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PI Controller Design for Suppressing Exogenous Disturbances

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Abstract—A novel approach is proposed to suppress bounded exogenous disturbances in linear control systems using a PI controller. The approach is based on reducing the original problem to a nonconvex matrix optimization problem. A gradient method for finding the controller's parameters is derived and its justification is provided. The corresponding recurrence procedure is rather effective and yields quite satisfactory controllers in terms of engineering performance criteria. This paper continues a series of the author's research works devoted to the design of feedback control laws from an optimization point of view.

Keywords: linear system, exogenous disturbances, PI controller, optimization, Lyapunov equation, gradient method, Newton's method, convergence

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1. INTRODUCTION

The recent paper [1] introduced a novel (optimization-based) approach to the classical problem of suppressing bounded nonrandom exogenous disturbances. This problem is posed as follows. Consider a linear control system described by

$$\begin{aligned} \dot{x} &= Ax + Bu + Dw, \quad x(0) = x_0, \\ y &= C_1 x, \\ z &= C_2 x + B_1 u \end{aligned}$$

with the state vector $x(t) \in \mathbb{R}^n$, the measured output $y(t) \in \mathbb{R}^l$, the controlled output $z(t) \in \mathbb{R}^r$, the control vector $u(t) \in \mathbb{R}^p$, and a measured disturbance $w(t) \in \mathbb{R}^m$ that is bounded at each time instant t :

$$|w(t)| \leqslant 1 \quad \text{for all } t \geqslant 0. \tag{1}$$

It is required to choose a stabilizing state-feedback $u = Kx$ or output-feedback $u = Ky$ control (if it exists) to reduce the "peak" of the output $z(t)$, i.e., the value $\max_t |z(t)|$.

Within the approach presented in [1], the original problem was reduced to a nonconvex matrix optimization problem. A gradient method for finding a static state-feedback or output-feedback control law of the system was developed, and its justification was given.

On the other hand, in [2], an optimization approach going back to [3] was applied to design a PID controller. The regular approach proposed therein involves solving a nonconvex matrix optimization problem to find the controller's parameters. The quality of this controller was evaluated by a quadratic criterion of the system output: the controller was tuned against the uncertainty in the initial conditions to make the system output uniformly small. As it turned out, the corresponding recurrence procedure is rather effective and yields controllers that are quite satisfactory in terms of engineering performance criteria.

This paper continues both of the research lines mentioned above: we design a PI controller for suppressing bounded exogenous disturbances in linear control systems by solving an optimization problem.

From this point onwards, the following notations are adopted: $|\cdot|$ is the Euclidean norm of a vector, $\|\cdot\|$ is the spectral norm of a matrix, $\|\cdot\|_F$ is the Frobenius norm of a matrix, $^{\mathrm{T}}$ stands for the transpose operation, tr means the matrix trace, I is an identity matrix of appropriate dimensions, and $\lambda_i(A)$ are the eigenvalues of a matrix A.

2. PROBLEM STATEMENT. THE METHOD OF INVARIANT ELLIPSOIDS

Consider a linear continuous-time control system described by

$$\begin{aligned} \dot{x} &= Ax + bu + Dw, \quad x(0) = x_0, \\ y &= c^{\mathrm{T}} x, \\ z &= Cx, \end{aligned} \tag{2}$$

where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, $D \in \mathbb{R}^{n\times m}$, and $c \in \mathbb{R}^n$, $C \in \mathbb{R}^{r\times n}$, with the state vector $x(t) \in \mathbb{R}^n$, the observed output $y(t) \in \mathbb{R}$, the controlled output $z(t) \in \mathbb{R}^r$, an exogenous disturbance $w(t) \in \mathbb{R}^m$ that satisfies the constraint (1), and the control vector $u(t) \in \mathbb{R}$ in the form of a PI controller

$$u(t) = -k_P y(t) - k_I \int_0^t y(\tau) d\tau. \tag{3}$$

The objective is to find the numerical parameters k_P and k_I of the controller (3) that stabilizes the closed loop system and suppresses the exogenous disturbances w by minimizing the bounding ellipsoid for the output z.

Let us conceptually recall the method of invariant ellipsoids; for details, see [4, 5]. Consider a linear continuous time-invariant dynamic system described by

$$\begin{aligned} \dot{x} &= Ax + Dw, \quad x(0) = x_0, \\ z &= Cx \end{aligned} \tag{4}$$

with the state vector $x(t) \in \mathbb{R}^n$, the output $z(t) \in \mathbb{R}^r$, and an exogenous disturbance $w(t) \in \mathbb{R}^l$ that satisfies the constraint (1). Assume that system (4) is stable (i.e., the matrix A is Hurwitz) and the pair (A, D) is controllable.

An ellipsoid centered at the origin is said to be *invariant* for system (4) if any of its trajectories evolving from a point inside the ellipsoid remains in this ellipsoid at any time instant under all admissible exogenous disturbances of the system.

When evaluating the effect of exogenous disturbances on the system output, it is natural to consider the minimal ellipsoids containing the system output (in a certain sense). Clearly, if an ellipsoid

$$\mathcal{E}_x = \left\{ x \in \mathbb{R}^n : \quad x^{\mathrm{T}} P^{-1} x \leqslant 1 \right\}, \quad P \succ 0, \tag{5}$$

is invariant, then the output of system (4) with $x_0 \in \mathcal{E}_x$ belongs to the so-called *bounding* ellipsoid

$$\mathcal{E}_z = \left\{ z \in \mathbb{R}^p : \quad z^{\mathrm{T}} (CPC^{\mathrm{T}})^{-1} z \leqslant 1 \right\}. \tag{6}$$

In the literature, the linear function $f(P) = \operatorname{tr} CPC^{\mathrm{T}}$ (the sum of the squares of the semi-axes of the bounding ellipsoid) is often considered a minimality criterion.

The paper [6] established an invariance criterion for ellipsoids in terms of linear matrix inequalities (LMIs). Let us formulate it as follows (see [4]).

Theorem 1. *Assume that the matrix A is Hurwitz, the pair (A, D) is controllable, and the matrix $P(\alpha) \succ 0$ satisfies the Lyapunov equation*

$$\left(A + \frac{\alpha}{2} I\right) P + P\left(A + \frac{\alpha}{2} I\right)^{\mathrm{T}} + \frac{1}{\alpha} DD^{\mathrm{T}} = 0$$

on the interval $0 < \alpha < 2\sigma(A)$.

Then the minimal bounding ellipsoid is obtained by minimizing the univariate function $f(\alpha) = \operatorname{tr} CP(\alpha)C^{\mathrm{T}}$ on the interval $0 < \alpha < 2\sigma(A)$; if α^* is the minimum point and x_0 satisfies the condition $x_0^{\mathrm{T}} P^{-1}(\alpha^*) x_0 \leqslant 1$, then the uniform estimate

$$|z(t)| \leqslant \sqrt{f(\alpha^*)}, \quad 0 \leqslant t < \infty,$$

holds.

3. SOLUTION APPROACH

Let us introduce an auxiliary scalar variable ξ as follows:

$$\dot{\xi} = y, \quad \xi(0) = 0.$$

With the extended state vector

$$g = \begin{pmatrix} x \\ \xi \end{pmatrix} \in \mathbb{R}^{n+1},$$

system (2) can be written as

$$\begin{aligned} \dot{g} &= \begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix} g + \begin{pmatrix} b \\ 0 \end{pmatrix} u + \begin{pmatrix} D \\ 0 \end{pmatrix} w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}, \\ y &= \begin{pmatrix} c^{\mathrm{T}} & 0 \end{pmatrix} g. \end{aligned} \tag{7}$$

According to (2) and (3), we have

$$\begin{aligned} u &= -k_P y(t) - k_I \int_0^t y(\tau) d\tau = -k_P c^{\mathrm{T}} x - k_I \xi \\ &= -k_P c^{\mathrm{T}} x - k_I \xi = -k_P \begin{pmatrix} c^{\mathrm{T}} & 0 \end{pmatrix} g - k_I \begin{pmatrix} 0 & 1 \end{pmatrix} g. \end{aligned} \tag{8}$$

The expression (8) with the more convenient notations $k_1 = k_P$ and $k_2 = k_I$ takes the form

$$u = -\begin{pmatrix} k_1 c^{\mathrm{T}} & k_2 \end{pmatrix} g. \tag{9}$$

Thus, system (7) with the feedback control law (9) is described by

$$\dot{g} = \begin{pmatrix} A - k_1 b c^{\mathrm{T}} & -k_2 b \\ c^{\mathrm{T}} & 0 \end{pmatrix} g + \begin{pmatrix} D \\ 0 \end{pmatrix} w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix}.$$

It can be represented as

$$\dot g = (\mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2)g + \begin{pmatrix} D \\ 0 \end{pmatrix} w, \quad g(0) = \begin{pmatrix} x_0 \\ 0 \end{pmatrix},$$

where

$$\mathcal{A}_0 = \begin{pmatrix} A & 0 \\ c^{\mathrm{T}} & 0 \end{pmatrix}, \quad \mathcal{A}_1 = \begin{pmatrix} -bc^{\mathrm{T}} & 0 \\ 0 & 0 \end{pmatrix}, \quad \mathcal{A}_2 = \begin{pmatrix} 0 & -b \\ 0 & 0 \end{pmatrix}.$$

Following the method of invariant ellipsoids, let the state g of system (7) belong to the invariant ellipsoid (5) generated by a matrix $P \in \mathbb{R}^{(n+1)\times(n+1)}$. We will minimize the size of the corresponding bounding ellipsoid (6) with respect to the output

$$z = Cx = \begin{pmatrix} C & 0 \end{pmatrix} g.$$

Due to Theorem 1, the associated problem is to minimize $\operatorname{tr}(C\ \ 0)P(C\ \ 0)^{\mathrm{T}}$ subject to the constraint

$$\left(\mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2 + \frac{\alpha}{2}I\right)P + P\left(\mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2 + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha}\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} = 0 \qquad (10)$$

with respect to the matrix variables $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$, the scalar variables k_1 and k_2, and the scalar parameter $\alpha > 0$. Given k_1, k_2, and α, the matrix P is found from equation (10); therefore, the independent variables are k_1, k_2, and α.

Consider the vector

$$k = \begin{pmatrix} k_1 \\ k_2 \end{pmatrix} \in \mathbb{R}^2$$

and the value

$$\operatorname{tr}\begin{pmatrix} C & 0 \end{pmatrix} P \begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}} + \rho|k|^2, \quad \rho \ll 1$$

as the performance criterion. Here, the second component is a control penalty (the coefficient $\rho > 0$ adjusts its significance) and ensures the coercivity of the objective function in k. (For details, see Section 5.)

Thus, the original problem (the design of a PI controller to suppress exogenous disturbances) has been reduced to the matrix optimization problem

$$\min f(k,\alpha), \quad f(k,\alpha) = \operatorname{tr} P\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + \rho|k|^2 \qquad (11)$$

subject to the constraint (10).

4. OPTIMIZATION OF THE FUNCTION $f(\alpha)$

Consider the problem

$$\min f(\alpha), \quad f(\alpha) = \operatorname{tr} PC^{\mathrm{T}}C,$$

subject to the constraint

$$\left(A + \frac{\alpha}{2}I\right)P + P\left(A + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha}DD^{\mathrm{T}} = 0$$

with respect to the matrix variable $P = P^{\mathrm{T}} \in \mathbb{R}^{n\times n}$ and the scalar parameter $\alpha > 0$. Assume that the matrix A is stable (Hurwitz).

As was shown in [1], minimization with respect to α can be effectively performed using Newton's method. Let us choose an initial approximation $0 < \alpha_0 < 2\sigma(A)$ and apply the iterative process

$$\alpha_{j+1} = \alpha_j - \frac{f'(\alpha_j)}{f''(\alpha_j)},$$

where

$$f'(\alpha) = \operatorname{tr} Y \left(P - \frac{1}{\alpha^2} DD^{\mathrm{T}} \right),$$

$$f''(\alpha) = 2 \operatorname{tr} Y \left(X + \frac{1}{\alpha^3} DD^{\mathrm{T}} \right),$$

and Y and X are the solutions of the Lyapunov equations

$$\left(A + \frac{\alpha}{2} I \right)^{\mathrm{T}} Y + Y \left(A + \frac{\alpha}{2} I \right) + C^{\mathrm{T}} C = 0$$

and

$$\left(A + \frac{\alpha}{2} I \right) X + X \left(A + \frac{\alpha}{2} I \right)^{\mathrm{T}} + P - \frac{1}{\alpha^2} DD^{\mathrm{T}} = 0,$$

respectively.

According to [1], the method converges globally (faster than the geometric progression with a coefficient of 1/2), with quadratic convergence in the neighborhood of the solution. It really requires at most 3–4 iterations to obtain a solution with high accuracy, unless the initial point is too close to the limits of the interval $(0, 2\sigma(A))$.

Thus, we have an efficient algorithm to perform minimization with respect to α in problem (11), (10): it suffices to replace the matrix A by $\mathcal{A}_0 + k_1\mathcal{A}_1 + k_2\mathcal{A}_2$, the matrix C by $\begin{pmatrix} C & 0 \end{pmatrix}$, and the matrix D by $\begin{pmatrix} D \\ 0 \end{pmatrix}$.

5. OPTIMIZATION OF THE FUNCTION $f(k)$

Introducing the convenient notation

$$\{\mathcal{A}, k\} = k_1\mathcal{A}_1 + k_2\mathcal{A}_2,$$

we accept the following hypothesis.

Assumption. *Let* $k_0 = \begin{pmatrix} k_1^0 \\ k_2^0 \end{pmatrix}$ *be a known stabilizing controller, i.e., the matrix* $\mathcal{A}_0 + \{\mathcal{A}, k_0\}$ *is Hurwitz.*

We will investigate the properties of the function

$$f(k) = \min_{\alpha} f(k, \alpha).$$

Lemma 1. *The function* $f(k)$ *is well-defined and positive on the set* $\mathcal{S}$ *of stabilizing controllers.*

The proofs of this and all subsequent results are given in Appendix 2.

Note that the set $\mathcal{S}$ can be nonconvex and disconnected whereas its boundaries can be nonsmooth.

Lemma 2. *The function $f(k,\alpha)$ is well-defined on the set of stabilizing feedback control laws k and for $0<\alpha<2\sigma(\mathcal{A}_0+\{\mathcal{A},k\})$. It is differentiable on this set, and the gradient is given by*

$$\frac{1}{2}\nabla_k f(k,\alpha)=\begin{pmatrix}\operatorname{tr} PY\mathcal{A}_1\\ \operatorname{tr} PY\mathcal{A}_2\end{pmatrix}+\rho k, \tag{12}$$

$$\nabla_\alpha f(k,\alpha)=\operatorname{tr} Y\left[P-\frac{1}{\alpha^2}\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}}\right], \tag{13}$$

where the matrices P and Y are the solutions of the Lyapunov equations

$$\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)P+P\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)^{\mathrm{T}}+\frac{1}{\alpha}\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}}=0$$

and

$$\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)^{\mathrm{T}}Y+Y\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)+\begin{pmatrix}C&0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C&0\end{pmatrix}=0, \tag{14}$$

respectively.

The function $f(k,\alpha)$ achieves minimum at an inner point of the admissible set that is determined by the conditions

$$\nabla_k f(k,\alpha)=0,\quad \nabla_\alpha f(k,\alpha)=0.$$

In addition, $f(k,\alpha)$ as a function of α is strictly convex on $0<\alpha<2\sigma(\mathcal{A}_0+\{\mathcal{A},k\})$ and achieves minimum at an inner point of this interval.

The Hessian of the function $f(k)$ has the following properties.

Lemma 3. *The function $f(k)$ is twice differentiable, and the action of its Hessian on an arbitrary vector[1] $e\in\mathbb{R}^2$ is given by*

$$\frac{1}{2}\left(\nabla^2_{kk}f(k)e,e\right)=\rho(e,e)+2\operatorname{tr} P'Y\{\mathcal{A},e\}, \tag{15}$$

where P' is the solution of the Lyapunov equation

$$\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)P'+P'\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)^{\mathrm{T}}+\{\mathcal{A},e\}P+P\{\mathcal{A},e\}^{\mathrm{T}}=0. \tag{16}$$

Remark 1. To obtain simple quantitative estimates in Lemmas 4 and 5 below, we incorporate the regularizing terms ε_1 and ε_2 into the optimization problem (11), (10) as follows:

$$\min f(k,\alpha),\quad f(k,\alpha)=\operatorname{tr} P\left(\begin{pmatrix}C&0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C&0\end{pmatrix}+\varepsilon_1 I\right)+\rho|k|^2,\quad \varepsilon_1\ll 1$$

subject to the constraint

$$\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)P+P\left(\mathcal{A}_0+\{\mathcal{A},k\}+\frac{\alpha}{2}I\right)^{\mathrm{T}}+\frac{1}{\alpha}\left[\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}}+\varepsilon_2 I\right]=0,\quad \varepsilon_2\ll 1. \tag{17}$$

The requirement of their introduction can be significantly weakened, but the current aim is to obtain the simplest and most obvious results.

[1] In the sense of the second derivative in a direction (the second directional derivative).

Lemma 4. *The function $f(k)$ is coercive on the set $\mathcal{S}$ (i.e., tends to infinity on its boundary) and, moreover,*

$$f(k) \geqslant \frac{\varepsilon_1}{4\sigma(\mathcal{A}_0+\{\mathcal{A},k\})\,(\|\mathcal{A}_0+\{\mathcal{A},k\}\|+\sigma(\mathcal{A}_0+\{\mathcal{A},k\}))}\|D\|_F^2, \tag{18}$$
$$f(k) \geqslant \rho|k|^2.$$

Corollary 1. *The level set*

$$\mathcal{S}_0 = \{k \in \mathcal{S}: \quad f(k) \leqslant f(k_0)\}$$

is bounded for any controller $k_0 \in \mathcal{S}$.

Corollary 2. *There exists a minimum point k_* on the set S and $\nabla f(k_*) = 0$.*

The gradient of the function $f(k)$ is not Lipschitz on the entire set $\mathcal{S}$, but it has this property on its subset $\mathcal{S}_0$. The corresponding result is presented below.

Lemma 5. *On the set $\mathcal{S}_0$, the gradient of the function $f(k)$ is Lipschitz with the constant*

$$L = \rho + \frac{8\sqrt{2n}f^2(k_0)}{\varepsilon_1\varepsilon_2^2}\left(\|\mathcal{A}_0\| + \max_i\|\mathcal{A}_i\|\sqrt{\frac{2}{\rho}f(k_0)}\right)^2\left(\frac{f^2(k_0)}{\varepsilon_1^2} + 2\max_i\|\mathcal{A}_i\|^2\right)\max\|\mathcal{A}_i\|_F. \tag{19}$$

These properties of the function $f(k)$ and its derivatives allow constructing a minimization method and justifying its convergence.

6. OPTIMIZATION ALGORITHM

We propose an iterative approach to solve problem (11). This approach is based on the application of the gradient method with respect to the variable k and Newton's method with respect to the variable α. The algorithm includes several steps as follows.

Algorithm 1 to minimize $f(k,\alpha)$:

1. Choose some values of the parameters $\varepsilon > 0$, $\gamma > 0$, $0 < \tau < 1$, and the initial stabilizing approximation k_0. Calculate $\alpha_0 = \sigma(\mathcal{A}_0 + \{\mathcal{A}, k_0\})$.
2. On the jth iteration, the values k_j and α_j are given.

   Calculate the matrix $\mathcal{A}_0 + \{\mathcal{A}, k_j\}$, solve the Lyapunov equations

$$\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)P + P\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha_j}\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}} = 0,$$
$$\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)^{\mathrm{T}}Y + Y\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right) + \begin{pmatrix}C & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C & 0\end{pmatrix} = 0,$$

   and find the matrices P and Y.

   Calculate the gradient

$$H_j = \nabla_k f(k_j, \alpha_j)$$

   from the relation

$$\frac{1}{2}\nabla_k f(k,\alpha) = \begin{pmatrix}\operatorname{tr} PY\mathcal{A}_1\\ \operatorname{tr} PY\mathcal{A}_2\end{pmatrix} + \rho k.$$

   If $|H_j| \leqslant \varepsilon$, then take k_j as the approximate solution.

3. Perform the gradient method step:

$$k_{j+1} = k_j - \gamma_j H_j.$$

Adjust the step length $\gamma_j > 0$ by fractionating γ until the following conditions are satisfied:

a. k_{j+1} is a stabilizing controller.

b. $f(k_{j+1}) \leqslant f(k_j) - \tau\gamma_j |H_j|^2$.

4. Minimize $f(k_{j+1}, \alpha)$ with respect to α (see Section 4) and find α_{j+1}. Revert to Step 2.

This method converges in the following sense.

Theorem 2. *In Algorithm 1, only a finite number of fractions are realized for γ_j at each iteration, the function $f(k_j)$ is monotonically decreasing, and its gradient vanishes with an exponential rate (like a geometric progression):*

$$\lim_{j\to\infty} |H_j| = 0.$$

Indeed, Algorithm 1 is well-defined at the initial point since k_0 is a stabilizing controller by the assumption. For sufficiently small γ_j, the function $f(k)$ monotonically decreases (moves in the direction of its antigradient); with this step adjustment, the values of k_j remain in the domain $\mathcal{S}_0$, where Lemma 5 ensures the Lipschitz property of the gradient. Thus, the gradient method for unconstrained minimization is convergent [7]. In particular, condition b) at Step 3 of Algorithm 1 will be satisfied after a finite number of fractions, and the gradient method will have gradient convergence with a linear rate.

Naturally, it is difficult to expect convergence to a global minimum: the domain of definition of $f(k)$ may even be disconnected.

7. EXAMPLE

Consider an illustrative example from the paper [8]. The transfer function has the form

$$G(s) = \frac{1}{(1+s)(1+\alpha s)(1+\alpha^2 s)(1+\alpha^3 s)}, \quad \alpha = 0.5.$$

MATLAB's procedure `tf2ss` gives the following matrices of system (4) in the state space:

$$A = \begin{pmatrix} -15 & -70 & -120 & -64 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \quad b = \begin{pmatrix} 1 \\ 0 \\ 0 \\ 0 \end{pmatrix}, \quad c = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 64 \end{pmatrix}.$$

Let us choose the matrix

$$D = \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ 0 & 0 \end{pmatrix}$$

and the controlled output matrix

$$C = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}.$$

We assign $\rho = 0.001$ and the stabilizing controller

$$k_0 = \begin{pmatrix} 1.7366 \\ 0.7734 \end{pmatrix}$$

as an initial one.

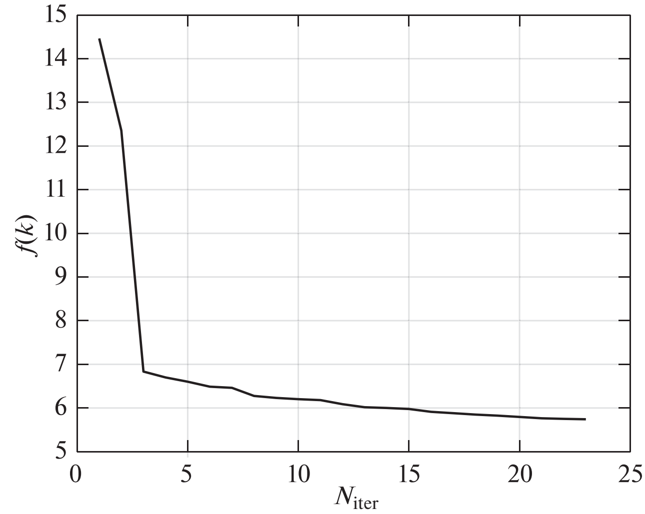

Fig. 1. Optimization procedure.

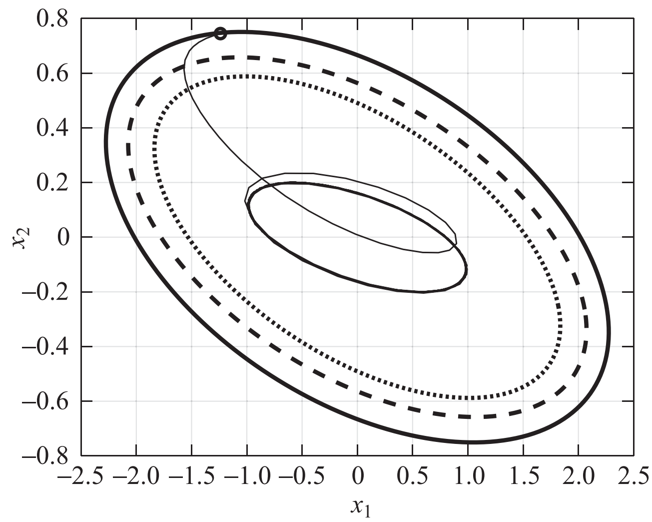

Fig. 2. Bounding ellipses.

The dynamics of the criterion $f(k)$ are demonstrated in Fig. 1. The process terminates with the PI controller with the gains

$$k_* = \begin{pmatrix} 0.2956 \\ 0.3514 \end{pmatrix}$$

and the corresponding bounding ellipse with the matrix

$$P_* = \begin{pmatrix} 5.1763 & -0.7885 \\ -0.7885 & 0.5635 \end{pmatrix}, \quad \operatorname{tr} P_* = 5.7398.$$

In Fig. 2, the solid line indicates the bounding ellipse and the trajectory of the closed loop system with the PI controller k_* under some admissible exogenous disturbance. Here, the dashed line shows the bounding ellipse for the closed loop system with the dynamic controller (see [4])

$$u = K\widehat{x},$$

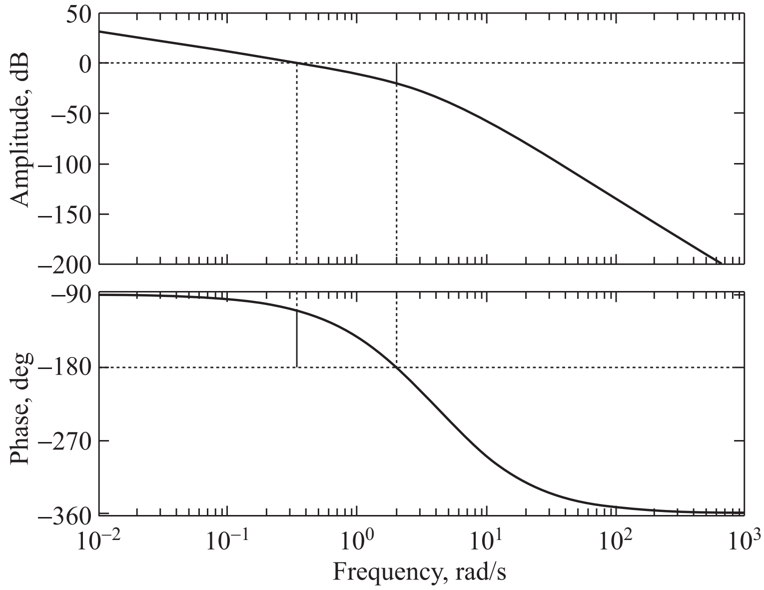

Fig. 3. The logarithmic amplitude-phase frequency response of the closed loop system.

where $\widehat{x}$ is an observer

$$\dot{\widehat{x}} = A\widehat{x} + bu + L(y - c^{\mathrm{T}}\widehat{x}), \quad \widehat{x}(0) = 0$$

with the matrices

$$K = \begin{pmatrix} -0.5154 & -2.6143 & -4.3786 & -2.4252 \end{pmatrix} \times 10^6, \qquad L = \begin{pmatrix} 0.0075 \\ -0.0225 \\ -0.0002 \\ 0.0189 \end{pmatrix}.$$

Finally, the dotted line in Fig. 2 presents the bounding ellipse for the closed loop system with the linear dynamic controller (see [4])

$$\begin{aligned} \dot{x}_r &= A_r x_r + B_r y, \quad x_r(0) = 0, \\ u &= C_r x_r + D_r y \end{aligned}$$

with the matrices

$$A_r = \begin{pmatrix} -0.1373 & -0.6748 & -1.0932 & -0.1035 \\ 0.0140 & 0.0688 & 0.1114 & -1.7096 \\ 0.0004 & 0.0019 & 0.0031 & -0.0509 \\ 0.0000 & 0.0000 & 0.0001 & -0.0007 \end{pmatrix} \times 10^5, \qquad B_r = \begin{pmatrix} -0.7528 \\ 2.7644 \\ 0.0821 \\ 0.0011 \end{pmatrix} \times 10^3,$$

$$C_r = \begin{pmatrix} -0.1135 & -0.5579 & -0.9037 & -2.9271 \end{pmatrix} \times 10^5, \quad D_r = 3.8176 \times 10^3.$$

Clearly, the PI controller leads to quite comparable results, being advantageous by simplicity and convenience of practical implementation. In addition, the PI controller has satisfactory characteristics.

The transfer function of the PI controller with the coefficients k_* has the form

$$G_{\mathrm{PID}}(s) = 0.2956 + \frac{0.3514}{s}.$$

The closed loop system with the PI controller k_* is stable by the Nyquist criterion; its minimal gain and phase margins are 20.6 dB and 70.3°, respectively (Fig. 3).

For comparison, choosing the initial stabilizing controller

$$\widetilde{k}_0 = \begin{pmatrix} 0.8882 \\ 0.6153 \end{pmatrix},$$

we obtain the PI controller with the gains

$$\widetilde{k}_* = \begin{pmatrix} 0.3277 \\ 0.3662 \end{pmatrix}$$

and the corresponding bounding ellipse with the matrix

$$\widetilde{P}_* = \begin{pmatrix} 5.0890 & -0.7854 \\ -0.7854 & 0.5721 \end{pmatrix}, \quad \operatorname{tr} \widetilde{P}_* = 5.6611.$$

The norms of the resulting controllers differ by 6.5% only whereas the bounding ellipses by less than 1.5% (in terms of the trace criterion).

All the calculations were carried out in MATLAB using `CVX` [9], a free package.

8. DISCUSSION

This paper has proposed a novel approach to designing a PI controller that optimally suppresses bounded exogenous disturbances in a linear control system. The approach is based on reducing the original problem to a nonconvex matrix optimization problem, which is further solved by the gradient method. Its justification has been provided as well.

Note that Theorem 2 establishes the convergence of this method only in the norm of the gradient of the objective function. However, according to numerical simulations, the method yields quite satisfactory PI controllers from an engineering point of view. At the same time, it seems important to consider meaningful particular formulations of the problem where the function $f(k)$ satisfies on the level set $\mathcal{S}_0$ the Polyak–Łojasiewicz condition [7]

$$\frac{1}{2}|\nabla f(k)|^2 \geqslant \mu \left(f(k) - f(k_*)\right)$$

with a constant $\mu > 0$ depending only on k_0 and the parameters of system (2). In this case, one could also speak of strong pointwise convergence, similar to what was shown in [3] for the linear quadratic problem with state-feedback control.

Finally, it would be interesting to extend this approach to the design of PID controllers, which will be the subject of subsequent publications.

APPENDIX A

The lemmas below contain well-known results necessary for the further presentation.

Lemma A.1 [1]. *Let X and Y be the solutions of the dual Lyapunov equations with a Hurwitz matrix A:*

$$A^{\mathrm{T}}X + XA + W = 0 \quad \textit{and} \quad AY + YA^{\mathrm{T}} + V = 0.$$

Then

$$\operatorname{tr}(XV) = \operatorname{tr}(YW).$$

Lemma A.2 [10].

1. *Matrices A and B of compatible dimensions satisfy the relations*

$$\|AB\|_F \leqslant \|A\|_F \|B\|,$$
$$|\operatorname{tr} AB| \leqslant \|A\|_F \|B\|_F,$$
$$\|A\| \leqslant \|A\|_F,$$
$$AB + B^{\mathrm{T}} A^{\mathrm{T}} \leqslant \varepsilon A A^{\mathrm{T}} + \frac{1}{\varepsilon} B^{\mathrm{T}} B \quad \text{for any } \varepsilon > 0.$$

2. *Nonnegative definite matrices A and B satisfy the relations*

$$0 \leqslant \lambda_{\min}(A)\lambda_{\max}(B) \leqslant \lambda_{\min}(A) \operatorname{tr} B \leqslant \operatorname{tr} AB \leqslant \lambda_{\max}(A) \operatorname{tr} B \leqslant \operatorname{tr} A \operatorname{tr} B.$$

Lemma A.3 [1]. *The solution P of the Lyapunov equation*

$$AP + PA^{\mathrm{T}} + Q = 0$$

with a Hurwitz matrix A and $Q \succ 0$ obey the bounds

$$\lambda_{\max}(P) \geqslant \frac{\lambda_{\min}(Q)}{2\sigma}, \quad \lambda_{\min}(P) \geqslant \frac{\lambda_{\min}(Q)}{2\|A\|},$$

where $\sigma = -\max_i \operatorname{Re} \lambda_i(A)$.

If $Q = DD^{\mathrm{T}}$ and the pair (A, D) is controllable, then

$$\lambda_{\max}(P) \geqslant \frac{\|u^* D\|^2}{2\sigma} > 0,$$

where

$$u^* A = \lambda u^*, \quad \operatorname{Re} \lambda = -\sigma, \quad \|u\| = 1,$$

i.e., u is the left eigenvector of the matrix A corresponding to the eigenvalue λ of the matrix A with the greatest real part. The vector u and the number λ can be complex-valued; here, u^* denotes the Hermitian conjugate.

APPENDIX B

Proof of Lemma 1. Indeed, if the matrix $\mathcal{A}_0 + \{\mathcal{A}, k\}$ is Hurwitz, then $\sigma(\mathcal{A}_0 + \{\mathcal{A}, k\}) > 0$ and there exists the solution $P \succcurlyeq 0$ of the Lyapunov equation (10) for $0 < \alpha < 2\sigma(\mathcal{A}_0 + \{\mathcal{A}, k\})$. Thus, the function $f(k, \alpha) > 0$ is well-defined and $f(k) > 0$ by Theorem 1. The proof of Lemma 1 is complete.

Proof of Lemma 2. The optimization problem has the form

$$\min f(k, \alpha), \quad f(k, \alpha) = \operatorname{tr} P \begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} C & 0 \end{pmatrix} + \rho |k|^2$$

subject to the constraint described by the Lyapunov equation

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2} I\right) P + P \left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2} I\right)^{\mathrm{T}} + \frac{1}{\alpha} \begin{pmatrix} D \\ 0 \end{pmatrix} \begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} = 0.$$

To differentiate with respect to k, we add the increment Δk and denote the corresponding increment of P by ΔP:

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k + \Delta k\} + \frac{\alpha}{2} I\right)(P + \Delta P) + (P + \Delta P)\left(\mathcal{A}_0 + \{\mathcal{A}, k + \Delta k\} + \frac{\alpha}{2} I\right)^{\mathrm{T}} + \frac{1}{\alpha}\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} = 0.$$

Let us apply linearization and subtract this and the previous equations to obtain

$$\begin{gathered}\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2} I\right)\Delta P + \Delta P\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2} I\right)^{\mathrm{T}} \\ + \{\mathcal{A}, \Delta k\}P + P\{\mathcal{A}, \Delta k\}^{\mathrm{T}} = 0.\end{gathered} \tag{B.1}$$

The increment of $f(k)$ is calculated by linearizing the corresponding terms:

$$\begin{aligned}\Delta f(k) &= \operatorname{tr}(P + \Delta P)\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + \rho|k + \Delta k|^2 \\ &\quad - \left(\operatorname{tr} P\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + \rho|k|^2\right) \\ &= \operatorname{tr}\Delta P\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + 2\rho k^{\mathrm{T}}\Delta k.\end{aligned}$$

Consider equation (14), dual to (B.1). Due to Lemma A.1, from equations (B.1) and (14) it follows that

$$\Delta f(k) = 2\operatorname{tr} Y\{\mathcal{A}, \Delta k\}P + 2\rho k^{\mathrm{T}}\Delta k.$$

Thus,

$$df(k) = 2\operatorname{tr} PY\sum_{i=1}^{2}\mathcal{A}_i dk_i + 2\rho\sum_{i=1}^{2} k_i dk_i,$$

which leads to (12).

The validity of (13) is demonstrated by analogy with [1, Lemma 1]. The proof of Lemma 2 is complete.

Proof of Lemma 3. The value $(\nabla^2_{kk} f(k)e, e)$ is calculated by differentiating $\nabla_k f(k)$ in the direction $e \in \mathbb{R}^2$. For this purpose, linearizing the corresponding terms and using the convenient notation

$$[\operatorname{tr} PY\mathcal{A}] = \begin{pmatrix} \operatorname{tr} PY\mathcal{A}_1 \\ \operatorname{tr} PY\mathcal{A}_2 \end{pmatrix},$$

we calculate the increment of $\nabla_k f(k)$ in the direction e:

$$\begin{aligned}\frac{1}{2}\Delta\nabla_k f(k)e &= \rho(k + \delta e) + [\operatorname{tr}(P + \Delta P)(Y + \Delta Y)\mathcal{A}] - (\rho k + [\operatorname{tr} PY\mathcal{A}]) \\ &= \rho(k + \delta e) + \left[\operatorname{tr}\,(P + \delta P'(k)e)\,(Y + \delta Y'(k)e)\,\mathcal{A}\right] - (\rho k + [\operatorname{tr} PY\mathcal{A}]) \\ &= \delta\left(\rho e + \left[\operatorname{tr}\,(PY'(k)e + P'(k)eY)\,\mathcal{A}\right]\right),\end{aligned}$$

where

$$\begin{aligned}\Delta P &= P(k + \delta e) - P(k) = \delta P'(k)e, \\ \Delta Y &= Y(k + \delta e) - Y(k) = \delta Y'(k)e.\end{aligned}$$

Thus, with $P' = P'(k)e$ and $Y' = Y'(k)e$, we have

$$\frac{1}{2}\left(\nabla^2_{kk} f(k)e, e\right) = (\rho e + [\,\mathrm{tr}\,(PY' + P'Y)\mathcal{A}], e)\,.$$

Furthermore, $P = P(k)$ is the solution of equation (17). We write it in increments in the direction $e:$

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta e\} + \frac{\alpha}{2}I\right)(P + \delta P')$$
$$+ (P + \delta P')\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta e\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} + \frac{1}{\alpha}\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} = 0$$

or

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)(P + \delta P') + (P + \delta P')\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}}$$
$$+ \delta\left(\{\mathcal{A}, e\}P + P\{\mathcal{A}, e\}^{\mathrm{T}}\right) + \frac{1}{\alpha}\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} = 0.$$

Subtracting equation (17) from this expression gives equation (16).

Similarly, $Y = Y(k)$ is the solution of the Lyapunov equation (14). We write it in increments in the direction $e:$

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta e\} + \frac{\alpha}{2}I\right)^{\mathrm{T}}(Y + \delta Y')$$
$$+ (Y + \delta Y')\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta e\} + \frac{\alpha}{2}I\right) + \begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} = 0,$$

or

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}}(Y + \delta Y') + (Y + \delta Y')\left(\mathcal{A}_0 + \{\mathcal{A}, k + \delta e\} + \frac{\alpha}{2}I\right)$$
$$+ \delta\left(\{\mathcal{A}, e\}^{\mathrm{T}}Y + Y\{\mathcal{A}, e\}\right) + \begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} = 0.$$

Subtracting equation (14) from this expression yields

$$\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right)^{\mathrm{T}} Y' + Y'\left(\mathcal{A}_0 + \{\mathcal{A}, k\} + \frac{\alpha}{2}I\right) + \{\mathcal{A}, e\}^{\mathrm{T}}Y + Y\{\mathcal{A}, e\} = 0. \tag{B.2}$$

From (16) and (B.2) it follows that

$$\mathrm{tr}\,P'Y\{\mathcal{A}, e\} = \mathrm{tr}\,PY'\{\mathcal{A}, e\},$$

so

$$\frac{1}{2}\left(\nabla^2_{kk} f(k)e, e\right) = \rho(e, e) + ([\,\mathrm{tr}\,(PY' + P'Y)\mathcal{A}], e) = \rho(e, e) + 2\,\mathrm{tr}\,P'Y\{\mathcal{A}, e\}.$$

The proof of Lemma 3 is complete.

Proof of Lemma 4. Consider a sequence of stabilizing controllers $\{k_j\} \in \mathcal{S}$ such that $k_j \to k \in \partial\mathcal{S}$, i.e., $\sigma(\mathcal{A}_0 + \{\mathcal{A}, k\}) = 0$. In other words, for any $\varepsilon > 0$ there exists a number $N = N(\varepsilon)$ such that

$$|\sigma(\mathcal{A}_0 + \{\mathcal{A}, k_j\}) - \sigma(\mathcal{A}_0 + \{\mathcal{A}, k\})| = \sigma(\mathcal{A}_0 + \{\mathcal{A}, k_j\}) < \epsilon$$

for all $j \geqslant N(\epsilon)$.

Let P_j be the solution of the Lyapunov equation (10) associated with the controller k_j :

$$\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)P_j+P_j\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)^{\mathrm{T}}+\frac{1}{\alpha_j}\left[\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}}+\varepsilon_2 I\right]=0.$$

Also, let Y_j be the solution of the dual Lyapunov equation

$$\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)^{\mathrm{T}}Y_j+Y_j\left(\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right)+\begin{pmatrix}C & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C & 0\end{pmatrix}+\varepsilon_1 I=0.$$

Using Lemma A.3, we have

$$\begin{aligned}
f(k_j)=&\ \operatorname{tr}P_j\left(\begin{pmatrix}C & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C & 0\end{pmatrix}+\varepsilon_1 I\right)+\rho|k_j|^2\geqslant \operatorname{tr}P_j\left(\begin{pmatrix}C & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C & 0\end{pmatrix}+\varepsilon_1 I\right)\\
=&\ \operatorname{tr}Y_j\frac{1}{\alpha_j}\left[\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}}+\varepsilon_2 I\right]\geqslant\frac{1}{\alpha_j}\lambda_{\min}(Y_j)\operatorname{tr}\left[\begin{pmatrix}D\\0\end{pmatrix}\begin{pmatrix}D\\0\end{pmatrix}^{\mathrm{T}}+\varepsilon_2 I\right]\\
\geqslant&\ \frac{1}{\alpha_j}\lambda_{\min}(Y_j)\left\|\begin{pmatrix}D\\0\end{pmatrix}\right\|_F^2\geqslant\frac{1}{\alpha_j}\frac{\lambda_{\min}\left(\begin{pmatrix}C & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C & 0\end{pmatrix}+\varepsilon_1 I\right)}{2\left\|\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right\|}\|D\|_F^2\\
\geqslant&\ \frac{\varepsilon_1}{4\sigma\left(\mathcal{A}_0+\{\mathcal{A},k_j\}\right)\left\|\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right\|}\|D\|_F^2\\
\geqslant&\ \frac{\varepsilon_1}{4\epsilon\left(\|\mathcal{A}_0+\{\mathcal{A},k_j\}\|+\epsilon\right)}\|D\|_F^2\xrightarrow[\epsilon\to 0]{}+\infty
\end{aligned}$$

since

$$0<\alpha_j<2\sigma\left(\mathcal{A}_0+\{\mathcal{A},k_j\}\right)$$

and

$$\left\|\mathcal{A}_0+\{\mathcal{A},k_j\}+\frac{\alpha_j}{2}I\right\|\leqslant\|\mathcal{A}_0+\{\mathcal{A},k_j\}\|+\frac{\alpha_j}{2}<\|\mathcal{A}_0+\{\mathcal{A},k_j\}\|+\sigma\left(\mathcal{A}_0+\{\mathcal{A},k_j\}\right).$$

On the other hand,

$$f(k_j)=\ \operatorname{tr}P_j\left(\begin{pmatrix}C & 0\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}C & 0\end{pmatrix}+\varepsilon_1 I\right)+\rho|k_j|^2\geqslant\rho|k_j|^2\xrightarrow[|k_j|\to+\infty]{}+\infty.$$

The proof of Lemma 4 is complete.

Proof of Corollary 2. The function $f(k)$ has a minimum point on the set $\mathcal{S}_0$ (as a continuous function on a compact set), but the set $\mathcal{S}_0$ shares no points with the boundary $\mathcal{S}$ due to (18). Finally, the function $f(k)$ is differentiable on $\mathcal{S}_0$ by Lemma 2, which concludes the proof of Corollary 2.

Proof of Lemma 5. Applying Lemma A.2 to (15) gives

$$\begin{aligned}
\frac{1}{2}\|\nabla^2_{kk}f(k)\|&=\frac{1}{2}\sup_{|e|=1}\left|\left(\nabla^2_{kk}f(k)e,e\right)\right|\leqslant\sup_{|e|=1}\rho(e,e)+2\sup_{|e|=1}|\operatorname{tr}P'Y\{\mathcal{A},e\}|\\
&=\rho+2\sup_{|e|=1}\|P'\|_F\|Y\{\mathcal{A},e\}\|_F\leqslant\rho+2\|P'\|_F\sup_{|e|=1}\|Y\|\|\{\mathcal{A},e\}\|_F\\
&\leqslant\rho+2\sqrt{2}\|P'\|_F\|Y\|\max_i\|\mathcal{A}_i\|_F
\end{aligned}$$

since

$$\|\{\mathcal{A},e\}\|_F=\left\|\sum_i\mathcal{A}_ie_i\right\|_F\leqslant\sum_i\|\mathcal{A}_i\|_F|e_i|\leqslant\max_i\|\mathcal{A}_i\|_F|e|_1\leqslant\sqrt{2}\max_i\|\mathcal{A}_i\|_F|e|.$$

Thus, it is necessary to estimate from above the value

$$\rho + 2\sqrt{2}\max_i \|\mathcal{A}_i\|_F \|P'\|_F \|Y\|.$$

For $\|Y\|$ we have the upper bound

$$\frac{\varepsilon_2}{\alpha}\|Y\| \leqslant \frac{1}{\alpha}\lambda_{\min}\left[\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} + \varepsilon_2 I\right] \operatorname{tr} Y \leqslant \operatorname{tr} Y \frac{1}{\alpha}\left[\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} + \varepsilon_2 I\right]$$
$$= \operatorname{tr} P\left(\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + \varepsilon_1 I\right) = f(k) - \rho|k|^2 \leqslant f(k) \leqslant f(k_0),$$

and consequently,

$$\|Y\| \leqslant \frac{\alpha}{\varepsilon_2} f(k_0). \tag{B.3}$$

An upper bound for α is established as follows:

$$\alpha < 2\sigma\left(\mathcal{A}_0 + \{\mathcal{A}, k\}\right) \leqslant 2\left\|\mathcal{A}_0 + \{\mathcal{A}, k\}\right\|$$
$$\leqslant 2\left(\|\mathcal{A}_0\| + \sum_i \|\mathcal{A}_i\||k_i|\right) \leqslant 2\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\||k|_1\right)$$
$$\leqslant 2\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{2}|k|\right) \leqslant 2\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{2}{\rho}f(k)}\right)$$
$$\leqslant 2\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{2}{\rho}f(k_0)}\right),$$

so

$$\|Y\| \leqslant \frac{2}{\varepsilon_2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{2}{\rho}f(k_0)}\right) f(k_0).$$

Now, let us estimate $\|P\|$ from above:

$$\varepsilon_1\|P\| \leqslant \lambda_{\min}\left(\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + \varepsilon_1 I\right)\|P\|$$
$$\leqslant \operatorname{tr} P\left(\begin{pmatrix} C & 0 \end{pmatrix}^{\mathrm{T}}\begin{pmatrix} C & 0 \end{pmatrix} + \varepsilon_1 I\right) = f(k) - \rho|k|^2 \leqslant f(k) \leqslant f(k_0),$$

which yields

$$\|P\| \leqslant \frac{f(k_0)}{\varepsilon_1}.$$

It remains to estimate from above the value $\|P'\|_F$. In view of Lemma A.2,

$$\lambda_{\max}\left(\{\mathcal{A}, e\}P + P\{\mathcal{A}, e\}^{\mathrm{T}}\right) = \left\|\{\mathcal{A}, e\}P + P\{\mathcal{A}, e\}^{\mathrm{T}}\right\| \leqslant \left\|P^2 + \{\mathcal{A}, e\}\{\mathcal{A}, e\}^{\mathrm{T}}\right\|$$
$$\leqslant \|P\|^2 + \|\{\mathcal{A}, e\}\|^2 \leqslant \frac{f^2(k_0)}{\varepsilon_1^2} + 2\max_i \|\mathcal{A}_i\|^2 \leqslant \xi\frac{\varepsilon_2}{\alpha} \leqslant \xi\frac{1}{\alpha}\lambda_{\min}\left[\begin{pmatrix} D \\ 0 \end{pmatrix}\begin{pmatrix} D \\ 0 \end{pmatrix}^{\mathrm{T}} + \varepsilon_2 I\right]$$

for

$$\xi = \frac{\alpha}{\varepsilon_2}\left(\frac{f^2(k_0)}{\varepsilon_1^2} + 2\max_i \|\mathcal{A}_i\|^2\right).$$

Therefore, the solution P' of the Lyapunov equation (16) satisfies the inequality

$$P' \preccurlyeq \xi P \preccurlyeq \frac{\alpha}{\varepsilon_2}\left(\frac{f^2(k_0)}{\varepsilon_1^2} + 2\max_i \|\mathcal{A}_i\|^2\right)\frac{f(k_0)}{\varepsilon_1} I$$
$$\preccurlyeq \frac{2f(k_0)}{\varepsilon_1\varepsilon_2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{2}{\rho}f(k_0)}\right)\left(\frac{f^2(k_0)}{\varepsilon_1^2} + 2\max_i \|\mathcal{A}_i\|^2\right) I.$$

Hence, it follows that

$$\|P'\|_F \leqslant \frac{2\sqrt{n}f(k_0)}{\varepsilon_1\varepsilon_2}\left(\|\mathcal{A}_0\| + \max_i \|\mathcal{A}_i\|\sqrt{\frac{2}{\rho}f(k_0)}\right)\left(\frac{f^2(k_0)}{\varepsilon_1^2} + 2\max_i \|\mathcal{A}_i\|^2\right). \tag{B.4}$$

Considering the bounds (B.3) and (B.4), we arrive at the relation (19). The proof of Lemma 5 is complete.

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