\mathrm T} (\mu\Psi + \nu\Upsilon) \begin{pmatrix} w_\Delta \\ z_\Delta \end{pmatrix} < 0,$$

where the increment of the function $V(x)$ in the first inequality is taken along the trajectory of system (4.1) with $w_a(t) \equiv 0$. We write these inequalities in matrix form and introduce the new matrix variable $Z = \Theta P$. Then replacing the matrix $\nu\Lambda$ with the matrix Λ without notational change and applying Schur's complement lemma lead to an important result.

Theorem 4.1. *The guaranteed estimates of the* γ_0, *generalized* H_2, *and* H_∞ *norms of the uncertain system* (2.1), (2.2) *with the control law* $u(t) = \Theta x(t)$, *where* $\Theta = ZP^{-1}$, *are below* γ *if the following LMIs are solvable in* $P > 0$, Z, $\Lambda \in \boldsymbol\Lambda$, *and* $\mu \geqslant 0$:

$$\begin{pmatrix} -P & * & * & * & * \\ \mathcal{F}_A & -P - \mu\Psi_{22} & * & * & * \\ \mathcal{F}_{C_\Delta} & -\mu\Psi_{12} & -\mu\Psi_{11} - \Lambda & * & * \\ \mathcal{F}_C & 0 & 0 & -I & * \\ 0 & \Lambda\eta^{\mathrm T}B_\Delta^{\mathrm T} & 0 & 0 & -\Lambda \end{pmatrix} \leqslant 0, \qquad \begin{pmatrix} P & * \\ B^{\mathrm T} & \gamma^2 I \end{pmatrix} > 0; \tag{4.4}$$

$$\begin{pmatrix} -P & * & * & * & * \\ \mathcal{F}_A & -P - \mu\Psi_{22} & * & * & * \\ 0 & B^{\mathrm T} & -I & * & \\ \mathcal{F}_{C_\Delta} & -\mu\Psi_{12} & 0 & -\mu\Psi_{11} - \Lambda & * \\ 0 & \Lambda\eta^{\mathrm T}B_\Delta^{\mathrm T} & 0 & 0 & -\Lambda \end{pmatrix} \leqslant 0, \qquad \begin{pmatrix} P & * \\ \mathcal{F}_C & \gamma^2 I \end{pmatrix} > 0 \tag{4.5}$$

and

$$\begin{pmatrix} -P & * & * & * & * & * \\ \mathcal{F}_A & -P-\mu\Psi_{22} & * & * & * & * \\ 0 & B^{\mathrm{T}} & -I & * & * & * \\ \mathcal{F}_{C_\Delta} & -\mu\Psi_{12} & 0 & -\mu\Psi_{11}-\Lambda & * & * \\ \mathcal{F}_C & 0 & 0 & 0 & -\gamma^2 I & * \\ 0 & \Lambda\eta^{\mathrm{T}}B_\Delta^{\mathrm{T}} & 0 & 0 & 0 & -\Lambda \end{pmatrix} < 0, \qquad (4.6)$$

where $\mathcal{F}_A = AP + B_u Z$, $\mathcal{F}_C = CP + DZ$, $\mathcal{F}_{C_\Delta} = C_\Delta P + D_\Delta Z$, *the elements of the matrices* Ψ *are given by* (2.11), *and the matrix* $\eta = \operatorname{diag}(\eta_1 I_{n_1}, \ldots, \eta_l I_{n_l})$ *is given by* (2.2).

The minimum values of γ^2 obtained using this theorem will be denoted by $\gamma^2(\widehat{\Delta}, \Theta)$, where the arguments are the corresponding system parameter matrix ($\widehat{\Delta}$ for the uncertain system and $\widehat{\Delta}^{(real)}$ for the real system) and the corresponding feedback parameter matrix ($\Theta^{(ab)}$ for the robust control law based on a priori and experimental data, $\Theta^{(a)}$ for the robust control law based on a priori data only, and $\Theta^{(b)}$ for the robust control law based on experimental data only). If only a priori data are used, then the guaranteed estimates of the norms $\gamma^2(\widehat{\Delta}, \Theta^{(a)})$ are found by solving these inequalities with $\mu = 0$; if only experimental data are used, then $\gamma^2(\widehat{\Delta}, \Theta^{(b)})$ are found by solving these inequalities with $\Lambda = 0$. It is clear that $\gamma^2(\widehat{\Delta}, \Theta^{(ab)}) \leqslant \min\{\gamma^2(\widehat{\Delta}, \Theta^{(a)}), \gamma^2(\widehat{\Delta}, \Theta^{(b)})\}$.

In the case of completely unknown state and control matrices of the system with the matrices of equation (2.1) given by (2.4) and $\Delta\Delta^{\mathrm{T}} \leqslant \eta^2 I$, Theorem 4.1 provides the guaranteed estimates of the corresponding norms for $\mathbf{\Lambda} = \{\lambda I : \lambda \geqslant 0\}$.

The inequalities in Theorem 4.1 serve for calculating the parameters of control laws and the norms in different scenarios by choosing appropriate blocks $\mathcal{F}_A$, $\mathcal{F}_C$, and $\mathcal{F}_{C_\Delta}$ and variables Λ and μ. In the next section, some of these scenarios will be implemented for an illustrative example. Also, the corresponding blocks $\mathcal{F}_A$, $\mathcal{F}_C$, and $\mathcal{F}_{C_\Delta}$ and variables Λ and μ in inequalities (4.4)–(4.6) will be presented.

5. ILLUSTRATIVE EXAMPLE

Consider the results of several experiments with one system of the form (2.1):

$$x(t+1) = \begin{pmatrix} 0.3 & 0.8 & -0.3 \\ -0.2+\delta & 0.6+\Delta_{11} & -0.1+\Delta_{12} \\ 0.5 & -0.2+\Delta_{21} & 0.9+\Delta_{22} \end{pmatrix} x(t) + \begin{pmatrix} 0.2 \\ 1+\delta \\ 0.5 \end{pmatrix} u(t) + w(t),$$

$$z(t) = \begin{pmatrix} I_3 \\ 0 \end{pmatrix} x(t) + \begin{pmatrix} 0_{3\times 1} \\ 0.2 \end{pmatrix} u(t),$$

where

$$B_\Delta = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad C_\Delta = I_3, \quad D_\Delta = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}, \quad \Delta = \begin{pmatrix} \Delta_1 & 0 \\ 0 & \Delta_2 \end{pmatrix},$$

$$\Delta_1 = \delta, \quad \Delta_2 = \begin{pmatrix} \Delta_{11} & \Delta_{12} \\ \Delta_{21} & \Delta_{22} \end{pmatrix}, \quad |\delta| \leqslant 0.12; \quad \Delta_2\Delta_2^{\mathrm{T}} \leqslant 0.19.$$

1. Based on a priori information only, we calculate the guaranteed estimates of the norms and parameter matrix of the corresponding robust control laws using the formula $\Theta^{(a)} = ZP^{-1}$ by

solving inequalities (4.4)–(4.6) with $\mathcal{F}_A = AP + B_u Z$, $\mathcal{F}_C = CP + DZ$, $\mathcal{F}_{C_\Delta} = C_\Delta P + D_\Delta Z$, $\eta = \mathrm{diag}\,(0.12; 0.19 I_2)$, $\mu = 0$, and the unknown variable $\Lambda \geqslant 0$:

$$\gamma_0^2(\widehat{\Delta}, \Theta_0^{(a)}) = 12.8095; \quad \Theta_0^{(a)} = (-0.4356;\ -0.6420;\ -0.3125),$$
$$\gamma_{g2}^2(\widehat{\Delta}, \Theta_{g2}^{(a)}) = 10.5935; \quad \Theta_{g2}^{(a)} = (-0.8498;\ -0.7996;\ -0.6503),$$
$$\gamma_\infty^2(\widehat{\Delta}, \Theta_\infty^{(a)}) = 49.2653; \quad \Theta_\infty^{(a)} = (-1.2373;\ -0.8204;\ -0.9710).$$

Suppose that the real system is described by the uncertain parameters $\delta^{(real)} = -0.05$, $\Delta_{11}^{(real)} = 0.2$, $\Delta_{12}^{(real)} = \Delta_{21}^{(real)} = 0$, and $\Delta_{22}^{(real)} = -0.1$ so that

$$\widehat{\Delta} = \widehat{\Delta}^{(real)} = \begin{pmatrix} 0 & 0 & 0 \\ -0.05 & 0.2 & 0 \\ 0 & 0 & -0.1 \end{pmatrix}, \tag{5.1}$$

whereas the state and control matrices of the real object are

$$A^{(real)} = A + \widehat{\Delta}^{(real)} C_\Delta, \quad B_u^{(real)} = B_u + \widehat{\Delta}^{(real)} D_\Delta.$$

Let us calculate the three norms of the closed loop system (the real object with the robust feedback control with the parameter matrix $\Theta^{(a)}$) by solving inequalities (4.4)–(4.6) with

$$\mathcal{F}_A = (A^{(real)} + B_u^{(real)}\Theta^{(a)})P, \quad \mathcal{F}_C = (C + D\Theta^{(a)})P, \quad \mathcal{F}_{C_\Delta} = 0,$$

$\Lambda = 0$, and $\mu = 0$:

$$\gamma_0^2(\widehat{\Delta}^{(real)}, \Theta_0^{(a)}) = 4.8319;$$
$$\gamma_{g2}^2(\widehat{\Delta}^{(real)}, \Theta_{g2}^{(a)}) = 5.1373;$$
$$\gamma_\infty^2(\widehat{\Delta}^{(real)}, \Theta_\infty^{(a)}) = 23.5459.$$

For comparison, here are the optimal values of these norms and parameter matrices of the optimal feedback control laws for the real system (if it were known), calculated using the formula $\Theta^{(real)} = ZP^{-1}$ by solving inequalities (4.4)–(4.6) with $\mathcal{F}_A = A^{(real)}P + B_u^{(real)}Z$, $\mathcal{F}_C = CP + DZ$, $\mathcal{F}_{C_\Delta} = 0$, $\Lambda = 0$, and $\mu = 0$:

$$\gamma_0^2(\widehat{\Delta}^{(real)}, \Theta_0^{(real)}) = 3.9569; \quad \Theta_0^{(real)} = (-0.0765;\ -0.9379;\ 0.0064),$$
$$\gamma_{g2}^2(\widehat{\Delta}^{(real)}, \Theta_{g2}^{(real)}) = 4.4024; \quad \Theta_{g2}^{(real)} = (-0.1369;\ -0.9249;\ -0.0741),$$
$$\gamma_\infty^2(\widehat{\Delta}^{(real)}, \Theta_\infty^{(real)}) = 10.4651; \quad \Theta_\infty^{(real)} = (-1.2547;\ -1.3605;\ -0.3919).$$

2. Consider the case of no a priori information about the possible range of unknown parameters of the object: experimental data are used instead. We calculate the guaranteed estimates of the norms and find the parameter matrices of the suboptimal robust feedback control laws using the formula $\Theta^{(b)} = ZP^{-1}$ by solving inequalities (4.4)–(4.6) with $\mathcal{F}_A = AP + B_u Z$, $\mathcal{F}_C = CP + DZ$, $\mathcal{F}_{C_\Delta} = C_\Delta P + D_\Delta Z$, $\Lambda = 0$, and the unknown variable $\mu \geqslant 0$. To obtain experimental data, we model equation (2.8) with the initial conditions $x_0 = (9; 5; -7)^{\mathrm{T}}$ under the uncertainties $\delta^{(real)} = -0.05$, $\Delta_{11}^{(real)} = 0.2$, $\Delta_{12}^{(real)} = \Delta_{21}^{(real)} = 0$, and $\Delta_{22}^{(real)} = -0.1$ so that $\widehat{\Delta} = \widehat{\Delta}^{(real)}$. The components of the control $u(t)$ and disturbance $w(t)$ vectors in the experiment are chosen as random variables with the uniform distribution on the intervals $[-1, 1]$

and $[-d, d]$, respectively, from a random-number generator. For $d = 0.1$ and $N = 100$, the results were as follows:

$$\gamma_0^2(\widehat{\Delta}, \Theta_0^{(b)}) = 9.2104; \quad \Theta_0^{(b)} = (-0.1087;\ -0.8626;\ -0.0074),$$
$$\gamma_{g2}^2(\widehat{\Delta}, \Theta_{g2}^{(b)}) = 11.0614; \quad \Theta_{g2}^{(b)} = (-0.1745;\ -1.0321;\ -0.0257),$$
$$\gamma_\infty^2(\widehat{\Delta}, \Theta_\infty^{(b)}) = 56.6811; \quad \Theta_\infty^{(b)} = (-0.6556;\ -1.3677;\ -0.0644).$$

For the real system with robust feedback control laws with the corresponding parameter matrices $\Theta^{(b)}$, solving inequalities (4.4)–(4.6) with

$$\mathcal{F}_A = (A^{(real)} + B_u^{(real)}\Theta^{(b)})P, \quad \mathcal{F}_C = (C + D\Theta^{(b)})P, \quad \mathcal{F}_{C_\Delta} = 0,$$

$\Lambda = 0$, and $\mu = 0$ yielded the following values of the norms:

$$\gamma_0^2(\widehat{\Delta}^{(real)}, \Theta_0^{(b)}) = 3.9640;$$
$$\gamma_{g2}^2(\widehat{\Delta}^{(real)}, \Theta_{g2}^{(b)}) = 4.4416;$$
$$\gamma_\infty^2(\widehat{\Delta}^{(real)}, \Theta_\infty^{(b)}) = 12.2661.$$

3. We design the suboptimal robust control law based on a priori information and the same experimental data for the real system (see above). For this purpose, we calculate the guaranteed estimates of the norms and find the parameter matrices of the robust feedback control laws using the formula $\Theta^{(ab)} = ZP^{-1}$ by solving inequalities (4.4)–(4.6) with $\mathcal{F}_A = AP + B_uZ$, $\mathcal{F}_C = CP + DZ$, $\mathcal{F}_{C_\Delta} = C_\Delta P + D_\Delta Z$, and the unknown variables $\Lambda \geqslant 0$ and $\mu \geqslant 0$:

$$\gamma_0^2(\widehat{\Delta}, \Theta_0^{(ab)}) = 8.2265; \quad \Theta_0^{(ab)} = (-0.1613;\ -0.7716;\ -0.0661),$$
$$\gamma_{g2}^2(\widehat{\Delta}, \Theta_{g2}^{(ab)}) = 8.9113; \quad \Theta_{g2}^{(ab)} = (-0.4617;\ -0.8835;\ -0.2449),$$
$$\gamma_\infty^2(\widehat{\Delta}, \Theta_\infty^{(ab)}) = 35.2885; \quad \Theta_\infty^{(ab)} = (-0.9790;\ -1.0324;\ -0.5212).$$

For the real system with robust feedback control laws with the parameter matrices $\Theta^{(ab)}$, the three norms calculated by solving inequalities (4.4)–(4.6) with

$$\mathcal{F}_A = (A^{(real)} + B_u^{(real)}\Theta^{(ab)})P, \quad \mathcal{F}_C = (C + D\Theta^{(ab)})P, \quad \mathcal{F}_{C_\Delta} = 0,$$

$\Lambda = 0$, and $\mu = 0$ took the following values:

$$\gamma_0^2(\widehat{\Delta}^{(real)}, \Theta_0^{(ab)}) = 4.0280;$$
$$\gamma_{g2}^2(\widehat{\Delta}^{(real)}, \Theta_{g2}^{(ab)}) = 4.5248;$$
$$\gamma_\infty^2(\widehat{\Delta}^{(real)}, \Theta_\infty^{(ab)}) = 14.3512.$$

Figures 1–3 show the guaranteed estimates of the γ_0, generalized H_2, and H_∞ norms, respectively, based on a priori information only, experimental data only, and a priori information together with experimental data, depending on the disturbance level d in the experiment; the lower horizontal lines correspond to the values of these norms for the real object whereas the upper ones to their values under robust control laws designed from a priori information only. Figure 4 plots the guaranteed estimate of the H_∞ norm obtained by the joint use of a priori information and experimental data with the disturbance level $d = 0.05$ as a function of the number of measurements N; the horizontal lines correspond to the H_∞ norm of the real object and the guaranteed estimate of the H_∞ norm obtained using a priori information only.

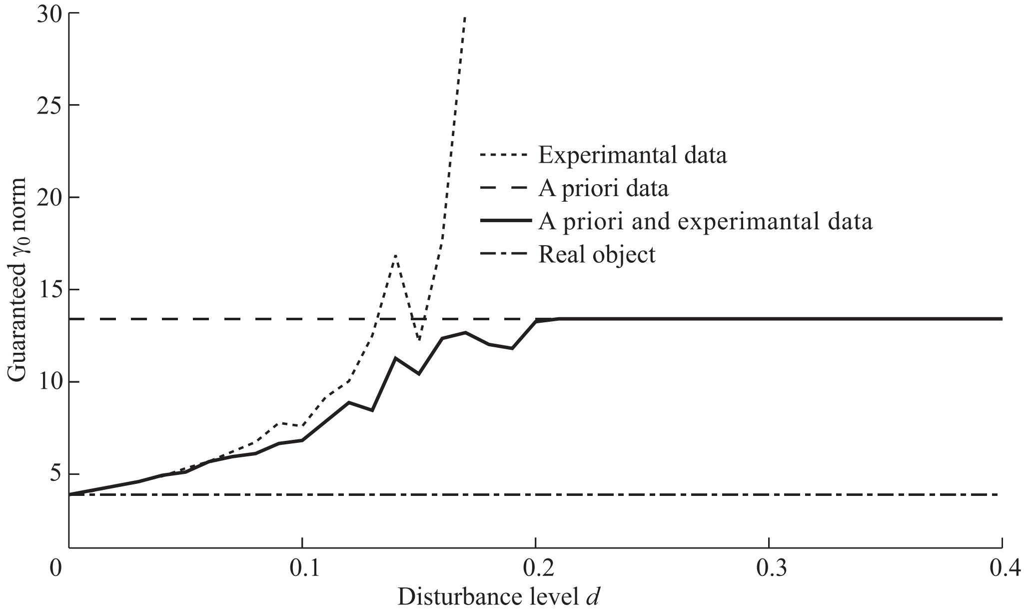

Fig. 1. The guaranteed estimates of the γ_0 norm as a function of the disturbance level in experimental data for different types of information used.

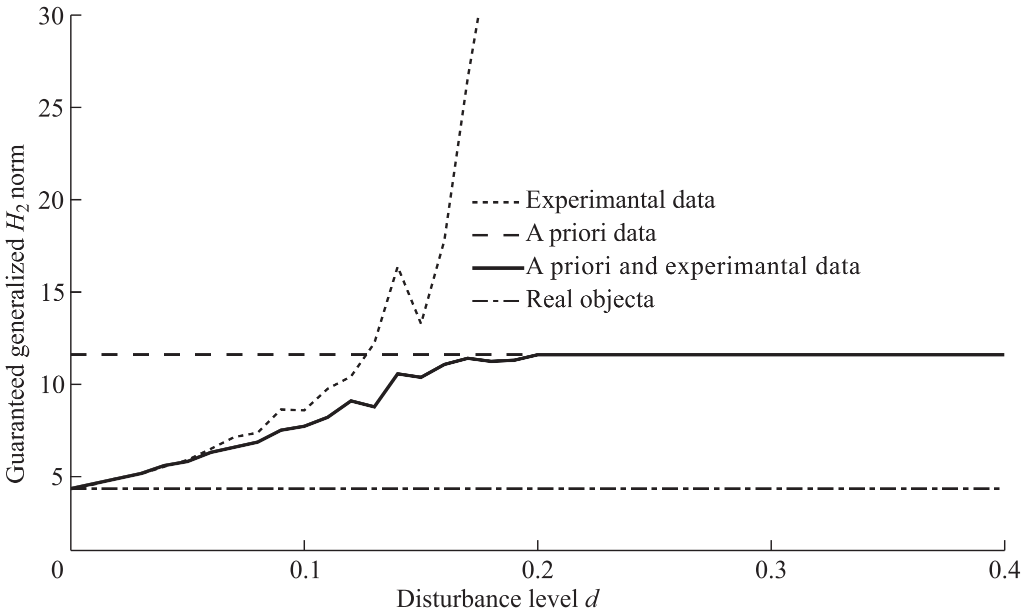

Fig. 2. The guaranteed estimates of the generalized H_2 norm as a function of the disturbance level in experimental data for different types of information used.

According to these results, if the disturbance level in the experiment is relatively small, then the guaranteed estimates of the norms of the closed-loop uncertain system designed using both a priori and experimental data are much smaller than their counterparts under the robust control laws designed using only a priori data or only experimental data. For example, the guaranteed estimates of the H_∞ norms of the closed-loop system with control laws designed using only a priori or only experimental data with the disturbance level $d = 0.1$ are $\gamma_\infty^2(\widehat{\Delta}, \Theta_\infty^{(a)}) = 49.2653$ and

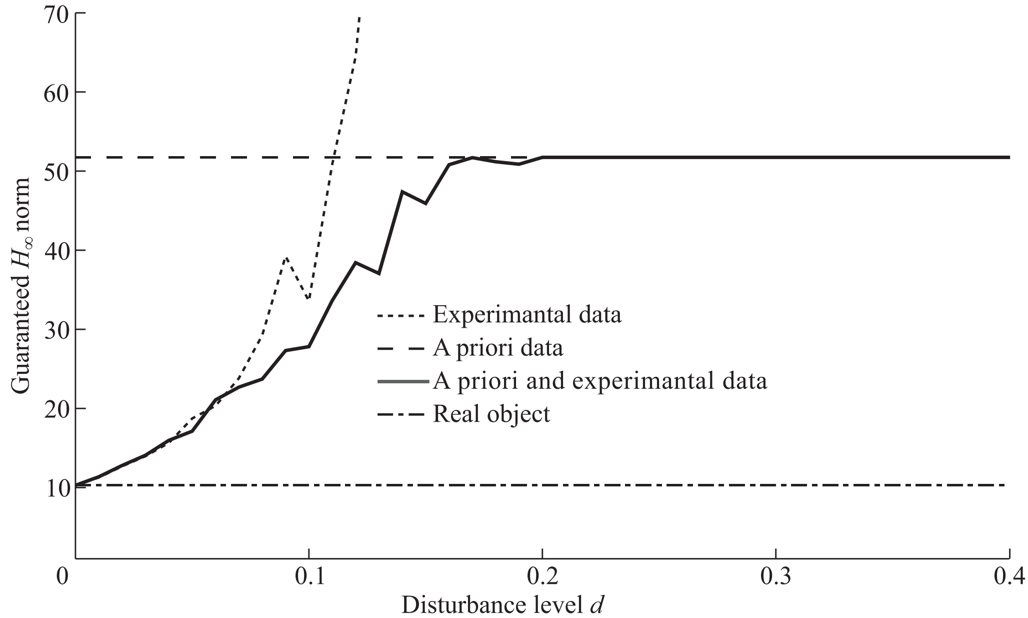

Fig. 3. The guaranteed estimates of the H_∞ norm as a function of the disturbance level in experimental data for different types of information used.

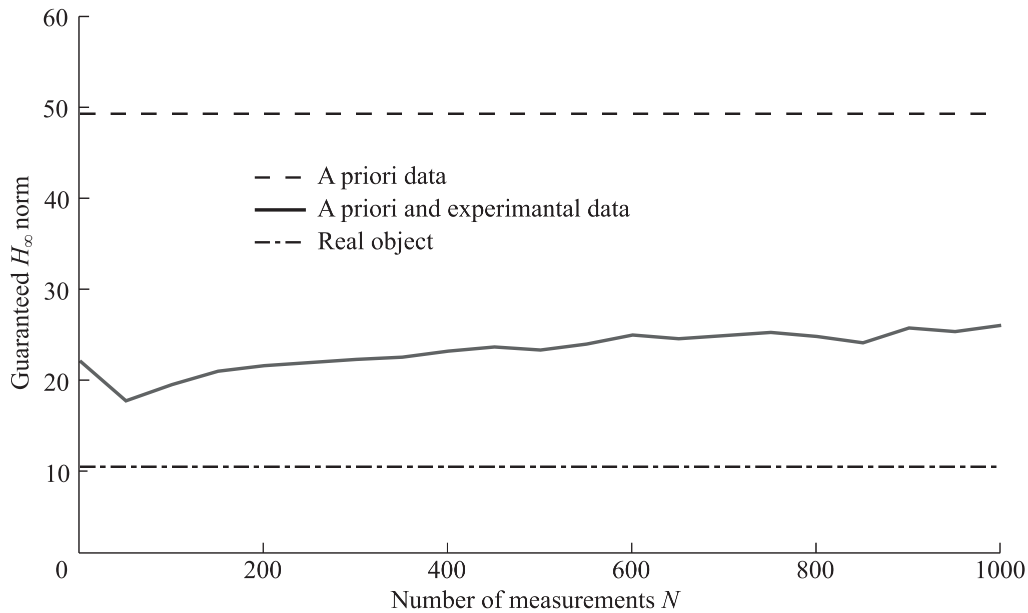

Fig. 4. The guaranteed estimate of the H_∞ norm with a given disturbance level in experimental data depending on the number of measurements.

$\gamma^2_\infty(\widehat{\Delta}, \Theta^{(b)}_\infty) = 56.6811$, respectively; when these a priori and experimental data are used together, the guaranteed estimate of the H_∞ norm equals $\gamma^2_\infty(\widehat{\Delta}, \Theta^{(ab)}_\infty) = 35.2885$. Note the effect of increasing the guaranteed estimates of the norms obtained from experimental data only. This effect can be explained as follows: the set of admissible models of the object consistent with the experimental data expands as the disturbance level increases, and the maximum value of the norm on this set grows accordingly. We emphasize another important feature: the range of disturbance levels in which the guaranteed estimate of the norm when using a priori and experimental data together is smaller than that when using a priori data only depends on the initial conditions and the chosen controls in the experiment; therefore, this range can be varied and even (apparently) planned. Furthermore, a large number of measurements are not required to obtain acceptable results (see Fig. 4).

6. CONCLUSIONS

This paper has proposed a novel design method for suboptimal robust control laws considering a priori information about the mathematical model of the object and, moreover, experimental data of modeling the object over a small time interval. When obtaining experimental data, neither the persistency of excitation (which ensures the identifiability of unknown parameters) nor data informativity for the corresponding control law is required. In this method, the use of additional information about the unknown parameters of the object obtained from experimental data significantly reduces the guaranteed estimates of the γ_0, generalized H_2, and H_∞ norms of the closed loop system.

APPENDIX

Proof of Lemma 3.1. We write the Lagrange function for this problem and express the optimal value of its dual function as

$$\min_{P_0\geqslant 0,\gamma^2\geqslant 0}\ \max_{K_x\geqslant 0,K_w\geqslant 0}\Big[\operatorname{tr} C_\Theta K_x C_\Theta^{\mathrm T}+\operatorname{tr} P_0(A_\Delta K_x A_\Delta^{\mathrm T}-K_x+BK_wB^{\mathrm T})+\gamma^2(1-\operatorname{tr} K_w)\Big]$$

$$=\min_{P_0\geqslant 0,\gamma^2\geqslant 0}\ \max_{K_x\geqslant 0,K_w\geqslant 0}\Big[\gamma^2+\operatorname{tr} K_x(A_\Delta^{\mathrm T}P_0A_\Delta-P_0+C_\Theta^{\mathrm T}C_\Theta)+\operatorname{tr} K_w(B^{\mathrm T}P_0B-\gamma^2 I)\Big].$$

This value is finite under inequalities (3.4); then the maximum is reached at $K_x = 0$ and $K_w = 0$. In this case, the optimal value of the dual problem coincides with $\lambda_{\max}(B^{\mathrm T}P_0B)$. Since the function is convex and there exists an interior point satisfying the constraint, the primal and dual problems have the same optimal value [18].

FUNDING

This work was supported by the Scientific and Educational Mathematical Center "Mathematics of Future Technologies," agreement no. 075-02-2023-945.

REFERENCES

1. Polyak, B.T. and Shcherbakov, P.S., *Robastnaya ustoichivost' i upravlenie* (Robust Stability and Control), Moscow: Nauka, 2002.

2. Petersen, I.R. and Tempo, R., Robust Control of Uncertain Systems: Classical Results and Recent Developments, *Automatica*, 2014, vol. 50, no. 5, pp. 1315–1335.

3. Andrievsky, B.R. and Fradkov, A.L., Speed Gradient Method and Its Applications, *Autom. Remote Control*, 2021, vol. 82, no. 9, pp. 1463–1518.

4. Annaswamy, A.A. and Fradkov, A.L., A Historical Perspective of Adaptive Control and Learning, *Annual Reviews in Control*, 2021, vol. 52, pp. 18–41.

5. De Persis, C. and Tesi, P., Formulas for Data-Driven Control: Stabilization, Optimality and Robustness, *IEEE Trans. Automat. Control*, 2020, vol. 65, no. 3, pp. 909–924.

6. Waarde, H.J., Eising, J., Trentelman, H.L., and Camlibel, M.K., Data Informativity: A New Perspective on Data-Driven Analysis and Control, *IEEE Trans. Automat. Control*, 2020, vol. 65, no. 11, pp. 4753–4768.

7. Berberich, J., Koch, A., Scherer, C.W., and Allgower, F., Robust Data-Driven State-Feedback Design, *Proc. Amer. Control Conf.*, 2020, pp. 1532–1538.

8. Waarde, H.J., Camlibel, M.K., and Mesbahi, M., From Noisy Data to Feedback Controllers: Nonconservative Design via a Matrix S-Lemma, *IEEE Trans. Automat. Control*, 2022, vol. 67, no. 1, pp. 162–175.

9. Bisoffi, A., De Persis, C., and Tesi, P., Data-Driven Control via Petersen's Lemma, *Automatica*, 2022, vol. 145, art. no. 110537.

10. Willems, J.C., Rapisarda, P., Markovsky, I., and De Moor, B., A Note on Persistency of Excitation, *Syst. Control Lett.*, 2005, vol. 54, pp. 325–329.

11. Yakubovich, V.A., S-procedure in Nonlinear Control Theory, *Vestn. Leningrad. Univ. Mat.*, 1977, vol. 4, pp. 73–93.

12. Petersen, I.R., A Stabilization Algorithm for a Class of Uncertain Linear Systems, *Syst. Control Lett.*, 1987, vol. 8, pp. 351–357.

13. Doyle, J.C., Analysis of Feedback Systems with Structured Uncertainties, *IEE Proc.*, 1982, vol. 129, part D(6), pp. 242–250.

14. Safonov, M.G., Stability Margins of Diagonally Perturbed Multivariable Feedback Systems, *IEE Proc.*, 1982, vol. 129, part D(6), pp. 251–256.

15. Kogan, M.M., Optimal Discrete-time H_∞/γ_0 Filtering and Control under Unknown Covariances, *Int. J. Control*, 2016, vol. 89, no. 4, pp. 691–700.

16. Wilson, D.A., Convolution and Hankel Operator Norms for Linear Systems, *IEEE Trans. Autom. Control*, 1989, vol. 34, no. 1, pp. 94–97.

17. Balandin, D.V., Biryukov, R.S., and Kogan, M.M., Minimax Control of Deviations for the Outputs of a Linear Discrete Time-Varying System, *Autom. Remote Control*, 2019, vol. 80, no. 12, pp. 345–359.

18. Boyd, S. and Vandenberghe, L., *Convex Optimization*, Cambridge: University Press, 2004.

This paper was recommended for publication by M.V. Khlebnikov, a member of the Editorial Board