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On Asymptotically Optimal Approach for Finding of the Minimum Total Weight of Edge-Disjoint Spanning Trees with a Given Diameter

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Abstract—We consider the intractable problem of finding several edge-disjoint spanning trees of the minimum total weight with a given diameter in complete undirected graph in current paper. The weights of edges of a graph are random variables from several continuous distributions: uniform, biased truncated exponential, biased truncated normal. The approximation algorithm with time complexity $\mathcal{O}(n^2)$, where n is number of vertices in graph, is proposed for solving this problem. The asymptotic optimality conditions for constructed algorithm is presented for each considered probabilistic distribution.

Keywords: minimum spanning tree with given diameter, approximation algorithm, probabilistic analysis, asymptotic optimality

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1. INTRODUCTION

The Minimum Spanning Tree (MST) Problem is one of well-known problems of discrete optimization. Is consists of finding spanning tree (connected acyclic subgraph on all vertices) of minimum weight in given edge-weighted graph $G = (V, E)$. Polynomial solvability of this problem was proved by construction of polynomial algorithms Boruvka (1926), Kruskal (1956), and Prim (1957). These algorithms have time complexities $\mathcal{O}(u \log n)$, $\mathcal{O}(u \log u)$, and $\mathcal{O}(n^2)$ respectively, where $u = |E|$ and $n = |V|$. It is interesting to note, that expected value of MST's weight in graph with random edge weights can be surprisingly small. For example, MST's weight with high probability is close to constant 2.02 for complete graph with edge weights from uniform distribution on interval $(0; 1)$ [1]. Similar results were obtained in [2, 3].

One possible generalization of the above problem is the bounded diameter version of the MST problem. The diameter of a tree is the number of edges in the longest simple path within the tree connecting a pair of vertices. This problem is as follows: given edge-weighted graph and parameter $d = d_n$, it is necessary to find MST in this graph with diameter bounded from above or below by the parameter d. Both problems are NP-hard in general formulation.

The bounded from above MST problem is polynomially solvable for diameters two or three, and NP-hard for any diameter between 4 and $(n-1)$, even for the edge weights equal to 1 or 2 [4, pp. 206]. The MST problem bounded from below is NP-hard, because its particular case for $d = n - 1$ is the problem "Hamiltonian Path" [4].

Recently, the authors of this article have began to study another modification of the MST problem with a bounded diameter, when the diameter of this tree is equal to a given number. It is noteworthy that the algorithm for solving such a problem can be transformed into an algorithm for solving a problem with a diameter bounded from above or from below. Thus, the scope of such a problem covers the scope of problems with a bounded diameter both from above and from below.

There are several applications for MST problem with bounded diameter from above in wireless ad-hoc networks [5], network design [6], in development of data compression algorithm [7] and distributed mutual exclusion algorithm [8] (for a detailed description see, for example, [9]).

The problem of finding several edge-disjoint spanning trees of minimum total weight with bounded from below diameter in complete graph arises in the theory of reliability of communication networks, when it is necessary to construct m-connected graph of minimum total weight for a set of objects excluding such configuration of graph, for which after failure of few nodes, the total structure of the graph becomes unreliable. Thus it is necessary to bound the diameter of constructed trees forming m-connected graph. It must be noted that in [10, 11] a probabilistic analysis of an approximation algorithm for this problem was carried out and conditions for its asymptotic optimality were obtained.

In [12, 13] the probabilistic analysis of polynomial algorithm is carried out and asymptotically optimal conditions for this algorithm were proposed for the problem of finding one and several MST with given in the case of complete directed graph. Unfortunately, the algorithm analysis is not accepted for the case of complete undirected graph. The appearance of the difficulty for probabilistic analysis in the case of undirected graph arises from the need to take into account the possible dependence between different objects (random variables) in the course of the algorithm.

We consider the problem of finding m edge-disjoint spanning trees of minimum total weight with a given diameter $d = d_n$ in complete undirected graph (this problem is denoted as m-d-UMST). The approximation algorithm for solving this problem and its conditions of asymptotic optimality are presented. Probabilistic analysis is accomplished for the case of complete edge-weighted undirected graph G without loops under assumption that weights of edges of a graph positive independent identically distributed random variables. The probability distribution functions (p.d.f.) of the weights of graph G are considered from three probabilistic distributions: uniform distribution $\mathrm{UNI}(a_n; b_n)$ on the finite segment $[a_n; b_n]$, as well as biased truncated distributions: exponential $\mathrm{EXP}(a_n, \lambda_n)$ and normal $\mathrm{NORM}(a_n, \sigma_n)$ on the unbounded semiopen interval $[a_n; \infty)$. The probabilistic density functions for these distributions are as follows

$$p(x) = \begin{cases} \dfrac{1}{b_n - a_n} & \text{if } a_n \leqslant x \leqslant b_n \text{ for } \mathrm{UNI}(a_n; b_n); \\ \dfrac{1}{\lambda_n} \exp\left(-\dfrac{x - a_n}{\lambda_n}\right) & \text{if } a_n \leqslant x < \infty \text{ for } \mathrm{EXP}(a_n, \lambda_n); \\ \dfrac{2}{\sigma_n\sqrt{2\pi}} \exp\left(-\dfrac{(x - a_n)^2}{2\sigma_n^2}\right) & \text{if } a_n \leqslant x < \infty \text{ for } \mathrm{NORM}(a_n, \sigma_n); \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

2. THE FINDING OF SEVERAL EDGE-DISJOINT SPANNING TREES OF MINIMUM TOTAL WEIGHT WITH GIVEN DIAMETER IN UNDIRECTED GRAPH

First of all, we formulate considered problem and then propose approximation algorithm for its solution.

Given complete n-vertex edge-weighted undirected graph $G = (V, E)$ and positive integer numbers $m \geqslant 2$, $d \geqslant 4$ such that $m(d + 1) \leqslant n$. The m-d-UMST is to find m edge-disjoint spanning trees $T_1, \ldots, T_m$ such that the diameter of each of them is equal to $d = d_n$ and their total weight is minimum. For solving this problem, the next deterministic algorithm is proposed.

Description of algorithm $\mathcal{A}$

Preliminary Step 0. In graph G, choose arbitrary $(n - m(d + 1))$-vertex subset V' and arbitrarily split remaining $m(d + 1)$ vertex into m subsets $V_1, V_2, \ldots, V_m$ with $(d + 1)$ vertices in each set.

Step 1. In each subgraph $G(V_s)$ $s = 1, \ldots, m$, beginning with arbitrary vertex construct $(d + 1)$-vertex Hamiltonian path P_s using greedy heuristic "go to the nearest unvisited vertex".

Put $T_s = P_s$, $s = 1, \ldots, m$.

Step 2. Hereafter we assume without loss of generality that d is odd (see remark 1 below). For each pair of paths P_i and P_j, $1 \leqslant i < j \leqslant m$, add vertices from P_j to T_i and from P_i to T_j in such a way that constructed subgraph consists of two edge-disjoint $2(d + 1)$-vertex subtree with diameter equals d. Each path P_s, $1 \leqslant s \leqslant m$, is considered as two halves (subpaths) P_s^1 and P_s^2, each of which contains one *end* vertex and $\frac{d-1}{2}$ *inner* vertices of path P_s totally $\frac{d+1}{2}$ vertices in each half.

Construction of edge-disjoint spanning trees T_i and T_j with help of vertices from halves P_i^1, P_i^2 and halves P_j^1, P_j^2 is described in following items 2.1–2.6.

2.1. Connect each inner vertex of P_i^1 by the shortest edge to the inner vertex of P_j^1. So we add this edge to T_j.

2.2. Connect each inner vertex of P_i^2 by the shortest edge to the inner vertex of P_j^2. We add this edge to T_j.

2.3. Connect each inner vertex of P_j^1 by the shortest edge to the inner vertex of P_i^2. Thus, we add this edge to T_i.

2.4. Connect each inner vertex of P_j^2 by the shortest edge to the inner vertex of P_i^1. We add this edge to T_i.

2.5. Connect each end vertex of the path P_i by the shortest edge to the inner vertex of the path P_j. We add this edge to T_j.

2.6. Connect each end vertex of the path P_j by the shortest edge to the inner vertex of the path P_i. So we add this edge to T_i.

Step 3. For $s = 1, \ldots, m$ each vertex of subgraph $G(V')$ is connected by the shortest edge to the inner vertex of the path P_s. Thus, we add this edge to corresponding tree T_s.

The construction of m edge-disjoint spanning trees $T_1, \ldots, T_m$ is completed (see example in Figs. 1–3).

Remark 1. In the case of even d algorithm must be slightly modified. On the Step 1 for the first chosen vertex it is necessary to find the closest vertex v_s, $s = 1, \ldots, m$ in d actions. The first chosen vertex is marked and used after all steps of algorithm. Hereafter all steps of the algorithm must be carried out for $d' = d - 1$, where the first vertex of each path is v_s, and after Step 3 marked vertices are connected with v_s, $s = 1, \ldots, m$. Thus, desired spanning trees are constructed with property that diameter of each tree equals exactly d, and time complexity of presented algorithm remains the same.

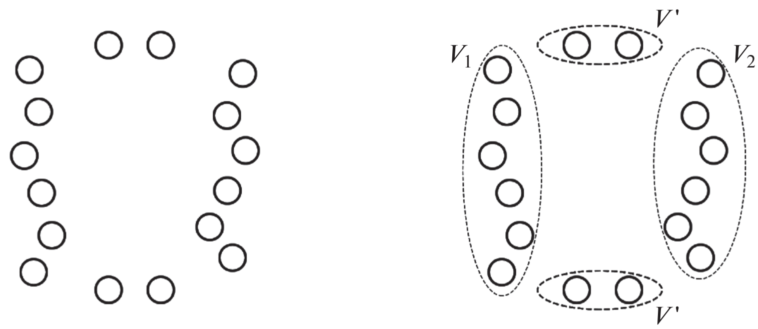

Fig. 1. Initial vertices of the graph and Step 0 of the work of the Algorithm $\mathcal{A}$ in 16-vertex complete graph, $m = 2$, $d = 5$.

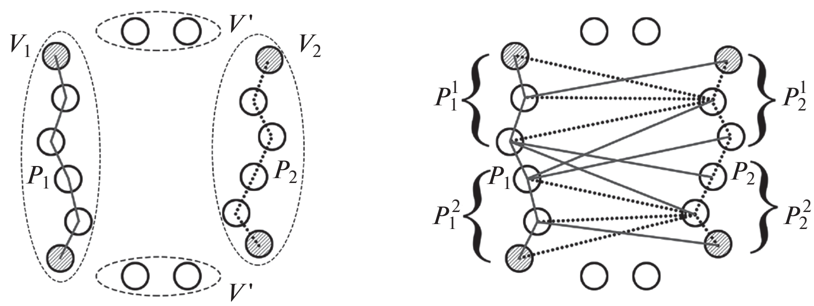

Fig. 2. Steps 1 and 2 of the work of the algorithm $\mathcal{A}$ in 16-vertex complete graph, $m = 2$, $d = 5$. The hatched vertices are end vertices. The solid edges belong to T_1. The dotted edges belong to T_2.

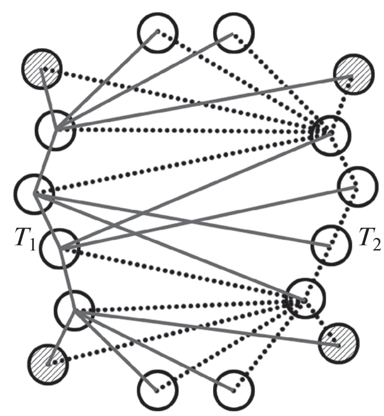

Fig. 3. Step 3 of the work of the Algorithm $\mathcal{A}$ in 16-vertex complete graph, $m = 2$, $d = 5$. The hatched vertices are end vertices. The solid edges belong to T_1. The dotted edges belong to T_2.

Let us introduce the notations: $W_{\mathcal{A}}$ is total weight of all spanning trees $T_1, \ldots, T_m$, which are constructed by algorithm $\mathcal{A}$, W_1, W_2, and W_3 are total weights of edges, which are added to the trees on Steps 1, 2, and 3 respectively. Then $W_{\mathcal{A}} = W_1 + W_2 + W_3$.

Let's formulate two statements concerning algorithm $\mathcal{A}$.

Statement 1. *Algorithm $\mathcal{A}$ constructs feasible solution for the m-d-*UMST.

Statement 2. *Time complexity of algorithm $\mathcal{A}$ is estimated by $\mathcal{O}(n^2)$.*

3. PROBABILISTIC ANALYSIS OF ALGORITHM $\mathcal{A}$

Let $F_A(I)$ and $OPT(I)$ be approximation (obtained using some algorithm A) and optimal value of objective function of problem on input I, respectively.

Definition 1. Algorithm A has estimates (performance guarantees) $(\varepsilon_n, \delta_n)$ on a set I of random inputs of a n-sized problem (where n is amount of input data required to describe the problem, see [4]), if

$$\mathbb{P}\{|F_A(I) - OPT(I)| > \varepsilon_n OPT(I)\} \leqslant \delta_n, \tag{2}$$

where $\varepsilon_n = \varepsilon_A(n)$ is an estimate of relative error of a solution obtained by algorithm A, $\delta_n = \delta_A(n)$ is an estimate of the failure probability of the algorithm, which is equal to the proportion of cases when the algorithm does not hold the relative error ε_n or does not produce any answer at all.

Definition 2 [14]. Approximation algorithm A is called asymptotically optimal on a class of input data of a problem, if there exist such performance guarantees that for all input I of size n

$$\varepsilon_n \to 0 \text{ and } \delta_n \to 0 \text{ as } n \to \infty.$$

Hereafter random variable, which is equal to minimum over k independent identically distributed random variables $\eta^1, \ldots, \eta^k$, is denoted as η_k.

According to the description of algorithm $\mathcal{A}$ for Steps 1–3 the following relations are true:

$W_1 = \sum\limits_{s=1}^{m} \sum\limits_{k=1}^{d} \eta_k$, since m paths $P_1, \ldots, P_m$ with d edges in each path are constructed on Step 1.

$W_2 = C_m^2\big(4\frac{d-1}{2}\eta_{(d-1)/2} + 4\eta_{(d-1)}\big)$, because connection of new edges to constructed set of spanning trees is carried out for each pair of paths from $C_m^2 = \frac{m(m-1)}{2}$ such pairs on corresponding items 2.1–2.6 of Step 2 as follows:

— firstly, each of $\frac{d-1}{2}$ inner vertices of one half of a path connected by shortest edge to each of $\frac{d-1}{2}$ inner vertices of half of another path;

— secondly, each end vertex of each path is connected by shortest edge to one of $d-1$ inner vertices of another path.

The multiplier 4 arises since inner vertices from two halves of one path are connected by shortest edges with inner vertices from two halves of another path for each pair of paths in items 2.1–2.4. In items 2.5–2.6, corresponding shortest edges connects 4 end vertices with inner vertices of considered paths.

$W_3 = m(n - m(d+1))\eta_{(d-1)}$, since each vertex over $n - m(d+1)$ vertices of set V' is connected by shortest edge to inner vertex of the m paths P_s, $1 \leqslant s \leqslant m$ on Step 3 considering $(d-1)$ inner vertices of each path.

Remark 2. It must be noted that for even d:

$$W_1 = m\sum_{k=1}^{d'} \eta_k, \quad W_2 = C_m^2\left(4\frac{d'-1}{2}\eta_{(d'-1)/2} + 4\eta_{(d'-1)}\right), \quad W_3 = m\left(n - m(d'+1)\right)\eta_{(d'-1)},$$

where $d' = d - 1 \geqslant 3$. Thus, replacing d by d' in all cases it is possible to accomplish probability analysis and prove all the proposed statements.

Hereafter we pass from random variables η, η_k to *normalized* random variables $\xi = \frac{\eta - a_n}{\beta_n}$, $\xi_k = \frac{\eta_k - a_n}{\beta_n}$, respectively, where $\beta_n = \begin{cases} b_n - a_n & \text{for UNI}(a_n; b_n); \\ \lambda_n & \text{for EXP}(a_n, \lambda_n); \\ \sigma_n & \text{for NORM}(a_n, \sigma_n). \end{cases}$

Let us consider random variables W_1, W_2, W_3:

$$W_1 = m\sum_{k=1}^{d}\eta_k = m\sum_{k=1}^{d}(\beta_n\xi_k + a_n) = mda_n + \beta_n m\sum_{k=1}^{d}\xi_k = mda_n + \beta_n W_1',$$

$$W_2 = C_m^2\left(4\frac{d-1}{2}\eta_{(d-1)/2} + 4\eta_{(d-1)}\right) = C_m^2\left(4\frac{d-1}{2}\left(\beta_n\xi_{(d-1)/2} + a_n\right) + 4\left(\beta_n\xi_{(d-1)} + a_n\right)\right)$$

$$= m(m-1)(d+1)a_n + \beta_n m(m-1)\Big((d-1)\xi_{(d-1)/2} + 2\xi_{(d-1)}\Big)$$

$$= \Big(m^2(d+1) - md - m\Big)\,a_n + \beta_n W_2',$$

$$W_3 = m(n - m(d+1))\eta_{(d-1)} = m(n-m(d+1))\left(\beta_n\xi_{(d-1)} + a_n\right)$$

$$= m(n-m(d+1))a_n + \beta_n m(n-m(d+1))\xi_{(d-1)} = \Big(mn - m^2(d+1)\Big)\,a_n + \beta_n W_3',$$

where W_1', W_2', W_3' are normalized random variables for W_1, W_2, W_3 respectively, and β_n is parameter of corresponding distribution.

So the following relation is obtained for the sum of the weights of the constructed spanning trees: $W_{\mathcal{A}} = m(n-1)a_n + \beta_n W_{\mathcal{A}}'$, where $W_{\mathcal{A}}' = W_1' + W_2' + W_3'$.

Lemma 1. *Algorithm $\mathcal{A}$ for the m-d-UMST in n-vertex complete graph with weights of edges from probabilistic distributions* ($\mathrm{UNI}(a_n; b_n)$, $\mathrm{EXP}(a_n, \lambda_n)$ *or* $\mathrm{NORM}(a_n, \sigma_n)$) *is the algorithm with the next estimate of relative error ε_n and failure probability δ_n:*

$$\varepsilon_n = \frac{2\beta_n}{m(n-1)a_n}\widehat{\mathbb{E}W_{\mathcal{A}}'}, \quad \delta_n = \mathbb{P}\{\widetilde{W_{\mathcal{A}}'} > \widehat{\mathbb{E}W_{\mathcal{A}}'}\}, \tag{3}$$

where β_n is parameter of corresponding distribution, $\widehat{\mathbb{E}W_{\mathcal{A}}'}$ is some upper bound for expected value $\mathbb{E}W_{\mathcal{A}}'$, $\widetilde{W_{\mathcal{A}}'} = W_{\mathcal{A}}' - \mathbb{E}W_{\mathcal{A}}'$.

Henceforth, the following statement from theory of probability is useful for probabilistic analysis of algorithm $\mathcal{A}$.

Theorem 1 [15]. *Let us consider random variables $X_1, \ldots, X_n$. We define positive constants T and $h_1, \ldots, h_n$ such that for all $k = 1, \ldots, n$ and $0 \leqslant t \leqslant T$ the inequality is true*

$$\mathbb{E}e^{tX_k} \leqslant e^{\frac{h_k t^2}{2}}. \tag{4}$$

Let $S = \sum\limits_{k=1}^{n} X_k$ and $H = \sum\limits_{k=1}^{n} h_k$. Then

$$\mathbb{P}\{S > x\} \leqslant \begin{cases} \exp\left\{-\dfrac{x^2}{2H}\right\}, & \text{if } \ 0 \leqslant x \leqslant HT, \\ \exp\left\{-\dfrac{Tx}{2}\right\}, & \text{if } \ x \geqslant HT. \end{cases}$$

Also the following statement will be useful for further analysis.

Statement 3. *For all integers $d \geqslant 3$ the following inequality is correct*

$$\sum_{k=1}^{d}\frac{1}{k} \leqslant \ln d + \frac{3}{4}.$$

It is assumed that d is odd and is defined on two semiopen intervals: case 1 ($\ln n \leqslant d < \frac{n}{\ln n}$) and case 2 ($\frac{n}{\ln n} \leqslant d < \frac{n}{m}$).

3.1. Probability Distribution UNI$(a_n; b_n)$

We pass from random variables η and η_k to normalized random variables $\xi = \frac{\eta - a_n}{b_n - a_n}$ and $\xi_k = \frac{\eta_k - a_n}{b_n - a_n}$ for uniform distribution UNI$(a_n; b_n)$.

Lemma 2. *For* $\mathbb{E}W'_{\mathcal{A}}$ *the following inequality holds*

$$\mathbb{E}W'_{\mathcal{A}} \leqslant m \ln d + \frac{2mn}{d}.$$

Lemma 3. *Let constants* $T = 1$ *and* $h_k = \frac{1}{(k+1)^2}$ *are defined. Then for* UNI$(a_n; b_n)$ *and biased random variables* $\widetilde{\xi_k} = \xi_k - \mathbb{E}\xi_k$ *the next inequalities are true* $\mathbb{E}e^{t\widetilde{\xi_k}} \leqslant e^{\frac{h_k t^2}{2}}$ *in Petrov's theorem* [15, pp. 54–55] *for each* $0 \leqslant t \leqslant T$ *and* $1 \leqslant k \leqslant d$.

Lemma 4. *In the case of* $\ln n \leqslant d < \frac{n}{m}$ *the following upper bound is correct*

$$H \leqslant \frac{mn}{d}$$

for sums of constants $h_k = \frac{1}{(k+1)^2}$, *which correspond to the added edges in the constructed trees.*

Lemma 5. *For the case of* $\ln n \leqslant d < \frac{n}{\ln n}$ *the next inequality holds*

$$\mathbb{E}W'_{\mathcal{A}} \leqslant \frac{3mn}{d} = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Lemma 6. *For* $\frac{n}{\ln n} \leqslant d < \frac{n}{m}$ *the following inequality is true:*

$$\mathbb{E}W'_{\mathcal{A}} \leqslant 3m \ln n = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

With help of previous lemmas, it is possible to prove the main result of this section.

Theorem 2. *Let parameter* $d = d_n$ *is defined as* $\ln n \leqslant d < \frac{n}{m}$. *Then algorithm* $\mathcal{A}$ *for the* m-d-UMST *with weights of edges from* UNI$(a_n; b_n)$ *is asymptotically optimal with failure probability* $\delta_n = n^{-m} \to 0$ *as* $n \to \infty$ *and the next conditions on scatter of weights of edges of graph* G

$$\frac{b_n}{a_n} = \begin{cases} o(d), & \textit{if} \quad \ln n \leqslant d < \dfrac{n}{\ln n}, \\ o\left(\dfrac{n}{\ln n}\right), & \textit{if} \quad \dfrac{n}{\ln n} \leqslant d < \dfrac{n}{m} \textit{ and } m < \ln n. \end{cases} \tag{5}$$

3.2. Probability Distribution EXP(a_n, λ_n)

We pass from random variables η, η_k to normalized random variables $\xi = \frac{\eta - a_n}{\lambda_n}$, $\xi_k = \frac{\eta_k - a_n}{\lambda_n}$ respectively. In the terms of this variables, p.d.f. is $\mathfrak{P}_\xi(x) = 1 - e^{-x}$ and probability density function is as follows (1)

$$p(\xi) = \begin{cases} e^{-\xi}, & \text{if } 0 \leqslant \xi < \infty, \\ 0 & \text{otherwise} \end{cases}$$

for random variable ξ. For random variable ξ_k, p.d.f. has the following form

$$\mathfrak{P}_{\xi_k}(x) = 1 - (1 - \mathfrak{P}_\xi(x))^k. \tag{6}$$

Lemma 7. *Mathematical expectation of random variable* ξ_k *equals* $\mathbb{E}\xi_k = 1/k$.

Lemma 8. *In the case of* $\mathrm{EXP}(a_n, \lambda_n)$*, the next upper bound is valid for expected value of a solution of algorithm* $\mathcal{A}$*:*

$$\mathbb{E}W'_{\mathcal{A}} \leqslant m \ln d + \frac{2mn}{d-1} = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Lemma 9. *Let* $T = \frac{1}{2}$*,* $h_k = \frac{3}{k^2}$*. Then for all* $1 \leqslant k \leqslant d$ *and* $0 \leqslant t \leqslant T$*, the conditions of Petrov's theorem* [15, pp. 54–55] *are true* $\mathbb{E}e^{t\widetilde{\xi}_k} \leqslant e^{\frac{h_k t^2}{2}}$ *for biased random variables* $\widetilde{\xi}_k = \xi_k - \mathbb{E}\xi_k$*.*

Lemma 10. *Let* $\ln n \leqslant d < \frac{n}{m}$*. Then for sufficiently large* n *the following upper bound is correct for the sums of constants* $h_k = \frac{3}{k^2}$*, which correspond to added edges in constructed trees*

$$H \leqslant \frac{3mn}{d-1}.$$

Lemma 11. *In the case of* $\ln n \leqslant d < \frac{n}{\ln n}$ *the next upper bound is valid*

$$\mathbb{E}W'_{\mathcal{A}} \leqslant \frac{3mn}{d-1} = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Lemma 12. *For* $\frac{n}{\ln n} \leqslant d < \frac{n}{m}$ *the inequality holds:*

$$\mathbb{E}W'_{\mathcal{A}} \leqslant 5m \ln n = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Using previous lemmas we can postulate the following theorem.

Theorem 3. *Let parameter* $d = d_n$ *be such that* $\ln n \leqslant d < \frac{n}{m}$*. Then algorithm* $\mathcal{A}$ *for solving the* m-d-UMST *with edge weights from* $\mathrm{EXP}(a_n, \lambda_n)$ *is asymptotically optimal with failure probability* $\delta_n = n^{-m} \to 0$ *as* $n \to \infty$ *under the conditions on scatter of weights of edges of graph* G*:*

$$\frac{\lambda_n}{a_n} = \begin{cases} o(d), & \text{if} \quad \ln n \leqslant d < \dfrac{n}{\ln n}, \\ o\left(\dfrac{n}{\ln n}\right), & \text{if} \quad \dfrac{n}{\ln n} \leqslant d < \dfrac{n}{m} \text{ and } m < \ln n. \end{cases} \tag{7}$$

3.3. Probability Distribution NORM(a_n, σ_n)

For distribution $\mathrm{NORM}(a_n, \sigma_n)$, we introduce normalized random variables $\xi = \frac{\eta - a_n}{\sigma_n}$ and $\xi_k = \frac{\eta_k - a_n}{\sigma_n}$ for corresponding weights of edges of graph instead of η and η_k.
For random variable ξ, the following probability density function according to (1) and p.d.f. are true

$$p(\xi) = \begin{cases} \sqrt{\dfrac{2}{\pi}} \exp\left(-\dfrac{\xi^2}{2}\right), & \text{if } 0 \leqslant \xi < \infty, \\ 0 & \text{otherwise.} \end{cases} \qquad \mathfrak{F}(x) = \sqrt{\frac{2}{\pi}} \int\limits_0^x \exp\left(-\frac{u^2}{2}\right) du.$$

Definition 3. We say that p.d.f. $\mathfrak{F}_1(x)$ dominates p.d.f. $\mathfrak{F}_2(x)$, if $\mathfrak{F}_1(x) \geqslant \mathfrak{F}_2(x)$ for all x.

Statement 4. *The p.d.f.* $\mathfrak{F}(x)$ *of normal random variable with parameter* σ_n *dominates exponential p.d.f. with parameter* $\lambda_n = 2\sigma_n$*:*

$$\mathfrak{F}(x) \geqslant \mathfrak{P}(x/2) \ \ \forall x \geqslant 0. \tag{8}$$

Lemma 13 [16]. *Let* $\chi_1, \ldots, \chi_k$ *are independent identically distributed random variables with p.d.f.* $F(x)$, $\widehat{F}(x)$ *is p.d.f. of random variable* $\chi = \min\limits_{i=1,\ldots,k} \chi_i$. *Also* $\zeta_1, \ldots, \zeta_k$ *are independent identically distributed random variables with p.d.f.* $G(x)$, *analogically* $\widehat{G}(x)$ *is p.d.f. of random variable* $\zeta = \min\limits_{i=1,\ldots,k} \zeta_i$. *Then for all* x

$$F(x) \leqslant G(x) \Rightarrow \widehat{F}(x) \leqslant \widehat{G}(x).$$

Lemma 14 [16]. *Let* $P_\vartheta, P_\omega, P_\zeta, P_\chi$ *are p.d.f.'s of random variables* $\vartheta, \omega, \zeta, \chi$ *respectively, where* ϑ *and* ζ *are independent, also* ω *and* χ *are independent too. Then*

$$(\forall x\ P_\vartheta(x) \leqslant P_\omega(x)) \wedge (\forall y\ P_\zeta(y) \leqslant P_\chi(y)) \Rightarrow (\forall z\ P_{\vartheta+\zeta}(z) \leqslant P_{\omega+\chi}(z)).$$

Lemma 15 [16]. *Let p.d.f.'s* $F(x)$ *and* $P(x)$ *such that* $F(x) \geqslant P(x)$ *for all* x. *Then performance guarantees* $(\varepsilon_\mathcal{A}, \delta_\mathcal{A})$ *for algorithm* $\mathcal{A}$ *on inputs with p.d.f.* $F(x)$ *are the same as for inputs with p.d.f.* $P(x)$.

Let us put $F(x) = \mathfrak{F}(x)$ and $P(x) = \mathfrak{P}(x/2)$. From Statement 4 and Lemmas 13–15 the following theorem implies for biased truncated normal distribution.

Theorem 4. *Let parameter* $d = d_n$ *be defined as* $\ln n \leqslant d < \frac{n}{m}$. *Then algorithm* $\mathcal{A}$ *for the* m-d-UMST *in* n*-vertex complete undirected graph with edge weights from unbounded semiopen interval* $[a_n; \infty)$ *according to* $\mathrm{NORM}(a_n, \sigma_n)$ *asymptotically optimal with failure probability* $\delta_n = n^{-m} \to 0$ *as* $n \to \infty$ *and the next asymptotically optimal conditions:*

$$\frac{\sigma_n}{a_n} = \begin{cases} o(d), & \text{if } \ln n \leqslant d < \dfrac{n}{\ln n}, \\ o\left(\dfrac{n}{\ln n}\right), & \text{if } \dfrac{n}{\ln n} \leqslant d < \dfrac{n}{m} \text{ and } m < \ln n. \end{cases}$$

4. CONCLUSION

In this work, deterministic approximation algorithm, which solves the problem of finding several edge-disjoint spanning trees with given diameter in edge-weighted complete undirected graph, has been presented. This algorithm finds feasible solution in time $\mathcal{O}(n^2)$, where n is number of vertices in graph. The probabilistic analysis has been carried out for several probabilistic distributions of weights of edges of graph: uniform $\mathrm{UNI}(a_n; b_n)$, biased truncated exponential $\mathrm{EXP}(a_n, \lambda_n)$, and biased truncated normal $\mathrm{NORM}(a_n, \sigma_n)$. Sufficient conditions of asymptotic optimality for this algorithm have been obtained in the case of each considered distribution. It would be interesting to investigate this problem on inputs with discrete probabilistic distributions. Also it would be useful to consider the problem of finding several edge-disjoint spanning trees of maximum total weight with given or bounded diameter.

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APPENDIX

Proof of Statement 1. Each edge-disjoint construction consists of n vertices and $(n-1)$ edges, since, first of all, the $(d+1)$-vertex path is constructed on Step 1, and then all remaining vertices are connected to it without increasing diameter of spanning tree on Step 2 and 3. At the end, m such constructions are made presenting feasible solution for the m-d-UMST.

Proof of Statement 2. Preliminary Step 0 requires $\mathcal{O}(n)$ elementary operations.

On Step 1, each path is constructed in $\mathcal{O}(d^2)$ time, so this step is carried out completely with time complexity $\mathcal{O}(md^2)$ or $\mathcal{O}(nd)$ (because $m(d+1) \leqslant n$).

Each pair of paths (P_i, P_j), $1 \leqslant i < j \leqslant m$ is interconnected with $\mathcal{O}(d^2)$ actions on items 2.1–2.4. For all $\frac{m(m-1)}{2}$ pairs of paths, it is required $\mathcal{O}(m^2d^2)$ or $\mathcal{O}(n^2)$ elementary operations.

Items 2.5–2.6 are carried out with time complexity $\mathcal{O}(md)$.

Step 3 requires $\mathcal{O}(mdn)$ or $\mathcal{O}(n^2)$ time for connection of $|G(V')| < n$ vertices by shortest edges with inner vertices of path P_s in each spanning tree T_s, $1 \leqslant s \leqslant m$.

Thus, total time complexity of algorithm $\mathcal{A}$ is equal to $\mathcal{O}(n^2)$.

Proof of Lemma 1. Let us consider inequality (2) for performance guarantees of the quality of the algorithm in relation to the considered case of the minimum problem.

$$\begin{aligned}
\mathbb{P}\Big\{W_{\mathcal{A}} - OPT(I) > \varepsilon_n OPT(I)\Big\} &= \mathbb{P}\Big\{W_{\mathcal{A}} > (1+\varepsilon_n)OPT(I)\Big\} \\
&\leqslant \mathbb{P}\Big\{W_{\mathcal{A}} > (1+\varepsilon_n)m(n-1)a_n\Big\} \\
&= \mathbb{P}\Big\{m(n-1)a_n + \beta_n W'_{\mathcal{A}} > (1+\varepsilon_n)m(n-1)a_n\Big\} \\
&= \mathbb{P}\Big\{W'_{\mathcal{A}} - \mathbb{E}W'_{\mathcal{A}} > \frac{\varepsilon_n m(n-1)a_n}{\beta_n} - \mathbb{E}W'_{\mathcal{A}}\Big\} \\
&= \mathbb{P}\Big\{\widetilde{W'_{\mathcal{A}}} > \frac{\varepsilon_n m(n-1)a_n}{\beta_n} - \mathbb{E}W'_{\mathcal{A}}\Big\} \\
\leqslant \mathbb{P}\Big\{\widetilde{W'_{\mathcal{A}}} > \frac{\varepsilon_n m(n-1)a_n}{\beta_n} - \widehat{\mathbb{E}W'_{\mathcal{A}}}\Big\} &= \mathbb{P}\Big\{\widetilde{W'_{\mathcal{A}}} > \widehat{\mathbb{E}W'_{\mathcal{A}}}\Big\} = \delta_n,
\end{aligned}$$

the penultimate equality is true for $\varepsilon_n = \frac{2\beta_n \widehat{\mathbb{E}W'_{\mathcal{A}}}}{m(n-1)a_n}$.

Proof of Statement 3. It is easy to understand that

$$\sum_{k=1}^{d} \frac{1}{k} \leqslant 1 + \frac{1}{2} + \frac{1}{3} + \int_3^d \frac{dx}{x} = \frac{11}{6} + \ln d - \ln 3 \leqslant \ln d + \frac{3}{4}.$$

Proof of Lemma 2. It is easy to establish that $\mathbb{E}\xi_k = \frac{1}{k+1}$ for inputs UNI$(a_n; b_n)$. Let us estimate from above each of the mathematical expectations of random variables W'_1, W'_2, and W'_3.

$$\mathbb{E}W'_1 = \sum_{s=1}^{m}\sum_{k=1}^{d} \mathbb{E}\xi_k = m\sum_{k=1}^{d}\frac{1}{k+1} \leqslant m\ln d,$$

penultimate inequality is correct for $d \geqslant 3$ due to the Statement 3 and relation

$$\sum_{k=1}^{d}\frac{1}{k+1} = \sum_{k=1}^{d}\frac{1}{k} - 1 + \frac{1}{d+1} \leqslant \sum_{k=1}^{d}\frac{1}{k} - \frac{3}{4} \leqslant \ln d.$$

$$\mathbb{E}W'_2 = C_m^2\left(4\frac{d-1}{2}\mathbb{E}\xi_{(d-1)/2} + 4\mathbb{E}\xi_{(d-1)}\right) = \frac{m(m-1)}{2}\left(\frac{4(d-1)/2}{(d-1)/2+1} + \frac{4}{d}\right) \leqslant 2m^2;$$

$$\mathbb{E}W'_3 = m(n - m(d+1))\mathbb{E}\xi_{(d-1)} = m\frac{n-m(d+1)}{d} \leqslant \frac{mn}{d} - m^2.$$

Summing three inequalities and taking into account that $m(d+1) \leqslant n$, we obtain

$$\mathbb{E}W'_{\mathcal{A}} = \mathbb{E}\Big(W'_1 + W'_2 + W'_3\Big) \leqslant m\ln d + 2m^2 + \frac{mn}{d} - m^2 \leqslant m\ln d + \frac{2mn}{d}.$$

Proof of Lemma 3. Let's estimate $\mathbb{E}e^{t\xi_k}$ from above using formula

$$\mathbb{E}e^{t\xi_k} = \sum_{i=0}^{\infty} \frac{t^i}{(k+1)\cdots(k+i)}$$

from monograph [17, pp. 129]. Introducing also the notations $\alpha = \frac{t}{k+1}$ and

$$Q_{k,t} = \frac{(k+1)}{(k+2)(1-\frac{t}{k+3})} \leqslant Q_{k,T} = \frac{(k+1)(k+3)}{(k+2)^2} < 1$$

for all $t \leqslant T$ and for all natural k, we obtain

$$\mathbb{E}e^{t\xi_k} = \sum_{i=0}^{\infty} \frac{t^i}{(k+1)\cdots(k+i)} \leqslant 1+\alpha+\alpha^2 Q_{k,t} \leqslant 1+\alpha+\alpha^2 \leqslant e^{\alpha+\frac{\alpha^2}{2}} = e^{t\mathbb{E}\xi_k} e^{\frac{h_k t^2}{2}},$$

since $\mathbb{E}\xi_k = \frac{1}{k+1}$ for inputs $\mathrm{UNI}(a_n; b_n)$.
Consequently,

$$\mathbb{E}e^{t(\xi_k - \mathbb{E}\xi_k)} = \mathbb{E}e^{t\widetilde{\xi}_k} \leqslant e^{\frac{h_k t^2}{2}},$$

where $\widetilde{\xi}_k = \xi_k - \mathbb{E}\xi_k$.

Proof of Lemma 4. In the case of $\ln n \leqslant d < \frac{n}{m}$, parameter H is equal to sum of H_1, H_2, and H_3 according to steps of algorithm $\mathcal{A}$. Taking into account the notations and estimates obtained earlier, we arrive at the following:

$$H_1 = m\sum_{k=1}^{d} h_k = m\sum_{k=1}^{d} \frac{1}{(k+1)^2} < \psi m,$$

where $\psi \approx 0.645$. Here we use Euler estimation for the sum of inverse squares $1+\frac{1}{2^2}+\frac{1}{3^2}+\frac{1}{4^2}+\ldots = \frac{\pi^2}{6} < 1.645$.

$$H_2 = 4C_m^2\left(\frac{d-1}{2}h_{(d-1)/2} + h_{(d-1)}\right) \leqslant 2m^2\left(\frac{(d-1)/2}{((d-1)/2+1)^2} + \frac{1}{d^2}\right)$$
$$= 2m^2\left(\frac{2(d-1)}{(d+1)^2} + \frac{1}{d^2}\right) \leqslant 4m^2\frac{d}{(d+1)^2}.$$

The last inequality holds for $d \geqslant 3$.

$$H_3 = m(n - m(d+1))h_{(d-1)} \leqslant \frac{mn}{d^2} - m^2\frac{d}{(d+1)^2}.$$

Since $n \geqslant m(d+1)$ and $m \geqslant 2$, we get

$$H = H_1+H_2+H_3 < \psi m + 4m^2\frac{d}{(d+1)^2} + \left(\frac{mn}{d^2} - m^2\frac{d}{(d+1)^2}\right)$$
$$\leqslant \frac{mn}{d}\left(\frac{d\psi}{n} + \frac{1}{d}\right) + 3m^2\frac{d}{(d+1)^2} \leqslant \left(\frac{\psi d}{2(d+1)} + \frac{1}{d} + \frac{3d^2}{(d+1)^3}\right)\frac{mn}{d}.$$

It is easy to verify that the expression in parentheses is less than 1 for all $d \geqslant 3$. Then we can obtain the next estimation $H \leqslant \frac{mn}{d}$.

Proof of Lemma 5. Taking into account that $\ln d \leqslant \ln n$ and $d < \frac{n}{\ln n}$, it is true that

$$\mathbb{E}W'_{\mathcal{A}} \leqslant m\ln d + \frac{2mn}{d} \leqslant m\ln n + \frac{2mn}{d} < m\frac{n}{d} + \frac{2mn}{d} = \frac{3mn}{d} = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Proof of Lemma 6. Because of Lemma 2, $\ln d \leqslant \ln n$, and $\frac{n}{d} \leqslant \ln n$, we get

$$\mathbb{E}W'_{\mathcal{A}} \leqslant m \ln d + \frac{2mn}{d} \leqslant 3m \ln n = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Proof of Theorem 2. First of all, it must be noted that in the course of the Algorithm $\mathcal{A}$ we have deal with random variables of the type ξ_k, $1 \leqslant k \leqslant d$. In the case of graphs with weights of edges from $\mathrm{UNI}(a_n; b_n)$ these biased variables satisfy the conditions $\mathbb{E}e^{t\widetilde{\xi}_k} \leqslant e^{\frac{h_k t^2}{2}}$ of Petrov's theorem [15, pp. 54–55] for constants $T = 1$ and $h_k = \frac{1}{(k+1)^2}$ (see Lemma 3).

We will carry out the proof of the theorem for two cases of possible semiopen intervals of parameter d.

$$\textit{Case 1:} \quad \ln n \leqslant d < \frac{n}{\ln n}.$$

According to Lemma 5 and formula (3) for relative error, we obtain

$$\varepsilon_n = \frac{2(b_n - a_n)}{m(n-1)a_n} \widehat{\mathbb{E}W'_{\mathcal{A}}} = \frac{2(b_n - a_n)}{m(n-1)a_n} \frac{3mn}{d} \leqslant \frac{6n}{(n-1)} \frac{b_n/a_n}{d}.$$

We can see that $\varepsilon_n \to 0$ as $n \to \infty$, if the following conditions are satisfied on scatter of weights of edges of graph G: $\frac{b_n}{a_n} = o(d_n)$.

Using Lemmas 1 and 5, we can estimate failure of probability:

$$\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > \widehat{\mathbb{E}W'_{\mathcal{A}}}\} = \mathbb{P}\Big\{\widetilde{W'_{\mathcal{A}}} > \frac{3mn}{d}\Big\}.$$

From Lemma 4 and inequality $d < \frac{n}{\ln n}$ it follows that $TH \leqslant \frac{mn}{d} < \frac{3mn}{d} = x$. According to Petrov's theorem [15, pp. 54–55], we get the next estimate of failure probability of algorithm $\mathcal{A}$: $\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > x\} \leqslant \exp\left\{-\frac{Tx}{2}\right\}$.

Since $\ln n < \frac{n}{d}$ and $\frac{Tx}{2} = \frac{3mn}{2d} > m \ln n$, then

$$\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > x\} \leqslant \exp\left\{-\frac{Tx}{2}\right\} < \exp(-m \ln n) = \frac{1}{n^m} \to 0 \text{ as } n \to \infty.$$

Therefore, in Case 1 Algorithm $\mathcal{A}$ gives asymptotically optimal solution for the m-d-UMST in graph with weights of edges from $\mathrm{UNI}(a_n; b_n)$.

$$\textit{Case 2:} \quad \frac{n}{\ln n} \leqslant d < \frac{n}{m}.$$

According to Lemma 6 and formula (3) we get the following equation for relative error:

$$\varepsilon_n = \frac{2(b_n - a_n)}{(n-1)a_n} \widehat{\mathbb{E}W'_{\mathcal{A}}} = \frac{2(b_n - a_n)}{m(n-1)a_n} 3m \ln n \leqslant \frac{6(b_n/a_n) \ln n}{(n-1)}.$$

It is clear that $\varepsilon_n \to 0$ as $n \to \infty$, if $\frac{b_n}{a_n} = o(\frac{n}{\ln n})$.

Now using Lemmas 1 and 6, we can estimate failure probability $\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > \widehat{\mathbb{E}W'_{\mathcal{A}}}\} = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > 3m \ln n\}$. If $T = 1$, $d \geqslant \frac{n}{\ln n}$ and bearing in mind Lemma 4, then the next inequality is valid: $TH \leqslant \frac{mn}{d} < 3m \ln n = x$. Since $\frac{Tx}{2} > m \ln n$ and Petrov's theorem [15, pp. 54–55] we obtain the following estimate for failure probability of algorithm $\mathcal{A}$: $\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > x\} \leqslant \exp\{-\frac{Tx}{2}\} \leqslant \exp(-m \ln n) = \frac{1}{n^m} \to 0$.

From this it follows that in the Case 2 Algorithm $\mathcal{A}$ gives asymptotically optimal solution for the problem m-d-UMST on n-vertex complete undirected graph with weights of edges from UNI$(a_n; b_n)$.

We conclude, that within the values of the parameter d for both cases, under conditions (5) we have that estimates of the relative error $\varepsilon_n \to 0$ and failure probability $\delta_n \to 0$ as $n \to \infty$.

Proof of Lemma 7. With reference to (6) we obtain

$$\mathbb{E}\xi_k = \int_0^\infty x d\mathfrak{P}_{\xi_k}(x) = \int_0^\infty xk(1-\mathfrak{P}_\xi(x))^{k-1} d\mathfrak{P}_\xi(x) = \int_0^\infty xke^{-kx}dx$$

$$= -x\, e^{-kx}\Big|_0^\infty + \int_0^\infty e^{-kx}dx = -\frac{1}{k}\, e^{-kx}\Big|_0^\infty = \frac{1}{k}.$$

Proof of Lemma 8. Let us estimate each expected value for random variables W_1', W_2', and W_3':

$$\mathbb{E}W_1' = \sum_{s=1}^m \sum_{k=1}^d \mathbb{E}\xi_k = m\sum_{k=1}^d \frac{1}{k} \leqslant m\left(\ln d + \frac{3}{4}\right)$$

taking into account Statement 3 and Lemma 7;

$$\mathbb{E}W_2' = C_m^2\left(4\frac{d-1}{2}\mathbb{E}\xi_{(d-1)/2} + 4\mathbb{E}\xi_{d-1}\right) = 2m(m-1)\left(1+\frac{1}{d-1}\right) \leqslant \frac{2d}{d-1}m^2 - 2m;$$

$$\mathbb{E}W_3' = m(n-m(d+1))\mathbb{E}\xi_{(d-1)} = m\frac{n-m(d+1)}{d-1} = \frac{mn}{d-1} - \frac{d+1}{d-1}m^2.$$

Adding the left and right parts of three ratios for $\mathbb{E}W_1'$, $\mathbb{E}W_2'$, $\mathbb{E}W_3'$ and bearing in mind that $m \leqslant \frac{n}{d+1}$, we get

$$\mathbb{E}W_{\mathcal{A}}' \leqslant m\ln d - \frac{5}{4}m + \frac{mn}{d-1} + m^2 \leqslant m\ln d + \frac{mn}{d-1} + \frac{mn}{d+1} \leqslant m\ln d + \frac{2mn}{d-1}.$$

Proof of Lemma 9. The following is true for quantities $\mathbb{E}e^{t\xi_k}$ according to formula (6).

$$\mathbb{E}e^{t\xi_k} = \int_0^\infty e^{tx} d\mathfrak{P}_{\xi_k}(x) = \int_0^\infty e^{tx}ke^{-kx}dx = \int_0^\infty ke^{-(k-t)x}dx$$

$$= -\frac{k}{k-t}e^{-(k-t)x}\Big|_0^\infty = \frac{1}{1-t/k} = \sum_{s=0}^\infty \left(\frac{t}{k}\right)^s \leqslant 1 + \frac{t}{k} + \left(\frac{t}{k}\right)^2 \frac{1}{1-t/k}.$$

Taking into account the inequality $\frac{t}{k} \leqslant \frac{1}{2}$, which is true under the conditions of the lemma, we estimate the value $\mathbb{E}e^{t\xi_k}$ from above:

$$\mathbb{E}e^{t\xi_k} \leqslant 1 + \frac{t}{k} + 2\left(\frac{t}{k}\right)^2 = 1 + \frac{t}{k} + \frac{1}{2}\left(\frac{t}{k}\right)^2 + \frac{3}{2}\left(\frac{t}{k}\right)^2$$

$$\leqslant \left(1 + \frac{t}{k} + \frac{1}{2}\left(\frac{t}{k}\right)^2\right)\left(1 + \frac{3}{2}\left(\frac{t}{k}\right)^2\right) \leqslant e^{t/k}\exp\left(\frac{3}{2}\left(\frac{t}{k}\right)^2\right) = e^{t\mathbb{E}\xi_k}\exp\left(\frac{h_k t^2}{2}\right),$$

because of Lemma 7 $\mathbb{E}\xi_k = \frac{1}{k}$ for EXP(a_n, λ_n). Consequently the conditions $\mathbb{E}e^{t\widetilde{\xi_k}} \leqslant e^{\frac{h_k t^2}{2}}$ of Petrov's theorem are true for constants $T = 1/2$, $h_k = 3/k^2$.

Proof of Lemma 10. For $\ln n \leqslant d < \frac{n}{m}$ parameter H is equal to sum of quantities H_1, H_2, and H_3 according to steps of algorithm $\mathcal{A}$. Taking into account previous notation and obtained estimations, we have

$$H_1 = m\sum_{k=1}^{d} h_k = m\sum_{k=1}^{d}\frac{3}{k^2} < 3(1+\psi)m < 5m,$$

where ψ equals Euler estimation for the sum of inverse squares minus 1 ($\psi \approx 0.645$).

$$H_2 = 4C_m^2\left(\frac{d-1}{2}h_{(d-1)/2} + h_{(d-1)}\right) = 6m(m-1)\left(\frac{2}{d-1} + \frac{1}{(d-1)^2}\right) \leqslant 6m^2\frac{2d-1}{(d-1)^2}.$$

$$H_3 = m(n - m(d+1))h_{(d-1)} = \frac{3}{(d-1)^2}\Big(mn - m^2(d+1)\Big) \leqslant \frac{3mn}{(d-1)^2} - 3m^2\frac{d+1}{(d-1)^2}.$$

With $n \geqslant m(d+1)$ and $m \geqslant 2$ we get

$$\begin{aligned} H = H_1 + H_2 + H_3 &< 5m + \frac{3m^2}{(d-1)^2}\Big((4d-2) - (d+1)\Big) + \frac{3mn}{(d-1)^2} \\ &= 3m\left(\frac{5}{3} + \frac{3m}{d-1}\right) + \frac{3mn}{(d-1)^2} \\ &\leqslant \frac{3n}{d+1}\left(\frac{5}{3} + \frac{3m}{d-1}\right) + \frac{3mn}{(d-1)^2} \leqslant \frac{3n}{d+1}\left(\frac{5m}{6} + \frac{3m}{d-1}\right) + \frac{3mn}{(d-1)^2} \\ &= \frac{3mn}{d-1}\left(\frac{5(d-1)}{6(d+1)} + \frac{3}{d+1} + \frac{1}{d-1}\right) \leqslant \frac{3mn}{d-1}. \end{aligned}$$

The last sign of inequality is due to the fact that, when n is sufficiently large, the value in parentheses is less than 1 since $d \geqslant \ln n$.

Proof of Lemma 11. Taking into account $\ln d \leqslant \ln n$, $d < \frac{n}{\ln n}$ and Lemma 8 we obtain

$$\mathbb{E}W'_{\mathcal{A}} \leqslant m\ln d + \frac{2mn}{d-1} \leqslant m\frac{n}{d} + \frac{2mn}{d-1} \leqslant \frac{3mn}{d-1} = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Proof of Lemma 12. According to Lemma 8 and inequality $\ln d \leqslant \ln n$ and $n \leqslant d\ln n$ we get

$$\mathbb{E}W'_{\mathcal{A}} \leqslant m\ln d + \frac{2mn}{d-1} \leqslant m\ln n + \frac{2md}{d-1}\ln n \leqslant 5m\ln n = \widehat{\mathbb{E}W'_{\mathcal{A}}}.$$

Proof of Theorem 3. First of all, let us note that random variables $\widetilde{\xi}_k = \xi_k - \mathbb{E}\xi_k$ satisfy conditions $\mathbb{E}e^{t\widetilde{\xi}_k} \leqslant e^{\frac{h_k t^2}{2}}$ of Petrov's theorem for constants $T = 1/2$ and $h_k = \frac{3}{k^2}$ (see Lemma 9).

Let's carry out the proof of the theorem for two cases of possible semiopen intervals of the value of the parameter d.

$$\textit{Case 1:}\quad \ln n \leqslant d < \frac{n}{\ln n}.$$

Bearing in mind Lemma 11 and formula (3) for relative error we obtain

$$\varepsilon_n = \frac{2\lambda_n}{m(n-1)a_n}\frac{3mn}{(d-1)} \leqslant \frac{6n}{(n-1)}\frac{\lambda_n/a_n}{(d-1)}.$$

So we can see that $n \to \infty$ as $\varepsilon_n \to 0$, if $\frac{\lambda_n}{a_n} = o(d_n)$.

Now using Lemmas 1 and 11, we can estimate failure probability:

$$\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > \widehat{\mathbb{E}W'_{\mathcal{A}}}\} = \mathbb{P}\Big\{\widetilde{W'_{\mathcal{A}}} > \frac{3mn}{d-1}\Big\} = \mathbb{P}\Big\{\widetilde{W'_{\mathcal{A}}} > \frac{3mn}{d-1}\Big\}.$$

For each edge with weight, which corresponds to random variable ξ_k, we define constants $T = 1/2$ and $h_k = \frac{3}{k^2}$.

From Lemma 10, it implies that $TH \leqslant \frac{3mn}{2(d-1)} < \frac{3mn}{d-1} = x$.

According to Petrov's theorem we get the next estimate for failure probability of algorithm $\mathcal{A}$:

$$\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > x\} \leqslant \exp\left\{-\frac{Tx}{2}\right\}.$$

Since $\frac{n}{d} > \ln n$, then $\frac{Tx}{2} = \frac{3mn}{2(d-1)} > m \ln n$. So we get that

$$\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > x\} \leqslant \exp\left\{-\frac{Tx}{2}\right\} < \exp(-m \ln n) = \frac{1}{n^m} \to 0 \text{ as } n \to \infty.$$

Thus, in Case 1, Algorithm $\mathcal{A}$ gives asymptotically optimal solution for the m-d-UMST in n-vertex complete undirected graph with weights of edges from $\mathrm{EXP}(a_n, \lambda_n)$.

Case 2: $\dfrac{n}{\ln n} \leqslant d < \dfrac{n}{m}.$

Knowing Lemma 12 and formula (3) for relative error ε_n we get

$$\varepsilon_n = \frac{2\lambda_n}{(n-1)a_n} \widehat{\mathbb{E}W'_{\mathcal{A}}} = \frac{2\lambda_n}{m(n-1)a_n} 5m \ln n \leqslant \frac{10(\lambda_n/a_n)\ln n}{n-1}.$$

It is clear that $\varepsilon_n \to 0$ as $n \to \infty$, if the conditions $\frac{\lambda_n}{a_n} = o\Big(\frac{n}{\ln n}\Big)$ are satisfied.

So using Lemmas 1 and 12, we can estimate failure probability

$$\delta_n = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > \widehat{\mathbb{E}W'_{\mathcal{A}}}\} = \mathbb{P}\{\widetilde{W'_{\mathcal{A}}} > 5m \ln n\}.$$

Putting constants h_k as in the Case 1, we set $T = 1/2$ and $x = 5m \ln n$.

Taking into account Lemma 10, quantities x, T, H, and $d \geqslant \frac{n}{\ln n}$, we arrive at the following inequality $TH \leqslant \frac{3mn}{2d} < 5m \ln n = x$.

Since $\frac{Tx}{2} > m \ln n$, according to Petrov's theorem we obtain the next estimate for failure probability of Algorithm $\mathcal{A}$:

$$\delta_n = \mathbb{P}\left\{\widetilde{W'_{\mathcal{A}}} > x\right\} \leqslant \exp\left\{-\frac{Tx}{2}\right\} \leqslant \exp(-m \ln n) = \frac{1}{n^m} \to 0 \text{ as } n \to \infty.$$

Consequently, in Case 2, Algorithm $\mathcal{A}$ gives asymptotically optimal solution for the m-d-UMST in n-vertex complete undirected graph with weights of edges from $\mathrm{EXP}(a_n, \lambda_n)$.

Therefore, for values of parameter d we have estimate of relative error $\varepsilon_n \to 0$ and failure probability $\delta_n \to 0$ as $n \to \infty$ in both cases under conditions (7).

Proof of Statement 4. We present the proof of this statement, as in [18]. The difference of left and right sides of inequality (8) is denoted as

$$h(x) = \sqrt{\frac{2}{\pi}} \int_0^x e^{-\frac{u^2}{2}} du - (1 - e^{-\frac{x}{2}}).$$

It is easy to check that for function $h(x)$ and its derivative

$$h'(x) = \sqrt{\frac{2}{\pi}}e^{-\frac{x^2}{2}} - \frac{1}{2}e^{-\frac{x}{2}}$$

the following is correct $h(0) = 0$, $\lim_{x\to\infty} h(x) = 0$, $h'(x) > 0$.
Since on positive positive semiaxis, the inequality $h'(x) = 0$ holds only in one unique point $x_0 = \frac{1}{2}(1 + \sqrt{1 + 12\ln(2) - 4\ln(\pi)} \geqslant 0$, we can conclude that $h(x) \geqslant 0$ as $x \geqslant 0$, which implies the validity of the statement.

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